# Unscented Kalman Filter

**A cheat sheet**

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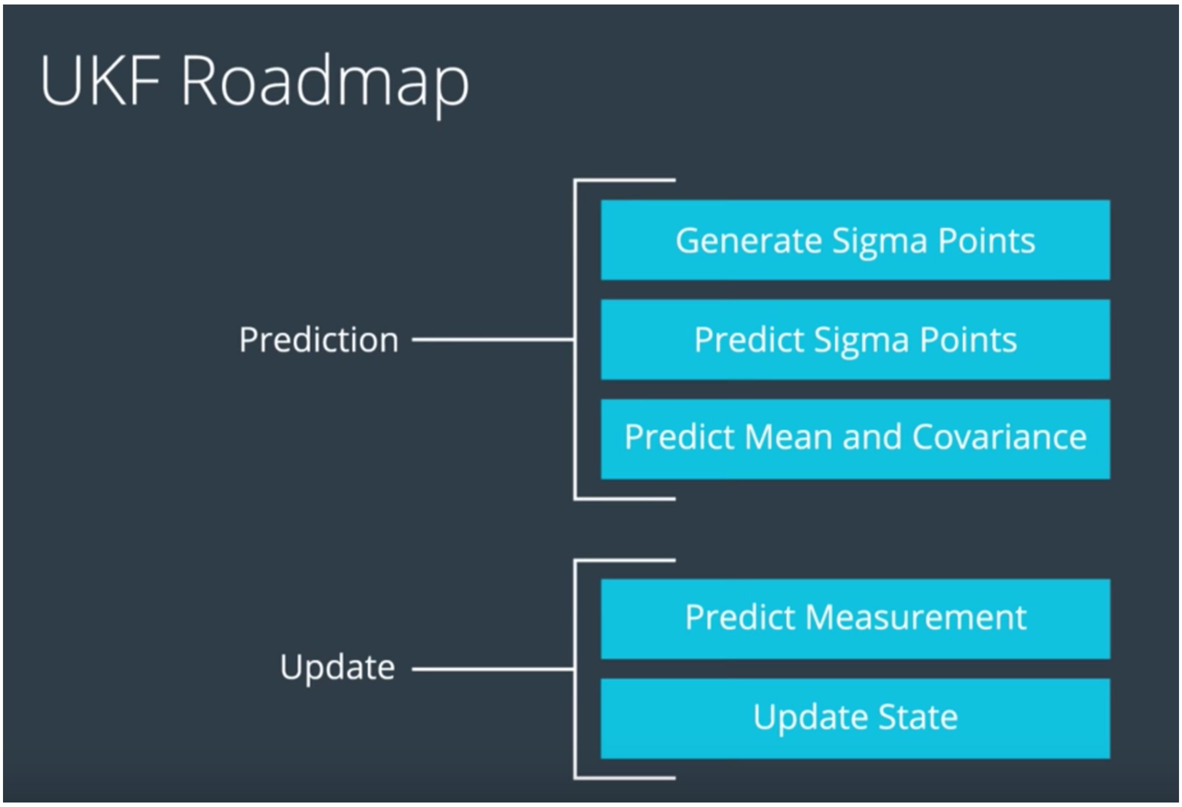
## Roadmap




### Rubric

https://review.udacity.com/#!/rubrics/783/view

## CTRV Model

### Helpful equations

$$x = \begin{bmatrix} p_x \\ p_y \\ v \\ \psi \\ \dot{\psi} \end{bmatrix}$$

If $\dot{\psi}_k$ is not zero

$$x_{k+1} = x_k + \begin{bmatrix} \frac{v_k}{\dot{\psi}_k}(sin(\psi_k + \dot{\psi}_k \Delta t) - sin(\psi_k)) \\ \frac{v_k}{\dot{\psi}_k}(-cos(\psi + \dot{\psi}_k \Delta t) + cos(\psi_k)) \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 cos(\psi_k)\nu_{a,k} \\ \frac{1}{2}(\Delta t)^2 sin(\psi_k)\nu_{a,k} \\ \Delta t \nu_{a,k} \\ \frac{1}{2}(\Delta t)^2 \nu_{\ddot{\psi},k} \\ \Delta t \nu_{\ddot{\psi},k} \end{bmatrix}$$

If $\dot{\psi}_k$ is zero

$$x_{k+1} = x_k + \begin{bmatrix} v_k cos(\psi_k)\Delta t \\ v_k sin(\psi_k)\Delta t \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 cos(\psi_k)\nu_{a,k} \\ \frac{1}{2}(\Delta t)^2 sin(\psi_k)\nu_{a,k} \\ \Delta t \nu_{a,k} \\ \frac{1}{2}(\Delta t)^2 \nu_{\ddot{\psi},k} \\ \Delta t \nu_{\ddot{\psi},k} \end{bmatrix}$$

## Generating Sigma Points

With state dimension as $n_x$[1]

$$\underbrace{\mathcal{X}_{k|k}}_{n_x \times (2n_x+1) \text{ matrix}} = [\quad \underbrace{x_{k|k}}_{1 \text{ column}} \quad \underbrace{x_{k|k} + \sqrt{(\lambda + n_x)P_{k|k}}}_{n_x \text{ columns}} \quad \underbrace{x_{k|k} - \sqrt{(\lambda + n_x)P_{k|k}}}_{n_x \text{ columns}} \quad ]$$

Remember that $x_{k|k}$ is the first column of the Sigma matrix
$x_{k|k} + \sqrt{(\lambda + n_x)P_{k|k}}$ are the second through $n_x + 1$ columns
$x_{k|k} - \sqrt{(\lambda + n_x)P_{k|k}}$ are the $n_x + 2$ through $2n_x + 1$ columns.

## UKF Augmentation

The process model $f$ is non-linear. We derive a new state from the previous state: $x_{k+1} = f(x_k, \nu_k)$, where $\nu_k$ is the noise vector. Note that $\nu_k$ is not additive and non-linearly affects the next state. Therefore we augment the state vector with the noise vector before feeding it into the noise model.

[1] $n_x = 5$ in the lesson

$$\text{Augmented State} = x_{a,k} = \begin{bmatrix} p_x \\ p_y \\ v \\ \psi \\ \dot{\psi} \\ \nu_a \\ \nu_{\ddot{\psi}} \end{bmatrix}$$

$$\text{Augmented Covariance Matrix} = P_{a,k|k} = \begin{bmatrix} P_{k|k} & 0 \\ 0 & Q \end{bmatrix}$$

### Augmented sigma points

With augmented state dimension as $n_a$[2]

$$\underbrace{\mathcal{X}_{a,k|k}}_{n_a \times (2n_a+1) \text{ matrix}} = [ \quad \underbrace{x_{a,k|k}}_{1 \text{ column}} \quad \underbrace{x_{a,k|k} + \sqrt{(\lambda + n_a)P_{a,k|k}}}_{n_a \text{ columns}} \quad \underbrace{x_{a,k|k} - \sqrt{(\lambda + n_a)P_{a,k|k}}}_{n_a \text{ columns}} \quad ]$$

## Sigma point prediction

Given the augmented sigma points $\mathcal{X}_{a,k|k}$ and the CTRV model $f$, we get a prediction for the sigma points of the next state:

$$\underbrace{\mathcal{X}_{k+1|k}}_{n_x \times n_\sigma} = f( \underbrace{\mathcal{X}_{a,k|k}}_{n_a \times n_\sigma} )$$

where $n_\sigma = 2n_a + 1$

## Predict Mean and Covariance

### Weights

$$w_i = \frac{\lambda}{\lambda + n_a}, \quad i = 1$$
$$w_i = \frac{1}{2(\lambda + n_a)}, \quad i = 2 \dots n_\sigma$$

### Predicted Mean

$$x_{k+1|k} = \sum_{i=0}^{n_\sigma} w_i \mathcal{X}_{k+1|k,i}$$

[2] $n_a = 7$ in the lesson

### Predicted Covariance

$$P_{k+1|k} = \sum_{i=0}^{n_\sigma} w_i (\mathcal{X}_{k+1|k,i} - x_{k+1|k})(\mathcal{X}_{k+1|k,i} - x_{k+1|k})^T$$

## Predict Radar Measurements

### State vector

$$x_{k+1|k} = \begin{bmatrix} p_x \\ p_y \\ v \\ \psi \\ \dot{\psi} \end{bmatrix}$$

### Measurement vector

$$z_{k+1|k} = \begin{bmatrix} \rho \\ \varphi \\ \dot{\rho} \end{bmatrix}$$

### Measurement model

For individual point

$$z_{k+1|k} = h(x_{k+1}) + \omega_{k+1}$$

For a set of points

$$\underbrace{\mathcal{Z}_{k+1|k}}_{n_z \times n_\sigma} = h(\underbrace{\mathcal{X}_{k+1|k}}_{n_x \times n_\sigma}) + \omega_{k+1}$$

where $n_z$ is the dimension of the measurement space and

$$\rho = \sqrt{p_x^2 + p_y^2}$$

$$\varphi = arctan(\frac{p_y}{p_x})$$

$$\dot{\rho} = \frac{p_x cos(\psi) v + p_y sin(\psi) v}{\sqrt{p_x^2 + p_y^2}}$$

### Predicted Measurement Mean

$$z_{k+1|k} = \sum_{i=1}^{n_\sigma} w_i \mathcal{Z}_{k+1|k,i}$$

### Predicted Covariance

$$S_{k+1|k} = \sum_{i=1}^{n_\sigma} w_i(\mathcal{Z}_{k+1|k,i} - z_{k+1|k})(\mathcal{Z}_{k+1|k,i} - z_{k+1|k})^T + R$$

$$R = E\{\omega_k \cdot \omega_k^T\} = \begin{bmatrix} \sigma_\rho^2 & 0 & 0 \\ 0 & \sigma_\varphi^2 & 0 \\ 0 & 0 & \sigma_{\dot{\rho}}^2 \end{bmatrix}$$

## UKF Update

### Cross-correlation Matrix

$$T_{k+1|k} = \sum_{i=1}^{n_\sigma} w_i(\mathcal{X}_{k+1|k,i} - x_{k+1|k})(\mathcal{Z}_{k+1|k,i} - z_{k+1|k})^T$$

### Kalman Gain

$$K_{k+1|k} = T_{k+1|k} S_{k+1|k}^{-1}$$

### State Update

Here $z_{k+1|k}$ is the predicted measurement and $z_{k+1}$ is the actual measurement.

$$x_{k+1} = x_{k+1|k} + K_{k+1|k}(z_{k+1} - z_{k+1|k})$$

### Covariance Matrix Update

$$P_{k+1} = P_{k+1|k} - K_{k+1|k} S_{k+1|k} K_{k+1|k}^T$$

## Parameter Consistency

| Process model | Process noise | Process noise covariance |
|---|---|---|
| $x_{k+1} = f(x_k, \nu_k)$ | $\nu_k = \begin{bmatrix} \nu_{a,k} \\ \nu_{\ddot{\psi},k} \end{bmatrix}$ | $Q = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_{\ddot{\psi}^2} \end{bmatrix}$ |

| Measurement model | Radar measurement noise | Measurement noise covariance |
|---|---|---|
| $z_{k+1} = h(x_{k+1}) + \omega_{k+1}$ | $\omega_k = \begin{bmatrix} \omega_{\rho,k} \\ \omega_{\varphi,k} \\ \omega_{\dot{\rho},k} \end{bmatrix}$ | $R = \begin{bmatrix} \sigma_\rho^2 & 0 & 0 \\ 0 & \sigma_\varphi^2 & 0 \\ 0 & 0 & \sigma_{\dot{rho}}^2 \end{bmatrix}$ |

## Normalized Innovation Squared (NIS)

$$\epsilon = (z_{k+1} - z_{k+1|k})^T \cdot S_{k+1|k}^{-1} \cdot (z_{k+1} - z_{k+1|k})$$

Here $\epsilon \sim \chi^2$. The following table shows expected values for $\epsilon$ for various degrees of freedom.

| df | $\chi^2.950$ | $\chi^2.900$ | $\chi^2.100$ | $\chi^2.050$ |
|---|---|---|---|---|
| 1 | 0.004 | 0.016 | 2.706 | 3.841 |
| 2 | 0.103 | 0.211 | 4.605 | 5.991 |
| 3 | 0.352 | 0.584 | 6.251 | 7.815 |
| 4 | 0.711 | 1.064 | 7.779 | 9.488 |
| 5 | 1.145 | 1.610 | 9.236 | 11.070 |