# Multiple regression and causal DAGs

ISTA 410 / INFO 510 - Bayesian Modeling and Inference

University of Arizona School of Information

September 13, 2021

# Outline

Last week:

- Model specification in PyMC3
- Inference with quadratic approximation via `quap`
- Posterior predictive sampling
- Linear regression

Today:

- Multiple regression
- Causal DAGs

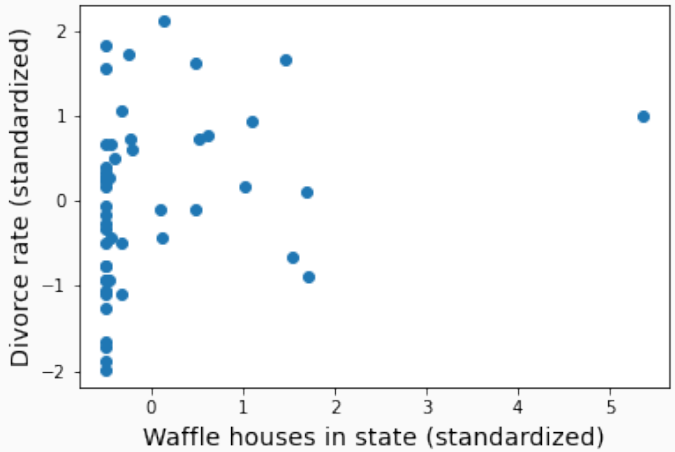
Starting chapter 5 of *Rethinking*

# Goals for this week

- Introduce multiple regression and explore its properties
  - Used correctly, these models can uncover masked associations or eliminate spurious correlations
  - Used carelessly, they can introduce spurious associations and lead to confusion
- Basics of causal inference
  - Directed acyclic graphs (DAGs)
  - DAG structures and causal relationships
  - Criteria for including/excluding variables

# Waffle House and divorce

Presence of Waffle Houses statistically correlated with divorce


- Are Waffle Houses dens of iniquity?

# It's not the waffles

It seems most plausible that:

- Waffle Houses don't cause divorce
- Some other property influences the presence of WH and also the rate of divorce

# It's not the waffles

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- Waffle Houses don't cause divorce
- Some other property influences the presence of WH and also the rate of divorce

- The US South

Anything idiosyncratic to the South will be associated with Waffle House.

# Correlation is not causation

- Truism from intro stats: correlation does not imply causation
- Causation does not imply correlation
- Causation implies *conditional correlation*
- Example (due to Rachael Meager):
  - Pressure on the gas pedal has a causal effect on your car's speed
  - If you go from a flat to a steep hill, you might increase pressure to keep speed constant

# Other predictors

What other features of the South might be responsible for the divorce rate?

# Other predictors

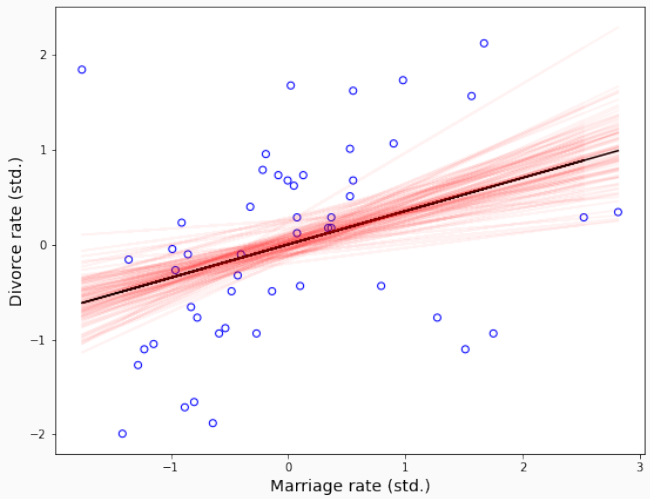
What other features of the South might be responsible for the divorce rate?

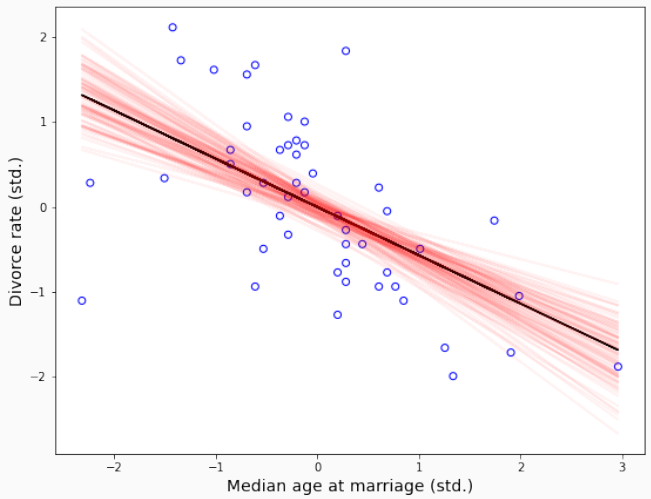
- The South has generally higher religiosity
- The South has generally higher poverty rates
- Two predictors that we'll use
  - Marriage rate
  - Median age at (first) marriage

# Marriage rate & age at marriage

- Marriage rate
  - You have to get married to get divorced (+)
  - Society values marriage, so opposes divorce (-)
- Median age at (first) marriage
  - Younger people make worse decisions
  - People change

# Marriage rate and median age at marriage




- Does higher marriage rate cause higher divorce rate?
- Does higher age at marriage cause lower divorce rate?

# DAGs

What's a DAG?

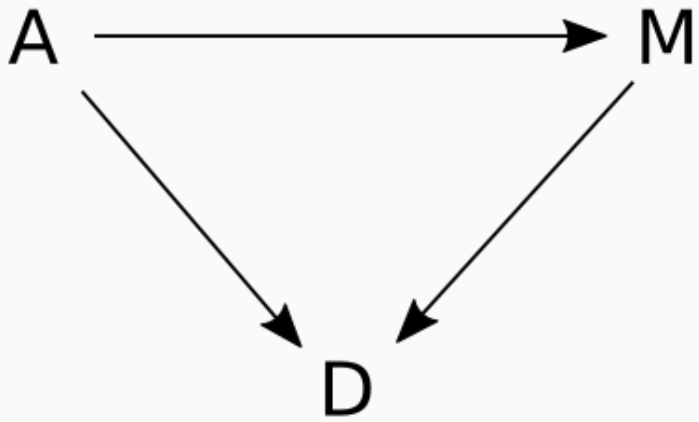
- Directed (edges are arrows)
- Acyclic (No directed loops)
- Graph (nodes and edges)

Use as a heuristic model for causal relationships

- Not a mechanical model – does not include explanation of how or why the causal relationship exists
- Not even a statistical model – does not specify probability distributions, only conditional dependence/independence

# Our first DAG

Here is a DAG representing a possible model for the relationships between age at marriage, marriage rate, and divorce rate:




What this DAG says:

1. *A* directly influences *D*
2. *M* directly influences *D*
3. *A* directly influences *M*

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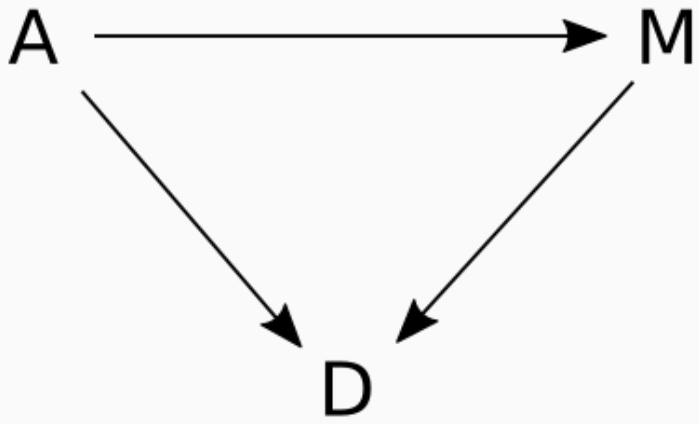



In this model, total causal effect of $A$ on $D$:

1. $A \to D$ – direct causal effect
2. $A \to M \to D$ – indirect effect

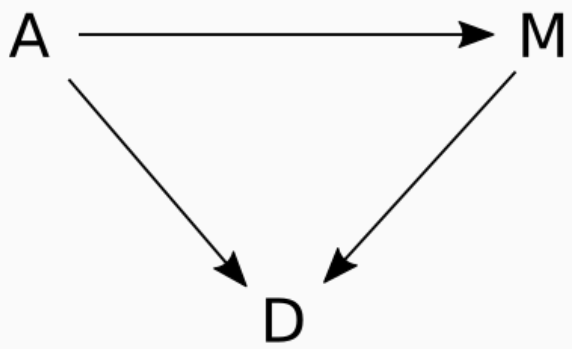
# Multiple regression and control

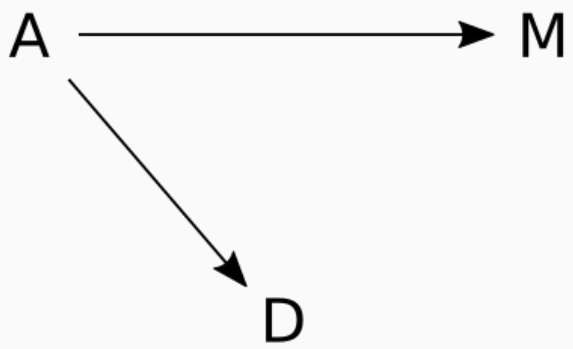
Multiple regression provides statistical "control." This means *conditioning on the information in one variable*, not *setting the value of one variable*

- Multiple regression answers: once we know all other predictors, how is each predictor associated with the outcome?
- To interpret the effect of statistical control, we need a clear model of what the causal relationships might be – this is what the DAG offers us

# Two competing DAGs

Here are two DAGs:




- Both DAGs are consistent with the inferences of our previous models
- Statistical association between $M$ and $D$ appears in both

# Multiple regression

Multiple regression estimates conditional associations:

- What is the value of a predictor, once we know the other predictors?
  - If we already know the marriage rate in a state, what do we learn from the age at marriage?
  - If we already know the age at marriage in a state, what do we learn from the marriage rate?

$$D_i \sim \text{Normal}(\mu_i, \sigma)$$
$$\mu_i = \alpha + \beta_M M + \beta_A A$$

# Multiple regression model

$$
\begin{aligned}
D_i &\sim \text{Normal}(\mu_i, \sigma) \\
\mu_i &= \alpha + \beta_M M + \beta_A A \\
\alpha &\sim \text{Normal}(0, 0.2) \\
\beta_M &\sim \text{Normal}(0, 0.5) \\
\beta_A &\sim \text{Normal}(0, 0.5) \\
\sigma &\sim \text{Exponential}(1)
\end{aligned}
$$

# Priors

What are these priors about?

- Working with standardized data

$$z_i = \frac{x_i - \bar{x}}{s}$$

- Each variable has mean 0, SD 1
- Intercept parameter should be 0:

$$\alpha \sim \text{Normal}(0, 0.2)$$

- Priors on slopes fairly vague

$$\beta_M \sim \text{Normal}(0, 0.5)$$
$$\beta_A \sim \text{Normal}(0, 0.5)$$

# Fitting the model

```
# Model with marriage rate and median age at marriage
with pm.Model() as linear_model:
    # Priors for model parameters
    alpha = pm.Normal('alpha', mu = 0, sigma = 0.2)
    betaA = pm.Normal('betaA', mu = 0, sigma = 0.5)
    betaM = pm.Normal('betaM', mu = 0, sigma = 0.5)
    sigma = pm.Exponential('sigma', 1)

    # Model equation
    mu = alpha + betaA * A + betaM * M

    # Observed variable and inference
    D_obs = pm.Normal('D_obs', mu = mu, sigma = sigma, observed = D)
    qp = quap()
```

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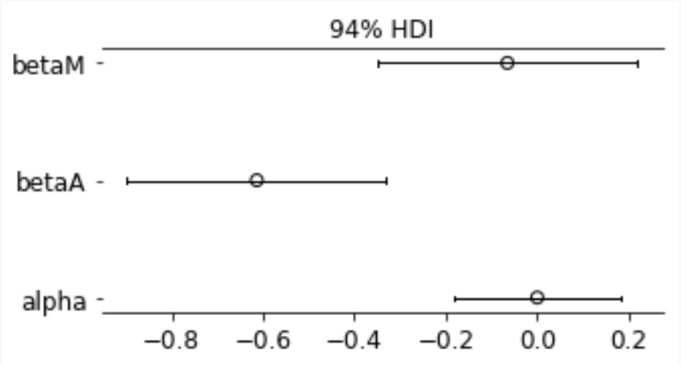
---

| | mean | sd | hdi_3% | hdi_97% |
|---|---|---|---|---|
| alpha | 0.000 | 0.098 | -0.184 | 0.184 |
| betaA | -0.614 | 0.151 | -0.897 | -0.330 |
| betaM | -0.065 | 0.151 | -0.349 | 0.218 |
| sigma | 0.793 | 0.079 | 0.645 | 0.941 |



# Comparing three model results

- Age only:

| | mean | sd | hdi_3% | hdi_97% |
|---|---|---|---|---|
| alpha | 0.000 | 0.098 | -0.185 | 0.185 |
| betaA | -0.568 | 0.110 | -0.775 | -0.361 |
| sigma | 0.796 | 0.079 | 0.648 | 0.945 |

- Marriage rate only:

| | mean | sd | hdi_3% | hdi_97% |
|---|---|---|---|---|
| alpha | 0.000 | 0.109 | -0.205 | 0.205 |
| betaM | 0.350 | 0.126 | 0.113 | 0.587 |
| sigma | 0.919 | 0.091 | 0.749 | 1.090 |

- Both:

| | mean | sd | hdi_3% | hdi_97% |
|---|---|---|---|---|
| alpha | 0.000 | 0.098 | -0.184 | 0.184 |
| betaA | -0.614 | 0.151 | -0.897 | -0.330 |
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| sigma | 0.793 | 0.079 | 0.645 | 0.941 |

# Exploring the result graphically

Two approaches to exploring our results graphically

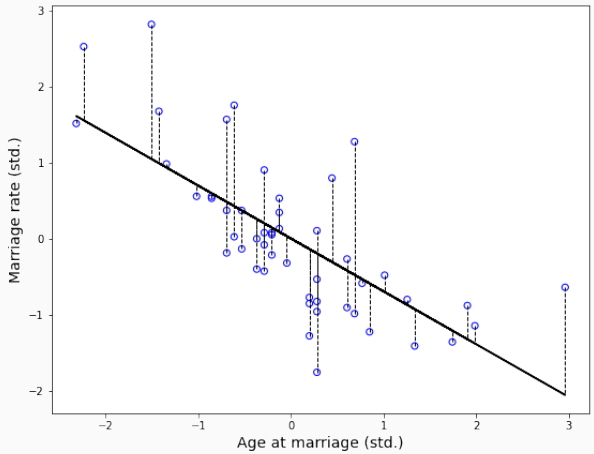
- Residual analysis – bad idea for analysis but useful for understanding the mechanics
  - Do a regression of marriage rate on age at marriage
  - Extract residuals and use them as inputs in a second regression
- Posterior predictive plots
  - Compare predicted divorce rate to actual divorce rate
  - Check overall model fit
  - Identify places where the model fails – can it be improved?

# Residual analysis

Do it both ways:

- Regress marriage rate on age at marriage
- Use the residuals as predictors for divorce
    - Idea: residuals represent "leftover" variability in marriage rate once age at marriage is known
    - If marriage rate has a direct effect on divorce, it will be visible here
- Do the same starting from a regression of age on marriage rate

# Residual analysis



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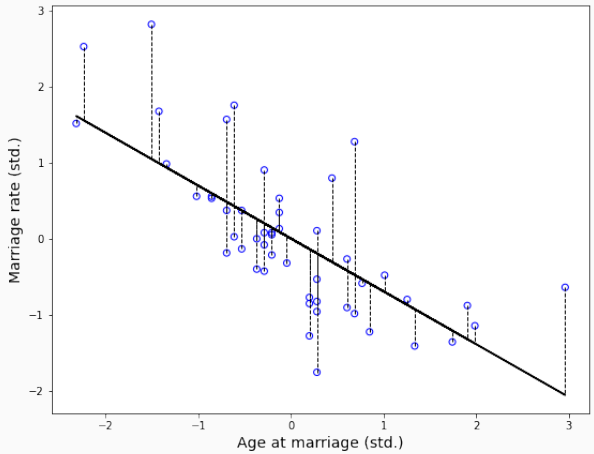


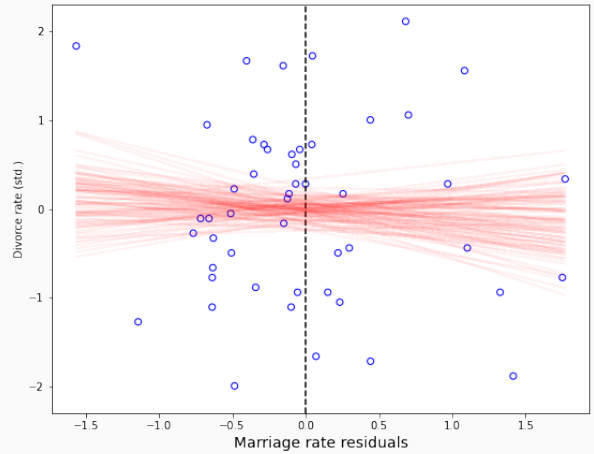

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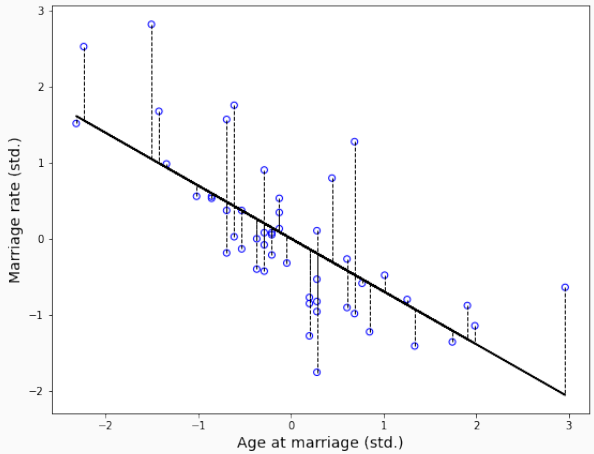


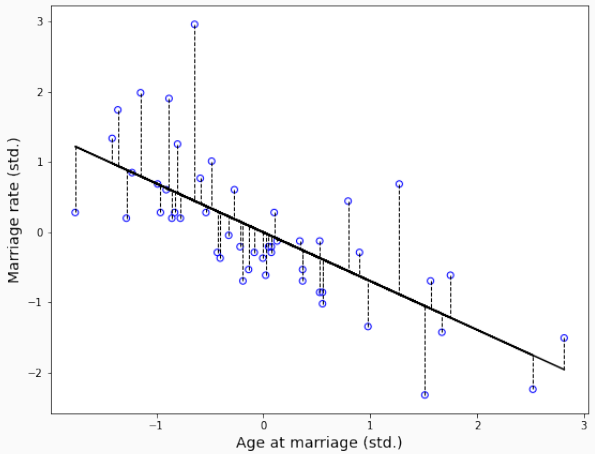


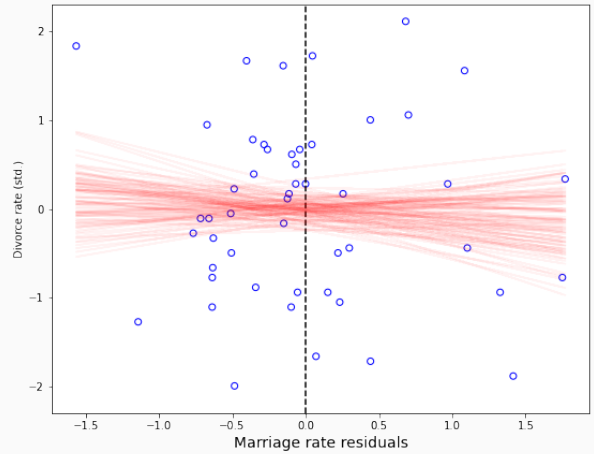

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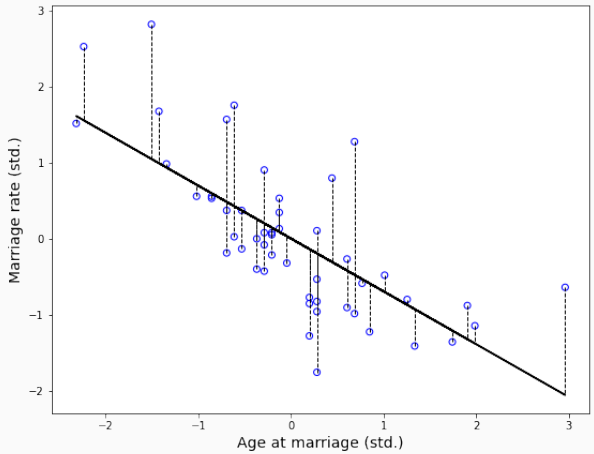
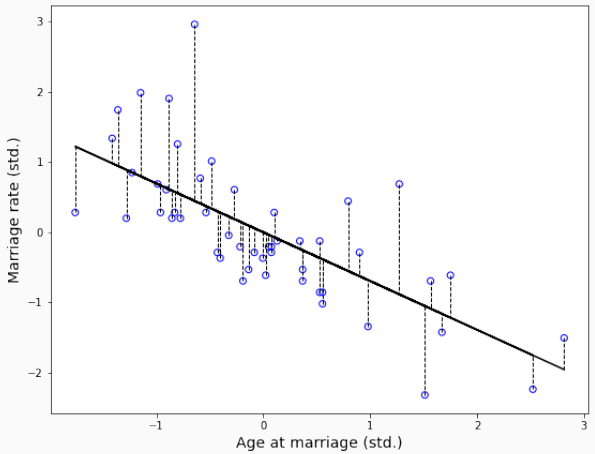
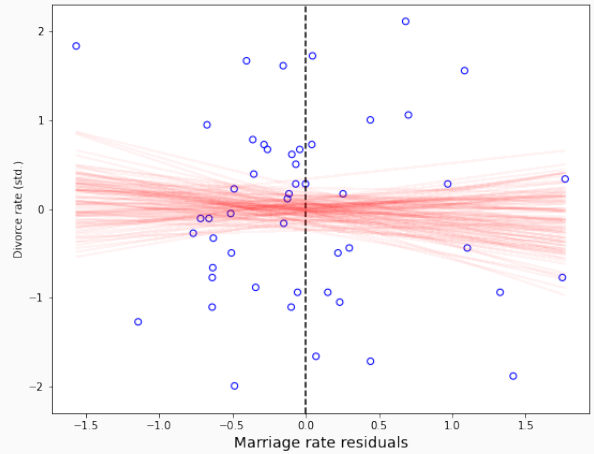
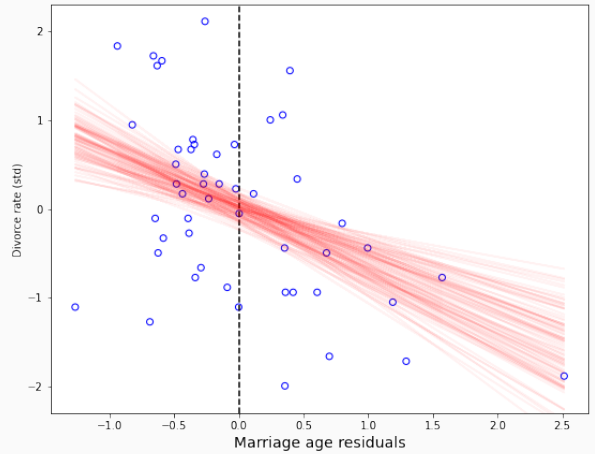


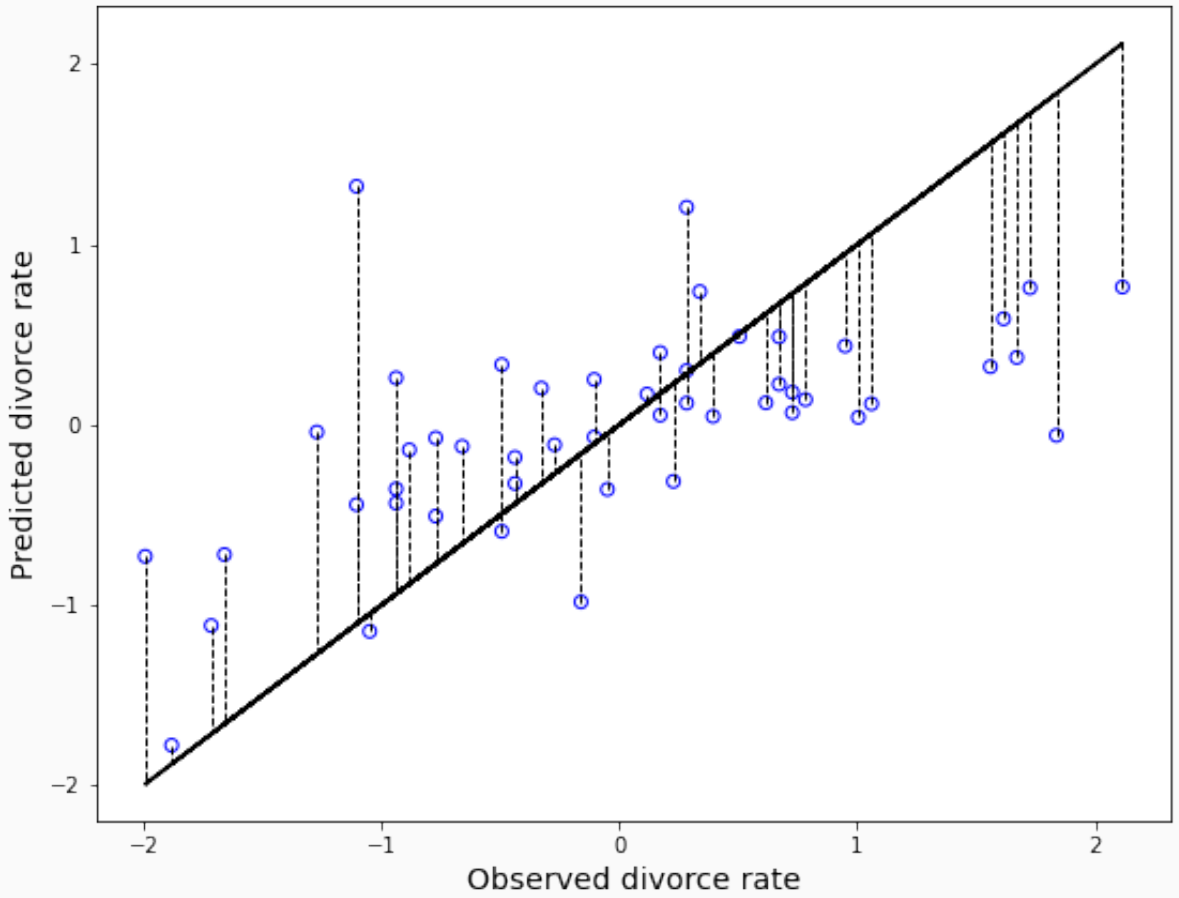



# Statistical "control"

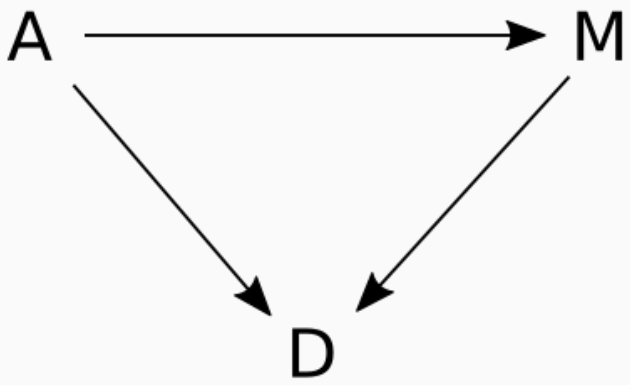
- Multiple regression often described as "controlling" for each variable
  - Not really control – we're not fixing input values in an experiment
  - Probabilistically: we are computing *conditional* distributions
  - Better way to think about this: "stratify"
- How is each predictor associated with the outcome, once all other predictors are known?

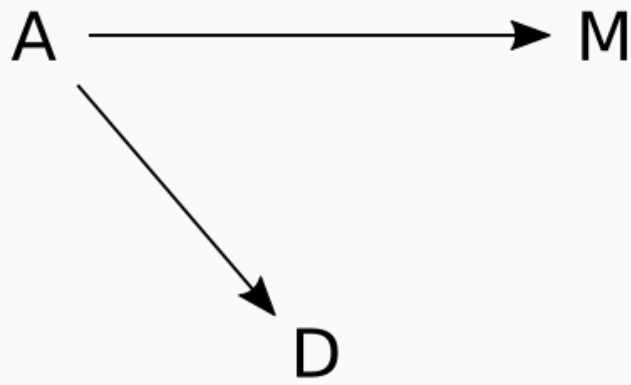
# Posterior predictive plot



# Summary

Recall our initial DAGs:

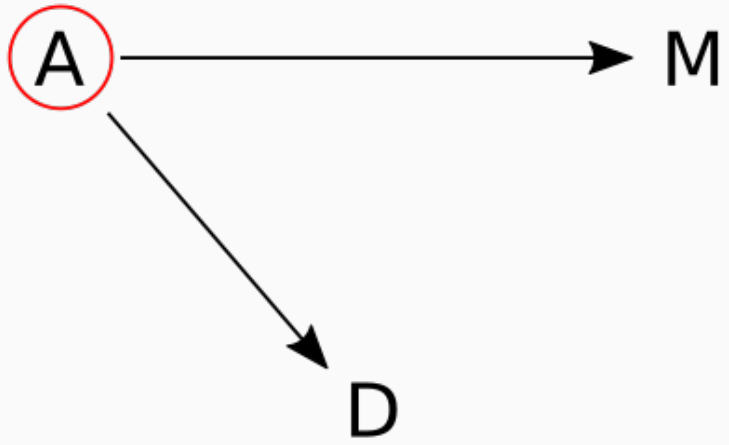



- Multiple regression reveals that the direct effect of $M$ on $D$ is weak/zero
- Note: if $A$ is unknown, $M$ is still useful; still carries *predictive* power
- If you wanted to stage interventions to reduce divorce, though, $M$ wouldn't be a good target

# Conditioning on $A$

Here, adding variables to the regression helped us determine the effect:




Conditioning on $A$ eliminated the spurious association of $M$ with $D$

In some cases, conditioning on a variable can *introduce* a spurious association – depends on the structure of the DAG

# Masked associations

- Previously: used multiple regression to suppress a spurious correlation
- Masked association:
  - There is really a direct causal effect of predictor on outcome
  - A simple regression shows no (or suppressed) association
- Typical situation:
  - Have two predictors that are associated with the outcome in the opposite direction
  - Two predictors are positively associated with one another
  - Effects cancel

# Urban foxes in London

Data set: urban foxes living in groups

- We want to monitor the health of groups of foxes living in the city
- Easy health metric: fox weight
  - Foxes with adequate food, no disease, etc. are heavier
  - Low weight is a marker for various health conditions (malnutrition, disease, age)
- Is the presence of food a good predictor for weight?
- Is the presence of food causally related to weight?

# What happens if you feed the foxes?

If food is added to an area, will the foxes get bigger?

- This is a causal question, not just a statistical question
- Difference: talks about an intervention
- Simplest thing to try: regress fox weight on average food

# Simple linear regression

Linear regression for fox weight:

$$
\begin{aligned}
w_i &\sim \text{Normal}(\mu_i, \sigma) \\
\mu_i &= \alpha + \beta_f \text{food} \\
\alpha &\sim \text{Normal}(0, 0.2) \\
\beta_f &\sim \text{Normal}(0, 0.5) \\
\sigma &\sim \text{Exponential}(1)
\end{aligned}
$$

# What does the fox model say?

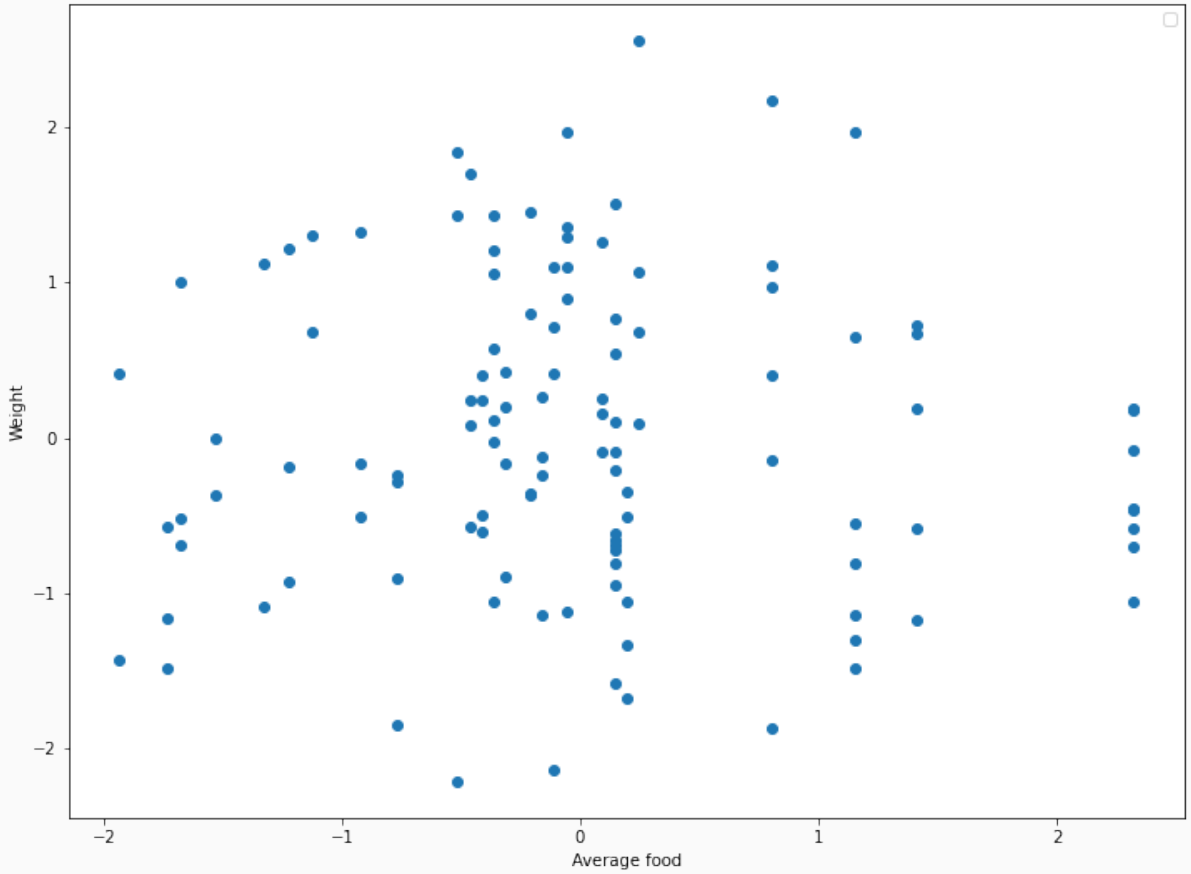
Here are the estimates from the fox model:

| | mean | sd | hdi_3% | hdi_97% |
|---|---|---|---|---|
| **bF** | -0.024 | 0.092 | -0.191 | 0.150 |
| **alpha** | -0.003 | 0.099 | -0.183 | 0.185 |
| **sigma** | 1.013 | 0.068 | 0.884 | 1.131 |

This is about as close to zero as we can get. Does this check out with the data?

# What does the fox model say?

Scatterplot of weight vs. average food:



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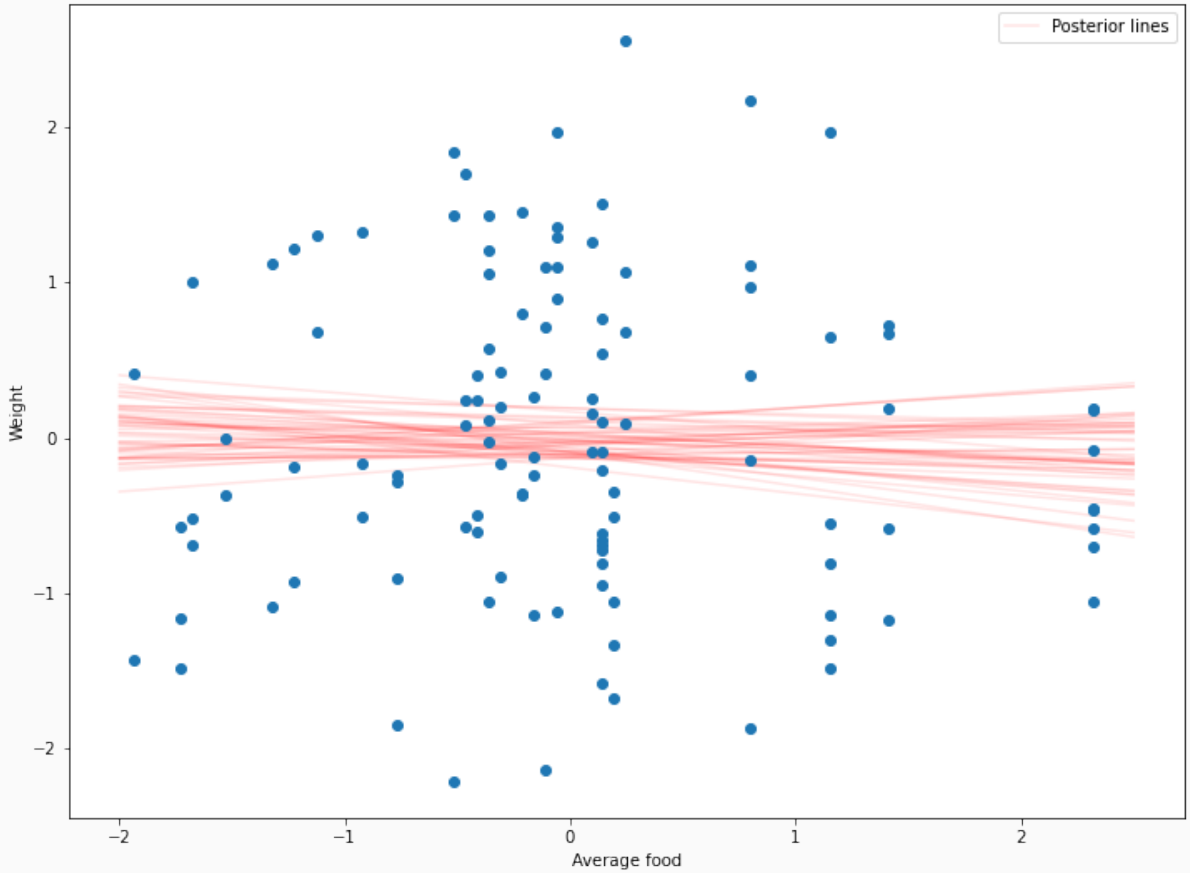

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The fox model tells us:

- No apparent association between food availability and fox weight
- But intuition tells us: if we provide more food, it must go somewhere!

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The fox model tells us:

- No apparent association between food availability and fox weight
- But intuition tells us: if we provide more food, it must go somewhere!
- More foxes

How can we check this? Include both variables.

# A multiple regression

$$
\begin{aligned}
w_i &\sim \text{Normal}(\mu_i, \sigma) \\
\mu_i &= \alpha + \beta_f \text{food} + \beta_g \text{groupsize} \\
\alpha &\sim \text{Normal}(0, 0.2) \\
\beta_f &\sim \text{Normal}(0, 0.5) \\
\beta_g &\sim \text{Normal}(0, 0.5) \\
\sigma &\sim \text{HalfCauchy}(1)
\end{aligned}
$$

# Multiple regression results

Here are the results from the multiple regression:

| | mean | sd | hdi_3% | hdi_97% |
|---|---|---|---|---|
| **bG** | -0.568 | 0.189 | -0.937 | -0.224 |
| **bF** | 0.475 | 0.188 | 0.153 | 0.859 |
| **alpha** | 0.001 | 0.090 | -0.176 | 0.160 |
| **sigma** | 0.967 | 0.068 | 0.854 | 1.103 |

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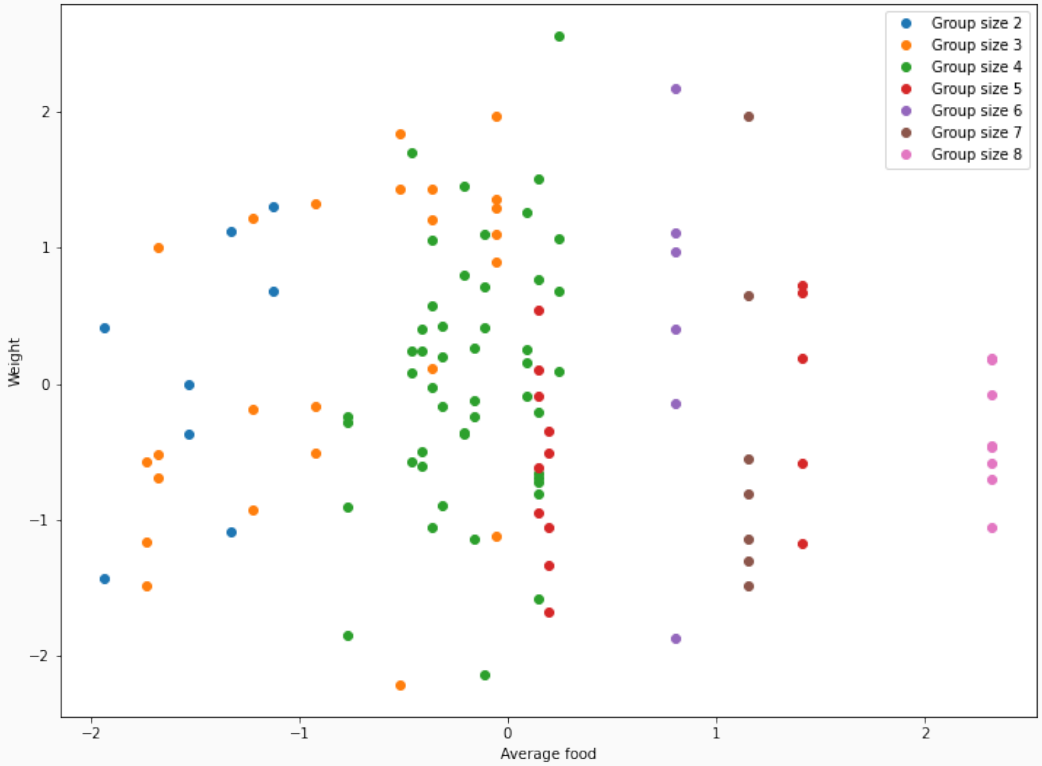
Can we see this in the scatter plot?

# Statistical control as stratification




The association between food and weight appears when the data is stratified by group size, but not before

# DAG for the fox model

Let's draw a DAG for the fox model:

# Direct vs. total effect

The graph for the fox model gives us a distinction between two effects of food on weight:

- direct effect: associated with the arrow from $F$ to $W$; effect of food on weight at fixed group size
    - estimated by the multiple regression, but not the simple regression
- total effect: associated with all paths from $F$ to $W$; effect of food on weight, including those mediated by changes in group size
    - estimated by the simple regression, but not the multiple regression

# Revealing and eliminating associations

In both cases we gained something by including the extra variable:

- In the marriage example, a spurious association is eliminated
- In the fox example, a masked association is revealed

This is the power of multiple regression: it thinks "hypothetically" about the variables

Beware: including the wrong variables can introduce spurious associations!

# Structure of DAGs

# What is a DAG?

What is a DAG?

- Directed acyclic graph
- Nodes are variables
- Directed arrows are causal associations

What can we use DAGs for? Probabilistic models, on two levels:

- probabilistic model for associations between variables
- metadata that guides choice of variables for inference

# Three technical slides

# Probabilistic model of a DAG

The probabilistic nature of a DAG is implied *conditional independence*.

Say we have $n$ variables $X_1, \ldots, X_n$. We can always write

$$p(x_1, \ldots, x_n) = \prod_i p(x_i | x_1, x_2, \ldots, x_{i-1})$$

(the chain rule). We are interested in the case where each $x_j$ is dependent on only some of the other variables:

$$p(x_i | x_1, \ldots, x_{i-1}) = p(x_i | pa_i)$$

where $PA_i$ is a subset of the remaining variables, called the "parents" of $X_i$.

# Graphical example

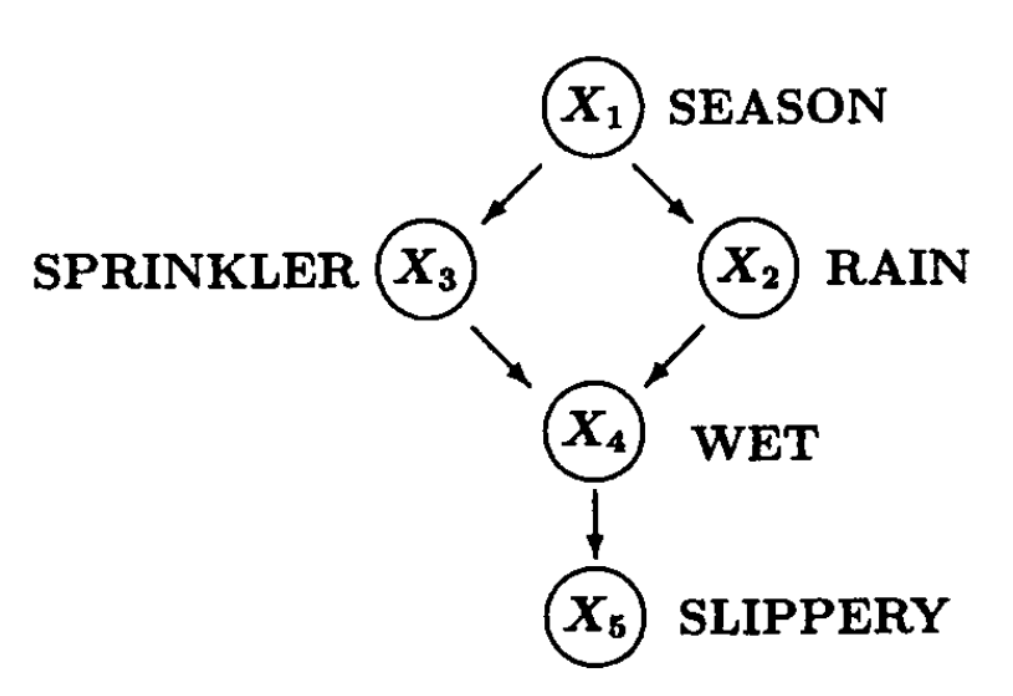



Figure from *Causality* by Judea Pearl

# Graphical example

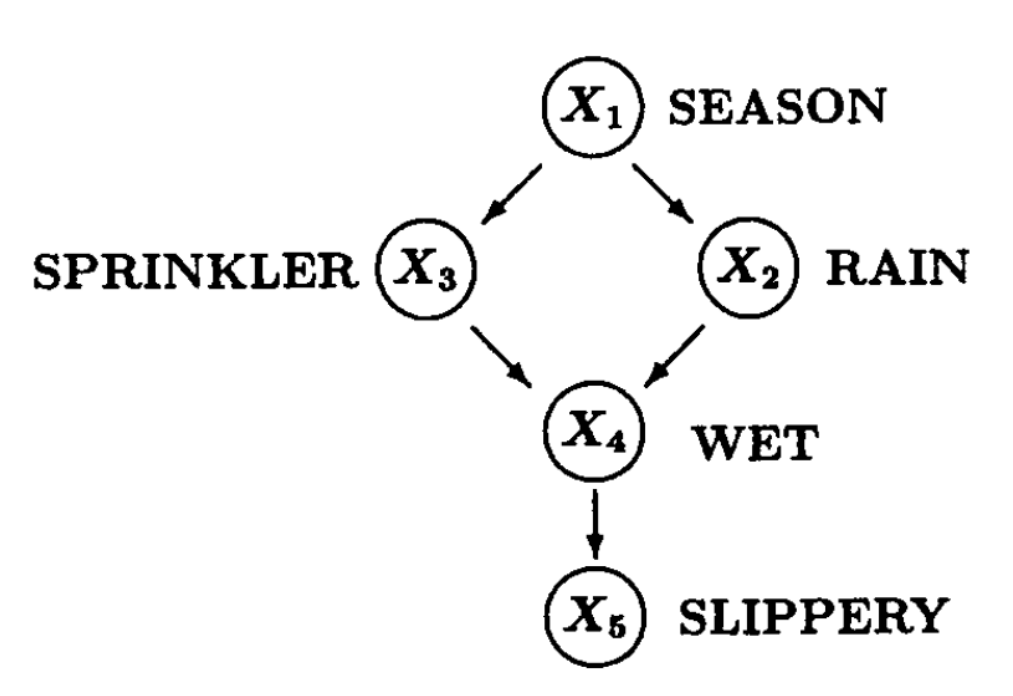



Figure from *Causality* by Judea Pearl

$$P(x_1, \ldots, x_5) = P(x_1)P(x_2|x_1)P(x_3|x_1)P(x_4|x_2, x_3)P(x_5|x_4)$$

# Controlling "flows"

When we're trying to estimate the effect of one variable on another:

- Control "flow" of information along paths
- Information flows along or against arrows
- Including a variable in the regression can either "block" or "open" paths

# Three basic paths

# Three basic paths

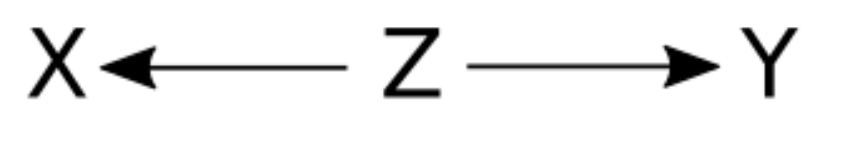
In a DAG, information flows along paths (both with and against the arrows).

A path from $X$ to $Y$ can be a direct path – an arrow between $X$ and $Y$. Or it can be an indirect path $X \leftrightarrow Z \leftrightarrow Y$ (or a concatenation of several of these).

Indirect paths can lead to confounding / spurious associations; to deal with this, we need to classify the different types of indirect paths.

# The "fork" path

The *fork* is the form most students learn as the sole definition of "confounding" in introductory classes: $X$ and $Y$ are confounded by their common cause, $Z$:




A statistical association exists between $X$ and $Y$ because they are both influenced by $Z$.

Example: $X$ is ice cream sales; $Y$ is drowning deaths; $Z$ is temperature

# The "fork" path

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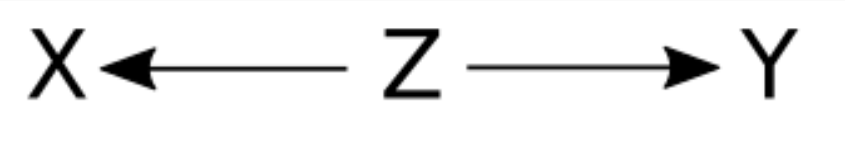
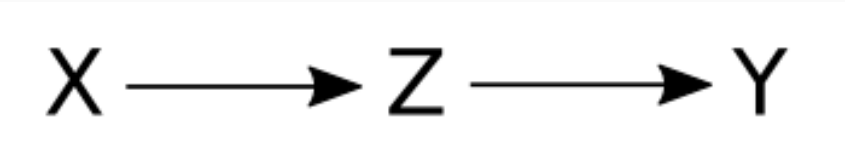



Conditional independence:

- DAG property means: conditional on $Z$, $X$ and $Y$ are independent.
- So, condition/stratify/control on $Z$ to block the path and estimate effect of $X$ on $Y$

# The "chain" path

The *chain* is a similar-looking form, where $Z$ sits in the middle of a causal path:



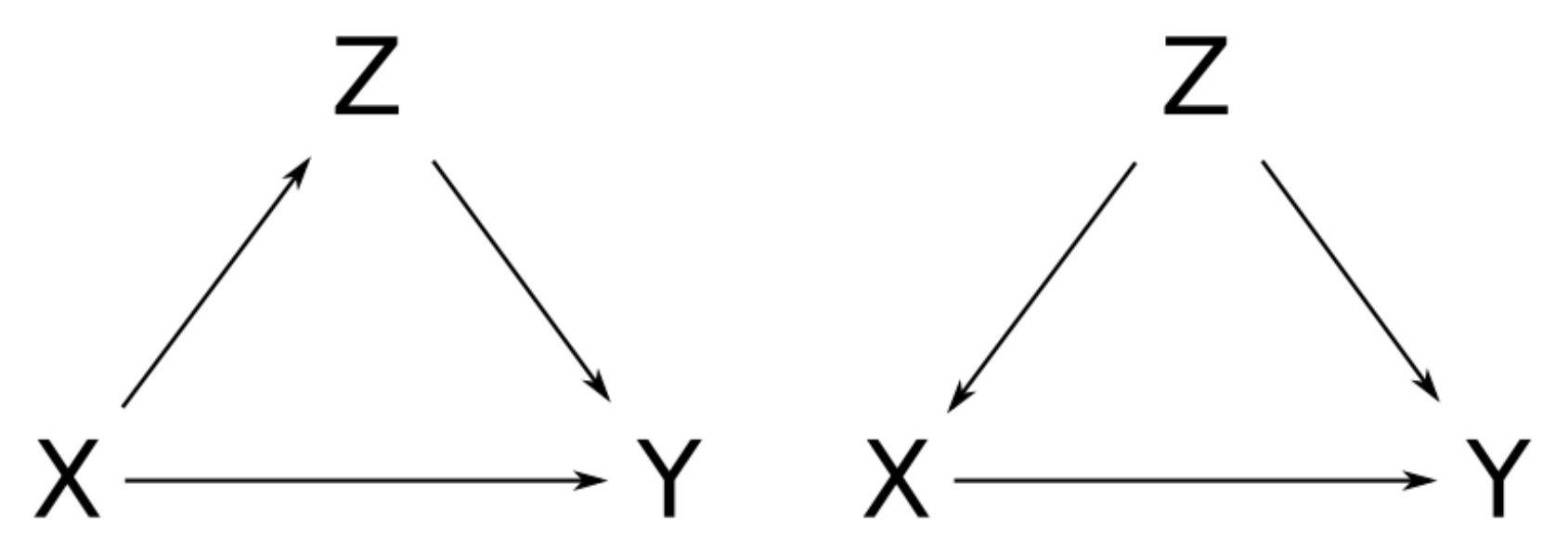

Typical case: $Z$ is an effect of $X$ that mediates the effect on $Y$

Example: $X$ is pesticide application; $Z$ is the pest population; $Y$ is crop yield.

Controlling for $Z$ blocks information flow along the path.

# When the data can't tell you

Multiple paths: should you include the variable $Z$ or not?




The data/model cannot tell you the difference between these, because they imply the same set of conditional independences

# The "collider" path

The third form is the *collider* or inverted fork, and it behaves quite differently:


In contrast to the fork or chain, information flows through the collider only when it *is* observed / controlled; controlling *unblocks* the path.

# Heuristic example


$X$: switch state on/off $Z$: light bulb on/off $Y$: power working/not working

The presence of power and the state of the switch are independent; but,

- turn on the switch and observe the light: it's off
- is the power working?

# The explaining-away effect

This property of colliders is responsible for a sometimes counterintuitive effect:

- "explaining away": observing one of the common causes
- Berkson's paradox: conditioning on a variable can introduce a spurious association

They're really the same effect; explaining away common in AI/ML; Berkson's paradox in statistics

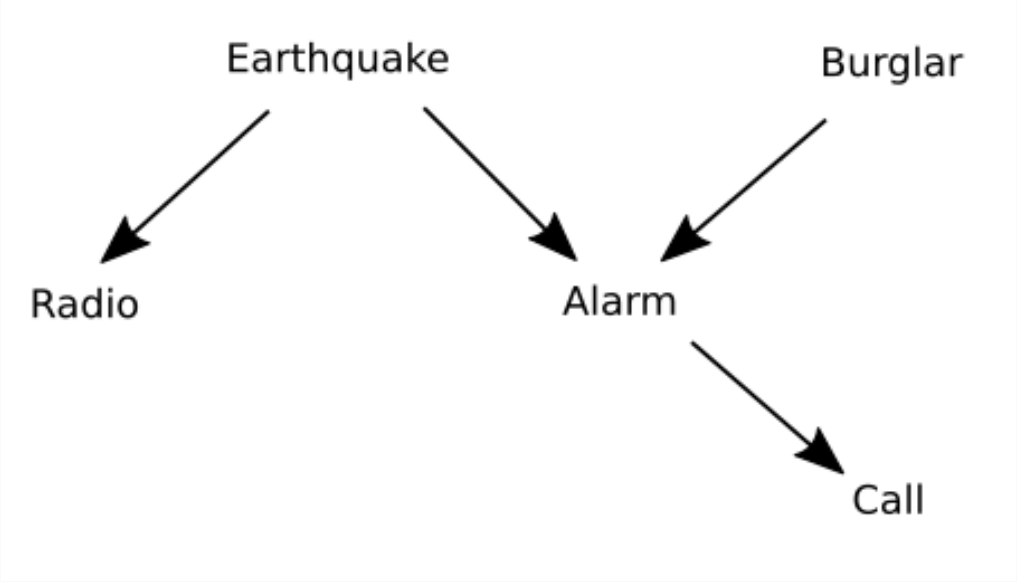
# Explaining away: the burglar alarm

From Judea Pearl by way of David MacKay:

> *Fred lives in Los Angeles and commutes 60 miles to work. Whilst at work, he receives a phone-call from his neighbour saying that Fred's burglar alarm is ringing. What is the probability that there was a burglar in his house today? While driving home to investigate, Fred hears on the radio that there was a small earthquake that day near his home. 'Oh', he says, feeling relieved, 'it was probably the earthquake that set off the alarm'. What is the probability that there was a burglar in his house?*

# Explaining away: the burglar alarm

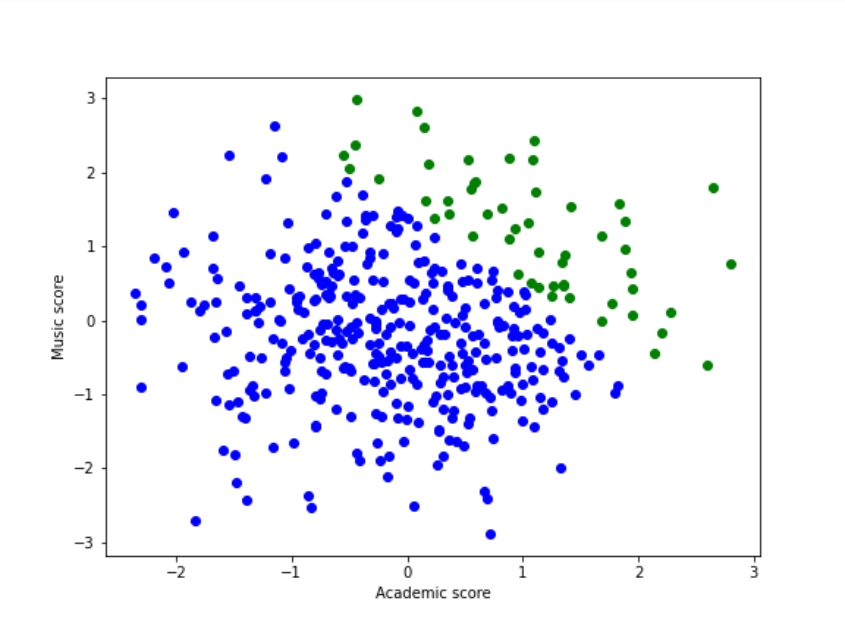
A DAG for the burglar alarm problem, showing the collider:




The alarm sits at a collider.

# Conditioning on colliders creates confounding

The spurious-association effect of conditioning on a collider:




Berkson's paradox a.k.a. *selection bias*

# Recent example

Recent example: risk factors for COVID-19

- Early studies of COVID-19 were based on observational studies
- Testing availability was low

This led to the potential for collider bias. Why?

- Any study of confirmed COVID-19 cases can only be applied to people who are tested (still true!)
- Data sets are implicitly conditional on having been tested

# Examining the effect of smoking

Example study: does smoking protect against severe disease?

- early observational data suggested a negative association between smoking and probability of severe COVID-19
- this is a surprising finding!

Implicit collider: who is getting tested in early 2020?

# Examining the effect of smoking

In the early stages of the pandemic, two groups of people were tested most commonly:

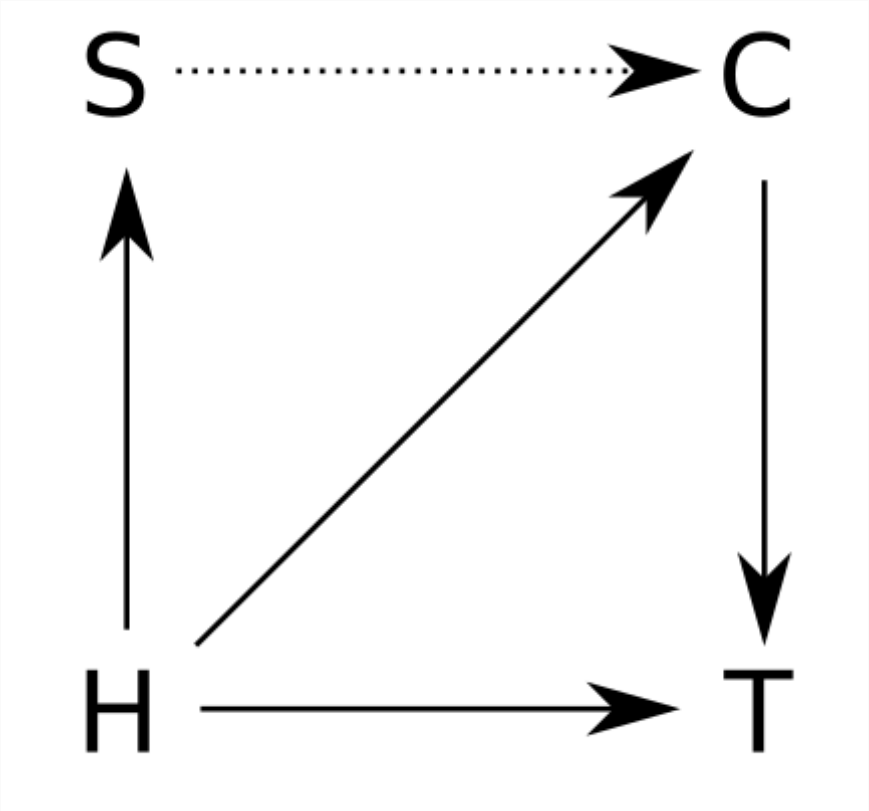
- people with severe disease
- healthcare workers

Conditioning on testing introduces an association between these two traits

Griffith et al., "Collider bias undermines our understanding of COVID-19 risk and severity" (Nature, 12 Nov 2020)

# A DAG for the smoking confound

Here is a DAG:



# The backdoor criterion

# $d$-separation

A (possibly undirected) path $p$ through a DAG $G$ is said to be *$d$-separated* or *blocked* by a set of nodes $Z$ if:

1. $p$ contains a chain $X_i \to M \to X_j$ or fork $X_i \leftarrow M \to X_j$ such that $M \in Z$; or,
2. $p$ contains a collider $X_i \to M \leftarrow X_j$ such that $M \notin Z$ and no descendent of $M$ is in $Z$.

(Why the descendant property? Look back at the burglar alarm.)

The $d$-separation (blocking) definition for paths leads to another definition, for sets of variables.

# The backdoor criterion

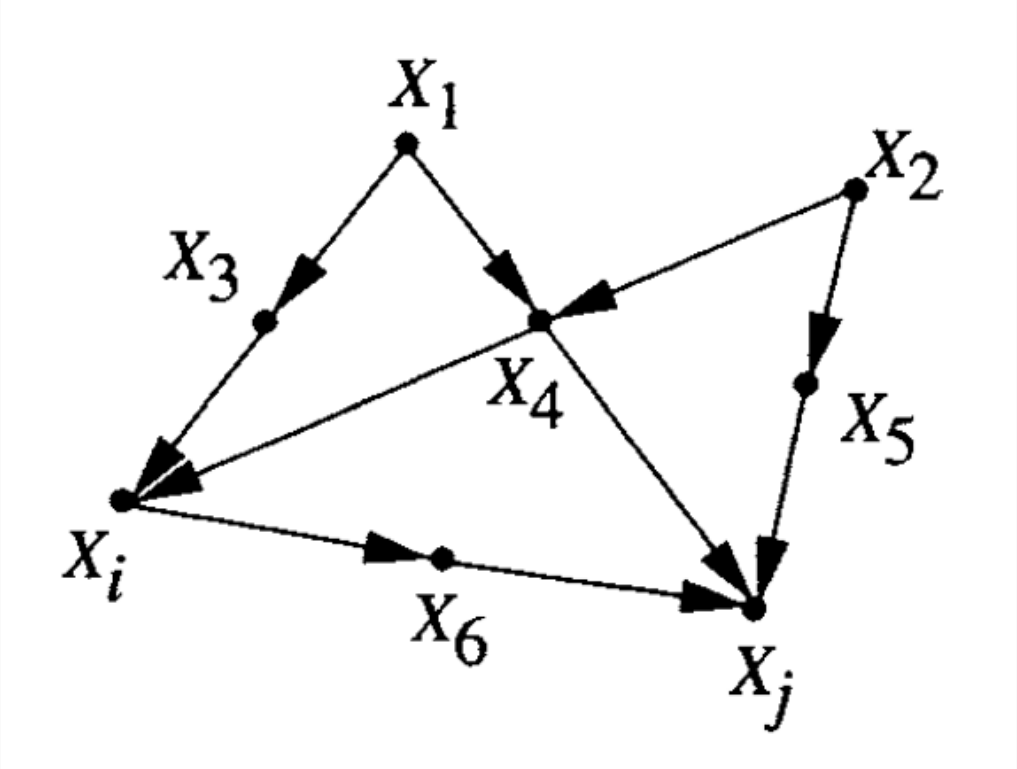
A related definition:

**Definiton**
*A set of variables $Z$ satisfies the backdoor criterion with respect to an ordered pair of variables $(X_i, X_j)$ in $G$ if:*

1. *no node in $Z$ is a descendent of $X_i$; and,*
2. *$Z$ blocks every path from $X_i$ to $X_j$ that contains an arrow into $X$.*

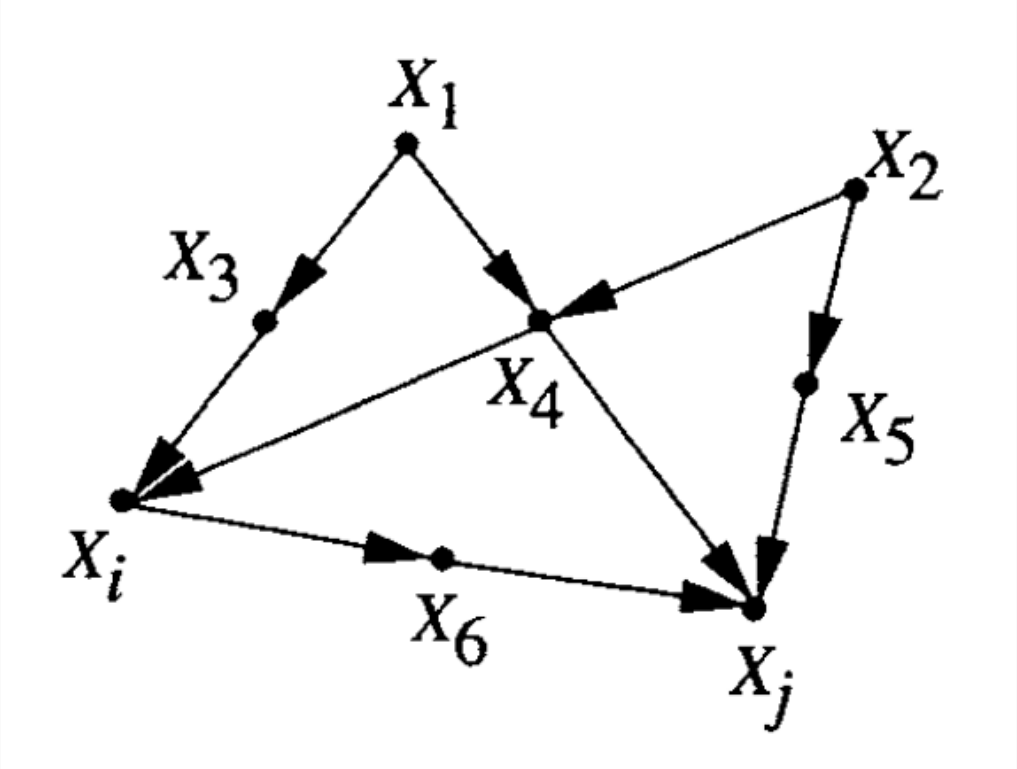
To estimate the causal effect of $X$ on $Y$, condition on a set of variables satisfying the backdoor criterion with respect to $(X, Y)$.

# Example




Which variables satisfy the backdoor criterion?

# Example



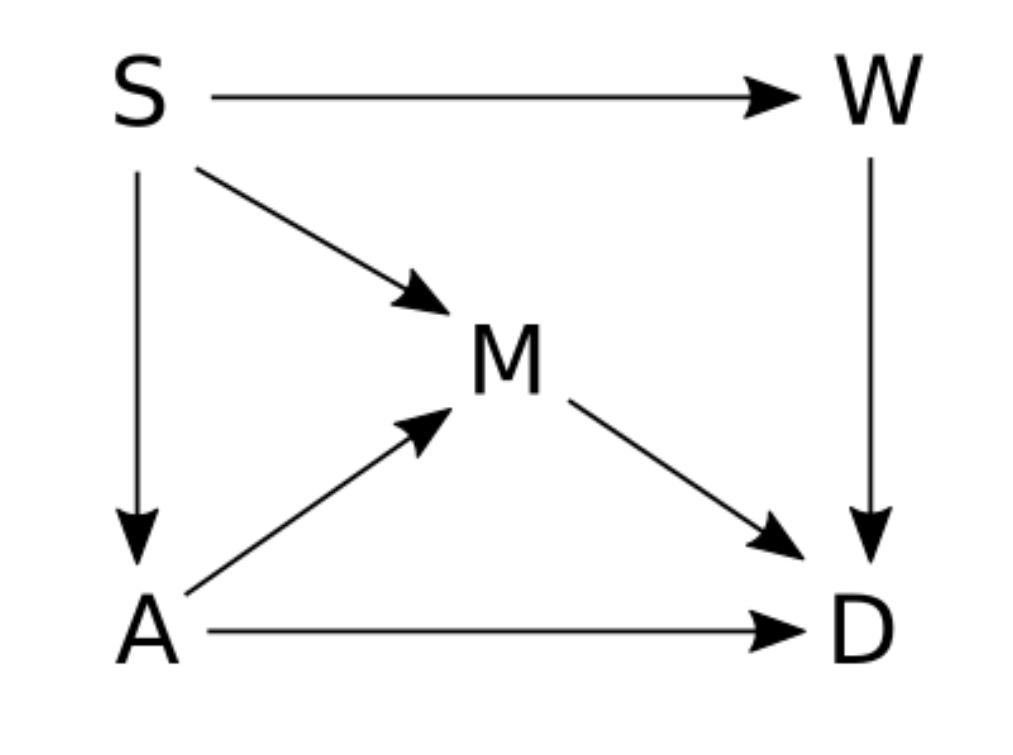

Which variables satisfy the backdoor criterion?

- $\{X_3, X_4\}$ or $\{X_4, X_5\}$
- Not $\{X_4\}$ (doesn't block every backdoor path), nor $\{X_6\}$ (descendent of $X_i$)

# Return to the Waffle House

A bigger DAG from the Waffle House example, including:

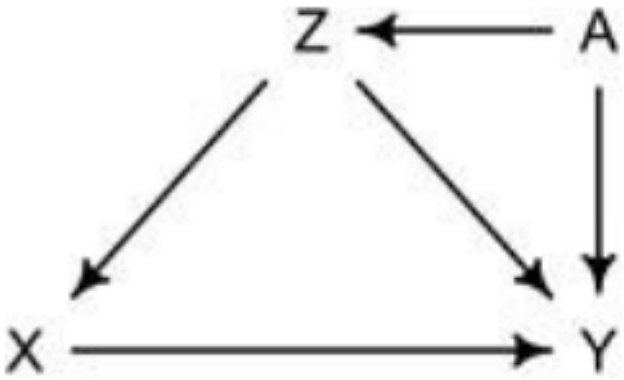
- $W$: number of Waffle Houses in the state
- $S$: Indicator variable for South

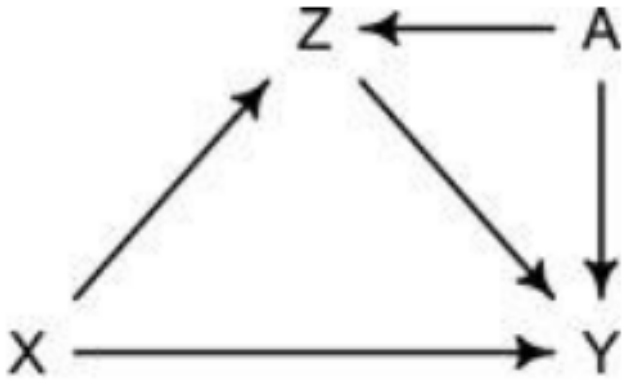


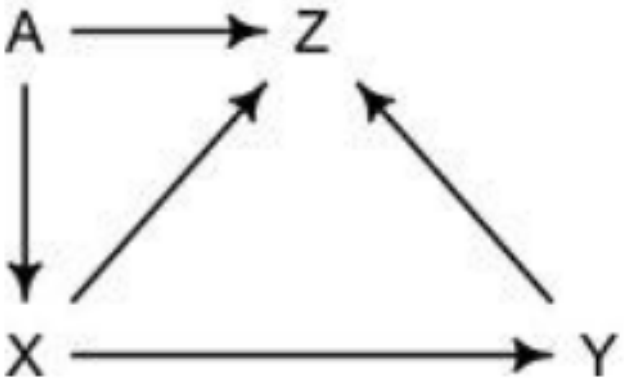

To estimate the direct effect of $W$ on $D$, what do we condition on?

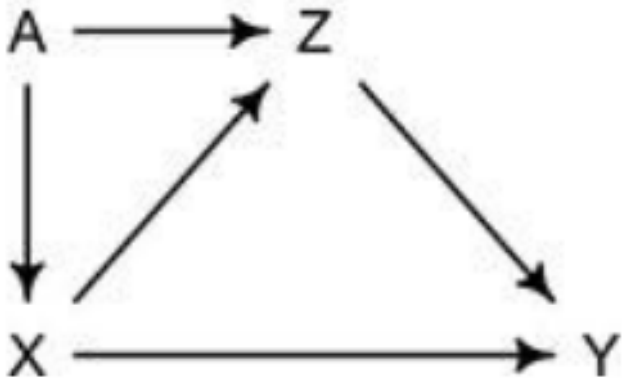
# Group exercise

For each DAG, which variable should be conditioned on to estimate total causal influence of $X$ on $Y$?



# Summary

Today:

- Multiple regression
- Causal DAGs

Next time:

- Structure of DAGs and the backdoor criterion