Exchangeability and more hierarchical models

ISTA 410 / INFO 510: Bayesian Modeling and Inference

U. of Arizona School of Information

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Outline

Last time:

- Bike lane example
- Hierarchical models
- Hyperprior selection

Now:

- Concept: exchangeability
- Hierarchical normal model

Recap

Bicycle traffic on neighborhood streets

Example from last time:

- Exercise 3.8 (and 5.13) in the textbook
- Data: observations of numbers of bicycles and other vehicles on neighborhood streets in Berkeley, CA
- Includes three classes of streets, with and without bike lanes
- We focus on one category: small streets with bike lanes

Goal: estimate the proportion of bicycle traffic

Fully pooled model

$$y_j \sim \mathrm{Binomial}(\theta, n_j)$$
$$\theta \sim \mathrm{Beta}(\alpha_0, \beta_0)$$

for fixed α_0, β_0.

- Choosing $\alpha_0 = 1, \beta_0 = 1$ gives a completely noninformative (flat) prior
- Weakly informative prior also reasonable, e.g. $\alpha_0 = 1, \beta_0 = 3$ for prior mean of 25% bicycle traffic

Fully separated model

As an alternative, we could treat each street as an independent entity:

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$$
\begin{aligned}
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\end{aligned}
$$

for fixed α_0, β_0.

- Exactly like the previous model, except we now have 10 independent θ_js for the 10 streets
- Same considerations for choice of prior

Call this the separate-effects model.

Setting up the model

A compromise: hierarchical model

μ - estimate of bike traffic on the "population" of similar streets.

η - controls variability among streets.

$$y_j \sim \text{Binomial}(\theta_j, n_j)$$

$$\theta_j \sim \text{Beta}(\alpha, \beta)$$

$$\mu := \frac{\alpha}{\alpha + \beta}$$ location parameter

$$\eta := \alpha + \beta$$ precision / inverse scale

$$p(\mu) \sim \text{Beta}(1, 3)$$

$$p(\eta) \sim \text{HalfCauchy}(1)$$

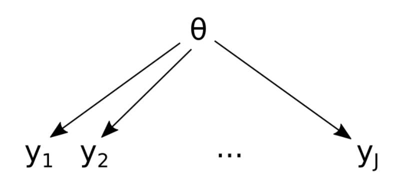
Note: the book uses:

$$p(\alpha, \beta) \propto (\alpha + \beta)^{-5/2}$$ §5.3 in BDA3

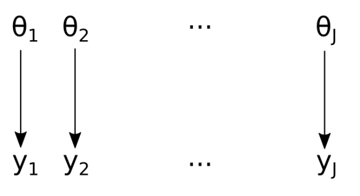
unif on $\left(\frac{\alpha}{\alpha+\beta}, (\alpha+\beta)^{-1/2} \right)$

Examining this graphically

Pooled model:



Separate model:

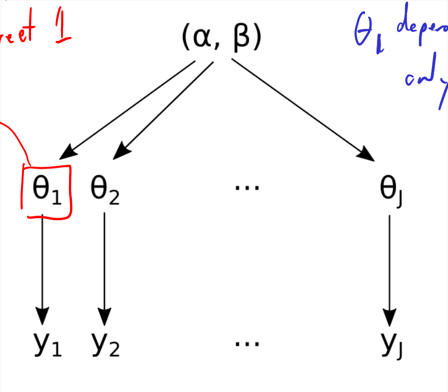


Examining this graphically

Hierarchical model combines the features of these two:

% of bike traffic on street 1

θ_1 depends on other θs only through (α, β).



Conditional: distribution of θ_1 given fixed α, β.

Marginal distribution: dist of θ_1 averaged over values of α, β.

Dice: distribution of # of 1s

(# of sides)

$\downarrow$

$\theta = p(1)$

$\downarrow$

$y = \%$ of 1s

Conditional:

fix # of sides,

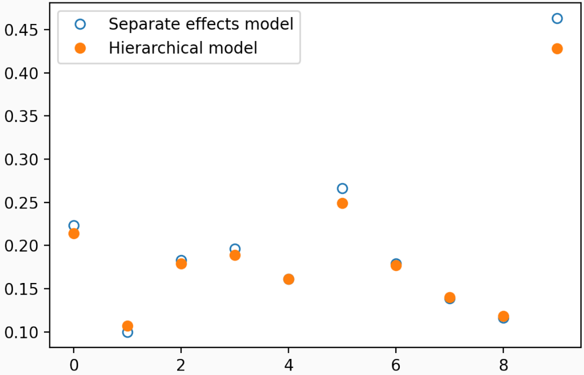
$\theta = (\text{\# of sides})^{-1}$

$y \sim N(\theta, se_\theta)$

Marginal: average over possible values for # of sides.

What is the difference in the results?

Let's compare point estimates:



Shrinkage and regularization

The shrinkage effect we see is a form of regularization:

- Most extreme observations “shrunk” toward a central value
- Amount of shrinkage tuned to relative sample size

Difference: we learned the strength of regularization from the data

Underfitting and overfitting

Another way to think about this, in terms of underfitting and overfitting:

- The pooled model: strong underfitting
- The separate-effects model: strong overfitting
- Hierarchical model: adaptive regularization

With enough observations the seperate effects model will estimate each street similarly to the hierarchical model.

Independence and exchangeability

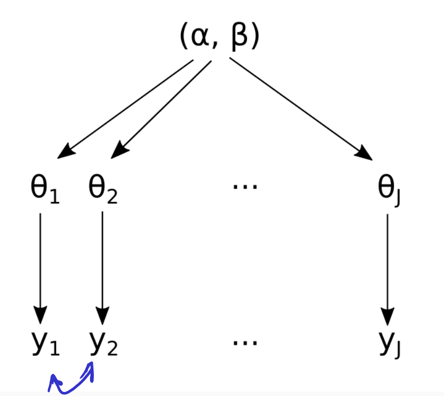
Independence of the θ_js

It's worth taking a moment to consider the independence properties of the parameters θ_j:

- In the hierarchical model, θ_js are not independent (they're independent in the separated model)
- However, they satisfy two weaker properties:
  - conditional independence
  - exchangeability

Conditional independence

The θ_js are not independent, but *given fixed* α, β, they are:



switching these does not change the model.

Exchangeability

A closely related concept is *exchangeability*, which justifies the use of the hierarchical model:

- Observations are *exchangeable* if the joint probability distribution is invariant to permutations of the index
- Roughly: we would have the same model if we relabeled the $y_1, y_2, \ldots$
- Exchangeability is also evident in the directed graph model

Levels of exchangeability

Table 3.3, p. 81 of book

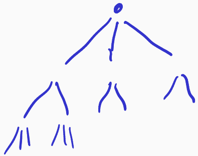
The full data set contains observations from a total of 58 streets:

- small residential streets, medium streets, and busy arterial streets
- streets with or without bike lanes

Evidently, if we label the streets $y_1, \dots, y_{58}$, they are not exchangeable.

But within the traffic/lane groups, the streets can be treated as exchangeable:

- Hierarchical model with several "levels"

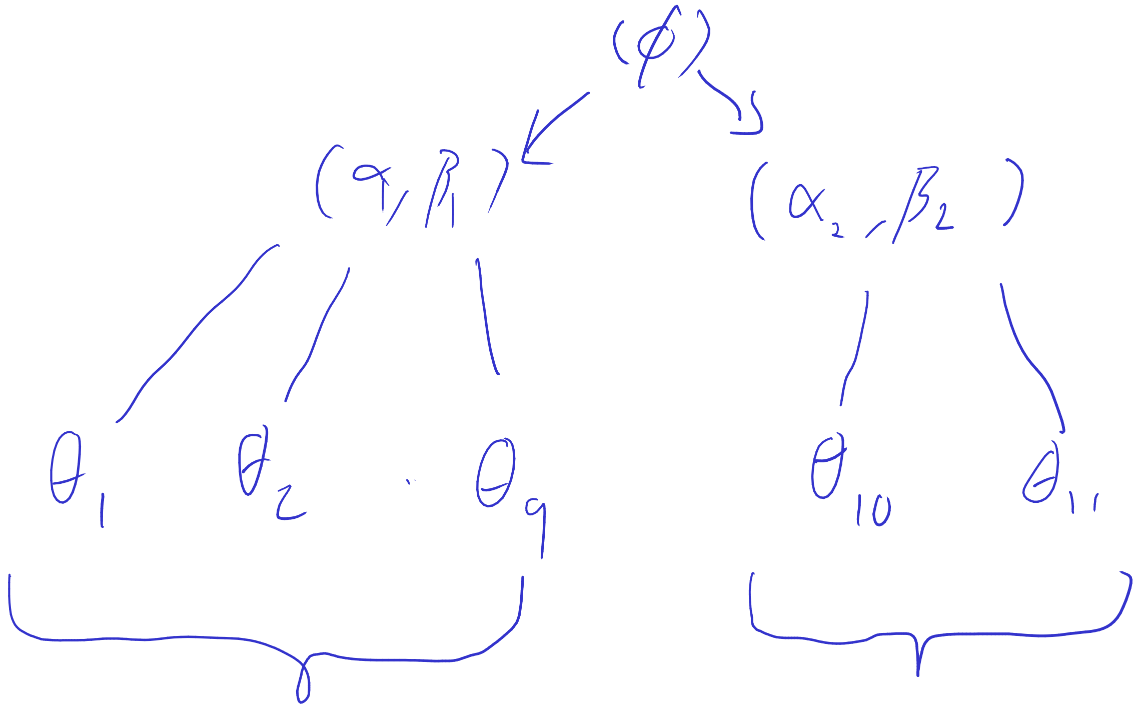


Ignorance implies exchangeability

These exemplify a broad practical idea: ignorance implies exchangeability.

- The less we know about a problem, the stronger a claim of exchangeability
- Example: a die with 6 sides
  - Initially all sides are exchangeable
  - Careful examination of the die might reveal imperfections, leading us to distinguish sides from one another
- If we don't know whether the streets have bike lanes, then they're exchangeable

  y_{10}
- If we know that y_10, the 10th sampled street, is University Blvd., then it shouldn't be treated as exchangeable with the others (we know geographic factors affecting bicycle traffic)



streets connecting to university campus entrances

Hierarchical normal model

Example: 8 schools

(Rubin, 1981)

Example: SAT coaching effectiveness

- SAT design intent: short term coaching should not improve outcomes significantly
- nonetheless, schools implement coaching programs
- examine effectiveness of coaching programs

Experiment:

closely related test

- All students pre-tested with PSAT
- Some students coached
- Coaching effects y_i estimated with linear regression
- Data is at the school level, not individual

Example: 8 schools

Data:

School	Effect	SE
A	28	15
B	8	10
C	-3	16
D	7	11
E	-1	9
F	1	11
G	18	10
H	12	18

estimate SE

estimated treatment effect

The model

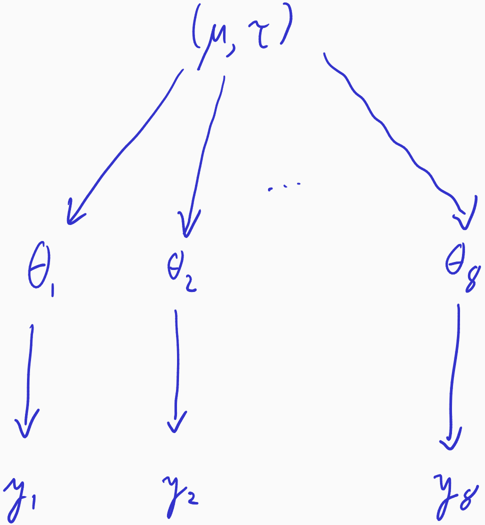
Normals at all levels:

$$
\begin{aligned}
y_j &\sim \mathrm{Normal}(\theta_j, SE_j) \leftarrow \text{observed data} \\
\theta_j &\sim \mathrm{Normal}(\mu, \tau) \leftarrow \text{per-school average effects} \\
\mu &\sim \mathrm{Normal}(\mu_0, \sigma_0) \\
\tau &\sim \mathrm{HalfCauchy}(5)
\end{aligned}
\Bigg\} \text{ priors for hyperparameters } \mu, \tau, .
$$

Notice: take SE to be known, only interested in estimating θ_j.

$\mu_0 = 0, \ \sigma_0 = 5$

Draw the model



Computation and computational difficulties

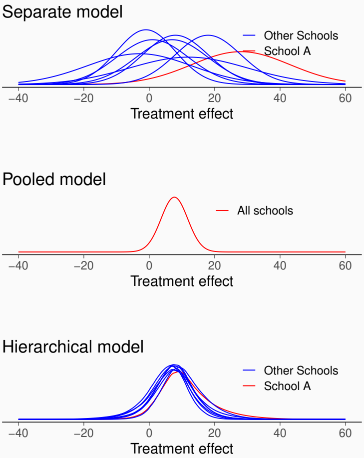
This model is easy to conceptualize, and structurally similar to the bike lane model.

But:

- The hierarchical normal model has some computational challenges
- Difficult for MCMC samplers to explore without re-parameterization

Let's take a look in PyMC3...

Results

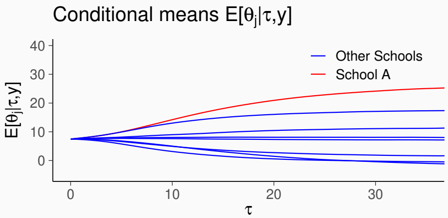


(graphics courtesy Aki Vehtari)

Hierarchical model as a compromise

Remember the (hyper)parameter τ

If we condition on τ:



Hierarchical model is "partial pooling" – compromise between total pooling and separate effects

Amount of pooling controlled by τ; hierarchical model learns this from the data.

Summary

Next week:

- MCMC - what is it?
- How do modern MCMC methods work?
- Diagnosing sampling problems