More covariance and Gaussian processes

ISTA 410 / INFO 510: Bayesian Modeling and Inference

U. of Arizona School of Information

April 14, 2021

Outline

Last time:

- Varying effects models; covariance between intercepts and slopes
- Multivariate normal distributions and covariance matrices

Today:

- Covariance that varies with space or time
- Gaussian processes

Example: political trends

US presidential voting and age

Example due to Gelman and Ghitza:

- Generational model of partisan preferences
- Model influence of political events on preference

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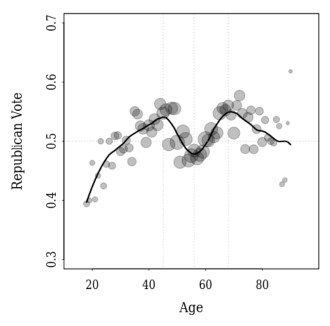
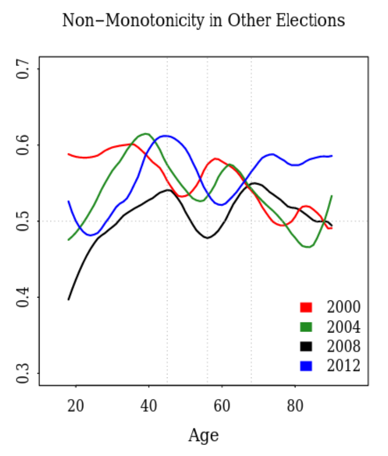


It's a cliche in US politics that older voters are more conservative

US presidential voting and age

What about other elections?



US presidential voting and age

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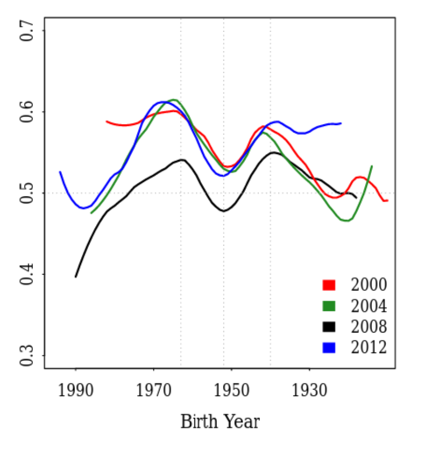


Modeling the birth year effect

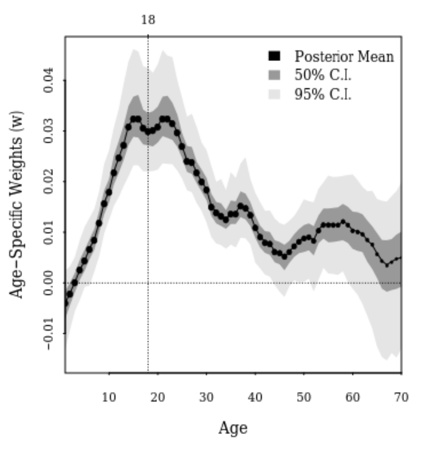
Survey respondents (voters) binned into groups by

- Birth year
- Year of election observed
- race/region $\in$ {minority, Southern white, non-Southern white}

Used to estimate "age weights": how much the political situation at a given age influences

Priors set on age weights to enforce similarity among nearby ages

Age effects


18
Posterior Mean
50% C.I.
95% C.I.
Age−Specific Weights (w)
−0.01
0.00
0.01
0.02
0.03
0.04
10
20
30
40
50
60
70
Age

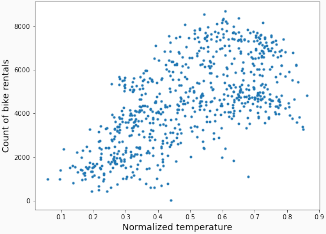
Example: bike share data

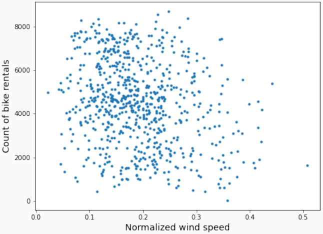
Bike share programs

- Bike share programs: short-term rentals for bicycles
- Have data on count of renters, along with daily weather data

Goal: estimate influence of temperature, wind speed

Make a plot to check reasonableness





Simple Poisson regression model

We have count data, so use Poisson regression:

$$
\begin{aligned}
y_j &\sim \mathrm{Poisson}(\lambda_j) \\
\log \lambda_j &= \alpha + \beta_T T_j + \beta_w w_j \\
\alpha &\sim \mathrm{Normal}(0, 5) \\
\beta_T &\sim \mathrm{Normal}(0, 1) \\
\beta_w &\sim \mathrm{Normal}(0, 1)
\end{aligned}
$$

Results and predictive check

- Summary:

	mean	sd	hdi_3%	hdi_97%
alpha	7.812	0.002	7.808	7.817
beta_temp	1.450	0.003	1.444	1.456
beta_wind	-0.823	0.008	-0.837	-0.809

Results and predictive check

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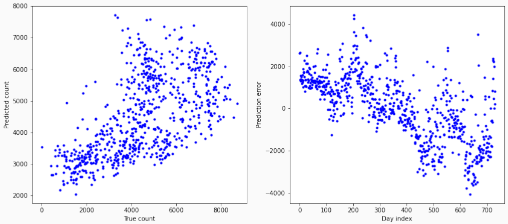
- Posterior predictive error vs. date:



Adding in varying intercepts

The posterior predictions show mediocre fit to data – in particular, prediction error clearly follows a trend over time.

Add in varying intercepts by month:

$$
\begin{aligned}
y_j &\sim \mathrm{Poisson}(\lambda_j) \\
\log \lambda_j &= \alpha_{\mathrm{month}(j)} + \beta_T T_j + \beta_w w_j \\
\beta_T &\sim \mathrm{Normal}(0, 1) \\
\beta_w &\sim \mathrm{Normal}(0, 1) \\
\alpha_{\mathrm{month}(j)} &\sim ?
\end{aligned}
$$

Varying intercepts

We could simply use our usual strategy and do something like:

$$\alpha \sim \text{Normal}(\mu, \tau)$$

with some hyperpriors on μ, τ

- Usual multilevel strategy oriented around the idea of exchangeable groups
- Share information among groups
- Exchangeability: the model doesn't change if we permute the index of the groups
- Time points not really exchangeable

Varying intercepts

Alternative:

- Sample varying intercepts from a multivariate normal with correlations
- Here, we can impose some structure on the correlations:
  - Months closer in time are more similar
  - Months closer in time should have higher correlations
- How do we impose this? Put it into the covariance matrix

New model

Make α a multivariate normal:

$$\boldsymbol{\alpha} \sim \mathrm{MVNormal}\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathsf{K} \right)$$

- Covariance between α_i, α_j should depend on how close months i and j are in time
- So, K_{ij} should be a function of i, j

Covariance function

Set:

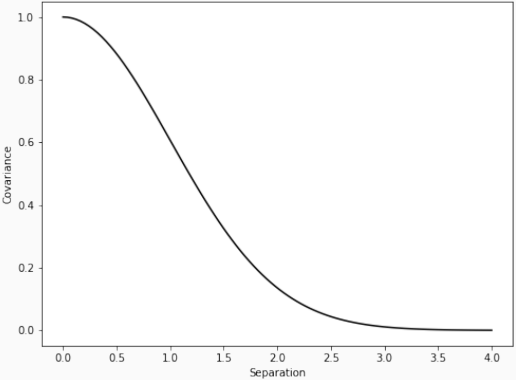
$$\mathsf{K}_{ij} = \eta^2 \exp\left(-\frac{(i-j)^2}{2\ell^2}\right) + \sigma^2\delta_{ij}$$

Parameters:

- η^2 – magnitude of correlations
- ℓ^2 – length scale
- σ^2 – self variance
    - Even if your model doesn't need this, a small amount useful for numerical stability

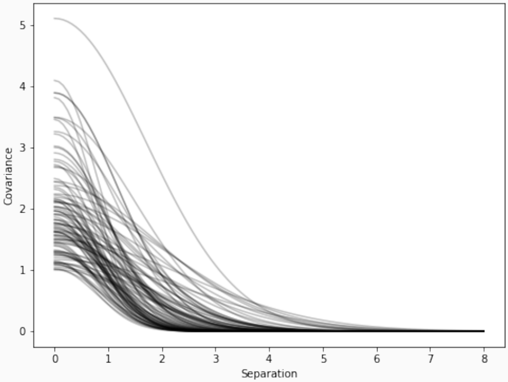
What does this look like?

What this means is the covariance between two αs is a function of their separation:



What does this look like?

Parameterized by varying η^2, ℓ^2:

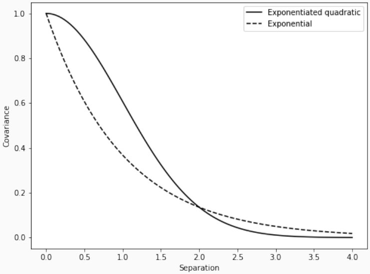


What about a different functional relationship?

The formula from before:

$$\mathsf{K}_{ij} = \eta^2 \exp\left(-\frac{(i-j)^2}{2\ell^2}\right) + \sigma^2\delta_{ij}$$

is an exponentiated quadratic; what about another form?



What to add to the model?

```
with pm.Model() as bike_model:
    beta_temp = pm.Normal('beta_temp', 0, 2)
    beta_wind = pm.Normal('beta_wind', 0, 2)
    eta = pm.Exponential('eta', 1)
    ls = pm.Exponential('ls', 4)

    Kij = (eta ** 2) * pm.math.exp(-(separation ** 2) / (ls ** 2)) + 0.01 * np.
    k = pm.MvNormal('k', mu=tt.zeros(24), cov=Kij, shape = 24)

    theta = pm.math.exp(k[bikes['month_index']] + beta_temp * bikes['temp']
            + beta_wind * bikes['windspeed'])

    y_ = pm.Poisson('y', theta, observed = bikes['cnt'])
```

separation is a 24×24 matrix with `i, j` entry equal to $|i - j|$

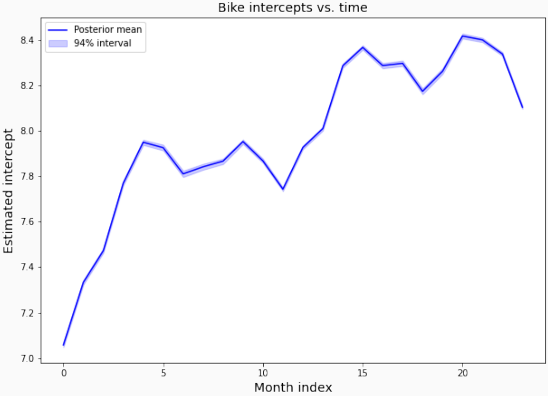
Results

Results from a summary table:

	mean	sd	hdi_3%	hdi_97%
beta_temp	0.965	0.008	0.951	0.982
beta_wind	-0.694	0.008	-0.708	-0.680
alpha[0]	7.058	0.005	7.048	7.069
alpha[1]	7.334	0.005	7.324	7.344
alpha[2]	7.472	0.005	7.463	7.482
alpha[3]	7.769	0.005	7.759	7.779
alpha[4]	7.949	0.006	7.938	7.960

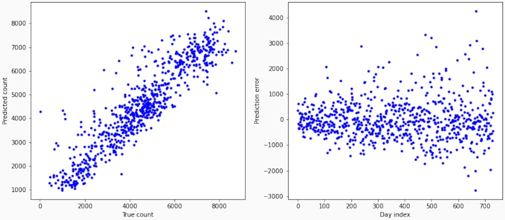
Intercepts over time

We can look at the intercepts estimated as a function of the month:



Posterior predictive check

Same predictive check as before:



Gaussian processes as random functions

Time grouping in the bike example

In the bike share example:

- We used k_{month} as our varying intercept
- Why monthly?

Time grouping in the bike example

In the bike share example:

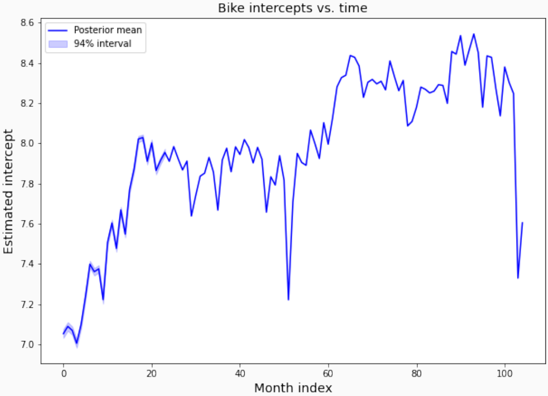
- We used k_{month} as our varying intercept
- Why monthly?

Try weekly instead:

- 105 varying intercepts
- Same approach: 105-dimensional multivariate normal; covariance matrix built in the same way

Intercepts over time

Now we get more resolution on the intercepts:



Gaussian process regression

- Weekly and monthly versions identical in spirit, just with different data resolution for the intercepts
- Unified way to think of this:

$$\log \lambda_j = \alpha(t_j) + \beta_T T_j + \beta_w w_j$$

  where α is a continuous function of time
- We're not trying to estimate a vector from observations of each component
- We're trying to estimate a function from several observations of function values

GP: the definition

A Gaussian process is a random *function* – i.e., we're really talking about a probability distribution on a space of functions.

The feature that makes a GP a GP: if you pick any n values of x, then the vector of function values $(\mu(x_1), \mu_(x_2), \ldots, \mu(x_n))$ has a multivariate normal distribution:

$$(\mu(x_1), \ldots \mu_(x_n)) \sim \mathrm{Normal}((m(x_1), \ldots, m(x_n)), K(x_1, \ldots, x_n))$$

The GP is determined by its mean function m and covariance K.

GP: the definition

Typically, the covariance matrix is determined by a function called the *kernel* $k(x, x')$.

- $k(x, x')$ determines how much the value of $\mu(x)$ depends on $\mu(x')$.
- Common (not universal) property: $k(x, x')$ depends on the distance between x, x'
- Idea: we're looking for continuous functions, so the values of $\mu(x), \mu(x')$ should be close if x, x' are close; but if they're far apart

Squared exponential covariance

Very common choice: squared exponential covariance function:

$$k(x, x') = \eta^2 \exp\left(-\frac{(x - x')^2}{2\ell^2}\right)$$

Covariance is high when $x - x'$ is small, falls off at longer ranges.

Hyperparameters:

- η: the maximum covariance
- ℓ: the *length scale*, controls how quickly covariance decays

In practice

How this is realized in practice:

- We have a set of observations $f(x_i)$
- GP property says

$$f(x_i) \sim \mathrm{MvNormal}((m(x_1), \ldots, m(x_n)), K(x_1, \ldots, x_n))$$

- So we evaluate the covariance function $k(x, x')$ at each pair of observed x values and use that to build a covariance matrix
- The Gaussian process distribution

$$\mathcal{GP}(\mu(\S), \|(\S, \S'))$$

  is really a prior distribution on the space of continuous functions

Summary

Summary:

- Many data sets naturally include observations that should be correlated based on, e.g. time or distance
- Including these correlations amounts to estimating an underlying function
- $\mathcal{GP}$ is a prior distribution on a space of functions, parameterized by a mean function and covariance

Next time:

- More Gaussian process regression
- Various covariance functions