Single Parameter Models

ISTA 410 / INFO 510 - Bayesian Modeling and Inference

University of Arizona School of Information

February 1, 2021

Outline

Last time:

- Joint distribution of several variables
- Conditional probability and independence
- Marginal distributions and marginalization

Today:

- One parameter models:
  - Beta-binomial models and conjugate priors
  - Normal model with known variance
- Summarizing posterior inferences

Recap: that one homework problem

The dice problem

Recall:

- Have a box containing four dice: 1 d4, 2 d6s, 1 d12
- Experiment: select a die at random, roll it 1000 times
- Outcome variable: proportion of 1s

Why this problem? Think about data-generating processes with an underlying, unknown parameter.

The dice problem

The marginal distribution of the count of 1s is a mixture of binomials:

$$y \sim 0.25 \times \mathrm{Binomial}(1000, 1/4) + 0.5 \times \mathrm{Binomial}(1000, 1/6) + 0.25 \times \mathrm{Binomial}(1000, 1/12)$$

Let's sketch this...

The dice problem

The dice problem

Beta-binomial model

Using a Bayesian model

Steps:

- Set up a probabilistic model for the observed data, dependent on un-observed parameters
- Apply a *prior distribution* to the parameters, representing our knowledge before observing data
- Apply Bayes' theorem to update the distribution of the parameters, resulting in a *posterior distribution*
- Summarize relevant results

Bayes' theorem

Recall Bayes' theorem:

$$\Pr(H|E) = \frac{\Pr(E|H)\Pr(H)}{\Pr(E)}$$

Terminology:

- $\Pr(H|E)$ – *posterior probability*
- $\Pr(H)$ – *prior probability*
- $\Pr(E|H)$ – *likelihood*
- $\Pr(E)$ – *normalizing constant*

Bayes' theorem with densities

Most commonly we have probability density functions that depend on unknown parameters:

- y – data
- θ – parameters

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

$$p(\theta|y) \propto p(y|\theta)p(\theta)$$

The normalizing constant $p(y)$ is gotten by marginalizing over θ by computing $\int p(y|\theta)p(\theta)d\theta$; this integral may be intractable, so we work with the proportionality statement.

Binomial model

If we are observing binary categorical outcomes, a binomial likelihood makes sense. $\mathrm{Binomial}(n, \theta)$ is the distribution of the count of "successes" in n independent trials with a fixed probability θ of success.

$$p(y \text{ successes}|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

In the dice problem, θ was an unknown parameter with a distribution

$$\begin{aligned} \Pr(\theta = 1/4) &= 1/4 \\ \Pr(\theta = 1/6) &= 1/2 \\ \Pr(\theta = 1/12) &= 1/4 \end{aligned}$$

Binomial model

Suppose we ran our experiment and got 234 1s. Then, we can plug into Bayes' theorem:

$$\Pr(\theta = 1/4 | y = 234) = \frac{\Pr(y = 234 | \theta = 1/4)\Pr(\theta = 1/4)}{\Pr(y = 234)}$$

$$\frac{(0.25)(1/4)^{234}(3/4)^{766}}{[(0.25)(1/4)^{234}(3/4)^{766} + (0.5)(1/6)^{234}(5/6)^{766} + (0.25)(1/12)^{234}(11/12)^{766}]}$$

which comes out to about 1. (So we can be very certain that we are rolling the d4.)

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So, this is the discrete version. What if we don't have four discrete physical dice, but an unknown continuous θ?

A continuous prior

A common choice of prior for a binomial likelihood is a beta distribution:

$$\theta \sim \mathrm{Beta}(\alpha, \beta)$$

where $\alpha, \beta > 0$ are chosen ahead of time.

Beta distribution: defined on $[0, 1]$ by the PDF

$$p(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$B(\alpha, \beta)$ is the normalizing constant, called a *Beta function*. There are formulas for it but not important for us right now.

What is the data-generating process?

The generative procedure now:

1. Draw a value of θ from $\mathrm{Beta}(\alpha, \beta)$
2. Draw a value of y from $\mathrm{Binomial}(n, \theta)$

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- The dice problem: y's distribution was a finite mixture of binomials, weighted by the PMF of θ
- Now: y's distribution is an infinite mixture of binomials, weighted by the PDF of θ

Conjugate prior

One reason for the choice of beta prior: *conjugacy*

A distribution $p(\theta)$ is conjugate to a likelihood $p(y|\theta)$ if the posterior distribution $p(\theta|y)$ is a member of the same family as $p(\theta)$:

In the beta-binomial model,

$$p(\theta|y) = \frac{1}{p(y)} \frac{1}{B(\alpha, \beta)} p(k|\theta) \left(\begin{array}{c} n \\ k \end{array} \right) \theta^{y} (1-\theta)^{n-y} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

The leading three factors don't depend on θ, so we absorb them into a single constant.

Conjugate prior

Now:

$$\begin{aligned} p(\theta|y) &= \frac{1}{Z}\theta^{y}(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &= \frac{1}{Z}\theta^{\alpha+y-1}(1-\theta)^{\beta+(n-y)-1} \end{aligned}$$

Since the dependence of the density on θ is that of a beta distribution with parameters $(\alpha+y, \beta+(n-y))$, the constant Z must be the corresponding beta function, and

$$\theta|y \sim \mathrm{Beta}(\alpha+y, \beta+(n-y))$$

Computationally very convenient! Convenience less important these days than it used to be, though.

Posterior distribution

So, in a beta-binomial model:

$$y \sim \mathrm{Binomial}(n, \theta)$$
$$\theta \sim \mathrm{Beta}(\alpha, \beta)$$

if we observe y successes and $n - y$ failures, the posterior distribution of θ is

$$\theta | y \sim \mathrm{Beta}(\alpha + y, \beta + (n - y))$$

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Interpretation: we may think of the prior parameters $\alpha - 1, \beta - 1$ as *pseudocounts*

Summarizing inferences; example

Inferences from the posterior

The posterior distribution is the primary product of inference; it contains all that we know about the parameter after incorporating prior and data.

In practice, often want to distill out some summary statistics:

- posterior mean – expected value of θ under the posterior distribution
- posterior intervals – 95% common, but arbitrary. Note difference between highest density and central intervals
- maximum a posteriori estimate – often not a good choice, especially if the model has many parameters

Example: Pfizer's vaccine trial

Prominent recent example: beta-binomial model in analysis of Pfizer's COVID-19 vaccine

Trial procedure:

- Study participants divided randomly into two "arms": control/placebo and vaccine
- Control arm given placebo, vaccine arm given vaccine
- Watch both groups and count cases, running the analysis when a predetermined number of cases is observed

Beta-binomial model

Defining parameters:

- π_c: probability that a control subject becomes ill
- π_v: probability that a vaccinated subject becomes ill
- Derived quantity: Vaccine efficacy:

$$VE = 1 - \frac{\pi_v}{\pi_c}$$

Parameter for the model:

$$\theta = \frac{1 - VE}{2 - VE} = \frac{\pi_v}{\pi_v + \pi_c}$$

Measures the probability that a case came from the vaccine arm

Pfizer's prior

Let y be the number of cases that come from the vaccinated group.

The model:

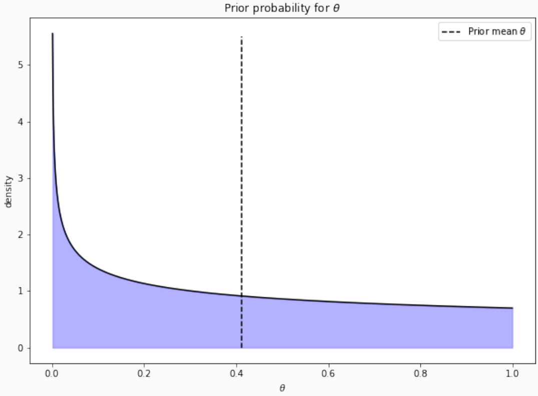
$$y \sim \mathrm{Binomial}(\theta, n)$$
$$\theta \sim \mathrm{Beta}(0.700102, 1)$$

Prior was stated in Pfizer's press release. No specific reason given for these parameters, but:

- VE at prior mean θ is 30%
- fairly uninformative: 95% interval is about $(-26.2, 0.995)$.

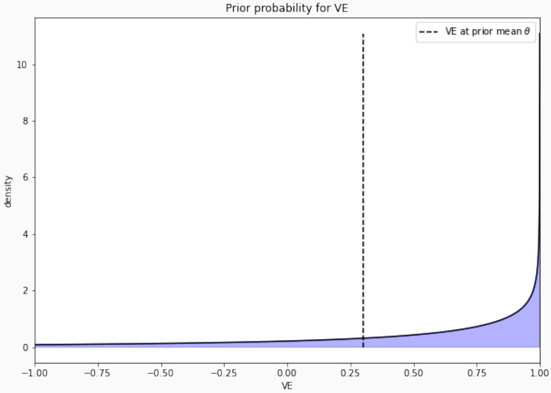
Pfizer's prior

On the θ scale:



Pfizer's prior

On the VE scale



What's the data?

The result of the study submitted to the FDA to obtain an emergency use authorization had a total of 170 observed cases, 8 of which were in the vaccine arm. So:

$$\theta | y \sim \mathrm{Beta}(0.700102 + 8, 1 + 162)$$

Let's examine this graphically...

Summary

Today:

- Beta-binomial model
- Posterior summary statistics

Next time:

- Normal model with known variance
- Some considerations for choosing priors

HW1 on D2L tonight! Due Friday, 2/12.