Operational Research Project

Christophoros Bekos

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1 Problem statement

Read "Project_OR_2018.pdf"

2 Modeling the solution

2.1 Definition of parameters and constants

We assume that :

1. x_{ij} = amount of gold (10^3Kg) mined by mine i during the year j (variable)
2. max_i = maximum amount of gold (10^3Kg) mined by mine i during any year (parameter)
3. c_{ij} = operating cost of mine i (contributions that must be paid) for year j (parameter)
4. PF_j = the profit (million \$) per each ton of the final mix for year j, in terms of present value (parameter)
5. $y_{ij} = \begin{cases} 0, & \text{mine i is not used during year j} \\ 1, & \text{otherwise} \end{cases} \quad (integer\ \ variable)$
6. $cost_{ij} = \begin{cases} 1, & \text{we pay the taxes of mine i for year j} \\ 0, & \text{otherwise} \end{cases} \quad (binary\ \ variable)$
   This is a helper variable, which is used in order to properly model constraint $[C1]$.
7. π_i = the quality of the gold mined in the mine i (parameter)
8. Π_j = the required mix quality for the year j (parameter)

Having that said , if we convert mine operation costs to present value , then matrix c if filled as shown below :

Cost table (c)

	Year				
Mine	1	2	3	4	5
1	5.0000	4.5455	4.1322	3.7566	3.4151
2	4.0000	3.6364	3.3058	3.0053	2.7321
3	4.0000	3.6364	3.3058	3.0053	2.7321
4	5.0000	4.5455	4.1322	3.7566	3.4151

Table 1: cost (million \$) for operating mine through years , computed in terms of present value

We also define the contents of the profit vector :

Profit vector (PF)

	Year				
	1	2	3	4	5
Profit per ton	10.0000	9.0909	8.2645	7.5131	6.8301

Table 2: profit (million $) per each ton of the final mix , in terms of present value

2.2 Definition of the objective function

In general , we want to maximize z , where z is the total profit of each year j

$$z = \sum_{j=1}^{5} \ (income_j \text{ - } costs_j)$$

Using a more mathematical way to describe income and costs , we could write :

$$total\_income = \sum_{j=1}^{5} PF_j \sum_{i=1}^{4} x_{ij}$$

and

$$total\_costs = \sum_{j=1}^{5} \sum_{i=1}^{4} c_{ij} cost_{ij}$$

One can observe that by defining income and costs like this , we may violate constraint [C1] , since we use all 4 mine in order to calculate yearly profit (while maximum number of active mine is 3) and also we consider as costs the term c_{ij} even if a mine i may be close permanently after some year j. However there is no mistake, these constraints will be taken into account by implementing the [I_C1] .

2.3 Definition of the constraints

- [C1] : Each year the sum of active mine cannot be larger than 3 . While mine i may remain closed for year j , contributions for j must be paid , in order the company to be able to re-open the mine some other year k (k>j).
- [I_C1] This is the mathematical implementation of constraint [C1] , among with some explanations .
  If we take in account [C1] , the total costs (after 5 years) for mine i will be :

$$Total\_Costs_{ij} = \begin{cases} \sum_{j=1}^{k-1} c_{ij} & \text{if } \sum_{j=k}^{5} y_{ij} = 0 \\ \\ \sum_{j=1}^{5} c_{ij}, & \text{otherwise} \end{cases}$$

  Generally we may define : $Total\_Costs_{ij} = \sum_{j=1}^{5} c_{ij} cost_{ij}$ and the above equation continues to have to be fulfilled.
  This can be easily understood if we consider that we would be forced to pay every year's contributions in order the mine i to be active at year 5 , but if we wouldn't use mine i after year 3 , then we should only pay contributions for the first three years. Now we should propose a way of modeling the above equation in terms of linear programming (must be modeled as a constraints). Using the [A1] we may write :

$$\sum_{j=1}^{5} c_{ij} cost_{ij} - \sum_{j=1}^{k-1} c_{ij} \leq M \sum_{j=k}^{5} y_{ij} \ \ \forall (i, k)$$

  Notice that the above inequality must be satisfied for each i and for each k = 1 ,...,5 , which gives us 20 inequalities.

Also the following constraint must be fulfilled :

$$\sum_{j=1}^{5} c_{ij}cost_{ij} - \sum_{j=1}^{k-1} c_{ij} \geq 0 \ \ \forall i$$

Now we need to ensure that if a mine i is not used during year j , then the income will not take in account any profits, x_{ij} from that mine . This can be written as :

$$x_{ij} = \begin{cases} 0 & \text{if } y_{ij} = 0 \\ \\ x_{ij} & \text{if } y_{ij} = 1 \end{cases}$$

As explained before , the above equation will be modeled as a multitude of constraints :

$$x_{ij} \leq My_{ij} \text{ and } x_{ij} \geq 0 \ \ \forall(i,j)$$

Now we need to ensure that our binary variables (y_{ij}) will satisfy the constraint :

$$\sum_{i=1}^{4} y_{ij} \leq 3 \ \ \forall j = 1, ..., 5$$

Also, by definition of y_{ij} and $cost_{ij}$ the following constraint must be fulfilled

$$cost_{ij} >= y_{ij} \ \ \forall(i,j)$$

- [C2] : The amount of gold mined from mine i ,during any year j, cannot be more than max_i
- [I_C2] : The mathematical implementation of constraint [C2]

$$x_{ij} \leq max_i \ \ \forall j = 1, ..., 5$$

- [C3] : For each year j , the final mix (each mine has a different quality of gold) will have quality equal to the weighted average of the quantities of blended minerals (using as weights the quantities of mixed minerals) . Each year's j final quality must equal to Π_j .
- [I_C3] : The mathematical implementation of constraint [C3] :
for year j the percentage of participation in the final quality, of the mine i is equal to $\frac{\pi_i x_{ij}}{\sum_{i=1}^{4} x_{ij}}$.
Having that in mind, we could model C3 as :

$$\frac{\sum_{i=1}^{4} \pi_i x_{ij}}{\sum_{i=1}^{4} x_{ij}} = \Pi_j \ \ \forall j = 1, ..., 5$$

2.4 Final model formulation

maximize :

$$z = \sum_{j=1}^{5} PF_j \sum_{i=1}^{4} x_{ij} - \sum_{j=1}^{5}\sum_{i=1}^{4} c_{ij}cost_{ij} \ \ (million \ \ \$)$$

s.t.:

$$\sum_{i=1}^{4} y_{ij} \leq 3 \ \ \forall j = 1, ..., 5 \qquad (C1.1)$$

$$\sum_{j=1}^{5} c_{ij}cost_{ij} - \sum_{j=1}^{k-1} c_{ij} \leq M \sum_{j=k}^{5} y_{ij} \ \ \forall(i,k) \qquad (C1.2)$$

$$cost_{ij} >= y_{ij} \ \ \forall(i,j) \qquad (C1.21)$$

$$x_{ij} \leq My_{ij} \text{ and } x_{ij} \geq 0 \quad \forall(i,j) \qquad (C1.3)$$

$$\sum_{j=1}^{5} c_{ij} cost_{ij} - \sum_{j=1}^{k-1} c_{ij} \geq 0 \ \ \forall i \qquad (C1.4)$$

$$x_{ij} \leq max_i \ \ \forall j = 1, ..., 5 \qquad (C2.1)$$

$$\sum_{i=1}^{4} \pi_i x_{ij} = \Pi_j \sum_{i=1}^{4} x_{ij} \quad \forall j = 1, ..., 5 \qquad (C3.1)$$

3 Ampl implementation

The proj1.dat file contains the values of parameters c,PF,π (named pi) and Π (named Pi). Also we define two sets , named Mines and Years , which contain the indexes for different mines (1,...,4) and for each year (1,...,5) respectively. File proj1.mod is used in order to model the objective function and the constraints , as follows :

3.1 Objective function

```
maximize total_profit: sum{j in Years}(PF[j]*(sum{i in Mines}x[i,j] ))
               - sum{j in Years}((sum{i in Mines}c[i,j]*cost[i,j] ));
```

3.2 Constraint C1.1

```
subject to c11 {j in Years}:
   sum {i in Mines} y[i,j] <= 3;
```

3.3 Constraint C1.2

```
subject to c12 {i in Mines, k in Years }:
   sum{j in Years}c[i,j]*cost[i,j] - sum{j in Years : j <= k-1}c[i,j] <= sum{j
       in Years : j>=k} (bigM*y[i,j]);
```

3.4 Constraint C1.21

```
subject to c121 {i in Mines,j in Years }:
   cost[i,j] >= y[i,j];
```

3.5 Constraint C1.3

```
subject to c13 {i in Mines, j in Years}:
   x[i,j] <= bigM*y[i,j];
```

3.6 Constraint C1.4

```
subject to c14 {i in Mines, k in Years }:
   sum{j in Years}c[i,j]*cost[i,j] - sum{j in Years : j <= k-1}c[i,j] >= 0;
```

3.7 Constraint C2.1

```
subject to c2 {i in Mines,j in Years}:
   x[i,j] <= max_x[i];
```

3.8 Constraint C3.1

```
subject to c3 {j in Years}:
   (sum{i in Mines}pi[i]*x[i,j]) = Pi[j]*(sum{i in Mines}x[i,j]);
```

4 Results

One can test the model by typing the commands :
reset; model proj1.mod; data proj1.dat; options solver cplex; solve;
By typing "display total_profit , y , x ;" we may observe the best solution :

X_{best}

	Year				
Mine	1	2	3	4	5
1	2	0	1.95	0.125	2
2	0	2.5	0	2.5	2.16667
3	1.3	1.3	1.3	0	1.3
4	2.45	2.2	0	3	0

Table 3: Amount of gold (tones) that must be mined by each mine in order to achieve the best result (maximize total profit)

Y_{best}

	Year				
Mine	1	2	3	4	5
1	1	0	1	1	1
2	0	1	0	1	1
3	1	1	1	0	1
4	1	1	0	1	0

Table 4: The values for y_{ij} that lead to the best solution

Using the above values for matrices x and y all constraints are fulfilled
and total_profit = 145.8611925 million $ (in terms of present value).

5 Appendix

- [A1] Modeling a double sided equation as constraint in IP problem (maximization only)
  Let's consider the following double equation :

$$x = \begin{cases} A & \text{if } y = 0 \\ \\ x & \text{otherwise} \end{cases}$$

  where y,A and B can be variables . The above equation can be modeled as a constraint into an IP problem :

$$x - A \leq My \text{ and } x - A \geq 0$$

  where M is a sufficiently large quantity. This would be enough in problems where we want to maximize an objective z . If y = 0 , then x - A becomes less or equal to zero, and since x - A can only be greater or equal to zero, this means that every time y = 0 ,x is forced to take the value of A.