Fast* Implementations of Finite Fields and Elliptic Curves in Lean 4

Matej Penciak

Lurk Lab

March 23, 2023

Slides available at:
https://github.com/mpenciak/LLL2023

Outline

1. Motivation: What we're building at Lurk Lab
2. A bit about zero knowledge cryptography
3. Optimizing FF arithmetic
   - Addition chains for fast exponentiation
   - Batched inversion
   - Montgomery form
4. Optimizing elliptic Curve arithmetic
   - Choice of coordinates
   - Efficient endomorphisms
   - Multi-scalar multiplication
5. Brief demo!
6. Lessons learned and future outlook

Motivation

Who am I? Software engineer and aspiring cryptographer at Lurk Lab (Formerly Yatima).

Motivation

Who am I? Software engineer and aspiring cryptographer at Lurk Lab (Formerly Yatima).

What is Lurk Lab? We're building the Lurk programming language, and an ecosystem of tools around it. Examples: The Yatima compiler and content addresser, a Lean typechecker, and the cryptography to generate "Lurk proofs"

Motivation

Who am I? Software engineer and aspiring cryptographer at Lurk Lab (Formerly Yatima).

What is Lurk Lab? We're building the Lurk programming language, and an ecosystem of tools around it. Examples: The Yatima compiler and content addresser, a Lean typechecker, and the cryptography to generate "Lurk proofs"

What is Lurk? Lurk is a turing complete programming language in the style of Lisp which generates zero knowledge proofs of execution.

Motivation

Who am I? Software engineer and aspiring cryptographer at Lurk Lab (Formerly Yatima).

What is Lurk Lab? We're building the Lurk programming language, and an ecosystem of tools around it. Examples: The Yatima compiler and content addresser, a Lean typechecker, and the cryptography to generate "Lurk proofs"

What is Lurk? Lurk is a turing complete programming language in the style of Lisp which generates zero knowledge proofs of execution.

If I run a Lurk program `f`, I can prove things like:

- I have run the computation of `f` applied to `u` and gotten `v`
- I know an input `u` to `f` such that `f(u) = v` without revealing `u`
- I can commit ahead of time to a program `f` and run it on any inputs $\{$`u`$\}$ you provide with proofs of execution.

A little about zero knowledge proofs

Two components to a zero knowledge proof system:

- An interactive proof scheme (the “information-theoretic” component)
- A commitment scheme (the “cryptographic” component)

A little about zero knowledge proofs

Two components to a zero knowledge proof system:

- An interactive proof scheme (the "information-theoretic" component)
- A commitment scheme (the "cryptographic" component)

An **IP scheme** is a protocol between two parties `P` (prover) and `V` (verifier) where (the potentially untrustworthy) `P` can provide a convincing argument to `V` that they know some fact/have performed some computation.

A little about zero knowledge proofs

Two components to a zero knowledge proof system:

- An interactive proof scheme (the “information-theoretic” component)
- A commitment scheme (the “cryptographic” component)

An **IP scheme** is a protocol between two parties `P` (prover) and `V` (verifier) where (the potentially untrustworthy) `P` can provide a convincing argument to `V` that they know some fact/have performed some computation.

A **commitment scheme** is an interactive protocol for one party to commit to some data (a vector, polynomial, etc...) without revealing its value. The commitment can be opened later, and it should be computationally infeasible for the committer to change their committed value

An example IP

P and V agree to two graphs G_1 and G_2, and P wants to prove to V that they know an isomorphism $\phi : G_1 \to G_2$ without revealing the isomorphism.

An example IP

`P` and `V` agree to two graphs G_1 and G_2, and `P` wants to prove to `V` that they know an isomorphism $\phi : G_1 \to G_2$ without revealing the isomorphism.

1. `P` chooses a random permutation σ of G_1, and sends $H = \sigma(G_1)$ to `V`
2. `V` chooses randomly between $\{1, 2\}$ and sends the choice i to `P`
3. `P` responds with an isomorphism $\phi_i : G_i \to H$
4. `V` checks that in fact $\phi_i(G_i) \cong H$
5. Repeat until `V` is sufficiently convinced

An example IP

P and V agree to two graphs G_1 and G_2, and P wants to prove to V that they know an isomorphism $\phi : G_1 \to G_2$ without revealing the isomorphism.

1. P chooses a random permutation σ of G_1, and sends $H = \sigma(G_1)$ to V
2. V chooses randomly between $\{1, 2\}$ and sends the choice i to P
3. P responds with an isomorphism $\phi_i : G_i \to H$
4. V checks that in fact $\phi_i(G_i) \cong H$
5. Repeat until V is sufficiently convinced

This protocol is **complete**, **knowledge sound**, and **zero knowledge**

An example IP

`P` and `V` agree to two graphs G_1 and G_2, and `P` wants to prove to `V` that they know an isomorphism $\phi : G_1 \to G_2$ without revealing the isomorphism.

1. `P` chooses a random permutation σ of G_1, and sends $H = \sigma(G_1)$ to `V`
2. `V` chooses randomly between $\{1, 2\}$ and sends the choice i to `P`
3. `P` responds with an isomorphism $\phi_i : G_i \to H$
4. `V` checks that in fact $\phi_i(G_i) \cong H$
5. Repeat until `V` is sufficiently convinced

This protocol is **complete**, **knowledge sound**, and **zero knowledge**

It is not: **non-interactive**, or **succinct**
The goal is to build **Z**ero **K**nowledge **S**uccinct **Ar**guments of **K**nowledge (ZK SNARKs) in Lean.

The basic building blocks for cryptography

Work over finite fields, and encode our computations in so-called "arithmetic circuits".

The level of security necessary is measured in the number of "bits" of security. To prevent a dishonest prover from forging a fraudulent proof, a good estimate is ≈ 128 bits of security.

The basic building blocks for cryptography

Work over finite fields, and encode our computations in so-called "arithmetic circuits".
The level of security necessary is measured in the number of "bits" of security. To prevent a dishonest prover from forging a fraudulent proof, a good estimate is ≈ 128 bits of security.

Example 1: The fastest (publicly) known factorization/discrete logarithm algorithm is the general number field sieve method with complexity

$$\exp\left(\left(\sqrt[3]{\frac{64}{9}} + o(1)\right)(log n)^{\frac{1}{3}}(\log\log n)^{\frac{2}{3}}\right)$$

To achieve ≈ 128 bits of security we need to work over finite fields of size $\approx 2^{2048}$

Elliptic curve cryptography

Example 2: Let E be an elliptic curve with a cyclic subgroup of points of size N.
The best (publicly) known discrete logarithm algorithm (given $P = a \cdot G$, find $a \in \mathbb{N}$) is the Pollard-ρ method which has complexity $O(\sqrt{N})$.
To achieve ≈ 128 bits of security we only need a curve with $\approx 2^{256}$ number of points.

Example 2: Let E be an elliptic curve with a cyclic subgroup of points of size N.
The best (publicly) known discrete logarithm algorithm (given $P = a \cdot G$, find $a \in \mathbb{N}$) is the Pollard-ρ method which has complexity $O(\sqrt{N})$.
To achieve ≈ 128 bits of security we only need a curve with $\approx 2^{256}$ number of points.

Hasse's bound, makes it "easy" to find elliptic curves with enough points – if E is an elliptic curve defined over a finite field with p elements, then the number of points N is bounded by:

$$|N - (p + 1)| \leq 2\sqrt{p}$$

When we discuss curves, we will see some magical curves with extra nice properties

Big numbers in Lean

Most computers don't natively support arithmetic for numbers larger than $2^{64} - 1$, stored in the Lean datatype `UInt64`.

Lean 4's `Nat` and `Int` types are built on multiprecision arithmetic libraries written in C(++), and linked in via the `@[extern "..."]` attribute

```
@[extern "lean_nat_add"]
protected def Nat.add : (@& Nat) → (@& Nat) → Nat
    | a, Nat.zero   => a
    | a, Nat.succ b => Nat.succ (Nat.add a b)
```

Big numbers in Lean

Most computers don't natively support arithmetic for numbers larger than $2^{64} - 1$, stored in the Lean datatype `UInt64`.
Lean 4's `Nat` and `Int` types are built on multiprecision arithmetic libraries written in C(++), and linked in via the `@[extern "..."]` attribute

```
@[extern "lean_nat_add"]
protected def Nat.add : (@& Nat) → (@& Nat) → Nat
    | a, Nat.zero   => a
    | a, Nat.succ b => Nat.succ (Nat.add a b)
```

which calls the C function defined in `lean.h`

```
static inline lean_obj_res
    lean_nat_add(b_lean_obj_arg a1, b_lean_obj_arg a2) {
     . . .
}
```

Multiprecision arithmetic libraries

Lean uses the **G**NU **M**ulti**P**recision library on Linux based systems. Mac and Windows get a home-brewed version written in the Lean 4 C++ runtime code.

Multiprecision arithmetic libraries

Lean uses the **G**NU **M**ulti**P**recision library on Linux based systems. Mac and Windows get a home-brewed version written in the Lean 4 C++ runtime code.

Both implementations represent `Nats` as arrays of `UInt64s`, and define arithmetic in base 2^{64}. If n is the number of "limbs" (`uint64_ts` in C) then the efficiency of some of the most commonly employed algorithms are:

1. Schoolbook multiplication: $O(n^2)$
2. Karatsuba multiplication: $O(n^{1.58})$
3. Strassen FFT algorithm: $O(n \log n \log \log n)$

Multiprecision arithmetic libraries

Lean uses the **G**NU **M**ulti**P**recision library on Linux based systems. Mac and Windows get a home-brewed version written in the Lean 4 C++ runtime code.

Both implementations represent `Nats` as arrays of `UInt64s`, and define arithmetic in base 2^{64}. If n is the number of "limbs" (`uint64_ts` in C) then the efficiency of some of the most commonly employed algorithms are:

1. Schoolbook multiplication: $O(n^2)$
2. Karatsuba multiplication: $O(n^{1.58})$
3. Strassen FFT algorithm: $O(n \log n \log \log n)$

We tried using fixed precision arithmetic based off Lean's `ByteArray` type instead of `Nat`.
This turned out to be $\approx 100\times$ slower than `Nat`, so we quickly dropped that plan.

A comment on benchmarking and testing

Testing and benchmarking is important for non-formalized systems. For testing we have built a testing framework called `LSpec` which is expressive and easy to use.

A comment on benchmarking and testing

Testing and benchmarking is important for non-formalized systems. For testing we have built a testing framework called `LSpec` which is expressive and easy to use.

```
#lspec check "add_comm"
    (forall n m : Nat, n + m = m + n)
#lspec check "not true"
    (forall n m : Nat, n^2 + m^2 >= n^3 + m)
```

```
? add_comm
x not true
===================
Found problems!
n := 2
m := 0
issue: 8 <= 4 does not hold
(1 shrinks)
```

A comment on benchmarking and testing

For benchmarking we have a rudimentary suite of tools for running benchmarks:

```
import YatimaStdLib.Benchmark
import YatimaStdLib.AddChain

open Benchmark

def f1 : Nat → Nat := (37 ^ .)
def f2 : Nat → Nat := Exp.fastExp 37
def addChainBench : Comparison f1 f2 where
    inputs := #[10000000, 15000000,  ...]

def main (args : List String) : IO UInt32 :=
    benchMain args addChainBench.benchmark
```

Benchmarking continued...

This can be run as `Benchmarks-AddChain --num 50 --log out.bench` and times the benchmarks and writes to the file:

```
10000000: f:213993503 vs g:236587657
15000000: f:315046597 vs g:335218176
20000000: f:457342988 vs g:537591730
25000000: f:598272104 vs g:681122849
30000000: f:670924537 vs g:743308727
35000000: f:786821581 vs g:940891049
40000000: f:950924114 vs g:1103080520
45000000: f:1108323608 vs g:1213150412
50000000: f:1242277288 vs g:1288141664
```

Back to fields

The arithmetic typeclasses we have (so far):

- `Ring`: Basic arithmetic: $+$, $*$, and so on
- `Field`: Ring together with inversion
- `PrimeField`: Arithmetic, plus pre-computations and methods to improve efficiency
- `NewField`: Automatically generated instance of a `PrimeField` with special methods related to Montgomery form
- `GaloisField`: Fields obtained as extension fields of a base field

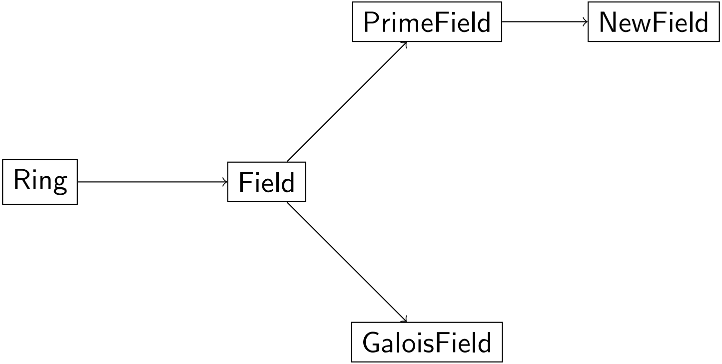
Back to fields

The arithmetic typeclasses we have (so far):

- `Ring`: Basic arithmetic: $+$, $*$, and so on
- `Field`: Ring together with inversion
- `PrimeField`: Arithmetic, plus pre-computations and methods to improve efficiency
- `NewField`: Automatically generated instance of a `PrimeField` with special methods related to Montgomery form
- `GaloisField`: Fields obtained as extension fields of a base field

No `Prop` valued fields! Nothing is formalized (yet).

Field typeclasses



Example classes

```
class Ring (R : Type _) extends Add R, Mul R, Sub R,
        HPow R Nat R, BEq R, Coe Nat R where
    zero : R
    one : R
class PrimeField (K : Type _) extends Field K where
    char : Nat -- characteristic
    sqrt : K → Option K -- Fast `sqrt` implementation
    content : Nat -- char.log2
    twoAdicity : Nat x Nat -- `(s, r)` where `p-1 = 2^s * r'
    legAC : Array ChainStep  -- Pre-computed AddChains to
    frobAC : Array ChainStep -- calculate .^p and .^(p-1)/2
    fromNat : Nat → K -- to and from `Nat`' methods
    natRepr : K → Nat
    batchedExp : K → Array Nat → Array K
    batchedInv : Array K → Array K
```

AddChains, batched arithmetic, and Montgomery form

AddChains: Exponentiation is a commonly used operation, but can be very slow.
Naive implementation: $a^N = a \cdot a^{N-1}$ is $O(N)$ (too slow)

AddChains, batched arithmetic, and Montgomery form

AddChains: Exponentiation is a commonly used operation, but can be very slow.
Naive implementation: $a^N = a \cdot a^{N-1}$ is $O(N)$ (too slow)

Square and multiply method requires $\approx \frac{3}{2} \log N$ operations
($\log N$ squarings plus the Hamming weight of N's binary expansion)
(**WARNING**: susceptible to side channel attacks)

AddChains, batched arithmetic, and Montgomery form

AddChains: Exponentiation is a commonly used operation, but can be very slow.
Naive implementation: $a^N = a \cdot a^{N-1}$ is $O(N)$ (too slow)

Square and multiply method requires $\approx \frac{3}{2} \log N$ operations
($\log N$ squarings plus the Hamming weight of N's binary expansion)
(**WARNING**: susceptible to side channel attacks)

For commonly used exponents (p and $\frac{p-1}{2}$): pre-compute a minimal length "addition chain": $[n_1, n_2, n_3, ..., n_r = N]$ where each $n_i = n_k + n_l$ for $k, l < i$.

AddChains, batched arithmetic, and Montgomery form

AddChains: Exponentiation is a commonly used operation, but can be very slow.
Naive implementation: $a^N = a \cdot a^{N-1}$ is $O(N)$ (too slow)

Square and multiply method requires $\approx \frac{3}{2} \log N$ operations
($\log N$ squarings plus the Hamming weight of N's binary expansion)
(**WARNING**: susceptible to side channel attacks)

For commonly used exponents (p and $\frac{p-1}{2}$): pre-compute a minimal length "addition chain": $[n_1, n_2, n_3, ..., n_r = N]$ where each $n_i = n_k + n_l$ for $k, l < i$.

Example: Calculating the Frobenius modulo the prime:
`0x40000000000000000000000000000000224698fc094cf91b992d30ed00000001`
381 field ops using square and multiply, and 337 field ops using a minimal addition chain.

Batched inversion

Batched inversion:
Inversion is another costly operation (inversion in $\mathbb{Z}/p\mathbb{Z}$ is $O(\log p)$ using the extended Euclidean algorithm)
To invert $[n_1, n_2, n_3, \ldots, n_r]$, calculate $N_j = \prod_i^j n_i$ for $j = 1, \ldots, r$ caching the results. Invert N_r. Recover all of the n_i^{-1} by multiplying with the cached results.

Batched inversion

Batched inversion:
Inversion is another costly operation (inversion in $\mathbb{Z}/p\mathbb{Z}$ is $O(\log p)$ using the extended Euclidean algorithm)
To invert $[n_1, n_2, n_3, \ldots, n_r]$, calculate $N_j = \prod_i^j n_i$ for $j = 1, \ldots, r$ caching the results. Invert N_r. Recover all of the n_i^{-1} by multiplying with the cached results.

Example: Inverting the set $[2, 3, 5]$.

1. Calculate $N_1 = 2$, $N_2 = 6$, $N_3 = 30$.
2. Invert N_3 : $N_3^{-1} = 30^{-1}$, let $t = N_3^{-1}$
3. Multiply $t \cdot N_2 = 30^{-1} \cdot 6 = 5^{-1}$, and $t = t \cdot n_3 = 6^{-1}$
4. Proceed as before: $t \cdot N_1 = 6^{-1} \cdot 2 = 3^{-1}$, and $t = t \cdot n_2 = 2^{-1}$

This method replaces r field inversions with 1 field inversion and $O(3r)$ field multiplications.

Montgomery form

Problem: How should we calculate $n \cdot m$ in $\mathbb{F}_p$ when we use `Nat` representatives?
Naive algorithm: Calculate $a \cdot b \mod p$, but this requires division by p for every multiplication.

Montgomery form

Problem: How should we calculate $n \cdot m$ in $\mathbb{F}_p$ when we use `Nat` representatives?
Naive algorithm: Calculate $a \cdot b \mod p$, but this requires division by p for every multiplication.

Intuition behind the solution: Replace reduction modulo p with reduction modulo R for some other R.
Let $R > p$ be a number co-prime to p for which division and reduction modulo R is efficient.
In practice: Choose R to be a power of 2 greater than p.
Represent $n \in \mathbb{Z}/p\mathbb{Z}$ in "Montgomery form" $[n]_w = n \cdot R \mod p$.
Montgomery reduction is an efficient algorithm that calculates $nR^{-1} \mod p$ using only arithmetic modulo R.

Montgomery form

Problem: How should we calculate $n \cdot m$ in $\mathbb{F}_p$ when we use `Nat` representatives?
Naive algorithm: Calculate $a \cdot b \mod p$, but this requires division by p for every multiplication.

Intuition behind the solution: Replace reduction modulo p with reduction modulo R for some other R.
Let $R > p$ be a number co-prime to p for which division and reduction modulo R is efficient.
In practice: Choose R to be a power of 2 greater than p.
Represent $n \in \mathbb{Z}/p\mathbb{Z}$ in "Montgomery form" $[n]_w = n \cdot R \mod p$.
Montgomery reduction is an efficient algorithm that calculates $nR^{-1} \mod p$ using only arithmetic modulo R.

Addition: $[n]_w + [m]_w = (n+m) \cdot R \mod p = [n+m]_w$
Multiplication: $[n]_w \cdot [m]_w = (n \cdot m)R^2 \mod p = [n \cdot m]_w R \mod p$
$\longrightarrow$ Montgomery reduce $[n \cdot m]_w$

NewField implementation

We implemented a `macro_rule` that expands the following declaration

```
new_field TestField with
    prime: 2011
    generator: 7
    root_of_unity: 2010 -- optional
```

into a type `TestField` with pre-computations and arithmetic, instances, and methods implemented.

NewField implementation

We implemented a `macro_rule` that expands the following declaration

```
new_field TestField with
    prime: 2011
    generator: 7
    root_of_unity: 2010 -- optional
```

into a type `TestField` with pre-computations and arithmetic, instances, and methods implemented.

Not advised to be used yet: Missing some key GMP methods to make Montgomery reduction more efficient (`mpz_tdiv_r_2exp`).

Part of Montgomery reduction of $a \cdot R \in \mathbb{N}$ involves working modulo R, which is not more efficient in Lean at this time.

Elliptic Curves

Immediate goals are to work with curves defined over fields of large characteristic, so short Weierstrass form is fine:

```
/--
Curves with Weierstrass form satisfying the equation
`y² = x³ + a x + b` for a prime field `F` such that
`char K > 3`
-/
structure Curve (F : Type _) [Field F] where
    a : F
    b : F
```

Arithmetic on curves

Arithmetic is more interesting:

```
class CurveGroup {F : outParam $ Type _} [Field F]
        (K : Type _) (C : outParam $ Curve F)  where
    zero : K
    inv : K → K
    add : K → K → K
    double : K → K
```

Arithmetic on curves

Arithmetic is more interesting:

```
class CurveGroup {F : outParam $ Type _} [Field F]
        (K : Type _) (C : outParam $ Curve F)  where
    zero : K
    inv : K → K
    add : K → K → K
    double : K → K
```

Separating the curve from its arithmetic is important to allow for different coordinate systems on the same curve.

Arithmetic on curves

Arithmetic is more interesting:

```
class CurveGroup {F : outParam $ Type _} [Field F]
        (K : Type _) (C : outParam $ Curve F)  where
    zero : K
    inv : K → K
    add : K → K → K
    double : K → K
```

Separating the curve from its arithmetic is important to allow for different coordinate systems on the same curve.

Projective coordinates avoid field inversions!
Affine coordinates are more compact on disk!
More exotic coordinate systems have other benefits.

Coordinate breakdown

Figure: Chart taken from *Handbook of Elliptic and Hyperelliptic Curve Cryptography*

Doubling		Addition	
Operation	Costs	Operation	Costs
$2\mathcal{P}$	$7\mathrm{M}+5\mathrm{S}$	$\mathcal{J}^m+\mathcal{J}^m$	$13\mathrm{M}+6\mathrm{S}$
$2\mathcal{J}^c$	$5\mathrm{M}+6\mathrm{S}$	$\mathcal{J}^m+\mathcal{J}^c=\mathcal{J}^m$	$12\mathrm{M}+5\mathrm{S}$
$2\mathcal{J}$	$4\mathrm{M}+6\mathrm{S}$	$\mathcal{J}+\mathcal{J}^c=\mathcal{J}^m$	$12\mathrm{M}+5\mathrm{S}$
$2\mathcal{J}^m=\mathcal{J}^c$	$4\mathrm{M}+5\mathrm{S}$	$\mathcal{J}+\mathcal{J}$	$12\mathrm{M}+4\mathrm{S}$
$2\mathcal{J}^m$	$4\mathrm{M}+4\mathrm{S}$	$\mathcal{P}+\mathcal{P}$	$12\mathrm{M}+2\mathrm{S}$
$2\mathcal{A}=\mathcal{J}^c$	$3\mathrm{M}+5\mathrm{S}$	$\mathcal{J}^c+\mathcal{J}^c=\mathcal{J}^m$	$11\mathrm{M}+4\mathrm{S}$
$2\mathcal{J}^m=\mathcal{J}$	$3\mathrm{M}+4\mathrm{S}$	$\mathcal{J}^c+\mathcal{J}^c$	$11\mathrm{M}+3\mathrm{S}$
$2\mathcal{A}=\mathcal{J}^m$	$3\mathrm{M}+4\mathrm{S}$	$\mathcal{J}^c+\mathcal{J}=\mathcal{J}$	$11\mathrm{M}+3\mathrm{S}$
$2\mathcal{A}=\mathcal{J}$	$2\mathrm{M}+4\mathrm{S}$	$\mathcal{J}^c+\mathcal{J}^c=\mathcal{J}$	$10\mathrm{M}+2\mathrm{S}$
—	—	$\mathcal{J}+\mathcal{A}=\mathcal{J}^m$	$9\mathrm{M}+5\mathrm{S}$
—	—	$\mathcal{J}^m+\mathcal{A}=\mathcal{J}^m$	$9\mathrm{M}+5\mathrm{S}$
—	—	$\mathcal{J}^c+\mathcal{A}=\mathcal{J}^m$	$8\mathrm{M}+4\mathrm{S}$
—	—	$\mathcal{J}^c+\mathcal{A}=\mathcal{J}^c$	$8\mathrm{M}+3\mathrm{S}$
—	—	$\mathcal{J}+\mathcal{A}=\mathcal{J}$	$8\mathrm{M}+3\mathrm{S}$
—	—	$\mathcal{J}^m+\mathcal{A}=\mathcal{J}$	$8\mathrm{M}+3\mathrm{S}$
—	—	$\mathcal{A}+\mathcal{A}=\mathcal{J}^m$	$5\mathrm{M}+4\mathrm{S}$
—	—	$\mathcal{A}+\mathcal{A}=\mathcal{J}^c$	$5\mathrm{M}+3\mathrm{S}$
$2\mathcal{A}$	$\mathrm{I}+2\mathrm{M}+2\mathrm{S}$	$\mathcal{A}+\mathcal{A}$	$\mathrm{I}+2\mathrm{M}+\mathrm{S}$

Naiive vs. Actual implementation

```
let mut X₃ := a * Z₃
let mut Y₃ := b₃ * t₂
Y₃ := X₃ + Y₃
X₃ := t₁ - Y₃
Y₃ := t₁ + Y₃
Y₃ := X₃ * Y₃
X₃ := t₃ * X₃
Z₃ := b₃ * Z₃
t₂ := a * t₂
t₃ := t₀ - t₂
t₃ := a * t₃
t₃ := t₃ + Z₃
Z₃ := t₀ + t₀
t₀ := Z₃ + t₀
t₀ := t₀ + t₂
t₀ := t₀ * t₃
Y₃ := Y₃ + t₀
t₂ := Y * Z
t₂ := t₂ + t₂
t₀ := t₂ * t₃
X₃ := X₃ - t₀
Z₃ := t₂ * t₁
Z₃ := Z₃ + Z₃
Z₃ := Z₃ + Z₃
return (X₃, Y₃, Z₃)
```

Reason: Ensure Lean is evaluating the arithmetic expressions to minimize finite field operations.

Scalar multiplication and GLV optimization

Given $n \in \mathbb{N}$ and $P \in C$, want to calculate $n \cdot P$.

Scalar multiplication and GLV optimization

Given $n \in \mathbb{N}$ and $P \in C$, want to calculate $n \cdot P$.

If C has prime order q, and an efficiently computable endomorphism $\Phi : C \to C$ then $\Phi(P) = \Lambda \cdot P$ for some $\Lambda \in \mathbb{N}$.
Given $n \in \mathbb{N}$, calculate find n_1 and n_2 of size $O(\sqrt{n})$ such that

$$n = n_1 + \Lambda n_2 \mod q$$

Scalar multiplication and GLV optimization

Given $n \in \mathbb{N}$ and $P \in C$, want to calculate $n \cdot P$.

If C has prime order q, and an efficiently computable endomorphism $\Phi : C \to C$ then $\Phi(P) = \Lambda \cdot P$ for some $\Lambda \in \mathbb{N}$.
Given $n \in \mathbb{N}$, calculate find n_1 and n_2 of size $O(\sqrt{n})$ such that

$$n = n_1 + \Lambda n_2 \mod q$$

Use this to compute

$$n \cdot P = (n_1 + \Lambda n_2) \cdot P = n_1 \cdot P + n_2 \cdot \Phi(P)$$

Scalar multiplication and GLV optimization

Given $n \in \mathbb{N}$ and $P \in C$, want to calculate $n \cdot P$.

If C has prime order q, and an efficiently computable endomorphism $\Phi : C \to C$ then $\Phi(P) = \Lambda \cdot P$ for some $\Lambda \in \mathbb{N}$.
Given $n \in \mathbb{N}$, calculate find n_1 and n_2 of size $O(\sqrt{n})$ such that

$$n = n_1 + \Lambda n_2 \mod q$$

Use this to compute

$$n \cdot P = (n_1 + \Lambda n_2) \cdot P = n_1 \cdot P + n_2 \cdot \Phi(P)$$

If the curve C is defined as $y^2 = x^3 + b$ then $\Phi : (x : y : z) \mapsto (\zeta x : y : z)$ for $\zeta \in \mu_3$ works.

Scalar multiplication and GLV optimization

Given $n \in \mathbb{N}$ and $P \in C$, want to calculate $n \cdot P$.

If C has prime order q, and an efficiently computable endomorphism $\Phi : C \to C$ then $\Phi(P) = \Lambda \cdot P$ for some $\Lambda \in \mathbb{N}$.
Given $n \in \mathbb{N}$, calculate find n_1 and n_2 of size $O(\sqrt{n})$ such that

$$n = n_1 + \Lambda n_2 \mod q$$

Use this to compute

$$n \cdot P = (n_1 + \Lambda n_2) \cdot P = n_1 \cdot P + n_2 \cdot \Phi(P)$$

If the curve C is defined as $y^2 = x^3 + b$ then $\Phi : (x : y : z) \mapsto (\zeta x : y : z)$ for $\zeta \in \mu_3$ works.

This optimization was subject to the patent `US7995752B2` *Method for accelerating cryptographic operations on elliptic curves* which expired in 2020

Some very magical curves

The curve $y^2 = x^3 + 5$ defined over over $\mathbb{F}_p$ and $\mathbb{F}_q$ with primes
$p =$
`0x40000000000000000000000000000000224698fc094cf91b992d30ed00000001`
(Pallas) and
$q =$
`0x40000000000000000000000000000000224698fc0994a8dd8c46eb2100000001`
(Vesta)

Some very magical curves

The curve $y^2 = x^3 + 5$ defined over over $\mathbb{F}_p$ and $\mathbb{F}_q$ with primes
$p =$
`0x40000000000000000000000000000000224698fc094cf91b992d30ed00000001`
(Pallas) and
$q =$
`0x40000000000000000000000000000000224698fc0994a8dd8c46eb2100000001`
(Vesta)

Nice properties of the Pasta curves:

1. $\#(E_{\text{Pallas}}) = q$ and $\#(E_{\text{Vesta}}) = p$
2. Efficiently computable endomorphism for the GLV optimization.
3. Large 2-adicity ($p - 1$ is divisible by 2^{32})
4. Both primes are of the form $2^{255} + \epsilon$ which helps with reduction in $\mathbb{F}_p$
5. Both have low-degree isogenies with non-zero j-invariant (helps with generating random points on the curves).

Multiscalar Multiplication

The multi-exponentiation problem: Given a set of list of integers $[n_1, n_2, \ldots, n_r]$ and a list of group elements $[g_1, g_2, \ldots, g_r]$ calculate $\prod_i^r g_i^{n_i}$.
Multi-exponentiation in the context of elliptic curve groups asks to calculate $\sum_i^r n_i \cdot G_i$ for points $G_i \in C$.

Multiscalar Multiplication

The multi-exponentiation problem: Given a set of list of integers $[n_1, n_2, \ldots, n_r]$ and a list of group elements $[g_1, g_2, \ldots, g_r]$ calculate $\prod_i^r g_i^{n_i}$.
Multi-exponentiation in the context of elliptic curve groups asks to calculate $\sum_i^r n_i \cdot G_i$ for points $G_i \in C$.

Used throughout ECC (signature schemes, commitment schemes, zero knowledge proof schemes).
In some protocols, $\approx 80\%$ of time in generating zero knowledge proofs is spent calculating multiscalar multiplications.

Typical r varies greatly by application, but can get up to sizes like 2^{28}.

Pippenger's algorithm

Pippenger's algorithm is one of the most efficient optimizations over the naive implementation.
Suppose the elliptic curve group order is approximately b bits large. Pick a "window size" c ($\log r$ turns out to be optimal) so that $b \approx k \cdot c$.

1. Decompose each $n_i = n_{i,0} + n_{i,1} \cdot 2^c + n_{i,2} \cdot 2^{2c} + \ldots + n_{i,k} \cdot 2^{kc}$
2. Split up the large b-bit MSM into k separate c-bit MSMs:

$$T_i = \sum_j^r n_{j,i} G_i$$

3. Solve each c-bit MSM and recombine as $M = \sum_i^k 2^i \cdot T_i$.
(perfect opportunity for parallelization)

Pippenger algorithm

We have reduced to the case of an MSM $T = \sum_i^r n_i \cdot G_i$ where each n_i is at most c-bits.

1. Keep track of $2^c - 1$ "buckets", and place each G_i into the n_i bucket.
2. Sum up all the sets of points to get 2^c bucket sums S_j for $j = 0, \ldots, 2^c - 1$
3. $T = S_{2^c-1} + (S_{2^c-1} + S_{2^c-2}) + \ldots + (S_{2^c-1} + S_{2^c-2} + \ldots + S_0)$

The Lean implementation provides an $\approx 10\times$ speedup over the naive implementation.

Further optimizations exist when each G_i is a fixed shape relative to a generator for the elliptic curve group.

A common case is when $G_i = \tau^i \cdot G$ for $i = 0, \ldots, r$ and G is a fixed generator for the elliptic curve group.

A small Demo!

All of the above can be found:

`https://github.com/yatima-inc/FFaCiL.lean`

`https://github.com/yatima-inc/yatimastdlib.lean`

Lessons learned

1. Squeezing performance out of Lean can sometimes feel like squeezing water from a stone.
2. Lean 4 does not have “zero cost abstractions”.
3. Lean 4 is early in its life, so there are few guides written down for writing efficient code.
4. There are some significant performance pits to fall in if one isn't careful with the reference counting.

Lessons learned

1. Squeezing performance out of Lean can sometimes feel like squeezing water from a stone.
2. Lean 4 does not have "zero cost abstractions".
3. Lean 4 is early in its life, so there are few guides written down for writing efficient code.
4. There are some significant performance pits to fall in if one isn't careful with the reference counting.
5. Lean 4 is incredibly expressive, it's easy to prototype complex structures and algorithms that *just work* on the first try.
6. The resulting code is beautiful, and very easy to read.
7. With time, more FFI work, and more effort I'm certain performance can be competitive.

Future goals

1. (short term) Optimize, optimize, optimize! (deeper GMP integration, more curve forms, ...).
2. (short-medium term) Expand `FFaCiL` to include more elliptic curve cryptography (isogenies!).
3. (short-medium term) Actually take advantage of dependent types.
4. (medium term) Begin exploring the formalization of provable security (formalizing adversarial games, and complexity).
5. (long term) Expand the Yatima Standard Library with more number theoretic primitives.
6. (Long term) Begin the process of formally verifying these algorithms!
7. (Epochs from now) Combine all of these efforts together into a `cryptolib` with formally verified cryptographic algorithms.

Thank you for your time!

Thanks!

References I

- Amin, Nada et al. *LURK: Lambda, the Ultimate Recursive Knowledge*. Cryptology ePrint Archive, Paper 2023/369. https://eprint.iacr.org/2023/369. 2023. URL: https://eprint.iacr.org/2023/369.
- Bergeron, F et al. "Addition chains using continued fractions". In: *Journal of Algorithms* 10.3 (1989), pp. 403–412. ISSN: 0196-6774. DOI: https://doi.org/10.1016/0196-6774(89)90036-9. URL: https://www.sciencedirect.com/science/article/pii/0196677489900369.
- Bernstein, Daniel J. "Pippenger's Exponentiation Algorithm". In: 2002.
- Bernstein, Daniel J. et al. *Faster batch forgery identification*. Cryptology ePrint Archive, Paper 2012/549. https://eprint.iacr.org/2012/549. 2012. URL: https://eprint.iacr.org/2012/549.

References II

Cohen, H. et al. *Handbook of Elliptic and Hyperelliptic Curve Cryptography*. Discrete Mathematics and Its Applications. CRC Press, 2005. ISBN: 9781420034981. URL: https://books.google.com/books?id=w6b0yhURTkQC.

Gallant, Robert P., Robert J. Lambert, and Scott A. Vanstone. "Faster Point Multiplication on Elliptic Curves with Efficient Endomorphisms". In: *Advances in Cryptology — CRYPTO 2001*. Ed. by Joe Kilian. Berlin, Heidelberg: Springer Berlin Heidelberg, 2001, pp. 190–200. ISBN: 978-3-540-44647-7.

Rosulek, Mike. *The Joy of Cryptography*. https://joyofcryptography.com. URL: https://joyofcryptography.com.

References III

1. https://electriccoin.co/blog/the-pasta-curves-for-halo-2-and-beyond/
2. https://github.com/zcash/pasta_curves
3. https://github.com/zcash/pasta

Check us out at: https://github.com/lurk-lab/