Homework #2: Laplacian and Multivariable Integration by Parts

Solutions

1 Rotational Invariance of Laplacian

Suppose that a 3D scalar function $v : \Re^3 \to \Re$ is related to a 3D scalar function $u : \Re^3 \to R$ by a rotation as follows:

$$v(x,y,z) = u\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)$$

Show by explicit calculation of the Laplacian that $\Delta v(x,y,z) = \Delta u\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)$

Solution:

We start by computing the first derivatives v_x, v_y, and v_z.

$$\begin{aligned}
v_x(x,y,z) &= \left.\frac{u_x + u_y + u_z}{\sqrt{3}}\right|_{\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)} \\
v_y(x,y,z) &= \left.\frac{-2u_x + u_y + u_z}{\sqrt{6}}\right|_{\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)} \\
v_z(x,y,z) &= \left.\frac{-u_y + u_z}{\sqrt{2}}\right|_{\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)}
\end{aligned}$$

Now we compute the second derivatives v_{xx}, v_{yy}, and v_{zz} by differentiating these first derivatives again.

$$\begin{aligned}
v_{xx}(x,y,z) &= \left.\frac{\left(\frac{u_x+u_y+u_z}{\sqrt{3}}\right)_x + \left(\frac{u_x+u_y+u_z}{\sqrt{3}}\right)_y + \left(\frac{u_x+u_y+u_z}{\sqrt{3}}\right)_z}{\sqrt{3}}\right|_{\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)} \\
&= \left.\frac{u_{xx} + u_{yx} + u_{zx} + u_{xy} + u_{yy} + u_{zy} + u_{xz} + u_{yz} + u_{zz}}{3}\right|_{\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)} \\
&= \left.\frac{1}{3}(u_{xx} + u_{yy} + u_{zz}) + \frac{2}{3}(u_{xy} + u_{xz} + u_{yz})\right|_{\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)} \\
v_{yy}(x,y,z) &= \left.\frac{-2\left(\frac{-2u_x+u_y+u_z}{\sqrt{6}}\right)_x + \left(\frac{-2u_x+u_y+u_z}{\sqrt{6}}\right)_y + \left(\frac{-2u_x+u_y+u_z}{\sqrt{6}}\right)_z}{\sqrt{6}}\right|_{\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)} \\
&= \left.\frac{4u_{xx} - 2u_{yx} - 2u_{zx} - 2u_{xy} + u_{yy} + u_{zy} - 2u_{xz} + u_{yz} + u_{zz}}{6}\right|_{\left(\frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{2}{3}u_{xx}+\frac{1}{6}u_{yy}+\frac{1}{6}u_{zz}-\frac{2}{3}u_{xy}-\frac{2}{3}u_{xz}+\frac{1}{3}u_{yz}\right|_{\left(\frac{x}{\sqrt{3}}-\frac{2y}{\sqrt{6}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}-\frac{z}{\sqrt{2}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}+\frac{z}{\sqrt{2}}\right)} \\
v_{zz}(x,y,z) &= \left.\frac{-\left(\frac{-u_y+u_z}{\sqrt{2}}\right)_y+\left(\frac{-u_y+u_z}{\sqrt{2}}\right)_z}{\sqrt{2}}\right|_{\left(\frac{x}{\sqrt{3}}-\frac{2y}{\sqrt{6}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}-\frac{z}{\sqrt{2}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}+\frac{z}{\sqrt{2}}\right)} \\
&= \left.\frac{u_{yy}-u_{zy}-u_{yz}+u_{zz}}{2}\right|_{\left(\frac{x}{\sqrt{3}}-\frac{2y}{\sqrt{6}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}-\frac{z}{\sqrt{2}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}+\frac{z}{\sqrt{2}}\right)} \\
&= \left.\frac{1}{2}u_{yy}+\frac{1}{2}u_{zz}-u_{yz}\right|_{\left(\frac{x}{\sqrt{3}}-\frac{2y}{\sqrt{6}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}-\frac{z}{\sqrt{2}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}+\frac{z}{\sqrt{2}}\right)}
\end{aligned}
$$

Finally, we add them together to get the Laplacian

$$
\begin{aligned}
\Delta v(x,y,z) &= v_{xx}+v_{yy}+v_{zz}\big|_{(x,y,z)} \\
&= \left(\frac{1}{3}+\frac{2}{3}\right)u_{xx}+\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{2}\right)u_{yy}+\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{2}\right)u_{zz} \\
&\quad \left.+\left(\frac{2}{3}-\frac{2}{3}\right)u_{xy}+\left(\frac{2}{3}-\frac{2}{3}\right)u_{xz}+\left(\frac{2}{3}+\frac{1}{3}-1\right)u_{yz}\right|_{\left(\frac{x}{\sqrt{3}}-\frac{2y}{\sqrt{6}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}-\frac{z}{\sqrt{2}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}+\frac{z}{\sqrt{2}}\right)} \\
&= u_{xx}+u_{yy}+u_{zz}\big|_{\left(\frac{x}{\sqrt{3}}-\frac{2y}{\sqrt{6}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}-\frac{z}{\sqrt{2}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}+\frac{z}{\sqrt{2}}\right)} \\
&= \Delta u\left(\frac{x}{\sqrt{3}}-\frac{2y}{\sqrt{6}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}-\frac{z}{\sqrt{2}},\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{6}}+\frac{z}{\sqrt{2}}\right)
\end{aligned}
$$

2 Gauss-Green Theorem

Recall the Gauss-Green Theorem presented in class

$$
\int_\Omega u_{x_i}\,d\mathbf{x}=\int_{\partial\Omega} uN_i\,ds
$$

where Ω denotes a subset of $\Re^n$ with boundary $\partial\Omega$, where u_{x_i} denotes the partial derivative of $u(\mathbf{x})$ with respect to the i'th component of $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, and where N_i denotes the i'th component of the outward unit normal $N=(N_1,N_2,\ldots,N_n)$ of the boundary $\partial\Omega$ and where ds denotes the area measure of the boundary. Exploit this to evaluate the integral

$$
\int_\Omega b\cdot\nabla u\,d\mathbf{x}
$$

for the 3D function $u(x,y,z)$ for the constant vector $b=(1,1.5,-2.25)$ where the region Ω denotes the unit cube with a vertex at the origin (0,0,0) and opposite vertex at (1,1,1). Assume that the value of $u$is zero along the three faces of the cube which share the origin vertex (0,0,0) and that the value of u is one along the other three faces of the cube which share the opposite vertex (1,1,1). Note that $d\mathbf{x}$ is a compact expression for $dx\,dy\,dz$. However, for full credit you must exploit the Gauss-Green theorem relationship to evaluate your answer. Do not simply write out the volume integral above as a triple nested integral in the variables x, y, and $z$with integration limits from 0 to 1 each and then integrate the nested set. By applying the theorem it is possible to determine the answer very easily without explicitly writing out such a triple integral (though you may wish to do so as a way to double-check your answer).

Solution

We start by expanding $b \cdot \nabla u$ into a weighted sum of partial derivatives, whose volume integral can in term be written as the same weighted sum of volume integrals (this is not the same as the triple nested integral you were instructed to avoid). We then apply the Gauss-Green theorem to each of these to obtain a weighted sum of surface integrals (note that in our expression below N_1, N_2, and N_3 denote the three components of the outward unit normal N.

$$\begin{aligned}\int_\Omega b \cdot \nabla u \, d\mathbf{x} &= \int_\Omega u_x + 1.5u_y - 2.25u_z \, d\mathbf{x} \\ &= \int_\Omega u_x \, d\mathbf{x} + 1.5 \int_\Omega u_y \, d\mathbf{x} - 2.25 \int_\Omega u_z \, d\mathbf{x} \\ &= \int_{\partial\Omega} uN_1 \, ds + 1.5 \int_{\partial\Omega} uN_2 \, ds - 2.25 \int_{\partial\Omega} uN_3 \, ds\end{aligned}$$

Now, noting that $u = 0$ on the three faces of the cube connected to the origin vertex (0,0,0), we may ignore these three faces in our surface integrals and refer to the 3 remaining faces of the cube, connected to the opposite vertex, as x face, y face and z face along the planes $x = 1$, $y = 1$, and $z = 1$ respectively. We will refer to these three faces together simply as "1 faces" in our expression. Since $u = 1$ along these remaining "1 faces", we may simplify the last expression to as follows.

$$\int_\Omega b \cdot \nabla u \, d\mathbf{x} = \int_{1 \text{ faces}} N_1 \, ds + 1.5 \int_{1 \text{ faces}} N_2 \, ds - 2.25 \int_{1 \text{ faces}} N_3 \, ds$$

Finally, since $N_1 = 1$ on the x face, $N_2 = 1$ on the y face, $N_3 = 1$ on the z face, and since each of these normal components are zero on the other faces, we may further simplify our surface integrals as follows.

$$\begin{aligned}\int_\Omega b \cdot \nabla u \, d\mathbf{x} &= \int_{x \text{ face}} 1 \, ds + 1.5 \int_{y \text{ face}} 1 \, ds - 2.25 \int_{z \text{ face}} 1 \, ds \\ &= (\text{area of } x \text{ face}) + 1.5(\text{area of } y \text{ face}) - 2.25(\text{area of } z \text{ face}) \\ &= 1 + 1.5 - 2.25 \\ &= .25\end{aligned}$$

3 Green's Formulas and Divergence Theorem

In class we derived the first of the two Green's formulas by using the Gauss-Green Theorem.

$$\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \nabla u \cdot N \, ds \tag{1}$$

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega u \Delta v \, dx + \int_{\partial\Omega} u \nabla v \cdot N \, ds \tag{2}$$

3.1 Derive the second Green's formula above

You should find it quite easy to derive by applying the following coordinate-wise integration by parts result of the Gauss-Green Theorem that we derived in class.

$$\int_\Omega u_{x_i} v \, d\mathbf{x} = -\int_\Omega u v_{x_i} d\mathbf{x} + \int_{\partial\Omega} u v N_i \, ds$$

Proof

Apply the integration by parts formula above, but substitute v_{x_i} in place of v

$$\int_{\Omega} u_{x_i} v_{x_i}\, d\mathbf{x} = -\int_{\Omega} u v_{x_i x_i} d\mathbf{x} + \int_{\partial\Omega} u v_{x_i} N_i\, ds$$

Now sum over the index i.

$$\begin{aligned}\int_{\Omega} \sum_{i=1}^{n} u_{x_i} v_{x_i}\, d\mathbf{x} &= -\int_{\Omega} u \sum_{i=1}^{n} v_{x_i x_i} d\mathbf{x} + \int_{\partial\Omega} u \sum_{i=1}^{n} v_{x_i} N_i\, ds \\ \int_{\Omega} \nabla u \cdot \nabla v\, d\mathbf{x} &= -\int_{\Omega} u \Delta v d\mathbf{x} + \int_{\partial\Omega} u \nabla v \cdot N\, ds\end{aligned}$$

3.2 Divergence Theorem

Use the Gauss-Green Theorem to derive the following Divergence Theorem

$$\int_{\partial\Omega} F \cdot N\, ds = \int_{\Omega} \nabla \cdot F\, dx$$

where F denotes a n-dimensional vector field over the set $\Omega \subset \Re^n$.

Proof

Let us denote the components of F by $(F^1, F^2, \ldots, F^n)$. Start by applying the Gauss-Green Theorem to a single component F_i as follows.

$$\int_{\partial\Omega} F^i N_i\, ds = \int_{\Omega} F^i_{x_i} d\mathbf{x}$$

Now sum of the index i.

$$\begin{aligned}\int_{\partial\Omega} \sum_{i=1}^{n} F^i N_i\, ds &= \int_{\Omega} \sum_{i=1}^{n} F^i_{x_i} d\mathbf{x} \\ \int_{\partial\Omega} F \cdot N\, ds &= \int_{\Omega} \nabla \cdot F\, dx\end{aligned}$$