Homework #3: Laplace and Transport PDE's for Thickness

Solutions

1 Thickness between two concentric circles

In this problem we will show how the combined Laplace and Transport PDE based approach to thickness analysis outlined in class applies to the case of two circles of radii $r_0 > 0$ and $r_1 > r_0$ both centered around the origin. For full credit, as you solve the various PDE's below, you must show how you arrived at the solution. It is not sufficient to merely demonstrate that your solutions satify the PDE's and boundary conditions.

1.1 Solve Laplace's equation with fixed boundary values

Note that the solution of Laplace's equation between two concentric circles is radially symmetric (rotationally invariant) around their centerpoint. Using the formulas derived while developing the Funadmental Solution to Laplace's Equation, find the exact analytical solution $u(x,y)$, expressed in terms of r_1 and r_0, for the PDE

$$\triangle u(x,y) = 0, \quad r_0 < \sqrt{x^2+y^2} < r_1$$

subject to boundary conditions

$$u(x,y) = \begin{cases} 0, & \sqrt{x^2+y^2} = r_0 \\ 1, & \sqrt{x^2+y^2} = r_1 \end{cases}$$

Solution:

Recall that all rotationally invariant solutions to Laplace's equation in two dimensions have the general form

$$u(x,y) = b \ln \sqrt{x^2+y^2} + c$$

where b and c are constants. We must choose these constants to give us the prescribed boundary conditions above. Plugging in $\sqrt{x^2+y^2} = r_0$ and $\sqrt{x^2+y^2} = r_1$ gives us the equations

$$\begin{aligned} b \ln r_0 + c &= 0 \\ b \ln r_1 + c &= 1 \end{aligned}$$

whose solution is $b = \frac{1}{\ln r_1 - \ln r_0}$ and $c = \frac{-\ln r_0}{\ln r_1 - \ln r_0}$. Substituting this into the general rotational invariant form yields

$$u(x,y) = \frac{\ln \sqrt{x^2+y^2} - \ln r_0}{\ln r_1 - \ln r_0}$$

1.2 Compute the tangent field

Next, using your solution above, compute the analytic expression for the tangent vector field $T(x,y) = \frac{\nabla u}{\|\nabla u\|}$ which will determine the characteristics for the subsequent transport PDE's.

Solution:

$$\nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \frac{1}{\ln r_1 - \ln r_0} \frac{1}{\sqrt{x^2+y^2}} \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2}} \\ \frac{1}{\ln r_1 - \ln r_0} \frac{1}{\sqrt{x^2+y^2}} \frac{1}{2} \frac{2y}{\sqrt{x^2+y^2}} \end{pmatrix} = \frac{1}{\ln(r_1/r_0)(x^2+y^2)} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\|\nabla u\| = \frac{\sqrt{x^2+y^2}}{\ln(r_1/r_0)(x^2+y^2)} = \frac{1}{\ln(r_1/r_0)\sqrt{x^2+y^2}}$$

$$T = \frac{\nabla u}{\|\nabla u\|} = \frac{\ln(r_1/r_0)\sqrt{x^2+y^2}}{\ln(r_1/r_0)(x^2+y^2)} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ y \end{pmatrix}$$

1.3 Construct and solve the transport PDE's

Next, using your solution for the tangent field T above, compute the analytic solutions $L_0(x,y)$ and $L_1(x,y)$ of the following transport/advection PDE's

$$\begin{aligned} \nabla L_o \cdot T &= +1 \\ \nabla L_1 \cdot T &= -1 \end{aligned}$$

with boundary conditions

$$\begin{aligned} L_0(x,y) &= 0, \quad \sqrt{x^2+y^2} = r_0 \\ L_1(x,y) &= 0, \quad \sqrt{x^2+y^2} = r_1 \end{aligned}$$

(correspondence trajectory lengths from each point to the inner and outer boundaries).

Solution

First we note that the tangent vector $T = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ y \end{pmatrix}$ computed above is simply the unit vector $\vec{r}$ in the radial direction if we think in terms of polar coordinates. As such, we may expressed the two transport PDE's as directional derivatives in the radial direction as follows.

$$\begin{aligned} \nabla L_o \cdot T &= \tfrac{\partial L_0}{\partial \vec{r}} = +1 \\ \nabla L_1 \cdot T &= \tfrac{\partial L_1}{\partial \vec{r}} = -1 \end{aligned}$$

whose solutions are therefore

$$\begin{aligned} L_0 &= r + c_0 = \sqrt{x^2+y^2} + c_0 \\ L_1 &= -r + c_1 = -\sqrt{x^2+y^2} + c_1 \end{aligned}$$

for constants c_0 and c_1. Now plugging this into the boundary conditions above gives the expressions

$$\begin{aligned} 0 &= r_0 + c_0 \\ 0 &= -r_1 + c_1 \end{aligned}$$

from which we can solve for the constants $c_0 = -r_0$ and $c_1 = r_1$. This yields the following expressions for L_0 and L_1.

$$\begin{aligned} L_0(x,y) &= \sqrt{x^2+y^2} - r_0 \\ L_1(x,y) &= r_1 - \sqrt{x^2+y^2} \end{aligned}$$

1.4 Compute Thickness

Finally, add the functions $L_1(x, y)$ and $L_2(x, y)$ that you computed above to obtain an expression for the thickness at any point (x, y) within the annulus.

$$\textbf{Thickness}(x, y) = L_0(x, y) + L_1(x, y)$$

Solution

$$\begin{aligned}
\text{Thickness}(x, y) &= L_0(x, y) + L_1(x, y) \\
&= \sqrt{x^2 + y^2} - r_0 + r_1 - \sqrt{x^2 + y^2} \\
&= r_1 - r_0
\end{aligned}$$