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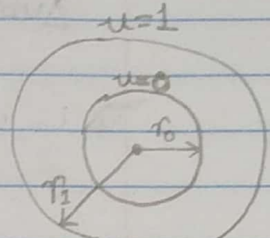
ECE 6560 Homework-3



(1.1) Using the equation derived in class while deriving the fundamental solution to Laplace's equation,

$$\frac{\ddot{v}(r)}{\dot{v}(r)} = \frac{1-n}{r}$$ (derivation upto this point is exactly same as lecture)

$$\therefore \frac{d}{dr}\left(\log \dot{v}(r)\right) = \frac{1-n}{r} \quad \ldots\ldots \left[\frac{d}{dr}\log \dot{v}(r) = \frac{1}{\dot{v}(r)}\frac{d}{dr}(\dot{v}(r))\right]$$

Integrating both sides w.r.t. r, we get,

$$\int \frac{d}{dr}\left(\log \dot{v}(r)\right) dr = \int \frac{1-n}{r}\, dr$$

$$\log \dot{v}(r) = (1-n)\log r + c$$

Taking exponentials on both sides, we get,

$$\dot{v}(r) = e^{\left[\log r^{(1-n)} + c\right]} = e^c r^{1-n} = b r^{1-n} \quad \ldots\ldots [b = e^c]$$

Once again integrating both sides wrt r, we get,

$$\int \frac{dv(r)}{dr} dr = b\int r^{1-n}\, dr$$

For this case, $n = 2$

$$\therefore v(r) = b\int \frac{1}{r}\, dr = b\log r + c$$

Using the given boundary conditions to find the constants of integration b and c

At $r = r_0$, $v(r_0) = u\left(x,y \,\middle|\, \sqrt{x^2+y^2} = r_0\right) = 0$

$$\therefore v(r_0) = 0 = b\log r_0 + c \Rightarrow \boxed{c = -b\log r_0} \rightarrow (1)$$

At $r = r_1$, $v(r_1) = u(x, y \mid \sqrt{x^2+y^2} = r_1) = 1$

$\therefore v(r_1) = 1 = b \log r_1 + c$

Substituting (1), we get,

$$v(r_1) = 1 = b \log r_1 - b \log r_0 = b(\log r_1 - \log r_0)$$

$$\therefore \frac{1}{b} = \log \frac{r_1}{r_0} \quad \text{or} \quad \boxed{b = \log(r_0/r_1)} \rightarrow (2)$$

Substituting (2) in (1), we get,

$$c = -\log\left(\frac{r_0}{r_1}\right) \log r_0$$

$$\therefore v(r) = u(x, y \mid r_0 < \sqrt{x^2+y^2} < r_1)$$

$$= \log\left(\frac{r_0}{r_1}\right) \log r - \log\left(\frac{r_0}{r_1}\right) \log r_0$$

$$\boxed{\therefore v(r) = u(x,y) = \log\left(\frac{r_0}{r_1}\right) \cdot \log\left(\frac{r}{r_0}\right)}$$

⟨1.2.⟩ $T(x,y) = \dfrac{\nabla u}{\|\nabla u\|}$ with $u = \log\left(\frac{r_0}{r_1}\right) \cdot \log\left(\frac{\sqrt{x^2+y^2}}{r_0}\right)$

where, $r = \sqrt{x^2+y^2}$

$$\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \log\left(\frac{r_0}{r_1}\right) \begin{bmatrix} \frac{\cancel{r_0}}{r} \cdot \frac{1}{\cancel{r_0}} \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cancel{2}x \\ \frac{\cancel{r_0}}{r} \cdot \frac{1}{\cancel{r_0}} \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cancel{2}y \end{bmatrix}$$

$$\nabla u = \log\left(\frac{r_0}{r_1}\right) \begin{bmatrix} \frac{x}{r^2} \\ \frac{y}{r^2} \end{bmatrix} \Rightarrow \|\nabla u\| = \sqrt{\left(\log \frac{r_0}{r_1}\right)^2 \left(\frac{x^2+y^2}{r^4}\right)}$$

$$= \sqrt{\left(\log \frac{r_0}{r_1}\right)^2 \cdot \frac{r^{\cancel{2}}}{r^{\cancel{4}^2}}}$$

$$\therefore \|\nabla u\| = \sqrt{\left(\log\frac{r_0}{r_1}\right)^2 \cdot \frac{1}{r^2}} = \frac{1}{r}\log\left(\frac{r_0}{r_1}\right) \quad \ldots\ldots \text{(given } r>0\text{)} \; (\text{for } r_0 < r < r_1)$$

$$\therefore T(x,y) = \frac{\nabla u}{\|\nabla u\|} = \begin{bmatrix} \dfrac{\cancel{\log\left(\frac{r_0}{r_1}\right)} \frac{x}{r^2}}{\frac{1}{\cancel{r}} \cancel{\log\left(\frac{r_0}{r_1}\right)}} \\ \dfrac{\cancel{\log\left(\frac{r_0}{r_1}\right)} \frac{y}{r^{\cancel{2}}}}{\frac{1}{\cancel{r}} \cancel{\log\left(\frac{r_0}{r_1}\right)}} \end{bmatrix}$$

$$\boxed{\therefore T(x,y) = \begin{bmatrix} \frac{x}{r} \\ \frac{y}{r} \end{bmatrix}}$$

<1.3.>

Consider the given transport PDE,

$$\nabla L_0(x,y) \cdot T = 1 \quad , \quad L_0(x,y) = 0 \text{ for } \sqrt{x^2+y^2} = r = r_0$$

We can the LHS as a directional derivative

$$\frac{\partial L_0}{\partial \vec{T}} = 1 = \nabla L_0 \cdot T = [L_{0x} \; L_{0y}]\begin{bmatrix} x/r \\ y/r \end{bmatrix} = \frac{x}{r} L_{0x} + \frac{y}{r} L_{0y}$$

Re-writing L_x as $L_x = L_r \cdot r_x$ and similarly re-writing L_y as $L_y = L_r \cdot r_y$ we get,

$$\frac{x}{r} L_{0r}\, r_x + \frac{y}{r} L_{0r}\, r_y = 1$$

But, $r_x = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x = \frac{x}{r}$ and similarly, $r_y = \frac{y}{r}$

$\therefore$ Equation above for L_0 becomes, $\frac{x^2}{r^2} L_{0r} + \frac{y^2}{r^2} L_{0r} = 1$

$\therefore L_{0_r}\left(\frac{x^2+y^2}{r^2}\right) = 1 \Rightarrow L_{0_r} = 1$ because $r = \sqrt{x^2+y^2}$

$\therefore$ Integrating both sides w.r.t r, we get,

$$\int \frac{\partial}{\partial r} L_0 \, dr = \int 1 \cdot dr$$

$$\therefore L_0 = r + c$$

Using the given boundary condition we have,

$L_0 = 0$ for $r = r_0$

$\therefore 0 = r_0 + c \Rightarrow c = -r_0$

$$\boxed{\therefore L_0(x,y) = r - r_0}$$

Similarly, for the other transport PDE,

$$\frac{\partial L_1}{\partial \vec{T}} = -1 = \frac{x}{r} L_{1x} + \frac{y}{r} L_{1y}$$

$$\therefore \frac{x}{r} L_{1_r} r_x + \frac{y}{r} L_{1_r} r_y = -1$$

$$\therefore L_{1_r} \cancel{\frac{(x^2+y^2)}{r^2}} = -1$$

Integrating both sides wrt and using the boundary condition,

$$L_1 = -r + c'$$

$\therefore 0 = -r_1 + c' \Rightarrow c' = r_1$

$$\boxed{\therefore L_1(x,y) = r_1 - r}$$

<1.4.> To find thickness at any point (x,y) we have

$$\text{Thickness}(x,y) = L_0(x,y) + L_1(x,y)$$
$$= \cancel{r} - r_0 + r_1 - \cancel{r}$$

$$\boxed{\therefore \text{Thickness}(x,y) = r_1 - r_0}$$