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ECE 6560 A
Homework #1: Transport PDE

1.) Given PDE:- $u_t + bu_x = e^t$
$u(x,0) = g(x)$ .... initial value

Writing the left hand side as a directional derivative

Let $\vec{v} = (b, 1)$.

$\therefore \frac{\partial u}{\partial \vec{v}} = \begin{bmatrix} b & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_t \end{bmatrix} = \vec{v} \cdot \nabla u = u_t + bu_x = e^t$

$\therefore \frac{\partial u}{\partial \vec{v}} = e^t$

In this case the characteristic lines will be straight lines (as in the homogeneous case) but the value of $u(x,t)$ will not be constant along the characteristic lines.

Parameterizing the evolution of $u(x,t)$ along the characteristic lines by s and let $(s=0)$ correspond to $u(x,t)$ such that $(s>0)$ corresponds to forward in time (beyond t) and $(s<0)$ corresponds to backward in time (before t)

$\therefore z(s) = u(x+bs, t+s) = u(\vec{x} + s\vec{v})$ where, $\vec{x} = [x, t]$ and $\vec{v} = [b, 1]$

$\therefore$ differentiating both sides, we get,

$$\dot{z}(s) = b\, u_x(x+bs, t+s) + 1 \cdot u_t(x+bs, t+s)$$
$$= e^{t+s} \quad \ldots\ldots \left[\begin{array}{l} \because u_t(x,t) + b\, u_x(x,t) = e^t \\ \therefore u_t(x+bs, t+s) + b\, u_x(x+bs, t+s) = e^{t+s} \end{array} \right]$$

$$\therefore z(s) = \int_0^s e^{t+s}\, ds + z(0)$$

Since we know the initial condition, we can pick the limits of integration that will allow us to incorporate the initial condition

If $s=0$ corresponds to $u(x,t)$ and $s=\hat{s}$ corresponds to $u(x+b\hat{s}, t+\hat{s})$, $s=-t$ should correspond to $z(-t) = u(x-bt, 0) = g(x-bt)$ .... (as per initial condition)

Let limits be $(0,-t)$ instead

$$\therefore z(0) = \int_{-t}^{0} e^{t+s}\,ds + z(-t)$$

$$\therefore u(x,t) = \int_{-t}^{0} e^{t+s}\,ds + g(x-bt)$$

$$= e^{t}\int_{-t}^{0} e^{s}\,ds + g(x-bt)$$

$$= e^{t}\left[e^{s}\right]_{-t}^{0} + g(x-bt)$$

$$= e^{t}\left[e^{0} - e^{-t}\right] + g(x-bt)$$

$$\boxed{\therefore u(x,t) = e^{t} - 1 + g(x-bt)}$$

Check to see if above solution satisfies the PDE,

$u_t = e^{t} + g_x(-b)$

$u_x = g_x(1)$

$\therefore u_t + bu_x = e^{t} - bg_x + bg_x = e^{t}$ ☑ PDE satisfied!

Check for initial condition,

$u(x,0) = e^{0} - 1 + g(x - b\cdot 0) = 1 - 1 + g(x) = g(x)$ ☑ I.C. satisfied!

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<2.> Homogeneous PDE with time varying coefficient

Given PDE :- $u_t - t u_x = 0$
$u(x,0) = g(x)$ ...... initial condition

Writing LHS as a directional derivative

$$\frac{\partial u}{\partial \vec{v}} = \begin{bmatrix} -t & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_t \end{bmatrix} \quad \ldots\ldots \text{ where } \vec{v} = \begin{bmatrix} -t & 1 \end{bmatrix}$$
$$= \vec{v} \cdot \nabla u = 0$$

Similar to problem 1, parameterizing by s along characteristic lines.

Since in this case the characteristics are not straight lines (as the coefficient of u_x is time varying), we can't use the same $z(s)$ as in problem 1.

We want to pick $z(s)$ such that $\dot{z}(s)$ represents the LHS of the given PDE.

So, if $\dot{z}(s) = -(t+s)\, u_x(x+f, t+s) + 1\, u_t(x+f, t+s)$
then as per given PDE

$$\dot{z}(s) = 0$$

To find f, $\quad -(t+s) = \frac{\partial}{\partial s}(x+f) = \frac{\partial f}{\partial s}$

$$\therefore \int (-t-s)\, ds = f + c$$

$$\therefore -ts - \frac{s^2}{2} = f + c$$

$\because u(x,t) \Leftrightarrow s=0 \Rightarrow$ at $s=0$, $f=0$

$$\therefore -t(0) - \frac{0^2}{2} = 0 + c \Rightarrow \boxed{c=0}$$

$\therefore f = -ts - \frac{s^2}{2}$

$\therefore z(s) = u\left(x - \left(ts + \frac{s^2}{2}\right), t+s\right)$

Now, similar to problem 1, since we know $u(x,0)$ substituting $s=-t$,

$$z(-t) = u\left(x - \left(t(-t) + \frac{(-t)^2}{2}\right), t-t\right)$$
$$= u\left(x - \left(\frac{t^2}{2} - t^2\right), 0\right) = u\left(x + \frac{t^2}{2}, 0\right)$$
$$= g(x + t^2/2)$$

Going back to $\dot{z}(s) = 0 \Rightarrow z(s) = \text{constant}$

$$\therefore z(s=0) = z(s=-t)$$
$$\boxed{\therefore u(x,t) = g\left(x + \frac{t^2}{2}\right)}$$

Testing solution with PDE,

$$u_t = g_x \frac{\partial}{\partial t}\left(x + \frac{t^2}{2}\right) = g_x t$$
$$u_x = g_x \frac{\partial}{\partial x}\left(x + \frac{t^2}{2}\right) = g_x$$

$\therefore u_t - t u_x = g_x t - t g_x = 0$ ☑ PDE satisfied

Testing with initial condition,

$u(x,0) = g(x + 0^2/2) = g(x)$ ☑ Initial condition satisfied

⟨3⟩ 2-D PDE

Given PDE :- $3u_t + 6u_x - 9u_y = x\cos t$

$u(x,y,0) = y^2 e^x$ ... initial condition

Since the coefficients are constant we can use the formula derived in class for higher dimensions

First rewriting the PDE as follows,

$$u_t + \frac{6}{3}u_x - \frac{9}{3}u_y = \frac{x}{3}\cos t \quad \text{(divide throughout by 3)}$$

$$\therefore u_t + 2u_x - 3u_y = \frac{1}{3}x\cos t$$

$$\therefore \frac{\partial u}{\partial t} + \vec{b}\cdot\nabla_{\vec{x}} u = f(\vec{x},t) \quad \text{where, } \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$\therefore$ using the general formula,

$$u(\vec{x},t) = g(\vec{x} - t\vec{b}) + \int_0^t f(\vec{x} + (s-t)\vec{b}, s)\, ds$$

$$\therefore u(x,y,t) = g(x-2t, y+3t) + \int_0^t f\left(x+2(s-t), y-3(s-t), s\right) ds$$

$$= (y+3t)^2 e^{x-2t} + \int_0^t \frac{1}{3}(x+2s-2t)\cos s\, ds$$

$$= (y^2 + 9t^2 + 6yt)\exp(x-2t) + \frac{(x-2t)}{3}\int_0^t \cos s\, ds + \frac{2}{3}\int_0^t s\cos s\, ds$$

$$\therefore u(x,y,t) = e^{x-2t}(y^2+9t^2+6yt) + \frac{(x-2t)}{3}\sin s\Big|_0^t + \frac{2}{3}\int_0^t s\cos s\, ds$$

$$\int_a^b \underset{s}{u}\, d\underset{\cos s}{v} = uv\Big|_a^b - \int_a^b v\,du$$

$$\therefore u(x,y,t) = (y^2+9t^2+6yt)e^{x-2t} + \frac{(x-2t)\sin t}{3} + \frac{2}{3}s\sin s\Big|_0^t - \frac{2}{3}\int_0^t \sin s\, ds$$

(using integration by parts)

$$= (y^2+9t^2+6yt)e^{x-2t} + \frac{(x-2t)}{3}\sin t + \frac{2t}{3}\sin t - \frac{2}{3}\Big[-\cos s\Big]_0^t$$

$$\boxed{\therefore u(x,y,t) = e^{x-2t}(y^2+9t^2+6yt) + \frac{(x-2t)}{3}\sin t + \frac{2}{3}t\sin t + \frac{2}{3}\cos t - \frac{2}{3}}$$

Testing solution with PDE,

$$u_t = e^{x-2t}(18t+6y) - 2(y^2+9t^2+6yt)e^{x-2t} + \frac{1}{3}(x-\cancel{2t})\cos t - \frac{2}{3}\sin t + \cancel{\frac{2}{3}t\cos t} + \cancel{\frac{2}{3}\sin t} - \cancel{\frac{2}{3}\sin t}$$

$$= e^{x-2t}(18t+6y-2y^2-18t^2-12yt) + (1/3)x\cos t - (2/3)\sin t$$

$$\therefore 3u_t = 3e^{x-2t}(18t+6y-2y^2-18t^2-12yt) + x\cos t - 2\sin t \longrightarrow (1)$$

$$u_x = (y^2+9t^2+6yt)e^{x-2t} + \sin t\cdot(1/3)$$

$$\therefore 6u_x = 6(y^2+9t^2+6yt)e^{x-2t} + 2\sin t \longrightarrow (2)$$

and $u_y = e^{x-2t}(2y+6t)$

$$\therefore -9u_y = -9e^{x-2t}(2y+6t) \longrightarrow (3)$$

$\therefore$ 1 + 2 + 3 $\Rightarrow$

$$3u_t + 6u_x - 9u_y = 3e^{x-2t}\left(18t + 6y - 2y^2 - 18t^2 - 12yt\right.$$
$$+ x\cos t - \cancel{2\sin t} + \cancel{2\sin t}$$
$$+ 6(y^2 + 9t^2 + 6yt)e^{x-2t}$$
$$- 9(2y + 6t)e^{x-2t}$$

$$= 3e^{x-2t}\left(\cancel{18t} + \cancel{6y} - \cancel{2y^2} - \cancel{18t^2} - \cancel{12yt}\right.$$
$$\left.+ \cancel{2y^2} + \cancel{18t^2} + \cancel{12yt} - \cancel{6y} - \cancel{18t}\right)$$
$$+ x\cos t$$

$\therefore 3u_t + 6u_x - 9u_y = x\cos t$ $\boxed{\checkmark}$ PDE satisfied!

Testing solution with initial condition,

$$u(x, y, 0) = e^{x-2(0)}\left(y^2 + \cancel{9(0)^2} + \cancel{6y(0)}\right) + \cancel{\frac{(x-2\cdot 0)\sin 0}{3}}$$
$$+ \cancel{\frac{2}{3}\cdot 0\cdot \sin 0} + \cancel{\frac{2}{3}\cos 0} - \cancel{\frac{2}{3}}$$
$$= y^2 e^x$$ $\boxed{\checkmark}$ initial condition satisfied!