ECE 6560 Marcus Pereira

Homework #4

⟨1.⟩ 4^{th} order central difference (1^{st} derivative)

Given equations are :-

(a) $f(x+2\Delta x) = f(x) + 2\Delta x f'(x) + 2\Delta x^2 f''(x) + \frac{4}{3}\Delta x^3 f'''(x) + \frac{2}{3}\Delta x^4 f^{(4)}(x) + \theta(\Delta x^5)$

(b) $f(x-2\Delta x) = f(x) - 2\Delta x f'(x) + 2\Delta x^2 f''(x) - \frac{4}{3}\Delta x^3 f'''(x) + \frac{2}{3}\Delta x^4 f^{(4)}(x) + \theta(\Delta x^5)$

(c) $f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{1}{2}\Delta x^2 f''(x) + \frac{1}{6}\Delta x^3 f'''(x) + \frac{1}{24}\Delta x^4 f^{(4)}(x) + \theta(\Delta x^5)$

(d) $f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{1}{2}\Delta x^2 f''(x) - \frac{1}{6}\Delta x^3 f'''(x) + \frac{1}{24}\Delta x^4 f^{(4)}(x) + \theta(\Delta x^5)$

We want to zero out columns 1, 3, 4 and 5 and sum the coefficients of column 2 to 1. Suppose columns 1, 3 & 5 can be zeroed out using ($\tilde{a}, -\tilde{a}, \tilde{b}$ and $-\tilde{b}$), which leaves us with columns 2 and 4. Consider column 4 first,

$$\frac{4}{3}\Delta x^3 \tilde{a} - \left(-\tilde{a}\frac{4}{3}\Delta x^3\right) + \frac{\tilde{b}}{6}\Delta x^3 - \left(-\tilde{b}\frac{1}{6}\Delta x^3\right) = 0$$

$$\Rightarrow \frac{8}{3}\Delta x^3 \tilde{a} + \frac{1}{3}\tilde{b}\Delta x^3 = 0 \quad \text{or} \quad \frac{8}{3}\tilde{a} + \frac{1}{3}\tilde{b} = 0 \quad (\Delta x > 0)$$

$$\therefore 8\tilde{a} + \tilde{b} = 0 \longrightarrow \text{I}$$

We also want column (2) to sum upto 1, we have

$$2\Delta x \tilde{a} - (-2\Delta x \tilde{a}) + \tilde{b}\Delta x - (-\tilde{b}\Delta x) = 1$$

$$4\tilde{a} + 2\tilde{b} = \frac{1}{\Delta x} \longrightarrow \text{II}$$

$$\therefore \text{I} - 2\,\text{II} \Rightarrow \begin{array}{r} 8\tilde{a} + \tilde{b} = 0 \\ -\;8\tilde{a} \pm 4\tilde{b} = 2/\Delta x \\ \hline -3\tilde{b} = -\frac{2}{\Delta x} \end{array} \Rightarrow \boxed{\tilde{b} = \frac{2}{3\Delta x}}$$

$$\therefore 8\tilde{a} = -\tilde{b} = -\frac{2}{3\Delta x} \Rightarrow \boxed{\tilde{a} = -\frac{1}{12\Delta x}}$$

Using $\tilde{a} = \frac{-1}{12\Delta x}$ and $\tilde{b} = \frac{2}{3\Delta x}$ in the following manner

$$(**) \quad ⓐ \times \tilde{a} + ⓑ \times (-\tilde{a}) + ⓒ \times \tilde{b} + ⓓ \times (-\tilde{b})$$

will zero out columns 1, 3 and 5 since coefficients will be equal and opposite and will also zero out column 4 and sum up column 2's coefficients to 1 as per above calculations.

$\therefore$ R.H.S $= f'(x)$

Using these to evaluate the L.H.S., we have,

$$\text{L.H.S} = \tilde{a} \cdot f(x+2\Delta x) - \tilde{a} \cdot f(x-2\Delta x) + \tilde{b} \cdot f(x+\Delta x) - \tilde{b} \cdot f(x-\Delta x)$$

$$= -\frac{1}{12\Delta x} f(x+2\Delta x) - \left(-\frac{1}{12\Delta x} f(x-2\Delta x)\right) + \frac{2}{3\Delta x} f(x+\Delta x) - \left(\frac{2}{3\Delta x} f(x-\Delta x)\right)$$

$$= \frac{1}{12\Delta x}\left(f(x-2\Delta x) - f(x+2\Delta x)\right) + \frac{2}{3\Delta x}\left(f(x+\Delta x) - f(x-\Delta x)\right)$$

$$= \frac{-1}{3}\left[\frac{f(x+2\Delta x) - f(x-2\Delta x)}{4\Delta x}\right] + \frac{4}{3}\left[\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}\right]$$

which proves the expression given!

As far as the higher order terms, each of ⓐ, ⓑ, ⓒ and ⓓ have $\Theta(\Delta x^5)$

However, when we multiply by coefficients as in $(**)$, since each of the coefficients has $\frac{1}{\Delta x}$, the higher order terms become $\Theta(\Delta x^4)$

This term is on the R.H.S, which becomes $-\Theta(\Delta x^4)$ on the L.H.S

But, this negative sign can be absorbed into the Big-O notation giving $\Theta(\Delta x^4)$ on the L.H.S

$\therefore$ We get L.H.S = R.H.S $\Rightarrow$

$$\boxed{f'(x) = \frac{4}{3}\left[\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}\right] - \frac{1}{3}\left[\frac{f(x+2\Delta x) - f(x-2\Delta x)}{4\Delta x}\right] + \Theta(\Delta x^4)}$$

⟨2.⟩ 4^{th} Order central difference (2^{nd} derivative)

Consider the 4 Taylor's expansions,

(a) $f(x+2\Delta x) = f(x) + 2\Delta x f'(x) + 2\Delta x^2 f''(x) + \frac{4}{3}\Delta x^3 f'''(x) + \frac{2}{3}\Delta x^4 f^{(4)}(x) + \frac{4}{15}\Delta x^5 f^{(5)}(x) + \Theta^3(\Delta x^6)$

(b) $f(x-2\Delta x) = f(x) - 2\Delta x f'(x) + 2\Delta x^2 f''(x) - \frac{4}{3}\Delta x^3 f'''(x) + \frac{2}{3}\Delta x^4 f^{(4)}(x) - (4/15)\Delta x^5 f^{(5)}(x) + \Theta(\Delta x^6)$

(c) $f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{1}{2}\Delta x^2 f''(x) + \frac{1}{6}\Delta x^3 f'''(x) + \frac{1}{24}\Delta x^4 f^{(4)}(x) + \frac{1}{120}\Delta x^5 f^{(5)}(x) + \Theta(\Delta x^6)$

(d) $f(x-\Delta x) = f(x) - \Delta x f'(x) + (1/2)\Delta x^2 f''(x) - (1/6)\Delta x^3 f'''(x) + (1/24)\Delta x^4 f^{(4)}(x) - (1/120)\Delta x^5 f^{(5)}(x) + \Theta(\Delta x^6)$

(e) $f(x) = f(x)$

We can zero out columns 2, 4 and 6 by multiplying with same coefficients of $(\tilde{a}, \tilde{a}, \tilde{b}, \tilde{b})$ as these columns have coefficients of equal magnitude but opposite sign

This leaves us with columns 1, 3 and 5 where all coefficients are positive.

For column 1, we would get $(2\tilde{a} + 2\tilde{b}) f(x)$

To eliminate this we can use equation (e) and multiply by coefficient $-(2\tilde{a} + 2\tilde{b})$ to cancel out coefficients of equations (a) to (d).

So, finally for columns 3 and 5 we solve for $\tilde{a}$ and $\tilde{b}$ such that coefficients of 3 sum upto 1 and those of 5 sum upto 0.

for col. 3, $2\Delta x^2 \tilde{a} + 2\Delta x^2 \tilde{a} + \frac{1}{2}\Delta x^2 \tilde{b} + \frac{1}{2}\Delta x^2 \tilde{b} = 1$

$$4\Delta x^2 \tilde{a} + \Delta x^2 \tilde{b} = 1$$

$$\text{or } 4\tilde{a} + \tilde{b} = \frac{1}{\Delta x^2} \rightarrow \text{(I)}$$

for col. 5, $\frac{2}{3}\cancel{\Delta x^4}\tilde{a} + \frac{2}{3}\cancel{\Delta x^4}\tilde{a} + \frac{1}{24}\cancel{\Delta x^4}\tilde{b} + \frac{1}{24}\cancel{\Delta x^4}\tilde{b} = 0 \quad \ldots (\Delta x^4 > 0)$

$$\frac{4}{3}\tilde{a} + \frac{1}{12}\tilde{b} = 0$$

$$16\tilde{a} + \tilde{b} = 0 \rightarrow \text{(II)}$$

$$4 \cdot \text{(I)} - \text{(II)} \Rightarrow \begin{array}{r} 16\tilde{a} + 4\tilde{b} = 4/\Delta x^2 \\ -16\tilde{a} + \tilde{b} = 0 \\ \hline 3\tilde{b} = \frac{4}{\Delta x^2} \end{array} \Rightarrow \boxed{\tilde{b} = \frac{4}{3\Delta x^2}}$$

$$\therefore\ 16\tilde{a} = -\tilde{b} = -\frac{4}{3\Delta x^2} \Rightarrow \tilde{a} = -\frac{4}{3 \cdot \cancel{16}_{4} \cdot \Delta x^2} = \boxed{-\frac{1}{12\Delta x^2}}$$

$\therefore$ (a) $\times \tilde{a}$ + (b) $\times \tilde{a}$ + (c) $\times \tilde{b}$ + (d) $\times \tilde{b}$ + (e) $\times (-2\tilde{a} - 2\tilde{b})$

will yield, $R.H.S = f''(x) + \frac{\theta(\Delta x^6)}{\Delta x^2}$

$$= f''(x) + \theta(\Delta x^4)$$

and $L.H.S = \tilde{a}\left[f(x+2\Delta x) + f(x-2\Delta x)\right] + \tilde{b}\left[f(x+\Delta x) + f(x-\Delta x)\right] - 2\tilde{a}f(x) - 2\tilde{b}f(x)$

$$= -\frac{1}{12\Delta x^2}\left[f(x+2\Delta x) + f(x-2\Delta x)\right] + \frac{4}{3\Delta x^2}\left[f(x+\Delta x) + f(x-\Delta x)\right] - 2\left(-\frac{1}{12\Delta x^2}\right)f(x) - 2\left(\frac{4}{3\Delta x^2}\right)f(x)$$

$$= \frac{1}{12\Delta x^2}\left[-f(x+2\Delta x) - f(x-2\Delta x) + 16f(x+\Delta x) + 16f(x-\Delta x) \quad 2f(x) - 32f(x)\right]$$

$$= \frac{1}{12\Delta x^2}\left[-f(x+2\Delta x) + 16f(x+\Delta x) - 30f(x) + 16f(x-\Delta x) - f(x-2\Delta x)\right]$$

Taking the $\theta(\Delta x^4)$ to LHS and absorbing the negative sign, the final expression is,

$$\boxed{f''(x) = \frac{-f(x+2\Delta x) + 16f(x+\Delta x) - 30f(x) + 16f(x-\Delta x) - f(x-2\Delta x)}{12\Delta x^2} + \theta(\Delta x^4)}$$

⟨3.⟩ 3rd order forward difference for 1st derivative

Consider the following Taylor's approximations,

$e_1: f(x) = f(x)$

$e_2: f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{1}{2}\Delta x^2 f''(x) + \frac{1}{6}\Delta x^3 f'''(x) + \theta(\Delta x^4)$

$e_3: f(x+2\Delta x) = f(x) + 2\Delta x f'(x) + 2\Delta x^2 f''(x) + \frac{4}{3}\Delta x^3 f'''(x) + \theta(\Delta x^4)$

$e_4: f(x+3\Delta x) = f(x) + 3\Delta x f'(x) + \frac{9}{2}\Delta x^2 f''(x) + \frac{9}{2}\Delta x^3 f'''(x) + \theta(\Delta x^4)$

We want to make column 1, 3 and 4 sum to zero and column 2 sum to 1.

Consider 2, 3 and 4.

for col. 2, $a\Delta x + b\,2\Delta x + c\,3\Delta x = 1$

$$a + 2b + 3c = \frac{1}{\Delta x} \rightarrow \text{I.}$$

for col. 3, $\frac{a}{2}\cancel{\Delta x^2} + 2b\cancel{\Delta x^2} + \frac{9c}{2}\cancel{\Delta x^2} = 0$ ...... $(\Delta x^2 > 0)$

$$a + 4b + 9c = 0 \rightarrow \text{II.}$$

for col. 4, $\frac{a}{6}\cancel{\Delta x^3} + \frac{4b}{3}\cancel{\Delta x^3} + \frac{9}{2}c\cancel{\Delta x^3} = 0$ ..... $(\Delta x^3 \neq 0)$

$$a + 8b + 27c = 0 \rightarrow \text{III}$$

$\text{III} - \text{II} \Rightarrow$

$$\begin{array}{r} a + 8b + 27c = 0 \\ -\ a + 4b + 9c = 0 \\ \hline 4b + 18c = 0 \end{array}$$

$$2b + 9c = 0 \Rightarrow b = -\frac{9}{2}c$$

$$\text{II} - \text{I} \Rightarrow \begin{array}{r} \cancel{a} + 4b + 9c = 0 \\ -\ \cancel{a} + 2b + 3c = \frac{1}{\Delta x} \\ \hline 2b + 6c = -\frac{1}{\Delta x} \end{array}$$

substituting $b = -\frac{9}{2}c$,

$$\cancel{2}\left(-\frac{9}{\cancel{2}}c\right) + 6c = -\frac{1}{\Delta x}$$

$$-9c + 6c = -\frac{1}{\Delta x}$$

$$-3c = -\frac{1}{\Delta x} \Rightarrow \boxed{c = \frac{1}{3\Delta x}}$$

$$\therefore b = -\frac{9}{2}c = -\frac{\cancel{9}^{3}}{2}\cdot\frac{1}{\cancel{3}\Delta x} = \boxed{-\frac{3}{2\Delta x}}$$

from II, $a = -4b - 9c = -\cancel{4}^{2}\left(\frac{-3}{\cancel{2}\Delta x}\right) - \cancel{9}^{3}\left(\frac{1}{\cancel{3}\Delta x}\right)$

$$= \frac{6}{\Delta x} - \frac{3}{\Delta x} = \boxed{\frac{3}{\Delta x}}$$

Using the following,

$e_1 \times -(a+b+c) + e_2 \times a + e_3 \times b + e_4 \times c$
will zero out columns 1, 3 and 4 and sum up coefficients of column 2 to 1.

$$\therefore R.H.S = f'(x) + \frac{\Theta(\Delta x^4)}{\Delta x} = f'(x) + \Theta(\Delta x^3)$$

and $L.H.S = -(a+b+c)f(x) + af(x+\Delta x) + bf(x+2\Delta x) + cf(x+3\Delta x)$

$$= -\left(\frac{3}{\Delta x} - \frac{3}{2\Delta x} + \frac{1}{3\Delta x}\right)f(x) + \frac{3}{\Delta x}f(x+\Delta x) - \frac{3}{2\Delta x}f(x+2\Delta x) + \frac{1}{3\Delta x}f(x+3\Delta x)$$

$$\therefore L.H.S = -\left(\frac{18-9+2}{6\Delta x}\right) f(x) + \frac{18}{6\Delta x} f(x+\Delta x)$$

$$-\frac{9}{6\Delta x} f(x+2\Delta x) + \frac{2}{6\Delta x} f(x+3\Delta x)$$

$$= \frac{2f(x+3\Delta x) - 9f(x+2\Delta x) + 18f(x+\Delta x) - 11f(x)}{6\Delta x}$$

$$\boxed{\therefore f'(x) = \frac{2f(x+3\Delta x) - 9f(x+2\Delta x) + 18f(x+\Delta x) - 11f(x)}{6\Delta x} + \theta(\Delta x^3)}$$