Homework #5: Von Neumann Stability Analysis

Solutions

1 Lax Method for 1-D Transport Equation

We saw in class that the following forward-time centered-space (FTCS) discretezation of the transport PDE $u_t + bu_x = 0$

$$u(x, t + \Delta t) = u(x, t) - b\Delta t \, \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x}$$

is unstable regardless of the chosen time step Δt**. A variant of this unstable FTCS scheme, known as the Lax Method, is to replace** $u(x, t)$ **on the right hand side, with the averge of its two neighbors as follows.**

$$u(x, t + \Delta t) = \frac{u(x + \Delta x, t) + u(x - \Delta x, t)}{2} - b\Delta t \, \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x}$$

Apply the DFT-based Von-Neumann analysis done in class to this discretization to determine whether it is stable, and if so, what is the required CFL condition (relationshiop between Δt **and** Δx**) to ensure stability. (You must show the derivation of your answer. Answers that are not derived, even if correct, will not be given credit.)**

Solution:

Applying the DFT to both sides (transforming the space variable x to a frequency variable ω) of the Lax update yields

$$\begin{aligned}
U(\omega, t + \Delta t) &= \frac{e^{j\omega\Delta x}U(\omega, t) + e^{-j\omega\Delta x}U(\omega, t)}{2} - b\Delta t \, \frac{e^{j\omega\Delta x}U(\omega, t) - e^{-j\omega\Delta x}U(\omega, t)}{2\Delta x} \\
&= \left(\frac{e^{j\omega\Delta x} + e^{-j\omega\Delta x}}{2} - j\frac{b\Delta t}{\Delta x} \, \frac{e^{j\omega\Delta x} - e^{-j\omega\Delta x}}{2j}\right) U(\omega, t) \\
&= \underbrace{\left(\cos(\omega\Delta x) - j\frac{b\Delta t}{\Delta x}\sin(\omega\Delta x)\right)}_{\text{amplification factor } \alpha(\omega)} U(\omega, t)
\end{aligned}$$

We now compute the squared complex amplitude of the amplification factor $\alpha(\omega)$ as follows.

$$|\alpha(\omega)|^2 = \cos^2(\omega\Delta x) + \left(\frac{b\Delta t}{\Delta x}\right)^2 \sin^2(\omega\Delta x)$$

Note that if the coefficient $\left(\frac{\Delta t\, b}{\Delta x}\right)^2$ multiplying the $\sin^2(\omega\Delta x)$ term is equal to one, then we have the sum of the cosine and sine squared, which is one. Thus if the coefficient exceeds one, then the sum will also exceed one, and if the coefficent is less than one, the sum will also be less than one. As such, the stability condition $|\alpha(\omega)|^2 \leq 1$ is equivalent to the condition $\left(\frac{\Delta t\, b}{\Delta x}\right)^2 \leq 1$ which in turn yields the following CFL condition

$$|b|\Delta t \leq \Delta x$$

2 Leap-Frog Method for 1-D Transport Equation

So far, all of the PDE discretizations discussed in class have used a first order, forward difference approxmation in time. Explicity schemes, however, can be created using higher order time derivative approximations if one or more previous values of the solution are saved (at $t-\Delta t, t-2\Delta t, \ldots$) as the scheme is iterated. The first such example is the second rder Leap-Frog method which uses a second order, central differences to approximate BOTH the time derivative as well as a the space derivative for $u_t + bu_x = 0$ as follows

$$\frac{u(x,t+\Delta t)-u(x,t-\Delta t)}{2\Delta t} + b\frac{u(x+\Delta x,t)-u(x-\Delta x,t)}{2\Delta x} = 0$$

yielding the following update which depends on both the current and previous values of u.

$$u(x,t+\Delta t) = u(x,t-\Delta t) - \frac{b\Delta t}{\Delta x}\left(u(x+\Delta x,t)-u(x-\Delta x,t)\right)$$

Apply DFT-based Von Neumann analysis to determine the stability of this scheme and, if stable, the necessary CFL condition. (HINT: You will end up with an expression relating the DFTs at three different times $t+\Delta t$, t, and $t-\Delta t$. To determine the amplification factor $\alpha(\omega)$ you may wish to substitute the relationships $U(\omega,t) = \alpha(\omega)U(\omega,t-\Delta t)$ and $U(\omega,t+\Delta t) = \alpha(\omega)U(\omega,t) = \alpha^2(\omega)U(\omega,t-\Delta t)$ into your expression and then solve for $\alpha(\omega)$.)

Solution

Applying the DFT to both sides (transforming the space variable x to a frequency variable ω) of the Leap-Frog update yields

$$\begin{aligned}
U(\omega,t+\Delta t) &= U(\omega,t-\Delta t) - \frac{b\Delta t}{\Delta x}\left(e^{j\omega\Delta x}U(\omega,t) - e^{-j\omega\Delta x}U(\omega,t)\right) \\
&= U(\omega,t-\Delta t) - 2j\frac{b\Delta t}{\Delta x}\left(\frac{e^{j\omega\Delta x}-e^{-j\omega\Delta x}}{2j}\right)U(\omega,t) \\
&= U(\omega,t-\Delta t) - 2j\frac{b\Delta t}{\Delta x}\sin(\omega\Delta x)\,U(\omega,t)
\end{aligned}$$

Since we have three different times involved, we cannot immediately get the amplification factor $\alpha(\omega)$ from this expression. Instead, we now substitute $U(\omega,t) = \alpha(\omega)U(\omega,t-\Delta t)$ and $U(\omega,t+\Delta t) = \alpha(\omega)U(\omega,t) = \alpha^2(\omega)U(\omega,t-\Delta t)$ into this expression and then solve for $\alpha(\omega)$ as follows.

$$\begin{aligned}
\alpha^2(\omega)U(\omega,t-\Delta t) &= \left(1 - 2j\alpha(\omega)\frac{b\Delta t}{\Delta x}\sin(\omega\Delta x)\right)U(\omega,t-\Delta t) \\
0 &= \alpha^2(\omega) + 2j\alpha(\omega)\frac{b\Delta t}{\Delta x}\sin(\omega\Delta x) - 1 \\
\alpha(\omega) &= -j\frac{b\Delta t}{\Delta x}\sin(\omega\Delta x) \pm \sqrt{1 - \left(\frac{b\Delta t}{\Delta x}\sin(\omega\Delta x)\right)^2}
\end{aligned}$$

Note that while the quadratic equation yields two different solutions for $\alpha(\omega)$, we are ultimately only intersted in the complex magnitude which is the same for both solutions whenever the square root is real (which is the case so long as the expression under the radical is non-negative). It is quite easy to see simlply by glancing that, in such a case, $|\alpha(\omega)|^2 = 1$ for all omega, which in turn means no attenuation or amplifcation of the DFT, and therefore a stable discretization scheme. As such, we simply write down the condition for the a non-negative expression under the radical as follows.

$$\begin{aligned}
\left(\frac{b\Delta t}{\Delta x}\sin(\omega\Delta x)\right)^2 &\leq 1 \\
\frac{|b|\Delta t}{\Delta x}|\sin(\omega\Delta x)| &\leq 1
\end{aligned}$$

Since the equality must be satisfied for all ω, we take the worst-case scenario where $|\sin(\omega\Delta x)| = 1$ to obtain our CFL condition.

$$|b|\Delta t \leq \Delta x$$

3 Fully Implicit Scheme for 1-D Heat Equation

In class we considered to explicit forward-Euler update scheme for the linear heat equation $u_t = bu_{xx}$ as follows

$$u(x, t+\Delta t) = u(x,t) + b\Delta t \left(\frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^2} \right)$$

where u_{xx} is approximated by a central difference at the time t to compute the updated value of u. If we, instead, replace this with the same central difference formula at the updated time $t + \Delta t$ we obtain an implicit update scheme (which requires the solultion of a linear system, as an explict point-by-point update is no longer possible).

$$u(x, t+\Delta t) = u(x,t) + b\Delta t \left(\frac{u(x+\Delta x, t+\Delta t) - 2u(x,t+\Delta t) + u(x-\Delta x, t+\Delta t)}{\Delta x^2} \right)$$

For the explicit forward Euler scheme, we derived the CFL stability condition $b\Delta t \leq \frac{1}{2}\Delta x^2$ (as well as the fact that b must be positive). Apply DFT-based Von Neumann analysis to this implicit version of the update scheme and determine the CFL stability condition (as well as whether or not b must still be positive to ensure stability).

Solution

Applying the DFT to both sides (transforming the space variable x to a frequency variable ω) of the implicit update yields

$$\begin{aligned}
U(\omega, t+\Delta t) &= U(\omega, t) + b\Delta t \left(\frac{e^{j\omega\Delta x} U(\omega, t+\Delta t) - 2U(\omega, t+\Delta t) + e^{-j\omega\Delta x} U(\omega, t+\Delta t)}{\Delta x^2} \right) \\
&= U(\omega, t) + 2\frac{b\Delta t}{\Delta x^2} \left(\cos(\omega\Delta x) - 1 \right) U(\omega, t+\Delta t) \\
U(\omega, t+\Delta t) &= \underbrace{\frac{1}{1 + 2\frac{b\Delta t}{\Delta x^2}\left(1 - \cos(\omega\Delta x)\right)}}_{\text{amplification factor } \alpha(\omega)} U(\omega, t)
\end{aligned}$$

Notice that the amplication factor $\alpha(\omega)$ is purely real, and since $1 - \cos(\omega\Delta x)$ is never negative, we can see that as long as $b > 0$, then the denominator is greater than or equal to one, and the amplification factor is therefore always a positive number less than 1 (and consquently stable) regardles of the size of the time step. As such, this implicit scheme is *unconditionally stable* for positive b (assuming your time step is in the forward direction). On the otherhand, if b is negative, then the denominator is less than 1 for most values of ω, and therefore the amplification factor is greater than 1, even for extremely small time steps, thereby making the scheme unstable (just as in the explicit forward Euler scheme) for negative b.