ECE 6560 Homework-2 Marcus Pereira

⟨1.⟩ Rotational Invariance of Laplacian

Given :- $v\begin{pmatrix} x \\ y \\ z \end{pmatrix} = u\begin{pmatrix} \frac{x}{\sqrt{3}} - \frac{2y}{\sqrt{6}} \\ \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{z}{\sqrt{2}} \\ \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{2}} \end{pmatrix} \overset{(say)}{:=} u\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}$

we want to show that : $u_{xx} + u_{yy} + u_{zz} = u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}} + u_{\tilde{z}\tilde{z}}$

we have, $\tilde{x} = \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{6}}y + 0z$

$\tilde{y} = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{6}}y - \frac{1}{\sqrt{2}}z$

$\tilde{z} = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{6}}y + \frac{1}{\sqrt{2}}z$

∴ We can define the following operators,

$$\frac{\partial}{\partial \tilde{x}}(\cdot) = \frac{1}{\sqrt{3}}\frac{\partial(\cdot)}{\partial x} - \frac{2}{\sqrt{6}}\frac{\partial}{\partial y}(\cdot)$$

$$\frac{\partial}{\partial \tilde{y}}(\cdot) = \frac{1}{\sqrt{3}}\frac{\partial}{\partial x}(\cdot) + \frac{1}{\sqrt{6}}\frac{\partial}{\partial y}(\cdot) - \frac{1}{\sqrt{2}}\frac{\partial}{\partial z}(\cdot)$$

and $\frac{\partial}{\partial \tilde{z}}(\cdot) = \frac{1}{\sqrt{3}}\frac{\partial}{\partial x}(\cdot) + \frac{1}{\sqrt{6}}\frac{\partial}{\partial y}(\cdot) + \frac{1}{\sqrt{2}}\frac{\partial}{\partial z}(\cdot)$

Consider, $u_{\tilde{x}} = \frac{1}{\sqrt{3}}v_x - \frac{2}{\sqrt{6}}v_y$ ..... $\left\{ \begin{array}{l} \because v(x,y,z) = u(\tilde{x},\tilde{y},\tilde{z}) \\ \text{We take partial on both sides wrt } \tilde{x} \end{array} \right.$

$$\therefore u_{\tilde{x}\tilde{x}} = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}v_{xx} - \frac{2}{\sqrt{6}}v_{yx}\right) - \frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{3}}v_{xy} - \frac{2}{\sqrt{6}}v_{yy}\right)$$

$$\therefore u_{\tilde{x}\tilde{x}} = \frac{1}{3}v_{xx} - \frac{\sqrt{2}}{3}v_{yx} - \frac{\sqrt{2}}{3}v_{xy} + \frac{2}{3}v_{yy} \longrightarrow (1)$$

Similarly, consider $u_{\tilde{y}} = \frac{1}{\sqrt{3}}v_x + \frac{1}{\sqrt{6}}v_y - \frac{1}{\sqrt{2}}v_z$

$$\therefore u_{\tilde{y}\tilde{y}} = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}v_{xx} + \frac{1}{\sqrt{6}}v_{yx} - \frac{1}{\sqrt{2}}v_{zx}\right) + \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{3}}v_{xy} + \frac{1}{\sqrt{6}}v_{yy} - \frac{1}{\sqrt{2}}v_{zy}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}v_{xz} + \frac{1}{\sqrt{6}}v_{yz} - \frac{1}{\sqrt{2}}v_{zz}\right)$$

$$\therefore u_{\tilde{y}\tilde{y}} = \frac{1}{3}v_{xx} + \frac{1}{3\sqrt{2}}v_{yx} - \frac{1}{\sqrt{6}}v_{zx} + \frac{1}{3\sqrt{2}}v_{xy} + \frac{1}{6}v_{yy} - \frac{1}{2\sqrt{3}}v_{zy} - \frac{1}{\sqrt{6}}v_{xz} - \frac{1}{2\sqrt{3}}v_{yz} + \frac{1}{2}v_{zz} \longrightarrow (2)$$

Finally, $u_{\tilde{z}} = \frac{1}{\sqrt{3}} v_x + \frac{1}{\sqrt{6}} v_y + \frac{1}{\sqrt{2}} v_z$

$$\therefore u_{\tilde{z}\tilde{z}} = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}} v_{xx} + \frac{1}{\sqrt{6}} v_{yx} + \frac{1}{\sqrt{2}} v_{zx}\right) + \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{3}} v_{xy} + \frac{1}{\sqrt{6}} v_{yy} + \frac{1}{\sqrt{2}} v_{zy}\right)$$

$$+ \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}} v_{xz} + \frac{1}{\sqrt{6}} v_{yz} + \frac{1}{\sqrt{2}} v_{zz}\right)$$

$$\therefore u_{\tilde{z}\tilde{z}} = \frac{1}{3} v_{xx} + \frac{1}{3\sqrt{2}} v_{yx} + \frac{1}{\sqrt{6}} v_{zx} + \frac{1}{3\sqrt{2}} v_{xy} + \frac{1}{6} v_{yy} + \frac{1}{2\sqrt{3}} v_{zy}$$

$$+ \frac{1}{\sqrt{6}} v_{xz} + \frac{1}{2\sqrt{3}} v_{yz} + \frac{1}{2} v_{zz} \longrightarrow (3.)$$

Adding (1.), (2.) and (3.), we get, (combine terms assuming symmetry. eg. $u_{xy} = u_{yx}$)

$$u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}} + u_{\tilde{z}\tilde{z}} = \left(\frac{1}{3} v_{xx} - \frac{2\sqrt{2}}{3} v_{xy} + \frac{2}{3} v_{yy}\right)$$

$$+$$

$$\left(\frac{1}{3} v_{xx} + \frac{(\sqrt{2})^2}{3\sqrt{2}} v_{xy} - \cancel{\frac{2}{\sqrt{6}} v_{xz}} + \frac{1}{6} v_{yy} + \frac{1}{2} v_{zz} - \cancel{\frac{1}{\sqrt{3}} v_{yz}}\right)$$

$$+$$

$$\left(\frac{1}{3} v_{xx} + \frac{(\sqrt{2})^2}{3\sqrt{2}} v_{xy} + \cancel{\frac{2}{\sqrt{6}} v_{xz}} + \frac{1}{6} v_{yy} + \frac{1}{2} v_{zz} + \cancel{\frac{1}{\sqrt{3}} v_{yz}}\right)$$

$$= \frac{3}{3} v_{xx} + \frac{3}{3} v_{yy} + \frac{2}{2} v_{zz} + v_{xy}\left(\frac{2\sqrt{2}}{3} - \frac{2\sqrt{2}}{3}\right)$$

$$\boxed{\therefore u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}} + u_{\tilde{z}\tilde{z}} = v_{xx} + v_{yy} + v_{zz}}$$

$$\Rightarrow \boxed{\Delta u(\tilde{x}, \tilde{y}, \tilde{z}) = \Delta v(x, y, z)}$$

Hence proved.

<2.> Gauss – Green Theorem :– $\int_{\Omega} u_{x_i} dx = \int_{\partial\Omega} u N_i ds$

we want to evaluate $\int_{\Omega} \vec{b} \cdot \nabla u \, d\bar{x}$ ..... $\bar{x} = (x, y, z)$

$$= \int_0^1 \int_0^1 \int_0^1 (u_x + 1.5 u_y - 2.25 u_z) \, dx \, dy \, dz$$

By using the theorem we have,

$$\int_{\Omega} \vec{b} \cdot \nabla u \, d\bar{x} = \int_{\Omega} u_x \, d\bar{x} + \int_{\Omega} 1.5 u_y \, d\bar{x} - \int_{\Omega} 2.25 u_z \, d\bar{x}$$

$$= \left[\int_{\partial\Omega} u N_x dS \right] + \left[1.5 \int_{\partial\Omega} u N_y dS \right] - \left[2.25 \int_{\partial\Omega} u_z N_z dS \right]$$

Given surface is a cube with one vertex at (0, 0, 0)
∴ Each of the above surface integrals can be written as a sum over area integrals of each of the faces

$$\Rightarrow \int_{\partial\Omega} u N_i ds = \int_{S_{x+}} u N_{x+} dS + \int_{S_{x-}} u N_{x-} dS + \int_{S_{y+}} u N_{y+} dS + \int_{S_{y-}} u N_{y-} dS + \int_{S_{z+}} u N_{z+} dS + \int_{S_{z-}} u N_{z-} dS$$

where, S_{x+} and S_{x-} are surfaces facing positive and negative x directions and N_{x+} and N_{x-} are respective outward normals to the surfaces.

Using same naming convention for S_{y+}, S_{y-}, N_{y+}, N_{y-}, S_{z+}, S_{z-}, N_{z+} and N_{z-}.

Back to given problem, considering each surface integral on R.H.S separately,

$$\int_{\partial S} u N_x dS = \int_{S_{x+}} u N_{x+} ds + \int_{S_{x-}} u N_{x-} dS + 0$$

The coefficients for y and z would be zero when considering the u_x integral.

Moreover, the coefficients $N_{x+} = 1$ and $N_{x-} = -1$ for the 2 faces in positive x facing and negative x facing directions.

$$\therefore \int_{\partial S} u\, N_x \, dS = \int_{S_{x+}} u \, dS - \int_{S_{x-}} u \, dS$$

Now, along the face facing positive x, $u = 1$ and $u = 0$ along the opposite face.

$$\therefore \int_{\partial S} u\, N_x \, dS = 1 \underbrace{\int_{S_{x+}} dS}_{\text{area of square side}} = 1(1)^2 = 1 \rightarrow (1)$$

Similarly, for $\int_{\partial S} 1.5\, u\, N_y \, dS = 1.5 \left(\int_{S_{y+}} u N_{y+} \, dS + \int_{S_{y-}} u N_{y-} \, dS \right)$

Same as above x and z coefficients of out normals are zero and $N_{y+} = 1$ (for face $u=1$) and $N_{y-} = -1$ (for face $u=0$)

$$\therefore \int_{\partial S} 1.5\, u N_y \, dS = 1.5 \left(\int_{S_{y+}} dS \right) = 1.5\,(1)^2 = 1.5 \rightarrow (2)$$

and doing same process for the 3rd integral, we have,

$$-2.25 \int_{\partial S} u_z N_z \, dS = -2.25 \rightarrow (3)$$

Adding (1), (2) and (3) gives,

$$\int_{\Omega} \vec{b} \cdot \nabla u \, d\bar{x} = 1 + 1.5 - 2.25 = \boxed{0.25}$$

Solving the triple integral to cross-check,

$$\int_0^1 \int_0^1 \int_0^1 (u_x + 1.5 u_y - 2.25 u_z)\, dx\, dy\, dz = \int_0^1 \int_0^1 \left(\int_0^1 -2.25 u_z \, dz \right) dx\, dy + \int_0^1 \int_0^1 \left(\int_0^1 u_x \, dx \right) dy\, dz + \int_0^1 \int_0^1 \left(\int_0^1 1.5 u_y \, dy \right) dx\, dz$$

$$= \int_0^1 \int_0^1 \left(-2.25\, u(x,y,z) \Big|_{z=0}^{z=1} \right) dx\, dy \quad \left\{ \begin{array}{l} \text{at face } z=1, u(x,y,1)=1 \\ \text{and} \\ \text{at face } z=0, u(x,y,0)=0 \end{array} \right\}$$

$$+$$

$$\int_0^1 \int_0^1 \left(u(x,y,z) \Big|_{x=0}^{x=1} \right) dy\, dz + \int_0^1 \int_0^1 \left(1.5\, u(x,y,z) \Big|_{y=0}^{y=1} \right) dx\, dz$$

Same for faces at $x=1$, $x=0$ and $y=1$ and $y=0$

$\therefore$ R.H.S $= -2.25[1-0] + 1[1-0] + 1.5[1-0]$

$= \boxed{0.25}$ This verifies the solution obtained by using Gauss - Green theorem.

⟨3.⟩ Green's formulas and divergence theorem

⟨3.1⟩ We have the equation for integration by parts

$$\int_\Omega u_{x_i} v\, dx = -\int_\Omega u v_{x_i}\, dx + \int_{\partial\Omega} u v N_i\, ds$$

replace v by v_{x_i}, the corresponding equation for integration by parts would be,

$$\int_\Omega u_{x_i} v_{x_i}\, dx = -\int_\Omega u (v_{x_i})_{x_i}\, dx + \int_{\partial\Omega} u v_{x_i} N_i\, dS$$

Doing this for each component x_i and summing the equations,

$$\sum_{i=1}^N \left(\int_\Omega u_{x_i} v_{x_i}\, dx \right) = -\sum_{i=1}^N \left(\int_\Omega u\, v_{x_i x_i}\, dx \right) + \sum_{i=1}^N \left(\int_{\partial\Omega} u v_{x_i} N_i\, dS \right)$$

$$\int_\Omega \sum_{i=1}^N (u_{x_i} v_{x_i})\, dx = -\int_\Omega u \left(\sum_{i=1}^N v_{x_i x_i} \right) dx + \int_{\partial\Omega} u \left(\sum_{i=1}^N v_{x_i} N_i \right) ds$$

each other the sums $\sum_i u_{x_i} v_{x_i}$ and $\sum_i v_{x_i} N_i$ can be replaced by dot products and $\sum_i v_{x_i x_i}$ is definition of Laplacian of v

Substituting the dot products and Laplacian, we get,

$$\boxed{\int_{\Omega} \nabla u \cdot \nabla v \, dx = -\int_{\Omega} u \, \Delta v \, dx + \int_{\partial\Omega} u \, \nabla v \cdot N \, ds}$$

where, $\nabla u = [u_{x_1}, u_{x_2}, \ldots\ldots, u_{x_N}]^T$

$\nabla v = [v_{x_1}, v_{x_2}, \ldots\ldots, v_{x_N}]^T$

$N = [N_1, N_2, \ldots\ldots, N_N]^T$

⟨3.2.⟩ To prove:- $\int_{\partial\Omega} F \cdot N \, ds = \int_{\Omega} \nabla \cdot F \, dx$ ..... Divergence Theorem

where, $F = [F_1, F_2, \ldots\ldots, F_N]^T$

$$RHS = \int_{\Omega} \nabla \cdot F \, dx = \int_{\Omega} \operatorname{div} F \, dx = \int_{\Omega} \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(F_i) \, dx$$

$$= \sum_{i=1}^{N} \int_{\Omega} F_{i\,x_i} \, dx$$

Using Gauss - Green Theorem we get,

$$RHS = \sum_{i=1}^{N} \int_{\Omega} F_{i\,x_i} \, dx = \sum_{i=1}^{N} \int_{\partial\Omega} F_i N_i \, dS$$

$$= \int_{\partial\Omega} \left(\sum_{i=1}^{N} F_i N_i \right) dS$$

$$= \int_{\partial\Omega} F \cdot \vec{N} \, dS$$ ..... where, $\vec{N} = [N_1, N_2, N_3, \ldots, N_N]^T$ and sum is replaced by dot product

$= L.H.S$

Hence proved.