ECE 6560 Homework #5 Marcus Pereira

⟨1⟩ Lax Method for 1-D Transport Equation

Consider the Lax based discretezation,

$$u(x, t+\Delta t) = \frac{u(x+\Delta x, t) + u(x-\Delta x, t)}{2} - b\Delta t \frac{u(x+\Delta x, t) - u(x-\Delta x, t)}{2\Delta x}$$

Taking the discrete Fourier transform on both sides, we have,

$$U(\omega, t+\Delta t) = \frac{1}{2}\left[e^{j\omega\Delta x} U(\omega,t) + e^{-j\omega\Delta x} U(\omega, t)\right] - \frac{b\Delta t}{2\Delta x}\left[e^{j\omega\Delta x} U(\omega,t) - e^{-j\omega\Delta x} U(\omega,t)\right]$$

$$= U(\omega,t)\left[\frac{e^{j\omega\Delta x} + e^{-j\omega\Delta x}}{2} - \frac{b\Delta t}{\Delta x} j\left(\frac{e^{j\omega\Delta x} - e^{-j\omega\Delta x}}{2j}\right)\right]$$

$$= U(\omega,t)\underbrace{\left[\cos\omega\Delta x - j\frac{b\Delta t}{\Delta x}\sin\omega\Delta x\right]}_{\text{amplification factor} := \alpha(\omega)}$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

For stability we want $|\alpha(\omega)| \le 1$ or $|\alpha(\omega)|^2 \le 1$

$$\therefore |\alpha(\omega)|^2 = \cos^2(\omega\Delta x) + \frac{b^2\Delta t^2}{\Delta x^2}\sin^2(\omega\Delta x) \le 1$$

Subtracting 1 from both sides gives,

$$-\left(1-\cos^2(\omega\Delta x)\right) + \frac{b^2\Delta t^2}{\Delta x^2}\sin^2(\omega\Delta x) \le 0$$

$$\therefore -\sin^2(\omega\Delta x) + \frac{b^2\Delta t^2}{\Delta x^2}\sin^2(\omega\Delta x) \le 0$$

$$\therefore \sin^2(\omega\Delta x)\left[\frac{b^2\Delta t^2}{\Delta x^2} - 1\right] \le 0$$

For the above inequality to be satisfied $\Delta x = 0$ will make $\sin^2(\omega \Delta x) = 0$. But, this is of no practical significance. Moreover, $\sin^2(\omega \Delta x) \geq 0$

So, we need $\frac{b^2 \Delta t^2}{\Delta x^2} - 1 \leq 0$

$$\Rightarrow \frac{b^2 \Delta t^2}{\Delta x^2} \leq 1 \quad \Rightarrow \boxed{b^2 \Delta t^2 \leq \Delta x^2}$$

This gives the necessary CFL condition to pick the discretezation to ensure stability of the numerical scheme.

Thus, Von-Neumann analysis shows that the Lax method is stable provided that the CFL condition is met.

<2.> Leap-Frog Method for 1-D transport Equation

This uses central difference approximation for both space and time derivatives, giving

$$u(x, t+\Delta t) = u(x, t-\Delta t) - \frac{b\Delta t}{\Delta x}\left(u(x+\Delta x, t) - u(x-\Delta x, t)\right)$$

Taking the discrete Fourier transform for the spatial variable,

$$(*) \quad U(\omega, t+\Delta t) = U(\omega, t-\Delta t) - \frac{b\Delta t}{\Delta x}\left(U(\omega,t)\, e^{j\omega\Delta x} - U(\omega,t)\, e^{-j\omega\Delta x}\right)$$

Let the amplification factor be such that it satisfies the following relationship, $U(\omega, t+\Delta t) = \alpha(\omega)\, U(\omega, t)$

then applying it to $U(\omega,t)$ gives, $U(\omega,t) = \alpha(\omega)\, U(\omega, t-\Delta t)$

Dividing each term of $(*)$ by $U(\omega,t)$ gives,

$$\frac{U(\omega, t+\Delta t)}{U(\omega,t)} = \frac{U(\omega, t-\Delta t)}{U(\omega,t)} - \frac{b\Delta t}{\Delta x}\left(e^{j\omega\Delta x} - e^{-j\omega\Delta x}\right)$$

$$\therefore \alpha(\omega) = \frac{1}{\alpha(\omega)} - \frac{b\Delta t}{\Delta x} 2j \underbrace{\frac{(e^{j\omega\Delta x} - e^{-j\omega\Delta x})}{2j}}_{\sin(\omega\Delta x)}$$

$$\therefore \alpha(\omega) = \frac{1}{\alpha(\omega)} - \frac{2b\Delta t}{\Delta x} j \sin\omega\Delta x$$

$$\therefore \alpha^2(\omega) = 1 - j\frac{2b\Delta t}{\Delta x} \cdot \sin(\omega\Delta x) \cdot \alpha(\omega)$$

$$\therefore \alpha^2(\omega) + \left(j\frac{2b\Delta t}{\Delta x} \sin\omega\Delta x\right)\alpha(\omega) - 1 = 0$$

$$\therefore \alpha(\omega) = \frac{-\frac{2jb\Delta t}{\Delta x}\sin\omega\Delta x \pm \sqrt{-\frac{4b^2\Delta t^2}{\Delta x^2}\sin^2(\omega\Delta x) + 4}}{2}$$

$$\alpha(\omega) = \underbrace{-j\frac{b\Delta t}{\Delta x}\sin(\omega\Delta x)}_{Im(\alpha(\omega))} \pm \underbrace{\sqrt{1 - \frac{b^2\Delta t^2}{\Delta x^2}\sin^2(\omega\Delta x)}}_{Re[\alpha(\omega)]}$$

For stability we want $|\alpha(\omega)| \leq 1$ or $|\alpha(\omega)|^2 \leq 1$

$$1 - \frac{b^2\Delta t^2}{\Delta x^2}\sin^2(\omega\Delta x) + \frac{b^2\Delta t^2}{\Delta x^2}\sin^2(\omega\Delta x) \leq 1$$

which gives $1 \leq 1$ implying that $|\alpha(\omega)|^2 \leq 1$ and hence the numerical scheme is stable.

To get CFL condition we can re-examine what allows us to get the scheme to be stable. In the analysis above $|\alpha(\omega)|^2 = 1$ only if the $Re(\alpha(\omega))$ stays real i.e.

we need $1 - \frac{b^2\Delta t^2}{\Delta x^2}\sin^2(\omega\Delta x) \geq 0$ so that its square root is a real number

$$\therefore \frac{b^2\Delta t^2}{\Delta x^2}\sin^2(\omega\Delta x) \leq 1$$

Again $\sin^2(\omega\Delta x)$ satisfies this only for the trivial

case of $\Delta x = 0$ (this is not practically usefful)
In the worst case $\sin^2(\omega \Delta x) = 1$

$\therefore$ we have $\frac{b^2 \Delta t^2}{\Delta x^2} \leq 1$

or $\boxed{b^2 \Delta t^2 \leq \Delta x^2}$ is the required CFL condition for stability

Note that when $1 - \frac{b^2 \Delta t^2}{\Delta x^2} \sin^2(\omega \Delta x) = 0$, the

$Re[\alpha(\omega)] = 0$ and $|\alpha(\omega)|^2 = \left(Im[\alpha(\omega)]\right)^2$

$\therefore |\alpha(\omega)|^2 = \frac{b^2 \Delta t^2}{\Delta x^2} \sin^2(\omega \Delta x)$ which should be ≤ 1

This would lead to the same CFL condition!

Comment about b :-

Since the CFL condition includes b^2, the sign of b will not affect the stability of the scheme. Thus, regardless of whether b is positive or negative so long as Δt is picked such that,

$b^2 \Delta t^2 \leq \Delta x^2$, the CFL condition will be met and the scheme will be stable!

<3> Fully Implicit Scheme for 1-D Heat Equation

$$u(x,t+\Delta t) = u(x,t) + \frac{b\Delta t}{\Delta x^2}\left[u(x+\Delta x, t+\Delta t) - 2u(x,t+\Delta t) + u(x-\Delta x, t+\Delta t)\right]$$

Taking discrete Fourier transform wrt the space variable,

$$U(\omega,t+\Delta t) = U(\omega,t) + \frac{b\Delta t}{\Delta x^2}\left[U(\omega,t+\Delta t)e^{j\omega\Delta x} - 2U(\omega,t+\Delta t) + U(\omega,t+\Delta t)e^{-j\omega\Delta x}\right]$$

$$\therefore U(\omega,t+\Delta t) - \frac{b\Delta t}{\Delta x^2}U(\omega,t+\Delta t)\left[e^{j\omega\Delta x} - 2 + e^{-j\omega\Delta x}\right] = U(\omega,t)$$

$$\therefore U(\omega,t+\Delta t)\left[1 - \frac{b\Delta t}{\Delta x^2}\left(e^{j\omega\Delta x} - 2 + e^{-j\omega\Delta x}\right)\right] = U(\omega,t)$$

$$\therefore U(\omega,t+\Delta t)\left[1 - \frac{2b\Delta t}{\Delta x^2}\left(\frac{e^{j\omega\Delta x} + e^{-j\omega\Delta x}}{2} - 1\right)\right] = U(\omega,t)$$

$$\therefore U(\omega,t+\Delta t)\underbrace{\left[1 - \frac{2b\Delta t}{\Delta x^2}\left(\cos\omega\Delta x - 1\right)\right]}_{\beta(\omega)} = U(\omega,t)$$

$$\therefore U(\omega,t+\Delta t) = \frac{1}{\beta(\omega)}U(\omega,t) = \alpha(\omega)U(\omega,t)$$

where, $\alpha(\omega) = \frac{1}{\beta(\omega)}$

for stability we want $|\alpha(\omega)| \leq 1$ or $|\alpha(\omega)|^2 \leq 1$

$\therefore$ we need $\left|\frac{1}{\beta(\omega)}\right|^2 \leq 1$ for stability

Since $\alpha(\omega)$ is real (i.e. $Im(\beta)=0$) in this case, we have,

$$|\cdot|^2 = \sqrt{Re(\cdot)^2 + Im(\cdot)^2} = \sqrt{Re(\cdot)^2} = Re(\cdot)$$

$$\therefore |\alpha(\omega)| = \frac{1}{1 - \frac{2\Delta t}{\Delta x^2} b (\cos \omega \Delta x - 1)} \leq 1$$

So, first let us check for stability.

Wkt, $(\cos \omega \Delta x - 1) \leq 0$
For $(\cos \omega \Delta x - 1) = 0$, the condition is satisfied easily.
When $(\cos \omega \Delta x - 1) < 0$, it depends on the sign of b.

if $b > 0$, the denominator will be (1 + "something") will satisfy the inequality. So, $\boxed{b > 0}$ is the CFL condition.
if $b < 0$, the denominator will be (1 - "something")
This "something" will be > 0
i.e.,

$$\tilde{\alpha} = \frac{2b\Delta t}{\Delta x^2} (\cos \omega \Delta x - 1) > 0 \quad \left\{ \text{for } (\cos \omega \Delta x - 1) < 0 \text{ and } b < 0 \right\}$$

In order to satisfy the inequality for stability, above $\tilde{\alpha}$ must be greater than 1.

$$\therefore \frac{2b\Delta t}{\Delta x^2} (\cos \omega \Delta x - 1) > 1$$

$$\text{or} \quad b\Delta t (\cos \omega \Delta x - 1) > \frac{\Delta x^2}{2}$$

in the worst case $\cos \omega \Delta x - 1 = -2$

$$\therefore -2b\Delta t > \frac{\Delta x^2}{2} \quad \text{or} \quad \boxed{-b\Delta t > \frac{\Delta x^2}{4}} \quad (*)$$

$\therefore$ Inspite of having a negative diffusion coefficient, the use of an implicit scheme still proves to be stable using Von-Neumann analysis.
$\Rightarrow$ b does not have to be positive for stability if the (*) CFL condition is met in the case when b is negative.