Homework #1: Transport PDE

Solutions

1 Nonhomogeneous with Constant Coefficients

Given the following PDE for the function $u(x,t)$ as well as initial value $g(x)$ at $t = 0$

$$\begin{aligned} u_t + bu_x &= e^t \\ u(x,0) &= g(x) \end{aligned}$$

derive the solution for $u(x,t)$ in terms of the function g. To obtain full credit, you should not just propose a solution and show that it solves the PDE nor should you simply plug this into the more general formula we derived in class (nor simply rewrite that derivation verbatim for that same more generalized form) for the nonhomogenous transport PDE. Instead, you should directly develop the solution for this specific PDE using the same principles from class. For partial credit, make sure you at least show how the PDE can be interpreted as a special directional derivative and how the solution of u evolves along this direction.

Solution:

First we write the PDE as a directional derivative as follows

$$\begin{aligned} \nabla u \cdot \begin{bmatrix} b \\ 1 \end{bmatrix} &= e^t \\ \frac{\partial u}{\partial B} &= e^t \qquad \text{where } B = \begin{bmatrix} b \\ 1 \end{bmatrix} \end{aligned}$$

allowing us to see that the PDE says that the rate of change of $u$as we move in direction B is simply 1. Thus, if we trace back a line from any point (x,t) in the backwards direction $-B$ until we intersect the $t = 0$ axis, we would simply add the elapsed time t to the initial condition value for u (given by the function g) at this point of intersection. To formalize this, we create a function $z(s)$ which gives us the value of u as we move along such a line. To do so, we need to make sure that the domain of $z(s)$ passes through a generic point (x,t) whose solution value we wish to construct, and that as we vary the parameter s we indeed move along this special direction B. Then taking the derivative of z will be the equivalent of taking the special directional derivative $\frac{\partial u}{\partial B}$ for u (whose value we know to be one).

$$\begin{aligned} z(s) &= u(x + sb, t + s) \\ z'(s) &= bu_x(x + sb, t + s) + u_t(x + sb, t + s) = e^{t+s} \end{aligned}$$

Solving for $z(s)$ is quite simple.

$$z(s) = \int_0^s z'(s)\, ds + z(0) = e^{t+s} - e^t + z(0)$$

We now plug in $s = -t$ into this solution for $z(s)$ to obtain (after slight rearrangement)

$$z(0) \quad = \quad z(-t) + e^t - 1$$

which in turn translates into the an equation for u by noting that $z(0) = u(x,t)$ and that $z(-t) = u(x-bt, 0)$

$$u(x,t) \quad = \quad u(x - bt, 0) + e^t - 1$$

Finally, we plug in our initial condition $g(x)$ to obtain the solution

$$u(x,t) \quad = \quad g(x - bt) + e^t - 1$$

2 Homogeneous with Time-Varying Coefficient

Using the exact same principles, but noting that the directional derivatives no longer occur along fixed/constant directions (and therefore no longer give rise to straight line characteristic trajectories), derive the solution $u(x,t)$ for the following PDE in terms of its initial condition g (only partial credit will be given if a solution is proposed and shown to satisfy the PDE rather than derived using the principles discussed in class).

$$\begin{aligned} u_t - t u_x &= 0 \\ u(x,0) &= g(x) \end{aligned}$$

Solution:

First we write the PDE as a directional derivative as follows

$$\begin{aligned} \nabla u \cdot \begin{bmatrix} -t \\ 1 \end{bmatrix} &= 0 \\ \frac{\partial u}{\partial B} &= 0 \qquad \text{where } B = \begin{bmatrix} -t \\ 1 \end{bmatrix} \end{aligned}$$

allowing us to see that the PDE says that the rate of change of $u$as we move in direction B is 0 (in other words, $\underline{u}$ is constant in the B direction). However, notice that the B direction changes with t. As such, we need to consider curves in the (x,t) plane whose rate of change in the t direction is fixed to 1 and whose rate of change in the x direction is equal to t. These curves are not straight lines as in the previous problem but are instead parabolas. Thus, if we trace backwards along such a parabola from any point (x,t) until we intersect the $t = 0$ axis, we would note that u remains constant along that parabola, and is therefore equal to the value of the initial condition (given by the function g) at this point of intersection. To formalize this, we create a function $z(s)$ which gives us the value of u as we move along such a parabola. To do so, we need to make sure that the domain of $z(s)$ passes through a generic point (x,t) whose solution value we wish to construct (we'll take this to happen when $s = 0$), and that as we vary the parameter s we indeed move along this special parabola whose slope is given by the time-varying direction vector B. Then taking the derivative of z will be the equivalent of taking the special directional derivative $\frac{\partial u}{\partial B}$ for u (whose value we know to be zero).

$$\begin{aligned} z(s) &= u(x - ts - \frac{1}{2}s^2, t + s) \\ z'(s) &= -(t+s)\, u_x(x - ts - \frac{1}{2}s^2, t+s) + u_t(x - ts - \frac{1}{2}s^2, t+s) = 0 \end{aligned}$$

Solving for $z(s)$ is quite simple.

$$z(s) = \text{constant}$$

Since $z(s)$ is constant we may write

$$z(0) = z(-t)$$

which in turn translates into the following equation for u

$$u(x,t) = u(x + t^2 - \frac{1}{2}t^2, 0) = u(x + \frac{1}{2}t^2, 0)$$

Finally, we plug in our initial condition $g(x)$ to obtain the solution

$$u(x,t) = g(x + \frac{1}{2}t^2)$$

3 Two Space Dimensions

Write out the solution $u(x, y, t)$ for the following PDE and initial condition. You may utilize the formulas (and their higher dimensional generalizations) derived in class rather than deriving the solution from scratch.

$$\begin{aligned} 3u_t + 6u_x - 9u_y &= x \cos t \\ u(x,y,0) &= y^2 e^x \end{aligned}$$

Solution:

First we divide both sides of the PDE by 3 so that the u_t term has a unit coefficient, and then we may apply the formula derived in class

$$u(\vec{x}, t) = g(\vec{x} - t\vec{b}) + \int_0^t f(\vec{x} + (s-t)\vec{b}, s)\, ds$$

noting that in this case $\vec{x} = (x, y)$; $\vec{b} = (2, -3)$; $f(x, y, t) = \frac{1}{3}x \cos t$; and $g(x, y) = y^2 e^x$. Substituting these expressions into the formula above yields

$$\begin{aligned} u(x,y,t) &= g(x - 2t, y + 3t) + \int_0^t f(x + 2(s-t), y - 3(s-t), s)\, ds \\ &= (y+3t)^2 e^{x-2t} + \int_0^t \frac{1}{3}\left(x + 2(s-t)\right) \cos s\, ds \\ &= (y+3t)^2 e^{x-2t} + \frac{1}{3}\left[(x-2t)\sin s + 2\left(\cos s + s \sin s\right)\right]_{s=0}^{s=t} \\ &= (y+3t)^2 e^{x-2t} + \frac{1}{3}x \sin t + \frac{2}{3}\cos t - \frac{2}{3} \end{aligned}$$

giving us the final form of the solution.

$$u(x,y,t) = (y+3t)^2 e^{x-2t} + \frac{1}{3}x \sin t + \frac{2}{3}\cos t - \frac{2}{3}$$

Note that we can verify our solution by first noting that when we plug in $t = 0$ we obtain our initial condition $u(x, y, 0) = y^2 e^x$ and when we plug our expression into the PDE we obtain

$$\begin{aligned}
3u_t + 6u_x - 9u_y &= 3\left(6(y+3t)e^{x-2t} - 2(y+3t)^2 e^{x-2t} + \frac{x}{3}\cos t - \frac{2}{3}\sin t\right) \\
&\quad +6\left((y+3t)^2 e^{x-2t} + \frac{1}{3}\sin t\right) \\
&\quad -9\left(2(y+3t)e^{x-2t}\right) \\
&= x\cos t
\end{aligned}$$