Independent Reading and Research

A Semester in Discrete Geometry

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Abstract

This document summarizes the topics I explored during an independent reading and research semester in discrete geometry, under the supervision of Dr. Andrew Newman. This paper introduces basic ideas in convex analysis and discrete geometry as well as basic geometric games. It also discusses the background of Cover's Theorem and offers a bijective proof for Cover's Theorem for $d = 2$. Finally, the paper concludes with a brief discussion of k-sets and order types. This paper represents the arc of the past semester and serves as a foundation for future work that I may do in this field.

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1 Introduction

This semester, I set out to get my first real experience in mathematical research — not necessarily to produce new results, but to better understand what the research process feels like and whether it is something I could see myself pursuing long-term. If I want to become a professor one day, research will inevitably be a major part of that path, and I wanted to test the waters in a focused but exploratory way.

My semester covered the likes of basic discrete geometry, geometric games on finite sets, and mainly a bijective proof of Cover's Theorem. What follows is a record of what I explored, what I learned, and what ideas I found most compelling. Some sections include partial results or sketches of arguments; others are more reflective or even speculative. Together, they represent the arc of the semester and serve as a foundation for future work that I may do in this field.

2 Convexity and Radon Partitions

There are a couple of fundamental concepts and definitions that are required before presenting the main problems and ideas. These definitions are taken from Matousek's *Lectures on Discrete Geometry.*

2.1 Convexity Basics

Definition 2.1. *A set $C \subset \mathbb{R}^d$ is* convex *if, for any two points $x, y \in C$, the line segment connecting them is entirely contained in C. That is,*

$$\forall \lambda \in [0,1], \quad \lambda x + (1-\lambda)y \in C.$$

The smallest convex set that contains a given set S is called its *convex hull*, which can be defined constructively using convex combinations:

Definition 2.2. *Given a finite set $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^d$, the* convex hull *of S, denoted* $\mathrm{conv}(S)$*, is the set of all convex combinations of the points in S. That is,*

$$\mathrm{conv}(S) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \,\middle|\, \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

It was useful to think of the convex hull as a "filled-in shape" formed by the points in S, and it is always a convex polytope when S is finite. Some analogies to wrapping a rubber band, in $\mathbb{R}^2$, around the vertices to create the boundary were helpful. Finally, we define one of the most common phrases heard in the realm of analysis: general position.

Definition 2.3. *A set of points $P \subset \mathbb{R}^d$ is said to be in* general position *if no $k+1$ points lie in a k-dimensional affine subspace, for any $1 \leq k < d$.*

The general position assumption is both convenient and reasonable. This assumption is convenient because k points will always define a $k-1$-dimensional flat and allows us to disregard many degenerative cases where 3 points lie on the same line, 4 on the same plane, etc. When we discuss inner tangents, this assumption allows us to satisfy certain uniqueness conditions rather easily. In terms of reasonability, we can imagine wiggling any points by some ϵ so that they no longer reside on this $k-2$-dimensional flat without impacting the overall geometry.

2.2 Half-spaces and Hyperplanes

Understanding half-spaces helped clarify how convex sets and partitions behave under linear separations.

Definition 2.4. *A* closed half-space *in $\mathbb{R}^d$ is a set of the form*

$$H = \{x \in \mathbb{R}^d \mid a \cdot x \leq b\}$$

for some fixed vector $a \in \mathbb{R}^d$ and scalar $b \in \mathbb{R}$. The set $h = \{x \in \mathbb{R}^d \mid a \cdot x = b\}$ is the associated hyperplane*, which forms the boundary of the half-space. We say that a half-space is open if the inequality is strict and open otherwise.*

Convex sets can be characterized as intersections of all the half-spaces containing the set. We can also immediately see a connection between half-spaces and linear classifiers in machine learning.

2.3 Radon's Theorem and Partitions

Many of the ideas we cover later have to do with when two convex sets intersect. We can formalize this idea and present Radon partitions and Radon's Theorem.

Definition 2.5. *Given a set $S \subset \mathbb{R}^d$, a* Radon partition *is a partition of S into two subsets A and B such that*

$$\operatorname{conv}(A) \cap \operatorname{conv}(B) \neq \emptyset.$$

Theorem 2.6 (Radon's Theorem)**.** *Any set of $d+2$ points in $\mathbb{R}^d$ in general position can be partitioned into two disjoint subsets whose convex hulls intersect.*

Radon's theorem guarantees the existence of such a partition whenever $|S| \geq d+2$. We can also formalize the idea that there is no intersection as an affine split.

Definition 2.7. *Given $A, B \subset \mathbb{R}^d$, we say that we can affinely split A and B if there exists a half-space H such that $A \subset H$ and $B \cap H = \emptyset$.*

With this definition, we can say that for any partition of a point set in $\mathbb{R}^d$, we have either an affine split or a Radon partition. This idea is a foundation for all the topics discussed.

3 Discrete Geometry Ideas

The following sections introduce some of the main concepts and ideas that were utilized in solving discrete geometry problems.

3.1 Gale Duality

Duality allows us to biject one format of a problem into an equivalent notion while preserving many important properties in this mapping, often simplifying the structure or highlighting new properties that are harder to see in the original formulation. One of the dualities that was utilized numerous times throughout the semester was Gale Duality.

Before immediately offering the definition, the motivation behind the Gale transform is that if we have slightly more than $d+1$ points, we can transform our problem into having n points in $n-d-1$-dimensional space which is likely much easier to work with.

Definition 3.1 (Gale Transform)**.** *Let $a = (a_1, a_2, \dots, a_n)$ be a sequence of $n \geq d+1$ points in $\mathbb{R}^d$ such that the affine hull of the a_i is all of $\mathbb{R}^d$. The* Gale transform *of a is another sequence $\bar{g} = (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)$ of n vectors in $\mathbb{R}^{n-d-1}$ constructed as follows:*

1. *Embed each $a_i \in \mathbb{R}^d$ into $\mathbb{R}^{d+1}$ by appending a 1 as the last coordinate: $\bar{a}_i = (a_i, 1) \in \mathbb{R}^{d+1}$.*
2. *Let A be the $(d+1) \times n$ matrix whose i-th column is $\bar{a}_i$. Then the row space of A spans a $(d+1)$-dimensional subspace $V \subset \mathbb{R}^n$.*
3. *Let $V^\perp$ be the orthogonal complement of V in $\mathbb{R}^n$, which has dimension $n-d-1$. Choose a basis $\{b_1, \dots, b_{n-d-1}\}$ for $V^\perp$, and let B be the $(n-d-1) \times n$ matrix whose rows are these basis vectors.*
4. *The j-th column of B is defined to be $\bar{g}_j \in \mathbb{R}^{n-d-1}$. The sequence $(\bar{g}_1, \dots, \bar{g}_n)$ is the Gale transform of $(a_1, \dots, a_n)$.*

The Gale transform encodes affine dependencies among the a_i as linear dependencies among the $\bar{g}_i$, and is defined up to linear transformation and choice of basis. It is a key tool for studying the combinatorial structure of point configurations.

Gale duality allows for the encoding of high-dimensional data / vectors into low dimensions with many structure-preserving properties. Most notably, a Radon partition in the original is a linear split in the Gale Transform.

3.2 Mapping to Polytopes and $f-$vectors

Some of the benefits of discrete geometry problems is the ability to map problems to geometric objects and see how certain questions translate to whether this object has a related geometric property. Thus, we will introduce some basic geometric properties that will be useful later.

Definition 3.2. *The* f-vector *of a d-dimensional polytope (or more generally, a polyhedral complex) is a tuple $(f_0, f_1, \dots, f_d)$, where f_k counts the number of k-dimensional faces in the structure.*

Here, f_0 is the number of vertices, f_1 is the number of edges, f_d is the number of top-dimensional faces, etc. With this in mind, we also conveniently have the Euler–Poincaré formula:

$$f_0 - f_1 + f_2 - \dots + (-1)^d f_d = \chi,$$

where χ is the Euler characteristic. Because the polytopes with which we work are isomoprhic to a sphere, we know that χ is 2 if d is even and 0 if d is odd. With this in mind, we can utilize this equality and the mapping of problems to a polytope to recover information via the remaining elements of the $f-$vector.

4 Geometric Games

4.1 Maker-Breaker / Intersector-Separator Games

First, we can broadly define the games to which we are referring: Maker-Breaker games.

Idea 4.1 (Maker-Breaker Games)**.** *A Maker-Breaker game is defined by a set of positions/points* X *and a family of winning-sets* $\mathcal{F} \subset \mathcal{P}(X)$*. It is played by two players, Maker and Breaker, who alternately take previously untaken elements of* X*. Maker's goal is to hold all elements of a winning-set, while Breaker wins if they prevent this. (Wikipedia: Maker-Breaker Games)*

With this general format for a type of game, we can now define a game on n points in a d-dimensional space.

Idea 4.2 (Intersector-Separator Game)**.** *Given* n *points in* $\mathbb{R}^d$ *in general position, two players alternate selecting points that are have not previously been selected. After all points have been selected, consider the partition corresponding to the two player's selected points. If this partition is a Radon partition, Intersector wins. If this partition is affinely separable, Separator wins.*

In game theory, we are always mainly concerned with winning strategies and related factors. We can immediately ask some of the following questions. Does Intersector vs. Separator going first impact the outcome? Given n, what bounds on d can sufficiently guarantee a winning strategy for a player? We are given many answers to these questions for free due to our Intersector-Separator game being a Maker-Breaker game. One can see that who is Maker and who is Breaker is entirely dependent on if we define winning-sets to be Radon partitions or affine splits. We will consider Maker to be Intersector and Breaker to be Separator for the following sections.

Because our Intersector-Separator game is a Maker-Breaker game, we can utilize any general findings and apply them here. These include pairing strategies, Erdős-Selfridge bound, and much more.

4.2 Pairing Strategies

One of the simplest general strategies in Maker-Breaker games is a *pairing strategy.* The idea is that Breaker can predefine disjoint pairs of elements in the game board (in our case, the point set) and guarantee that Maker can never fully claim a winning-set by always responding with the other half of any selected pair.

Idea 4.3 (Pairing Strategy)**.** *If the board* X *can be partitioned into disjoint pairs* (x_i, y_i) *such that every winning-set completely intersects at least one pair, then Breaker can always win by playing the unchosen member of any pair selected by Maker. This guarantees that Maker cannot claim an entire winning-set. Similarly, if every winning-set contains at least one element from every pair of some pairing, Maker has a winning strategy.*

Connecting back to our Intersector-Separator game, we can cite a useful result.

Corollary 4.4. *If there are less than* $d + 1$ *pairs, Separator does not have a pairing strategy. If there are more than d pairs, Intersector does not have a pairing strategy.*

While pairing strategies can be useful to guarantee winning strategies, the necessary conditions are incredibly strict and the majority of game outcomes are not defined by a player having a pairing strategy.

4.3 Erdős-Selfridge Bound

The Erdős-Selfridge bound provides a general condition under which Breaker has a guaranteed winning strategy.

Theorem 4.5 (Erdős–Selfridge)**.** *Let $\mathcal{F}$ be the family of winning-sets in a Maker-Breaker game played on a finite set X. If*

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

then Breaker has a winning strategy.

In our case, $\mathcal{F}$ consists of all point subsets that, when assigned to Player 1, would result in a Radon partition (i.e., an Intersector win). If the number of such subsets is small enough, or if their sizes are large enough (so $2^{-|A|}$ is small), then the inequality may hold and Player 2 (Separator) is guaranteed to win. We can also note that $|A| = \frac{n}{2}$, so this bound is entirely dependent upon the number of winning-sets of this specific size.

5 Cover's Theorem: Toward a Bijective Proof

Theorem 5.1 (Cover's Theorem)**.** *Given $x_1, ..., x_n \in \mathbb{R}^d$ in general position, let $C(n, d)$ be the number of affine splits for some partition of these points. Then,*

$$C(n,d) = \sum_{i=0}^{d} \binom{n-1}{i}$$

.

If we consider these as colorings as opposed to partitions, we get an extra factor of 2 but this is negligible for all intents and purposes. Given that we have 2^{n-1} total unoriented splits for these points, we immediately have the corollary that the number of Radon partitions must be $2^{n-1} - \sum_{i=0}^{d} \binom{n-1}{i}$. Recovering the formula for oriented splits is simply an extra multiple of 2 to pick an orientation.

The proof of Cover's Theorem is an inductive proof, following purely from the recursive relation that $C(n, d) = C(n-1, d) + C(n-1, d-1)$. Here is a brief sketch of the argument behind this:

Proof. Consider $n-1$ points in d dimensions and add one more point. For each of the separating hyperplanes, we either can pass through this new point or we cannot, creating two cases. If we can pass through this point, we can freely assign it to either set in the partition, giving us two splits from this one. If we cannot, we must assign it consistently with the current split, giving us one split. This is the same as counting the number of splits in d dimensions, $C(n-1, d)$, as this counts each case once, and then adding $C(n-1, d-1)$ as we can think of passing through the point as removing one degree of freedom and projecting one dimension lower. These will count each case twice and once, respectively, giving us the desired recurrence of $C(n, d) = C(n-1, d) + C(n-1, d-1)$.

With this recurrence relation, we can easily prove the formula holds via induction. [2] ∎

Corollary 5.2. *Given $x_1, \ldots, x_n \in \mathbb{R}^d$ in general position, let $C_{linear}(n, d)$ be the number of linear splits for some partition of these points. Then,*

$$C_{linear}(n,d) = \sum_{i=0}^{d-1} \binom{n-1}{i}$$

This follows from the idea that we lose a degree of freedom due to a bias or additive constant term when going from affine to linear splits. This is equivalent to projecting down a dimension.

While an inductive proof is sufficient, Cover's Theorem having such a simple formula begs the question of whether there is a bijective proof. With a bijective proof, we may be able to generate linear classifiers directly from our point set. Many machine learning algorithms loop over a dataset multiple times before reaching a linear classifier, so a bijective proof may offer some insight or improvement in efficiencies depending upon the complexity of the bijection.

5.1 Inner Tangents and Support Functions

Definition 5.3. *Let H^+ be a closed half-space containing* 0 *and H^- be a closed half-space not containing* 0*. An* inner tangent *of two convex hulls $K_1, K_2 \subset \mathbb{R}^d$ is the hyperplane $h = H^+ \cap H^-$ with the following properties: $K_1 \subset H^+$, $K_2 \subset H^-$, and both $K_1 \cap h \neq \emptyset$ and $K_2 \cap h \neq \emptyset$.*

Essentially, there is a hyperplane that puts two sets of points on opposite sides while intersecting both. This definition is helpful as it is an equivalent notion to a separating hyperplane with the added fact that the set of separating hyperplanes is infinite, whereas the set of inner tangents must be finite for a finite point set. Our goal is to establish a relationship between these inner tangents and affine splits as well as to develop some approach to counting these tangents. Our first step is to offer a definition in terms of a function we will call the support function.

Definition 5.4. *Let $h_K(\boldsymbol{u}) = \max\{\langle x, \boldsymbol{u}\rangle : x \in \operatorname{conv}(K)\}$. Define the support function $h : \mathbb{S}^{d-1} \to \mathbb{R}$ via*

$$h(\boldsymbol{u}) = h_{K_1}(\boldsymbol{u}) - h_{K_2}(-\boldsymbol{u})$$

The function $h_K(u)$ takes the hyperplane with normal vector u and sweeps it out to the furthest point in K. For $h(u)$, we get $h(u) = 0$ if and only if we have $h_{K_1}(u) = h_{K_2}(-u)$. This only occurs when the same hyperplane has all of K_1 below and all of K_2 above it. We can now see how zeros of our support function give us the hyperplanes that split any two sets.

5.2 Bijective Proof in 2D

We will prove Cover's Theorem for n points in $\mathbb{R}^2$ by creating a bijection between inner tangents and affine splits. First, note that for $d = 2$, we have $C(n, 2) = 1 + \binom{n}{2}$. Because there is only one way to trivially partition n points, our aim is to show that there are $\binom{n}{2}$ ways to non-trivially partition n points in $\mathbb{R}^2$.

Lemma 5.5. *For $K_1, K_2 \subset \mathbb{R}^2$ and $\operatorname{conv}(K_1) \cap \operatorname{conv}(K_2) = \emptyset$, $\exists! u_1, u_2 \in \mathbb{S}^1, u_1 \neq u_2$ such that $h(u_1) = h(u_2) = 0$.*

Proof. Because there is a trivial intersection between K_1 and K_2, we can scale and translate our sets so that K_1 has all positive $x-$values and K_2 negative (this choice is irrelevant). We know that $h((1, 0)) > 0$ and $h((-1, 0)) < 0$. We also know that h is a continuous function on $\mathbb{S}^1$ and thus satisfies the Intermediate Value Theorem. Thus, we are guaranteed at least two solutions to $h(\boldsymbol{u}) = 0$.

To show that there are exactly two, assume for the sake of contradiction that there are three inner tangents. These three lines either all meet at a point and partition the plane into six regions or have three points of intersection and partition the plane into seven regions. First, note that K_1

and K_2 must be completely contained in two distinct regions. Second, all lines must determine some part of the boundary of these regions to ensure that they are inner *tangents.* [1]

We will first deal with the case where the three inner tangents meet at a point. Because these lines intersect at a point, only two lines determine the boundary of a region except for the point of intersection. Thus, both K_1 and K_2 must include the intersection point to ensure that each line is a tangent. However, this contradicts K_1 and K_2 being disjoint.

Let us now consider the case where they intersect at three separate points, partitioning the plane into seven regions. This creates a triangle region in the middle. We see that there are only four regions that are bounded by all three lines: the triangle and the three regions that share an edge with this triangle. Again, both K_1 and K_2 must be completely contained in distinct regions. If the triangle and any other region are chosen, either of the lines that do not correspond to the shared edge will separate K_1 and K_2 and thus is not an inner tangent. If the triangle is not one of the two selected regions, both regions share an edge with the triangle. The line corresponding to the unshared edge will not separate K_1 and K_2 and thus is not an inner tangent. In both cases, we have reached a contradiction and conclude that there are exactly two unique inner tangents for any two disjoint convex hulls in $\mathbb{R}^2$.

□

Lemma 5.6. *There is a two-to-two correspondence between affine splits and inner tangents of a given partition in* $\mathbb{R}^2$.

Proof. By Lemma 5.5, we have that any affine split produces two inner tangents. If we consider choosing two points, these two points uniquely define a line that can be used to classify the remaining points. We can then arbitrarily assign the points intersected to opposite sides to guarantee a nontrivial split, producing two splits for any given pair of points. Thus, a split corresponds to exactly two inner tangents and an inner tangent corresponds to exactly two splits which implies the number of splits must be equal to the number of inner tangents. □

To wrap up our proof, we simply state that the number of inner tangents is equal to $\binom{n}{2}$ and we get our result for Cover's Theorem in $\mathbb{R}^2$.

Some code was written to validate this approach that can be used to generate splits via inner tangents and generalizes to higher dimensions. This may be helpful in examining whether and how inner tangents may play a role in $d-$ dimensions, as this idea may or may not have the ability to generalize.

6 Connections to k-Sets and Order Types

The concepts of $k-$ sets and order types were two that I found interesting but found late in the semester, so they were not thoroughly researched.

Definition 6.1. *Given a set $P \subset \mathbb{R}^d$ of n points in general position, a* k-set *is a subset $Q \subset P$ of size k such that there exists a hyperplane h that strictly separates Q from $P \setminus Q$. That is, all points of Q lie on one side of h, and all points of $P \setminus Q$ lie on the other.*

[1]The original was incredibly simplified using an argument thanks to an answer on this post.

The k-set problem is a classic question in discrete geometry: for fixed n and d, how many such k-sets can a point set have? There are known upper bounds in low dimensions (e.g. $O(nk^{1/3})$ in $\mathbb{R}^2$ [1]), but the problem remains largely open in higher dimensions. Regardless of bounds, the idea of k-sets captures the notion of separating structure in a point configuration, which is central to the types of games and questions I was looking at. Connecting this back, we see that Cover's Theorem is simply a summation over all values of k up to n. In addition, the Intersector-Separator game is an application of the k-set problem for $k = n/2$. Thus, many bounds that revolve around general $k-$set problems can be applied to our game above.

Definition 6.2. *Let P be a finite set of points in $\mathbb{R}^2$. The* order type *of P is a function that assigns to each ordered triple (p_i, p_j, p_k) the orientation (clockwise, counterclockwise, or collinear) of that triple. Two point sets have the same order type if there exists a bijection between them preserving the orientation of every triple.*

Order types are a way to capture the combinatorial structure of a point configuration without caring about exact coordinates. Two point sets of the same order type are essentially the same from the perspective of convexity, separation, and k-sets.

I found this idea most interesting – we can boil any point configuration down to some base combinatorial structure, translating an infinite number of possibilities to a finite number of outcomes. While the bounds are too loose to be of any help in a practical sense, the idea that certain problems, like Cover's Theorem, could be exhaustively verified for specific n and d is a result I enjoyed discovering. If drastic improvements in computation were made, these ideas could be used to gain insight in small cases.

7 Further Questions

- What is the complexity of generating all realizable affine splits for n points in d dimensions? While a bijective argument certainly seems feasibly for Cover's Theorem, creating a bijection could be equivalent to a direct algorithm in computing all realizable affine splits. I am curious if there is a reduction argument that reduces generating these realizable splits to some P vs. NP type of problem.

- Is the bijective proof for $d = 2$ generalizable? The inner tangent function maintains its properties; however, finding zeros becomes more challenging in higher dimensions. The proof also relies on $\binom{n}{2}$, which will not work as effectively for higher dimensions.

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