Politecnico di Milano

Homework #2 - 2D and 1D systems

Students: Vittoria Malaman [10610731] - Riccardo Martinelli [10456202] - Matteo Pettenò [10868930]

Course: Musical Acoustics – Professor: Fabio Antonacci
Due date: *October 27th, 2022*

1. Circular membrane characterization

It is given a circular membrane with radius 0.15 m, and tension 10 N/m. The unit surface weight is $\sigma = 0.07\,\mathrm{kg/m^2}$.

a) Compute the propagation speed in the membrane

The propagation speed c in a membrane is given by:

$$c = \sqrt{\frac{T}{\sigma}} = 11.95\,\mathrm{m/s}$$

b) Compute the frequency of the first eighteen modes for this membrane (ordered by frequency) and draw in Matlab the modeshapes of the first six ones

To find the eigenmodes' frequencies of a circular membrane with radius a is a well-known problem. It is solved by starting from the wave equation expressed in polar coordinates, where the origin of the reference frame coincides with the center of the membrane, and by fixing the boundary conditions; in our case we considered a membrane with a fixed edge (no displacement at the border). Thus if we identify the displacement with $z(r, \phi, t)$ the problem is depicted by:

$$\begin{cases} \frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2} \right) \\ 0 \leq r \leq a, 0 \leq \phi \leq 2\pi \\ z(a, \phi, t) = 0 \end{cases}$$

Exploiting the separation of variables method, the solutions can be expressed as $z(r, \phi, t) = R(r)\Phi(\phi)e^{j\omega t}$ and we get:

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2} \right) R = 0 \Rightarrow R(r) = J_m(kr)$$

and

$$\frac{d^2 \phi}{d\Phi^2} + m^2 \phi = 0 \Rightarrow \Phi(\phi) = Ae^{\pm jm\phi}$$

where $J_m(kr)$ are the first kind Bessel function of order m.
The overall complex displacement can be written as:

$$\tilde{z}(r, \phi, t) = \sum_m \sum_n c_{mn} \tilde{z}_{mn}(r, \phi) e^{j\omega_{mn} t}$$

where

$$\tilde{z}_{mn}(r, \phi) = \Phi(\phi)R(r) = Ae^{\pm jm\phi} J_m(k_n r) \tag{1}$$

is the expression of a natural mode of vibration in the circular membrane, for whom k_n is such that $k_n a = Z_n(J_m(kr))$ (i.e. the nth zero of $J_m(kr)$).
Recalling that $k_n = 2\pi f_n \cdot \sqrt{\frac{\sigma}{T}}$ and, since the zeros of the Bessel functions are known, we can compute the frequency of the modes as:

$$f_{mn} = \frac{Z_n(J_m(kr))}{2\pi a}\sqrt{\frac{T}{\sigma}}$$

To compute the first eighteen eigenfrequencies, we need to know the first eighteen zeros of the n-th order first kind Bessel functions. In order to achieve this in an automatic way, we used a $MATLAB$ function which, given the type and order of a Bessel function, uses Halley's method to calculate its first n zeros. Iterating this for the first ten orders, we built a 10×10 matrix containing in the j-th row the first ten zeros of the j-th order Bessel function: unrolling the matrix and sorting the outcome gave us what we needed. By substituting the values of the first eighteen zeros, the frequencies in the table below have been obtained, while the picture shows the modeshapes of the first six ones.

(m,n)	$Z_n(J_m(kr))$	f_{mn}
$(0,1)$	2.4048	30.50 Hz
$(1,1)$	3.8317	48.59 Hz
$(2,1)$	5.1356	65.13 Hz
$(0,2)$	5.5201	70.00 Hz
$(3,1)$	6.3802	80.91 Hz
$(1,2)$	7.0156	88.97 Hz
$(4,1)$	7.5883	96.23 Hz
$(2,2)$	8.4172	106.75 Hz
$(0,3)$	8.6537	109.74 Hz

(m,n)	$Z_n(J_m(kr))$	f_{mn}
$(5,1)$	8.7715	111.24 Hz
$(3,2)$	9.7610	123.79 Hz
$(6,1)$	9.9361	126.00 Hz
$(1,3)$	10.1735	129.02 Hz
$(4,2)$	11.0647	140.32 Hz
$(7,1)$	11.0864	140.59 Hz
$(2,3)$	11.6198	147.36 Hz
$(0,4)$	11.7915	149.54 Hz
$(8,1)$	12.2251	155.04 Hz

c) Limited to the frequency range where the first eighteen modes lie, and assuming that the drum is struck at coordinates $r = 0.075\,\mathrm{m}$, $\varphi = 15°$ and a displacement sensor is mounted at coordinates $r = 0.075\,\mathrm{m}$, $\varphi = 195°$, derive the displacement time signal read by the sensor, assuming that all modes are characterized by the same quality factor, equal to 25. Moreover, the impact force signal of the hammer is $f(t) = 0.1e^{-\frac{(t-0.03)^2}{0.01^2}}$

To compute the response at location x_j of the system when it is excited at the point x_k, the modal superposition approach has been exploited. From a mathematical point of view, this approach consists of switching from physical coordinates $z(x,t)$ to the modal ones $q_i(t)$ and limiting the transformation to the first n modes.

$$z(x_j,t) \approx \sum_{i=1}^{n} \Phi_i(x_j)q_i(t) \tag{2}$$

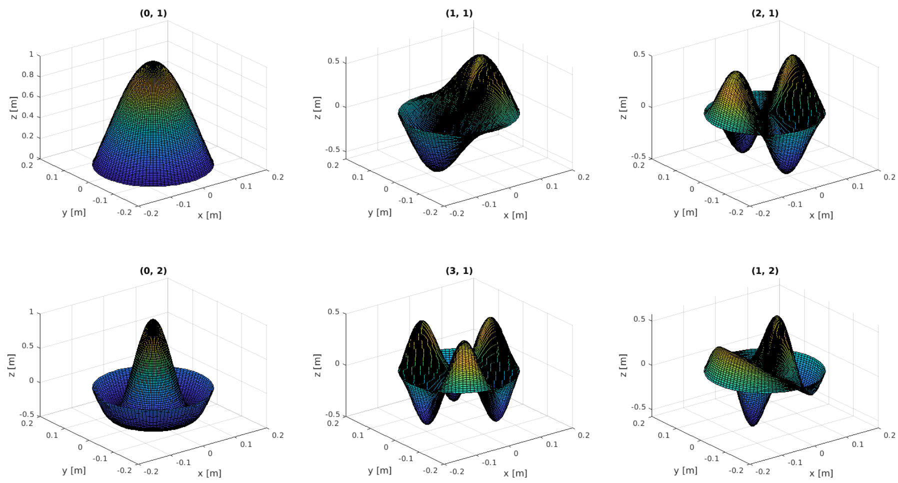

Figure 1: Modeshapes of the first six modes of the membrane

The equation above can be reformulated in matrix form, and adopted in the standard Lagrange's equations, to obtain the system's motion equations in principal (modal) coordinates. This approach allows us to approximate the continuous system (with infinite DOF) by a discrete one (with n DOF), and to efficiently model the system in the frequency range of interest. This can be done because of the orthogonality of eigenfunctions, i.e. the fact that the both the modal mass and the stiffness matrix are always diagonal, independently of the specific boundary conditions and associated eigenfunctions Φ_i. Under the hypothesis of structural or light damping, this consideration can be also extended to the damping matrix. It follows that the behavior of the system can be described by means of a system of n decoupled differential equations of the form

$$m_i\ddot{q}_i + c_i\dot{q}_i + k_i q_i = Q_i(t) = \Phi_i(x_k) f_k(t) \tag{3}$$

where every equation is analogous to that of a linear single-DOF vibrating system. Each single-DOF system corresponds to a modal coordinate q_i and is forced by the generalized force $Q_i(t)$, which stands for the excitation of the i-th mode produced by the external force $f_k(t)$. If we consider a sinusoidal excitation in the form $f_k(t) = F_{k0}e^{j\omega t}$, since the system is linear, its steady-state response will be in the form $z_j(t) = z_{j0}e^{j\omega t}$ and we can also consider a sinusoidal form for the modal coordinates $q_i(t) = q_{i0}e^{j\omega t}$. Substituting the latter in equation (3), we get:

$$(-\omega^2 m_i + j\omega c_i + k_i)q_{i0}e^{j\omega t} = \Phi_i(x_k)F_{k0}e^{j\omega t} \Rightarrow q_{i0} = \frac{\Phi_i(x_k)F_{k0}}{-\omega^2 m_i + j\omega c_i + k_i} \qquad i = 1, 2, ..$$

Then we can apply our modal coordinates $q_i(t)$ to (2) in order to obtain the physical displacement approximation:

$$z(x_j, t) = \sum_{i=1}^{n} \Phi_i(x_j)q_{i0}e^{j\omega t} = F_{k0}e^{j\omega t} \sum_{i=1}^{n} \frac{\Phi_i(x_j)\Phi_i(x_k)}{-\omega^2 m_i + j\omega c_i + k_i} \tag{4}$$

By looking at equation (4) and considering again that the displacement is sinusoidal as well, we can define the FRF (Frequency Response Function) as the ratio between the displacement at the point j and the exctitation force applied in k:

$$G(j\omega) = \frac{z_{j0}}{F_{k0}} = \sum_{i=1}^{n} \frac{\Phi_i(x_j)\Phi_i(x_k)}{-\omega^2 m_i + j\omega c_i + k_i}$$

We can refer to $G(j\omega)$ also as the system's receptance and express it as:

$$G(j\omega) = \sum_{i=1}^{n} \frac{\Phi_i(x_j)\Phi_i(x_k)}{m_i(\lambda_i - \omega^2)}$$

where m_i is the modal mass, $\lambda_i = w_i^2(1 + j\mu_i)$, w_i is the natural pulsation of the mode and $\mu_i = \frac{1}{Q_i}$ is the damping loss factor. Since our membrane is a 2D-system, x_j and x_k are expressed in polar coordinates and the modal mass can be computed with a double integration:

$$\begin{cases} m_i = \iint_S \sigma|\Phi_i|^2 \, dS = \int_0^a \int_0^{2\pi} \sigma|\Phi_i|^2 r \, dr \, d\phi \\ x_j = (r_j, \phi_j), \; x_k = (r_k, \phi_k) \end{cases}$$

For the membrane we have to consider that the modeshapes are invariant with respect to the rotation. This implies that the nodal diameters organize themselves to have the maximum displacement in correspondence of the excitation point (unless there is a nodal circle). We kept this into account by rotating the given excitation and measurement points in order to have the excitation point at the angle ϕ where the modeshape has its maximum value: as we can see from (1) $\tilde{z}_{mn}(r, \phi)$ is max when $\phi = 0°$, so we get $\phi_j = 0°$ and $\phi_k = 180°$.
The figure below shows the magnitude in dB of $G(j\omega)$ for our membrane: we can clearly see the peaks at some of the resonance frequencies.

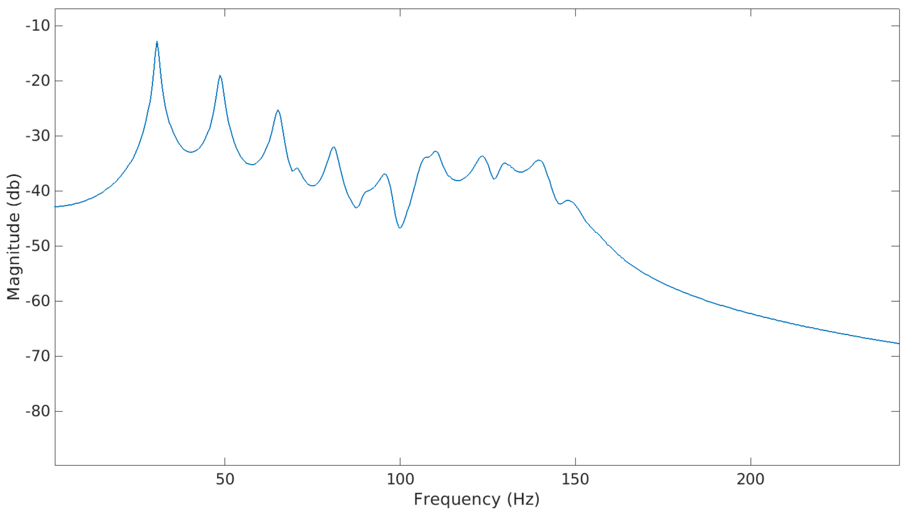

Figure 2: Membrane receptance

Finally the displacement's sensor signal (Fig. 5) can be estimated by means of the convolution between the given excitation force signal (Fig. 3) and the inverse Fourier Transform of the receptance (Fig. 4):

$$z_j(t) = \mathcal{F}^{-1}\{G(j\omega)\} * f_k(t)$$

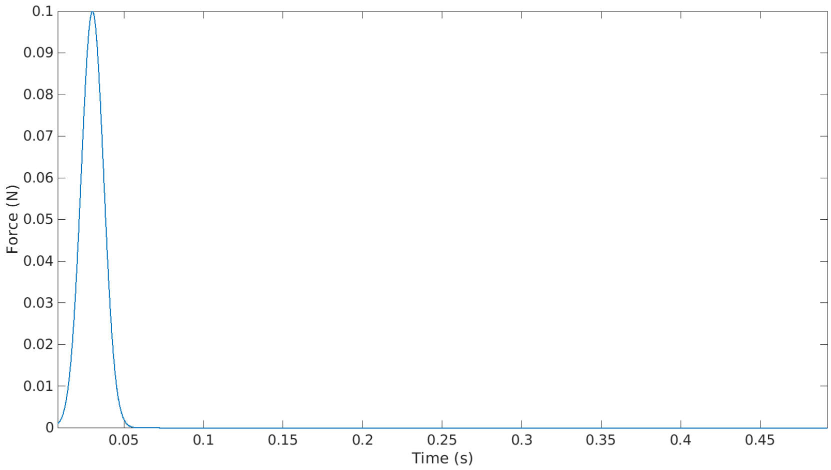

Figure 3: Excitation force signal

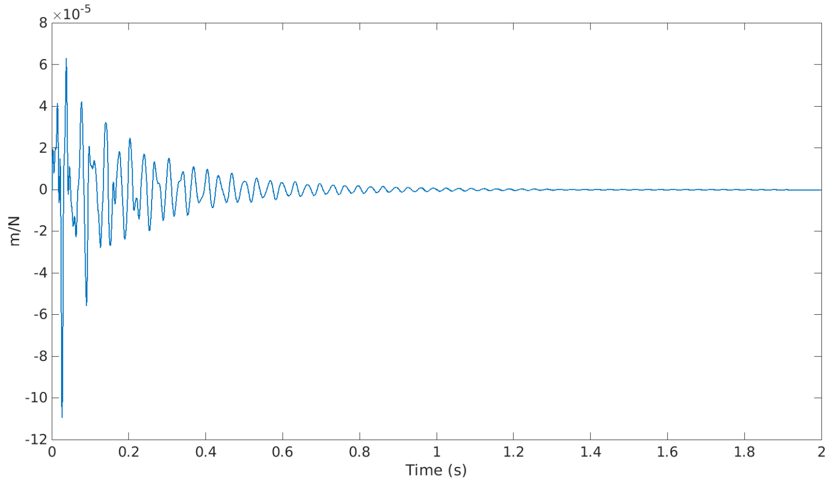

Figure 4: Membrane receptance IFT

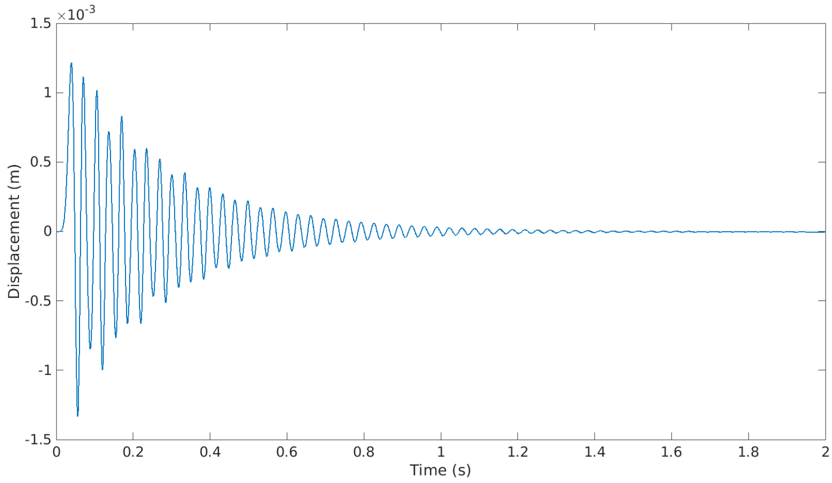

Figure 5: Sensor displacement signal

2. Circular plate characterization

Consider now a thin plate with clamped edges and with the same size of the membrane. The plate has a thickness of 1 mm, and it is made with aluminum (E=69 × 10^9 Pa, ρ=2700 kg/m^3, ν=0.334).

d) Compute the propagation speed of quasi-longitudinal and longitudinal waves

The speed of propagation c_L of quasi-longitudinal waves is given by:

$$c_L = \sqrt{\frac{E}{\rho(1-\nu^2)}} = 5363.24\,\mathrm{m/s}$$

Instead for longitudinal waves the speed of propagation c'_L is computed as:

$$c'_L = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} = 6199.16\,\mathrm{m/s}$$

e) Plot the propagation speed of the bending waves as a function of the frequency for the considered plate

The equation of motion of bending waves in thin plates is

$$\frac{\partial^2 z}{\partial t^2} + \frac{Eh^2}{12\rho(1-\nu^2)}\nabla^4 z = 0$$

It is a partial differential equation, whose solution $z(x, y, t)$ can be computed exploiting the separation of variables and assuming an harmonic temporal dependency.

$$z(x, y, t) = Z(x, y)e^{j\omega t}$$

Replacing this expression in the previous equation of motion we get

$$\nabla^4 Z - k^4 Z = 0$$

where the wave number k^2 is given by

$$k^2 = \frac{\sqrt{12}\omega}{c_L h} = \frac{\sqrt{12}\omega}{h}\sqrt{\frac{\rho(1-\nu^2)}{E}}$$

While the propagation speed v can be written as:

$$v(f) = \frac{\omega}{k} = \sqrt{\frac{\omega h c_L}{\sqrt{12}}} = \sqrt{\frac{2\pi}{\sqrt{12}}}\sqrt{f h c_L}$$

We notice that v depends on the thickness and it's proportional to the square root of the frequency, as shown in the graph below.

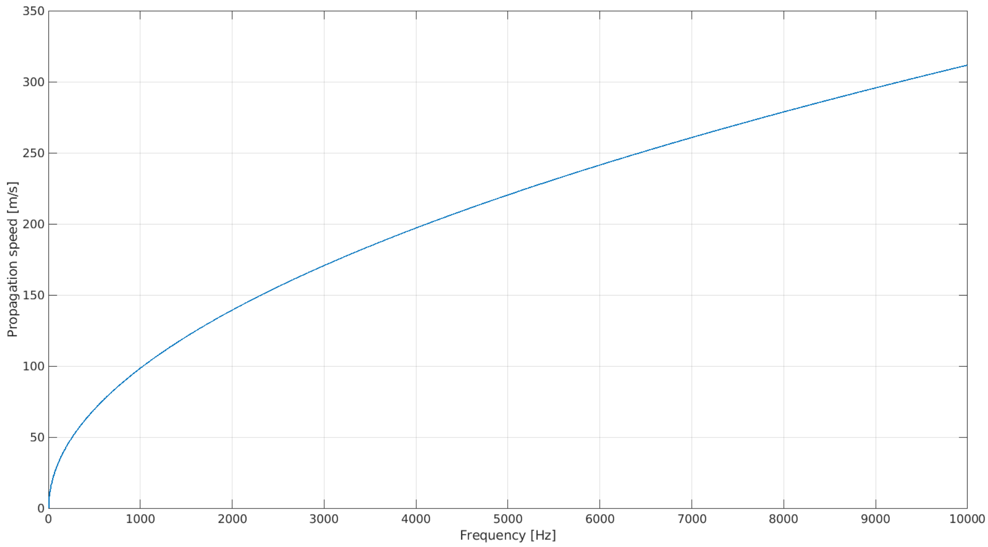

Figure 6: Propagation speed of bending waves as a function of the frequency

f) Find the modal frequencies of the first five bending modes of the plate

The wave numbers and the modal frequencies depend on the boundary conditions i.e. on the geometry of the plate (a = radius of the plate).
The latter has clamped edges, so the modal frequencies for the first five bending modes of the plate ordered by frequency can be computed as

$f_{01} = 0.4694 \cdot \frac{c_L h}{a^2} = 111.89\,\text{Hz}$
$f_{11} = 2.08 \cdot f_{01} = 232.73\,\text{Hz}$
$f_{21} = 3.41 \cdot f_{01} = 381.54\,\text{Hz}$
$f_{02} = 3.89 \cdot f_{01} = 435.25\,\text{Hz}$
$f_{31} = 5.00 \cdot f_{01} = 559.45\,\text{Hz}$

3. Interaction between coupled systems

Consider now that a string is attached to the considered plate, and its fundamental mode is tuned to the frequency of the first mode of the plate. The string is made with iron (ρ_s=5000 kg/m^3), its cross section is circular with a radius of 0.001 m, and its length is L=0.4 m. Due to internal losses and sound radiation, the plate at the frequency of the considered mode dissipates energy, and the merit factor is 50.

g) Compute the tension of the string so that its fundamental mode is tuned with the first mode of the soundboard

The frequency of the first mode ($n = 1$) of the plate modifies the tension of the string attached on it.
We compute the linear density of the string starting from its volume density ρ_s and cross-section $S_s = \pi r_s^2$

$$\mu = \rho_s S_s = \rho_s \pi r_s^2 = 0.0157\,\mathrm{kg/m}$$

From paragraph 2) we know that the frequency of the first mode f_{01} of the plate is equal to 111.89 Hz.
Since the string is tuned at the same frequency, we can compute its tension as

$$T = c^2\mu = \left(\frac{\omega_{01} L}{n\pi}\right)^2 \mu = 125.86\,\mathrm{N}$$

h) Compute now the frequencies of the first five modes of the string considering its stiffness. The Young modulus of the iron is $E_s = 200 \times 10^9\,\mathrm{Pa}$

When the string is non-ideal, its equation of motion takes into account not only the tension but also the stiffness of the string

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \frac{E_s S_s K^2}{\mu} \frac{\partial^4 y}{\partial x^4}$$

where E_s is the Young modulus and K is the gyration radius of the string.
Solving the equation of motion and imposing the appropriate boundary conditions, the natural frequencies of the string (which is attached to the plate and has clamped ends) are computed as:

$$f_n^* = n f_1 \sqrt{1 + Bn^2}\left[1 + \frac{2}{\pi}\sqrt{B} + \left(\frac{2}{\pi}\right)^2 B\right]$$

where $B = \pi E_s S_s K^2 / T L^2$ and f_1 is the natural frequency of the first mode of the string without stiffness.
Due to the stiffness, the first five modes of the string diverge from the ideal case, where the modes are multiples of the first mode: $f_n = n f_1$. The results are compared in the following tables, where its possible to notice how the frequencies are increased with respect to the case without stiffness.

Without stiffness
$f_1 = 111.89\,\mathrm{Hz}$
$f_2 = 2f_1 = 223.78\,\mathrm{Hz}$
$f_3 = 3f_1 = 335.67\,\mathrm{Hz}$
$f_4 = 4f_1 = 447.56\,\mathrm{Hz}$
$f_5 = 5f_1 = 559.45\,\mathrm{Hz}$

With stiffness
$f_1^* = 125.66\,\mathrm{Hz}$
$f_2^* = 260.19\,\mathrm{Hz}$
$f_3^* = 411.48\,\mathrm{Hz}$
$f_4^* = 585.93\,\mathrm{Hz}$
$f_5^* = 788.30\,\mathrm{Hz}$

i) Neglecting the stiffness of the string, compute the frequencies of the modes of the string- soundboard system considering the coupling between the plate and the string

The behaviour of the composed system (plate + string) differs from the one of the separate elements because of the exchange of vibrational energy caused by the coupling. We can distinguish two kind of coupling: the **weak** coupling, where the eigenfrequencies of a structure aren't impacted by the other one, and the **strong** one, where the interaction between the two structures modifies the eigenfrequencies of both. The nature of the interaction depends both on the characteristics of the two structures and the frequency.
Knowing the mass of the string $m = \rho_s \pi r_s^2 L = 0.0063\,\mathrm{kg}$, the mass of the plate $M = \rho_p \pi r_p^2 h = 0.1909\,\mathrm{kg}$ and the merit factor of the resonance of the soundboard $Q = 50$, the **weak coupling** for each n-th mode of the string occurs when:

$$\frac{m}{n^2 M} < \frac{\pi^2}{4Q^2} \tag{5}$$

Instead for the **strong coupling**:

$$\frac{m}{n^2 M} > \frac{\pi^2}{4Q^2} \tag{6}$$

If the two independent elements have the **same resonance frequency** $\omega_s = \omega_p = \omega_0$ and we're in case of:
- weak coupling $\Rightarrow$ the two frequencies not altered by the interaction
- strong coupling $\Rightarrow$ the original frequency is split symmetrically about ω_0 in two different resonance frequencies for each mode

If the original **uncoupled frequencies** of the string and the plate are **different** ($\omega_s \neq \omega_p$), both for weak and strong coupling the frequencies are split, but not symmetrically.
In our case, to obtain the resonance frequencies of the coupled system, we compared the first five modes of the plate to those of the string (without stiffness), computed in the previous paragraphs.
Evaluating the inequalities (5) and (6), we also calculated the adimensional weak-strong coefficient threshold of our system, as follows

$$\frac{\pi^2}{4Q^2} = 0.987 \cdot 10^3$$

Mode number	**Plate frequency**	**String frequency**	$\frac{m}{n^2M} \cdot 10^3$
$n = 1$	111.89 Hz	111.89 Hz	32.921
$n = 2$	232.73 Hz	223.78 Hz	8.320
$n = 3$	381.54 Hz	335.67 Hz	3.658
$n = 4$	435.25 Hz	447.56 Hz	2.058
$n = 5$	559.45 Hz	559.45 Hz	1.317

Looking at the table above, it's possible to notice that the first and the fifth modes are equal between the plate and the string ($\omega_n^p = \omega_n^s = \omega_n$).
In addition, the weak-strong coefficient is above the threshold for both the modes i.e. we have strong coupling. To estimate the new symmetrically split frequencies of the coupled system we considered the graph shown below (figure 7). To determine the value y_n on the vertical axis, we exploited the intersection between the weak-strong coefficient for each mode (in the horizontal axis) and the curve of the plate's quality factor ($Q = 50$). y_n is related to the coupled frequencies as follows:

$$y_n = \frac{\Omega_n^+ - \Omega_n^-}{\omega_n} \quad \Rightarrow \quad \begin{aligned} \Omega_n^+ &= \omega_n \cdot (1 + y_n/2) \\ \Omega_n^- &= \omega_n \cdot (1 - y_n/2) \end{aligned} \quad \Rightarrow \quad \begin{aligned} f_n^+ &= f_n \cdot (1 + y_n/2) \\ f_n^- &= f_n \cdot (1 - y_n/2) \end{aligned}$$

From figure (7) we can observe that, for the **fifth mode**, the graph is not consistent around the origin. In fact, it's not possible to properly define the intersection between the coefficient and the desired Q-curve. As a consequence, we can conclude that this mode leads to a case of weak coupling, without any frequency split.
The results are reported in the table below:

n	**Plate**	**String**	$\frac{m}{n^2M} \cdot 10^3$	$\frac{\Omega_n^+ - \Omega_n^-}{\omega_n}$	**Coupled frequencies**		
1	111.89 Hz	111.89 Hz	32.921	0.077	Strong	$f_1^- = 107.58$ Hz	$f_1^+ = 116.20$ Hz
5	559.45 Hz	559.45 Hz	1.317	0	Weak	$f_1^- = f_1^+ = 559.45$ Hz	

The frequencies of the other modes of the uncoupled systems are different with respect to each other ($\omega_n^p \neq \omega_n^s$) but once again, looking at inequalities (5) and (6), we can infer these are all cases of strong coupling as well. For these reasons we can evaluate the new resonance frequencies applying the same previous graphical criterion to the respective graph (figure 8), as shown below.

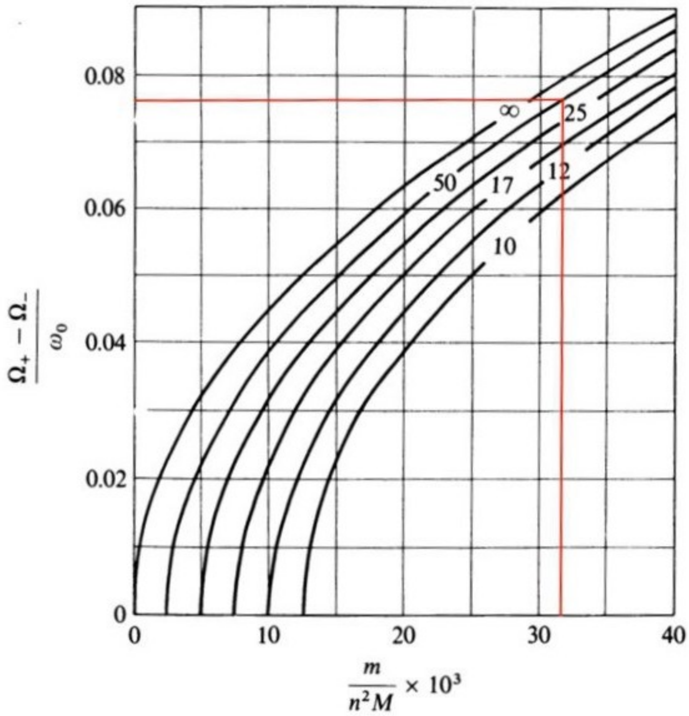

Figure 7: Normal mode splitting in case of strong coupling when $\omega_n^p = \omega_n^s = \omega_n$

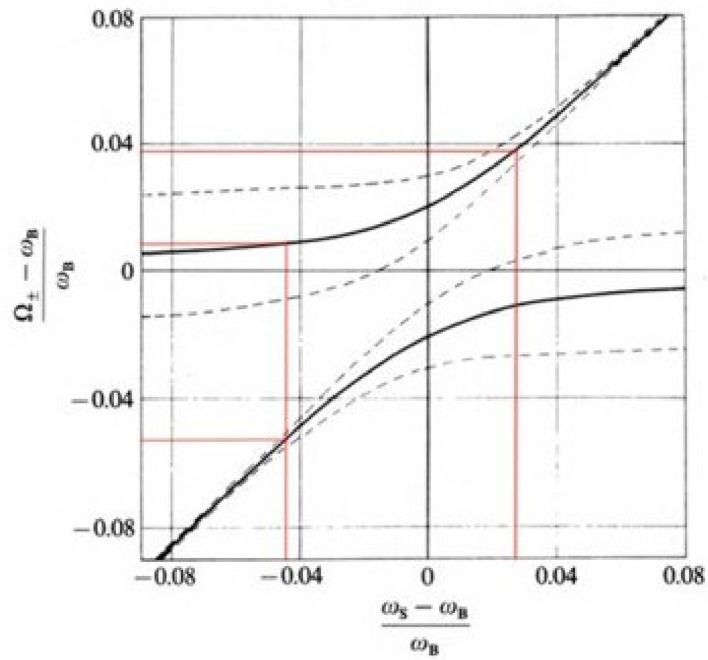

Figure 8: Normal mode splitting in case of strong coupling when $\omega_n^s \neq \omega_n^p$

Differently from before, this time we got two different values on the vertical axis and therefore we computed the new resonance frequencies exploiting the following relation:

$$x_n = \frac{\omega_n^s - \omega_n^p}{\omega_n^p} \quad \Rightarrow \quad y_n^{(1,2)} = \frac{\Omega_n^{(1,2)} - \omega_n^p}{\omega_n^p} \quad \Rightarrow \quad \begin{aligned} f_n^{(1)} &= f_n^p \cdot (1 + y_n^{(1)}) \\ f_n^{(2)} &= f_n^p \cdot (1 - y_n^{(2)}) \end{aligned}$$

For the **third mode** the value $x_3 = -0.1202$ lays off the graph, that's why it isn't shown in figure 8. In order to obtain the correspondent frequency values, we assumed an asymptotic trend for the lower curve, which resembles a straight line with a slope of 45°; as regards the higher curve, it tends to zero as the x-values decrease.
The results are reported in the table.

n	**Plate**	**String**	$\frac{\omega_s - \omega_p}{\omega_p}$	$\frac{\Omega_n^{(1)} - \omega_n^p}{\omega_n^p}$	$\frac{\Omega_n^{(2)} - \omega_n^p}{\omega_n^p}$	**Strong coupled frequencies**	
2	232.73 Hz	223.78 Hz	-0.0385	-0.046	0.010	$f_2^{(1)} = 222.02$ Hz	$f_2^{(2)} = 223.77$ Hz
3	381.54 Hz	335.67 Hz	-0.1202	-0.120	0.004	$f_3^{(1)} = 335.76$ Hz	$f_3^{(2)} = 383.07$ Hz
4	435.25 Hz	447.56 Hz	0.0283	-0.012	0.037	$f_4^{(1)} = 430.03$ Hz	$f_4^{(2)} = 451.35$ Hz