Politecnico di Milano

Homework #5 - Design of a recorder flute

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Course: Musical Acoustics – Professor: Fabio Antonacci
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1. First component: resonator

The resonator is shaped as a cone whose conical semiangle is 0.75°. The instrument is aimed at being a treble recorder, with a length of 0.45 m. For the sake of simplicity, we consider that only two finger holes are present.

Question 1: Find the diameter of the cone at the resonator head and foot so that the note produced when all the finger holes are closed is E4 ($f_0 = 329.63\,\text{Hz}$).

The resonator's diameters were computed starting from the formula of the input impedance for a cone of geometrical length L_r, with a throat opening of area S at a distance x_f from the conical vertex and open at the mouth:

$$Z_{IN} \approx jZ_0 \frac{\sin(kL')\sin\theta}{\sin(kL'+\theta)} \tag{1}$$

where:

$$\theta = \tan^{-1}(kx_f), \qquad Z_0 = \frac{\rho c}{S_f} \qquad \text{and} \qquad L' \approx L + \text{end corrections}$$

Knowing the resonance value ($\omega_0 = 2\pi f_0$), we computed the wave number k as $k = \omega_0/c$ and for the end corrections, we applied the unflanged-tube-correction both for the head and foot of the resonator:

$$L' = L_r + \Delta L_{hd} + \Delta L_f \qquad \text{with} \quad \Delta L_{hd} = L_r + 0.6 \cdot r_{hd}, \quad \Delta L_f = L_r + 0.6 \cdot r_f \tag{2}$$

where r_f and r_{hd} are the radii in correspondence of the foot and head of the resonator, expressed as function of x_f exploiting the following trigonometrical relations (fig. 1):

$$r_f = x_f \cdot \tan(\theta) \tag{3}$$
$$r_{hd} = (x_f + L_r) \cdot \tan(\theta) \tag{4}$$

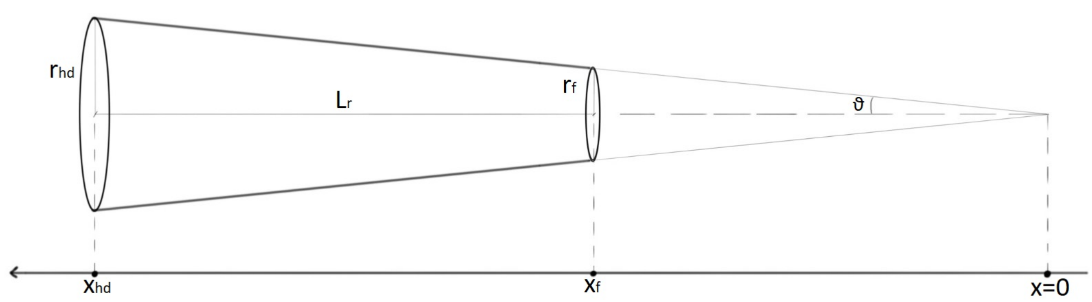

Figure 1: Scheme of the cone section

Once we highlighted the dependency of the input impedance on the distance from the vertex, we thought about which value of x_f granted the most reasonable working condition. Despite the cone approximation, being the recorder a jet-switch instrument, it'll work best

at an admittance maximum (i.e. impedance minimum) so we computed the x_f value which satisfies this condition and the corresponding r_f and r_{hd} using equations 3 and 4.
Although its mathematical straightforwardness, this approach led to inappropriate results for r_f and r_{hd}, which were over-sized for any actual recorder.
We therefore refined the above described method, limiting the ratio between the head and foot radii, so that it ranged between 0.5 and 0.7, in accordance with measures derived from actual instruments. The admittance behavior as function of r_f is shown in fig. 2: any value in the range limited by the vertical red lines will ensure a realistic geometry for the flute.

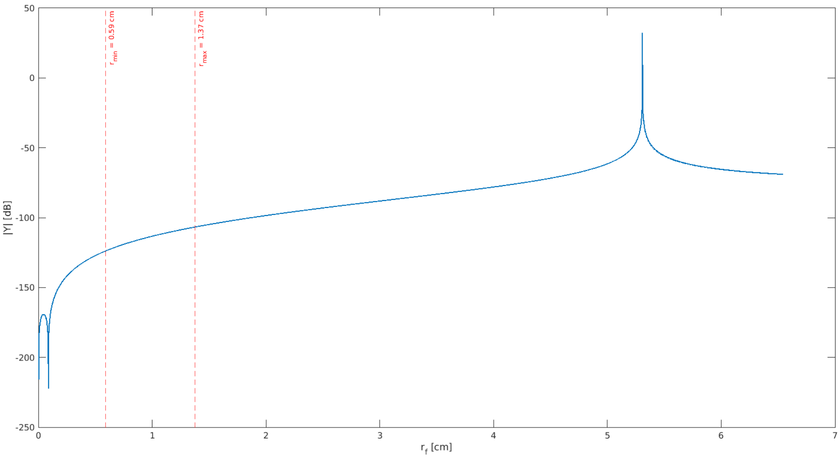

Figure 2: Admittance as function of r_f

We chose $r_{f_{max}}$ for the foot radius value and we obtained:

$$r_f = 1.37\,\text{cm} \Rightarrow d_f = 2 \cdot r_f = 2.74\,\text{cm}$$
$$r_{hd} = 1.96\,\text{cm} \Rightarrow d_{hd} = 2 \cdot r_{hd} = 3.92\,\text{cm}$$

This second approach led to much more realistic values for the recorder's dimension, at the expense of the working condition: the resonance doesn't occur exactly in correspondence of ω_0 as before, but is slightly shifted towards the right (fig. 3).
To try and solve this problem, we further improved our approach by modifying the assumptions at the basis of the previous description: the most relevant simplification that we made regards the formulation of the recorder's impedance. In our model we equally treated the corrections at the mouth and at the foot of the resonator. The unflanged-open-pipe model, however, well adjusts to the configuration at the resonator's foot, but oversimplifies the mouth one. The correct formulation for the instrument's impedance consists of the series between Z_{IN} and Z_{hd}, where the impedance of the mouth window Z_{hd} is an intertance $j\omega M$. The end correction for the mouth therefore becomes $\Delta L_{hd} = \frac{MS}{\rho}$. Since we were not given the values of the inertance M, we couldn't apply this formulation to our model, and had to rely on the one described beforehand.

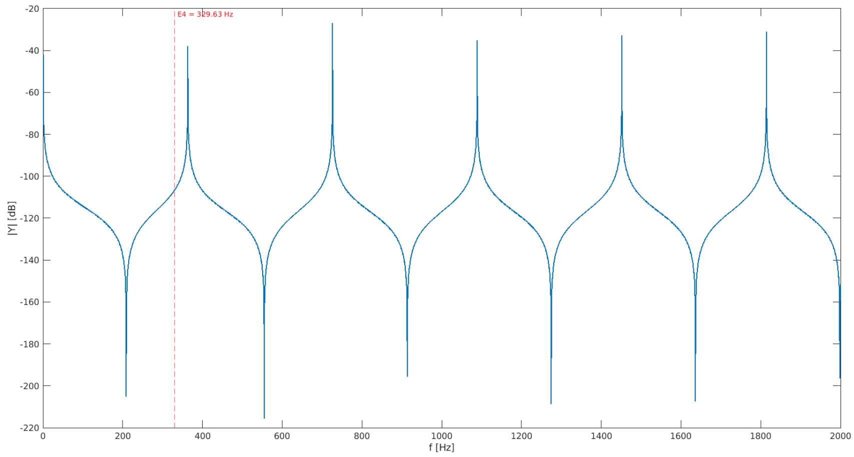

Figure 3: Admittance as function of frequency

Question 2: Find the position of the last finger hole (i.e. the one closest to the resonator foot) in order to produce the note F4# (369.99 Hz) when it is open. Consider the finger hole diameter to be equal to the bore diameter at the resonator foot (simplification).

The presence of an open hole changes the length of the air column, thus modifying the overall impedance of the system. To correctly compute the distance D_1 of the last finger hole from the resonator's open end, we followed an admittance maximization approach: as shown in figure **??** we can subdivide our pipe in two conical sections, at the right and left-hand side of the hole respectively.

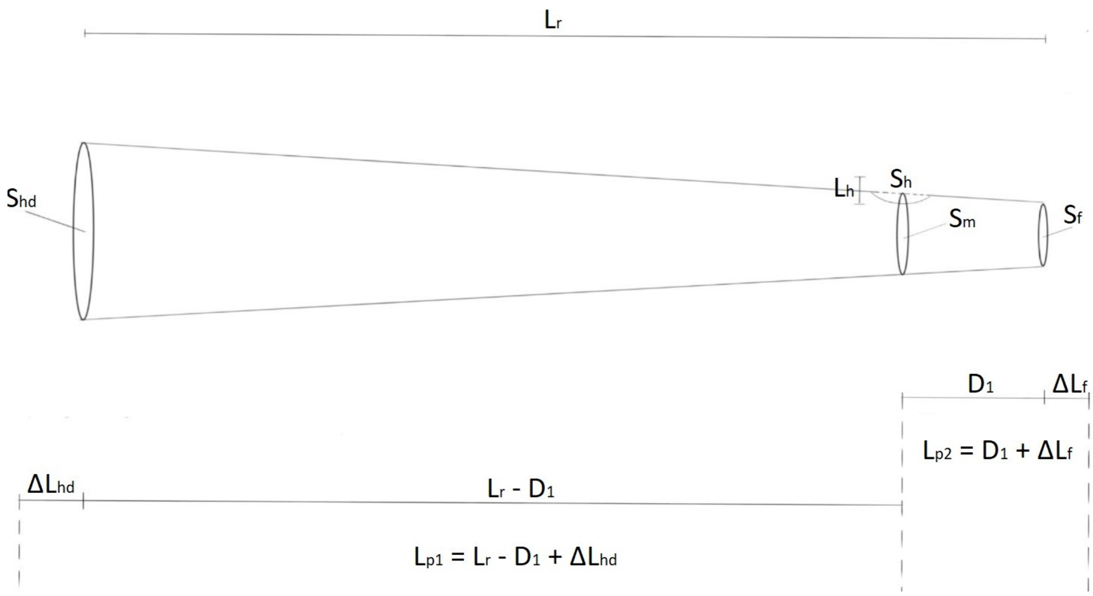

Figure 4: Scheme of the resonator with one hole

The impedances of the last conical section p_2 and of the hole are given by:

$$Z_{p2} = jZ_{0f}\frac{\sin(kL_{p2})\sin(\theta_1)}{\sin(kL_{p2}+\theta_1)}, \qquad \text{with} \quad Z_{0f} = \frac{\rho c}{S_f} \quad \text{and} \quad L_{p2} = D_1 + \Delta L_f \tag{5}$$

$$Z_h = jZ_{0h}\tan(kL_h), \qquad \text{with} \quad Z_{0h} = \frac{\rho c}{S_h}. \quad S_h = S_f \quad \text{and} \quad L_h = \Delta L_f \tag{6}$$

As we can see from equation (6), we assumed that the finger hole's diameter is equal to the resonator foot diameter ($S_h = S_f$) and that the hole has no chimney, that is: only the open-end correction is considered ($L_h = \Delta L_f$).
For the impedance of the first conical section p_1 we have to consider that it is connected to a load, given by the parallel of Z_{p_2} and Z_h, thus we have that:

$$Z_{p_1} = Z_{0m}\left[\frac{jZ_L[\sin(kL_{p_1}-\theta_2)/\sin\theta_2] + Z_{0hd}\sin(kL_{p_1})}{Z_L[\sin(kL_{p_1}+\theta_1-\theta_2)/(\sin\theta_1\sin\theta_2)] - jZ_{0hd}[\sin(kL_{p_1}+\theta_1)/\sin\theta_1]}\right] \tag{7}$$

where:

$$Z_{0m} = \frac{\rho c}{S_m}, \quad \text{with} \quad S_m = \pi r_m^2, \quad r_m = (x_f + D_1)\tan(\theta)$$

$$Z_{0hd} = \frac{\rho c}{S_{hd}}, \qquad L_{p1} = L_r - D_1 + \Delta L_{hd}, \qquad Z_L = Z_{p2} \parallel Z_h$$

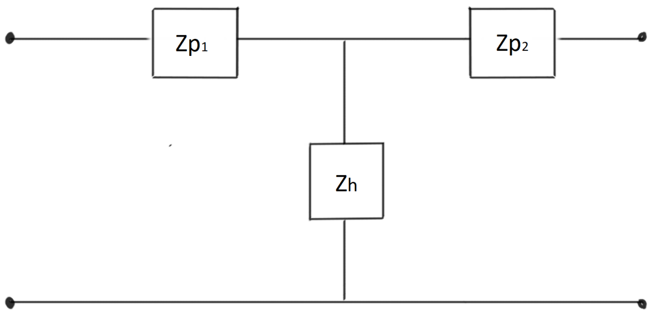

Figure 5: Equivalent impedance circuit

The equivalent impedance circuit is shown in figure 4 and the total impedance Z is given by:

$$Z = Z_{p_1} + (Z_{p2} \parallel Z_h) = Z_{p_1} + \left(\frac{Z_{p2}\cdot Z_h}{Z_{p2}+Z_h}\right) \tag{8}$$

The key point in this process is that impedances in equations (5), (6) and (7), can be expressed as function of D_1, so the admittance $Y = 1/Z$ is function of D_1 as well and we can follow the steps described in the previous section to determine its maxima and derive the optimum value for the distance (fig. 5). The admittance behavior as function of frequency is shown in fig. 6, with the first maximum in correspondence of the desired frequency.
This approach led to $D_1 = 2.08\,\text{cm}$.

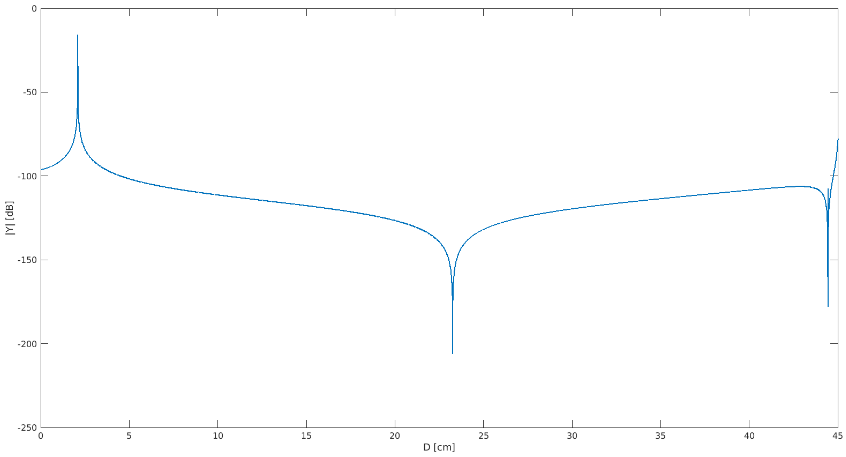

Figure 6: Admittance as function of D_1

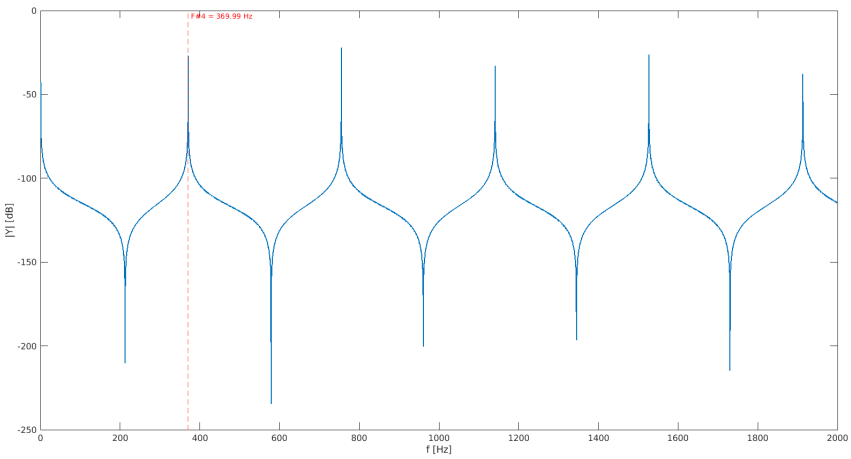

Figure 7: Admittance as function of frequency

Question 3: Find the position of the second last finger hole in order to produce the note G4# (415.3 Hz) when the two finger holes are open. Consider the finger hole diameter to be equal to the bore diameter at the resonator foot (simplification).

To find distance D_2 of the second last finger hole from the open end of the recorder, we slightly modified the operations of the previous point to adapt the admittance maximization process to this new configuration (fig. **??**). In order to reduce the complexity of our *Matlab* code, we defined the `conical_horn_impedance` function, which takes as input k, ρ, c, the conical section's length, the radii at the edges and the load impedance, and gives as output the impedance value, applying equation (7) to the desired section. We started from the calculation of the impedance of the conical section p_3, placing the last finger hole accordingly to the previously obtained results. Then we proceeded moving towards the left, and computed the impedances of the conical section p_1 and p_2, as function of distance. Unlike what we have seen for p_3, both p_1 and p_2 are affected from the presence of a load (Z_{L1} and Z_{L2} respectively), which was properly included in the above described

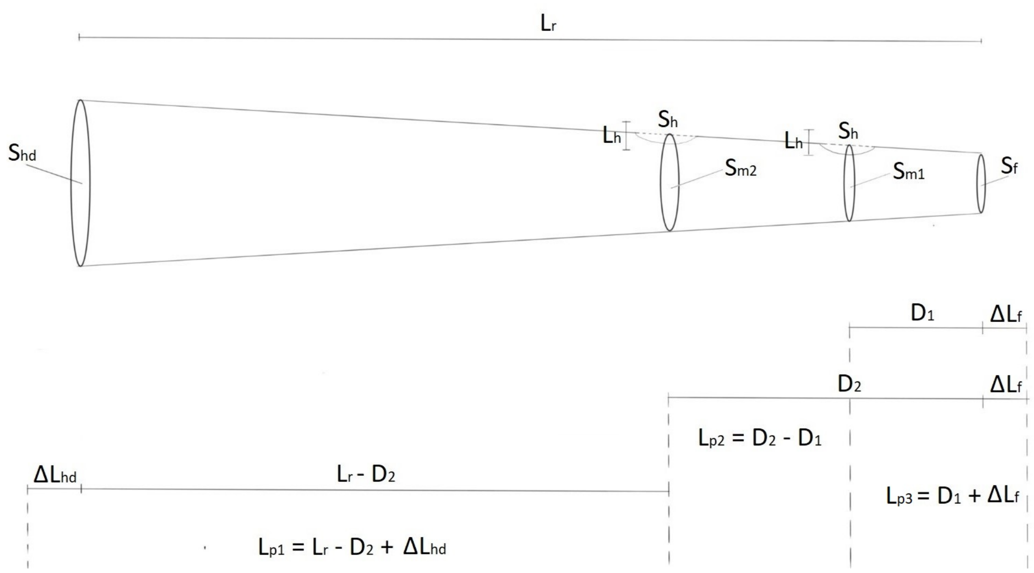

Figure 8: Scheme of the resonator with two holes

function. In particular we have that:

$$Z_{L2} = Z_{p_3} \parallel Z_h = \left(\frac{Z_{p_3} \cdot Z_h}{Z_{p_3} + Z_h}\right)$$
$$Z_{L1} = Z_D \parallel Z_h = \left(\frac{Z_D \cdot Z_h}{Z_D + Z_h}\right)$$

where

$$Z_D = Z_{p_2} + Z_{L2}$$

As we can see from fig. 7, the presence of the second hole corresponds to an additional branch with respect to the circuit shown in fig. 4.

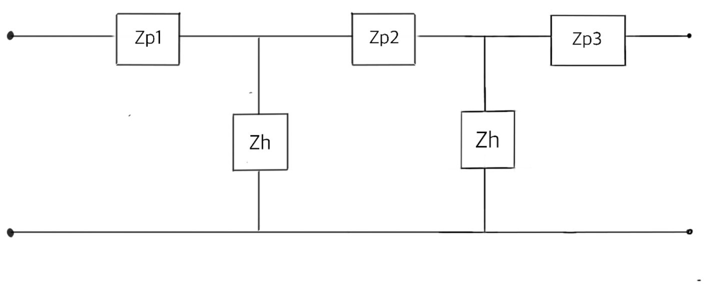

Figure 9: Equivalent impedance circuit

The overall impedance of the new configuration can be written as:

$$Z = Z_{p_1} + Z_{L1}$$

The remaining part of the computation is exactly the same as before: we obtained a formulation of Z as function of D_2 and then maximized the correspondent admittance

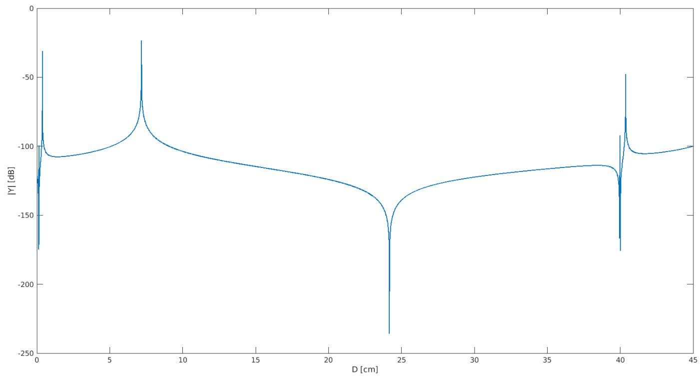

Figure 10: Admittance as function of D_2

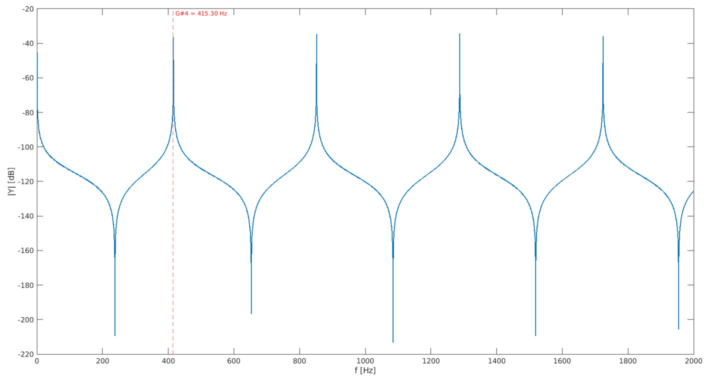

Figure 11: Admittance as function of frequency

($Y = 1/Z$). This approach led to an optimum value of the distance between the resonator's foot and the second last finger hole equal to $D_2 = 7.16\,\text{cm}$.
Fig. 8 and 9 show the admittance behavior as function of D_2 and frequency respectively: we can notice how one of the peaks corresponds to G4$\#$.

2. Second component: flue channel

The instrument is aimed at producing a spectrum whose centroid is at 1.7 kHz when the pressure difference between the player mouth and the flue channel entrance is 55 Pa.

Question 4: Find the flue channel thickness that complies with the above specifications. For this pair of thickness and jet velocity, compute the Reynolds number Re and assess the jet regime that is undergoing at the flue channel exit and in the instrument mouth (laminar, turbulent, etc).

To calculate the flue channel thickness, we computed of the central velocity of the jet Uj, starting with the pressure difference between the player mouth and the flue channel.

$$U_j = \sqrt{\frac{2L_f p}{\rho_0}} \tag{9}$$

From the position of the spectrum centroid f_{center}, under the Bernoulli's approximation of uniform velocity, we derived the flue channel thickness h as:

$$f_{center} = 0.3 \cdot \frac{U_j}{h} \Rightarrow h = 0.3 \cdot \frac{U_j}{f_{center}} = 1.687\,\mathrm{mm}$$

The corresponding Reynolds number is given by

$$Re = \frac{U_j h}{\nu} = 1.075 \times 10^3\,[-]$$

where $\nu = 1.5 \times 10^{-5}\,\mathrm{m^2/s}$ is the kinematic viscosity of air.
From this value of Re (< 2000) we can state that the jet remains laminar for a short distance at the flue channel exit, before turbulence develops.

Question 5: Consider that the flue channel length is $25\,\mathrm{mm}$. Find the thickness of the boundary layer at the flue channel exit for the specifications defined above (Question 4).

Because of viscosity, the tangential velocity is zero at the walls, and increases within a thin layer up to the value of U_j determined in equation 9. This transition layer is called boundary layer and its thickness L_f increases as the flow travels trough the channel.
L_f can be expressed as function of the flue channel length l as follows:

$$L_f(l) \approx \sqrt{\frac{\nu \cdot l}{U_j}} \approx 1.981 \times 10^{-4}\,\mathrm{m}$$