Politecnico di Milano

Homework #1 - Helmholtz Resonator

Students: Vittoria Malaman [10610731] - Riccardo Martinelli [10456202] - Matteo Pettenò [10868930]

Course: Musical Acoustics – Professor: Fabio Antonacci
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It is given a Helmholtz resonator. The internal volume of the resonator consists of a parallelepipedon, with sizes 25 cm (H) 25 cm (W) 18 cm (L). The neck of the resonator is cylindrical, with length 6 cm, and radius 2.5 cm.

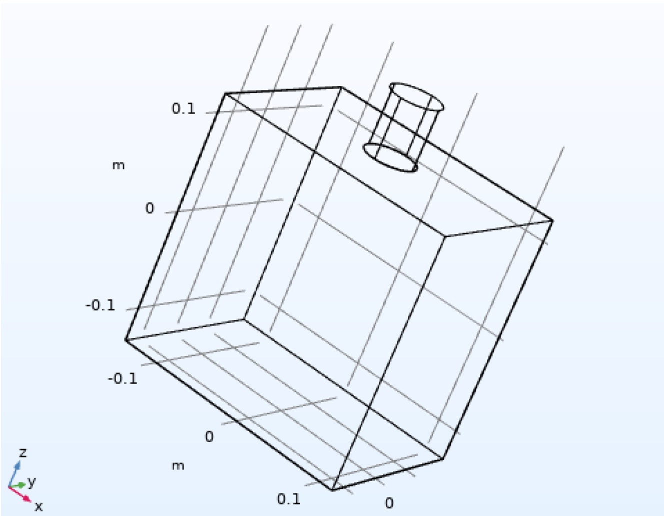

Figure 1: Helmholtz resonator geometry

The resonator's volume and the surface and length of its neck are given by:

$$V_0 = H \cdot W \cdot L = 25\,\text{cm} \times 25\,\text{cm} \times 18\,\text{cm} = 0.0113\,\text{m}^3$$
$$S = \pi \cdot r^2 = \pi \cdot (2.5 \cdot 10^{-2})^2 = 0.0020\,\text{m}^2$$
$$l = 6 \times 10^{-2}\,\text{m}$$

In the computations below we assumed an absolute temperature of $T = 20\,°\text{C}$ and the absolute pressure at sea level $p = 101.325\,\text{kPa}$.

a) Derive the resonance frequency of the resonator neglecting the virtual elongation of the neck.

In order to compute the resonance frequency of the Helmholtz resonator, the vibrating volume of air is compared to a piston that freely moves in the neck of the resonator.
The mass of the piston equals the mass of air confined in the neck and therefore can be written as:

$$m \approx \rho S l \tag{1}$$

where ρ is the density of air, S and l are the cross section and the length of the piston respectively.
When the piston is pushed down of a length x in the neck (i.e. when someone blows inside the resonator) both an increase in pressure and a decrease in volume occur. Since the process happens very fast, these two phenomena aren't proportional but follow the Poisson's equation for adiabatic processes reported below:

$$PV^{\gamma} = P_0 V_0^{\gamma}$$

Since $V = V_0 - Sx$, the expression above can be rephrased as follows:

$$P = P_0 V_0^{\gamma}(V_0 - Sx)^{-\gamma} = P_0 V_0^{\gamma}\left(1 - \frac{Sx}{V_0}\right)^{-\gamma} V_0^{-\gamma}$$

Exploiting the McLaurin expansion (being Sx very small compared to V_0) the following result is obtained:

$$P = P_0\left(1 + \gamma\frac{Sx}{V_0}\right) \tag{2}$$

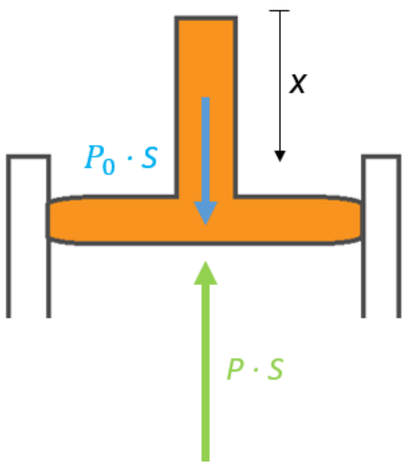

Figure 2: Net force acting on the piston

At this point, by taking into account the net force acting on the piston due to the different pressure levels P and P_0 existing inside and outside the resonator, it's possible to write that:

$$F = -S(P - P_0) \tag{3}$$

The equation above is obtained under the hypothesis that $\lambda >> r$.

By substituting equation (2) in (3), and recalling the second Newton's law:

$$m\frac{d^2x}{dt^2} = -S\left[P_0\left(1+\gamma\frac{Sx}{V_0}\right) - P_0\right]$$

$$m\frac{d^2x}{dt^2} = -\left(\frac{S^2P_0\gamma}{V_0}\right)x$$

$$m\frac{d^2x}{dt^2} = -K_{eq}x$$

The last equation resembles the one governing the simple harmonic motion; as a result, we can compute the natural frequency of the Helmholtz resonator as follows:

$$\omega^H = \sqrt{\frac{K_{eq}}{m}} = \sqrt{\frac{S^2P_0\gamma}{V_0m}}$$

Recalling that $m \approx \rho Sl$ and that the speed of sounds in fluids $c = \sqrt{\frac{\gamma P_0}{\rho}}$, it follows:

$$\omega^H = c\sqrt{\frac{S}{V_0l}}\,[\text{rad/s}] \quad \Rightarrow \quad f^H = \frac{c}{2\pi}\sqrt{\frac{S}{V_0l}}\,[\text{Hz}] \tag{4}$$

For our system we have $f^H = 92.6\,\text{Hz}$. We can also notice that:

$$\lambda = \frac{c}{f^H} = 3.68\,\text{m}$$

This results shows that the wavelength is much longer than the radius of the resonator and allows us to neglect any pressure variations inside the resonator's chamber, validating the previous demonstration.

b) Derive the resonance frequency of the resonator including the virtual elongation of the neck.

The previous calculus of the natural frequency of the resonator can be refined by including the virtual elongation of the neck.
At the core of this reasoning lies the proposal that the air immediately outside the ends of the neck takes part in the acoustic oscillations. This portion of air makes the neck appear to be acoustically longer than its physical length and the apparent length increase is called end correction.
As a result, the effective length can be calculated by $L^R = l^R + \delta_{in} + \delta_{ex}$, in which δ_{in} and δ_{ex} are the corrections mentioned above.
These corrections derive from the study of the input impedance Z_{IN} for a finite length of pipe as function of the characteristic impedance Z_0 and of a finite load impedance Z_L, whose value depends on the termination of the pipe (flanged or unflanged).

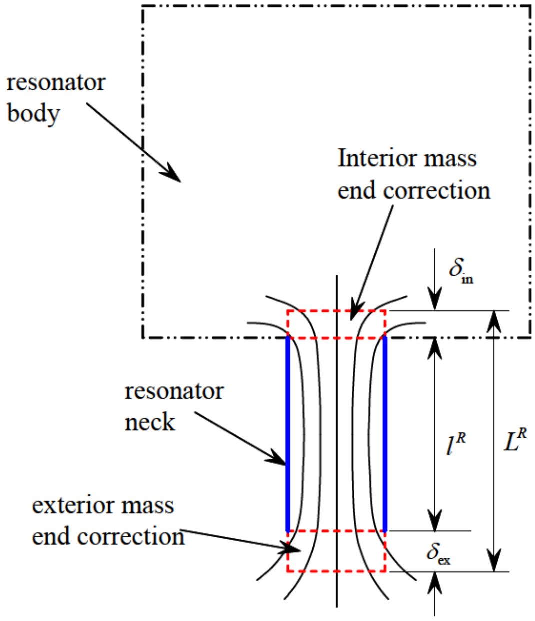

Figure 3: Internal and external end corrections

The strict mathematical demonstration combines the impedance analysis together with complex mathematical functions such as the Struve and Bessel ones, and leads to the following results:

$$\begin{cases} \delta_{in} = 0.85 \cdot r \left(1 - 1.25\frac{r}{R}\right) & flanged \\ \delta_{ex} = 0.61 \cdot r & unflanged \end{cases}$$

For our resonator r is the radius of the neck, while R approximates the radius of the chamber. It follows that $\frac{r}{R} \to 0$, $\delta_{ex} = 0.85 \cdot r$.

By substituting L^R inside equation (4), it follows that:

$$f_{vn}^H = \frac{c}{2\pi}\sqrt{\frac{S}{V_0 \cdot L^R}} = 72.98\,\text{Hz}$$

c) Let us consider that there is also a resistance in the system. Find the resistance value for which the system is critically damped, considering that all the other quantities are kept fixed.

When damping is added, the equation of motion describing the behavior of the system changes, and can be written as:

$$m\ddot{x} + R\dot{x} + Kx = 0 \tag{5}$$

where R is the value of the resistance.
By introducing the damping coefficient $\alpha = \frac{R}{2m}$ and recalling the definition of natural frequency ω_0, equation (5) can be rewritten as:

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0$$

A system is critically damped when $\alpha = \omega_0$. So in order to compute the resistance value R^* for which the system is critically damped we have that:

$$\alpha = \frac{R^*}{2m} \quad \Rightarrow \quad \omega_0 = \frac{R^*}{2m} \quad \Rightarrow \quad R^* = \omega_0 \cdot 2m$$

and for the systems in points a) and b) we obtained respectively:

$$R^* = 0.1650\,\mathrm{kg\,s^{-1}}$$
$$R^*_{VN} = 0.2093\,\mathrm{kg\,s^{-1}}$$

d) Let us now consider that the resistance is $R = 5 \cdot 10^{-4}$ kg/s. Derive the impedance of the system for the case when the virtual elongation is kept into account. Plot the impedance in Matlab.

In order to compute the impedance of our system, we assume it's subjected to an harmonic force $\tilde{F} = Fe^{j\omega t}$. The equation of motion governing the system therefore becomes:

$$m\ddot{\tilde{x}} + R\dot{\tilde{x}} + K\tilde{x} = \tilde{F}$$

Moving to the frequency domain, and solving for $\tilde{X}(\omega)$, we have:

$$-\omega^2 m\tilde{X} + j\omega R\tilde{X} + K\tilde{X} = \tilde{F} \quad \Rightarrow \quad \tilde{X}(\omega) = \frac{\tilde{F}}{K - \omega^2 m + j\omega R}$$

Since the system is linear, its response has the same frequency of the driving force and in general it can be written as $\tilde{x} = Ae^{j\omega t}$. Thus the system's velocity is given by:

$$\tilde{v} = \frac{d\tilde{x}}{dt} = j\omega\tilde{x} \quad \Rightarrow \quad \tilde{V} = j\omega\tilde{X} = \frac{j\omega\tilde{F}}{K - \omega^2 m + j\omega R}$$

The impedance Z of the system is defined as the ratio between the force applied to the system and its velocity:

$$Z = \frac{\tilde{F}}{\tilde{V}} = \frac{K - \omega^2 m + j\omega R}{j\omega} = R + j\left(\omega m - \frac{K}{\omega}\right) = R + jX$$

By computing the impedance Z as function of the frequency the trend below has been obtained.

It's possible to notice that the real part of the impedance, also known as mechanical resistance, is the one due to the presence of the dissipative element, while the imaginary one is also called mechanical reactance and it's null for $\omega = \omega_0$.
The plot above can also be discussed in terms of mass and spring-like behavior:
In particular, for $\omega > \omega_0$, the dominant term is $jm\omega$, therefore we have a mass-like resonance. On the other hand, for $\omega < \omega_0$, the dominant term is $\frac{-jK}{\omega}$, so we have a spring-like behavior.

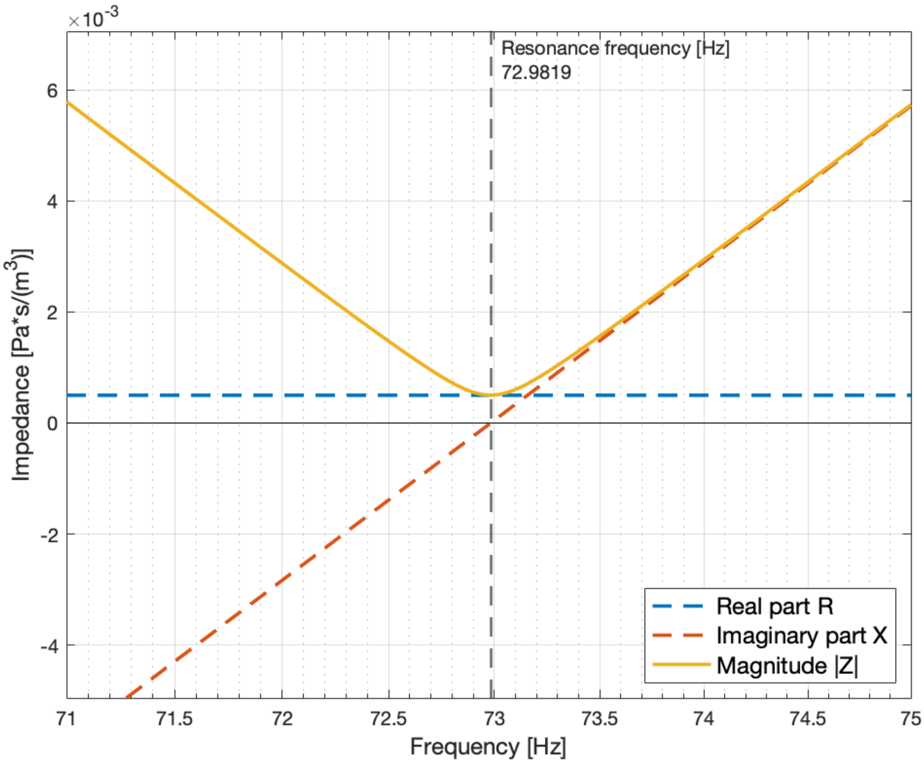

Figure 4: Plot of the impedance

e) Compute the Q factor and time decay factor for the above system.

The Q-factor (quality factor) is a dimensionless parameter that describes how the Helmholtz resonator is damped, comparing the spring and the damping forces:

$$Q = \frac{m\omega_0}{R} = \frac{K}{\omega_0 R} = 209.25$$

The decay time measures the rate at which the amplitude of the resonator decays over time. In particular it indicates the time that the maximum amplitude of the oscillation decreases by a factor $1/e$.
The decay time comes from the exponential behaviour of the oscillations of the system and it is calculated as:

$$\tau = \frac{1}{\alpha} = \frac{2m}{R} = 0.91\,\mathrm{s}$$

We notice that lower is the resonator resistance through the neck, higher is the decay time.

The Q-factor comes from the receptance of the system $\Theta(\omega)$, representing the ratio between the displacement and the force of the system in the frequency domain:

$$\Theta(\omega) = \frac{X(\omega)}{F(\omega)} \propto \frac{1}{\omega^2 - 2j\alpha\omega - \omega_0^2}$$

The Q-factor can be calculated also as the ratio between the central frequency ω_0, where the magnitude of $\Theta(\omega)$ reaches the maximum, and the bandwidth 2α between the frequencies where the magnitude of $\Theta(\omega)$ decreases by 3 dB:

$$Q = \frac{\omega_0}{2\alpha} = \frac{\tau\omega_0}{2} = 209.25$$

The result is the same previously computed.

f) Plot the Q factor and resonance frequency of the system as a function of the resistance R, for a range of R between 0 kg/s and 0.5 kg/s.

The plot in Figure **??** shows the damped resonance frequency in function of the resistance (red line):

$$f_d(R) = \frac{\omega_d(R)}{2\pi} = \frac{\sqrt{\omega_0^2 - \alpha(R)^2}}{2\pi}$$

If the resistance R is small, the damping coefficient α is also small (dashed yellow line) and the resonance frequency can be approximate to $\omega_d \approx \omega_0$ (or $f_d \approx f_0 = 72.9819\ Hz$): the red line representing f_d is close to the constant dashed green line of f_0.
The damped resonance frequency f_d decreases as resistance R increases, until the damping coefficient α has the value of the natural resonance frequency ω_0 (in the graph considering frequency axis in Hz: $\alpha/2\pi = \omega_0/2\pi = f_0$), when the system is critically damped and the frequency goes to 0.

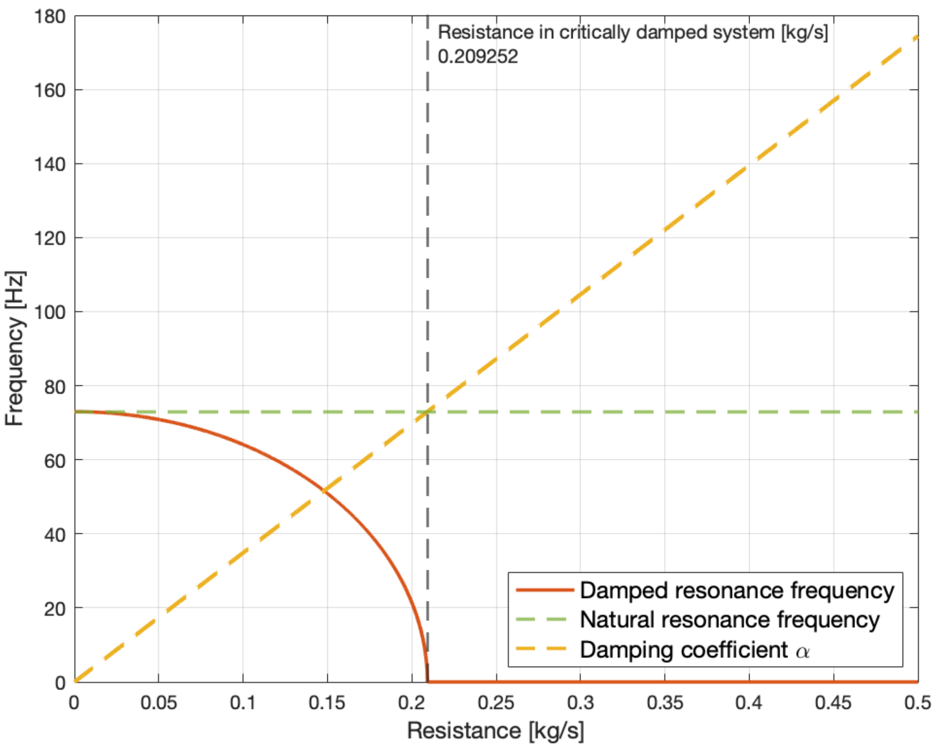

Figure 5: Resonance frequency of the system as function of the resistance

In Figure **??** we see the Q-factor in function of the resistance:

$$Q(R) = \frac{K}{\omega_0 R}$$

If $R \to 0$ (no resistance) the value of Q-factor goes to infinity: it decreases as the resistance increases.
Considering the interval between 0 kg/s and 0.5 kg/s we noticed that Q reaches zero immediately. In order to show its trend more clearly we only plotted it between 0 kg/s and 0.05 kg/s. Indeed only for small values of damping R we have a consistent and useful value of Q, as the resistance considered in the point d and e.

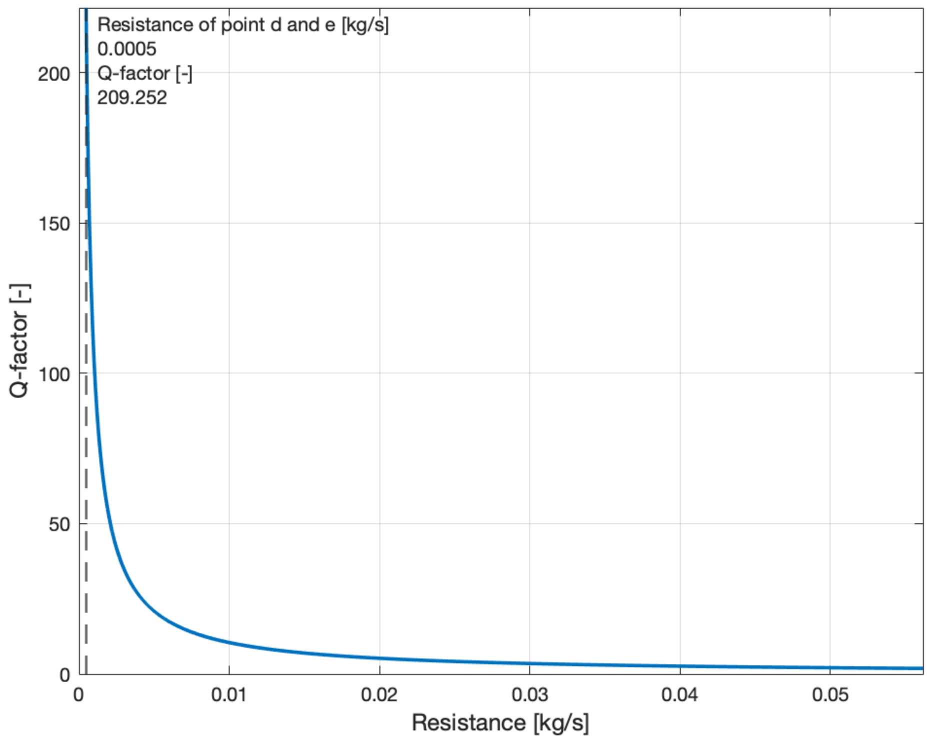

Figure 6: Q-factor as function of the resistance