Politecnico di Milano

Homework #4 - Design of a piano

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Course: Musical Acoustics – Professor: Fabio Antonacci
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1. Soundboard characterization

Consider a rectangular plate of dimensions 1 m × 1.4 m, to be used as a soundboard. The plate is assumed to be clamped.
The thickness of the soundboard is 1 cm, and the material is assumed to be Sitka Spruce and the grain is directed in the shorter direction of the plate (i.e. 1 m), while the other dimension corresponds to the radial direction.

a) Compute in *Comsol* the input impedance of the soundboard as a function of the position (x, y) and of the frequency f. Assume the reference frame to be centered in the lower left corner of the soundboard. Compute the impedance for 10 points in the x direction and 14 points in the y direction (i.e. sampling period 10 cm).

Geometry configuration

In order to properly model the soundboard in *Comsol*, we started from a blank model, then added a 3D Component and modelled the rectangular plate of the desired size and thickness, all within the geometry node.
We disposed the plate horizontally and aligned the grain with the y axis (longitudinal direction), as requested from the assignment. As a consequence, x corresponds to the radial direction, z to the tangential one.
As concerns the material, since it is orthotropic, we started from simple Spruce and then added the appropriate parameters inside the *Linear Elastic Material* node. The values we adopted for the Young's modulus, the Shear modulus and the Poisson's ratio are reported in table **??** where the subscripts L, R and T indicate longitudinal, radial and tangential directions, respectively.

Young's modulus (GPa)		
E_L	E_R	E_T
12.1	1.19	1.428
Shear modulus (GPa)		
G_{LR}	G_{LT}	G_{RT}
0.988	1.021	0.0828
Poisson's ratio		
ν_{LR}	ν_{LT}	ν_{RT}
0.03	0.422	0.019

Table 1: Sitka spruce structural parameters

In the *Solid Mechanics* node we also added fixed constraints (being the soundboard clamped) and Rayleigh damping. For the latter, the mass loss factor has been neglected, while the stiffness one has been set equal to $\beta = 2 \times 10^{-6}$ s.
For the mesh, we chose a free triangular pattern for the top surface, which we then swept along the thickness dimension. This simplification was possible since we were not interested in detailing the response for the clamped edges of the plate, and allowed us to gain a lower level of computational complexity.

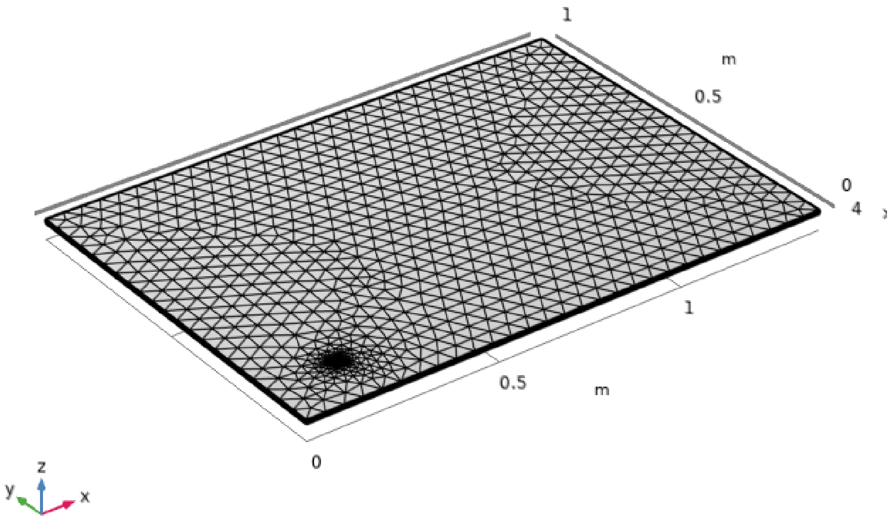


Figure 1: *Comsol* model of the soundboard

Load addition and study setup

Once the basic geometry was completed, we focused on the load application for the impedance calculation. To do so, we modelled a circle of radius $r = 0.5\,\mathrm{cm}$ on the soundboard, which represents the point in which the load is applied. Next, we added a *Boundary Load* to the circle in the *Solid Mechanics* node, and set the z component of the total force equal to $1\,\mathrm{N}$. The main advantage of using a boundary load instead of a point one is the lower impact it has on the overall mesh computation: the point load is more delicate, since it involves a singularity and leads to a much more anti symmetric total mesh, thus making the study more complex.
To compute the impedance for 14 points in the x direction and 10 points in the y direction, we implemented a parametric study so to move the load on a point grid. We carried out a *Frequency Domain Modal* study, to which we added the parametrization of the load application point. The study consists of two steps: an *Eigenfrequency study* step for computing the eigenfrequencies and eigenmodes of the structure and a second step for computing the modal response in the frequency domain. In the mode superposition analysis, the deformation of the structure is represented by a linear combination of the structure's eigenmodes. This means that the frequencies to be studied are limited by the frequencies of the computed eigenmodes and this noticeably reduces the computational time. We looked for 25 eigenfrequencies around 400 Hz, to include the lowest and highest notes required in the second point, then set the range of frequency for the second step from 15 to 800 Hz with a fixed step of 5. To further reduce the computational complexity of our study, we exploited the symmetry of the rectangular plate and confined the parametrization to only a quarter of the plate starting from its bottom-left corner (7x15 points).

The parametric sweep was therefore implemented selecting as parameters *xLoad* and *yLoad*, to be varied as follows:

$$xLoad \in [xOffset; xOffset + X_S \cdot (n/2 - 1)]$$
$$yLoad \in [yOffset; yOffset + Y_S \cdot (m/2 - 1)]$$

where X_S and Y_S are the spatial sampling steps along the x and y axis, n and m are the number of points for the longitudinal and radial direction respectively. *xOffset* and *yOffset* represent the distance of the points grid from the plate's edges. We placed the point grid away from the edges so to compute the study within the central region of the model, which is the most likely to vibrate.

Results

Starting from the parametrization of the load application point, we obtained the impedance value as the ratio between the total load applied over the circular surface S and the z component of the velocity.

$$Z = \int_S \frac{F_Z}{v_z} dS$$

We collected all the values in one table with four columns: xLoad, yLoad, frequency and impedance. We plotted the the results as function the position and frequency using a *Table Surface* plot (*Impedance (x,y)*), whose source is given by the above-mentioned table.

2. String pairing

Consider pairs of strings (approximation wrt reality!) mounted on the soundboard. The pairs are deputed to produce the notes F_2 ($f_0 = 349.23\,\text{Hz}$), A_4 ($f_0 = 440\,\text{Hz}$), C_5 ($f_0 = 523.25\,\text{Hz}$), E_5 ($f_0 = 659.25\,\text{Hz}$), G_5 ($f_0 = 783.99\,\text{Hz}$).

b) Design the shape of the bridge for producing the above notes, so that the input impedance at the force application points is not at a maximum. In the design, assume that the bridge transfer energy to the soundboard at the point where the string is mounted.

To export our previous results for the bridge design phase, we used *Matlab Livelink*: by connecting *Matlab* to the *Comsol* server we were able to directly load the *Comsol* model in *Matlab* and to access the model data and the relative impedance results by means of the `mphopen`, `mphtable` and `mphevaluate` functions.
Once we stored the data, we focused on the strings to be modelled. In order to properly mount them on the soundboard, starting with the frequencies we were given, we considered their closest integer multiple of five, in accordance with the frequency step we chose for our *Comsol* model. After defining the frequency array, we assigned a different colour to each frequency.
The basic idea for bridge design phase was to compute a set of local impedance minima for each desired frequency and to recreate a points grid (representing the soundboard) to visualize the most suitable position for the bridge location.
Since our *Comsol* analysis was circumscribed to one quarter of the soundboard, we had

to translate the impedance computation to the remaining surface. To do so, we used two nested for loops: the outermost iterates among the frequencies, extracting fifteen points of local minima for each frequency and storing them in the Z matrix, together with the corresponding impedance value; the innermost extends the evaluation to the whole plate, exploiting its symmetry axis. Once the analysis covers the total surface, the impedance values for each frequency are stored inside the *soundboard* matrix, together with the points' coordinates, the correspondent mobility and the RGB indices for each chosen color.
To simultaneously plot the impedance minima for all the frequencies avoiding the overlapping between different points, we used a 3D scatter plot, specifying the jitter (spacing of points) along the x and y directions. Once we were able to map the impedance minima as function of frequency among the whole surface, we arbitrary chose the bridge final shape and position, combining our results with the well-known structure of a real piano. Due to the rectangular shape of the soundboard, we took as reference an upright piano. We disposed the bridge considering the frequency range we're interested in (which extends for about the two central octaves), and disposed the strings from left to right, following an ascending frequency order. The images below show the selection of points we chose from the grid and the corresponding 3D model.

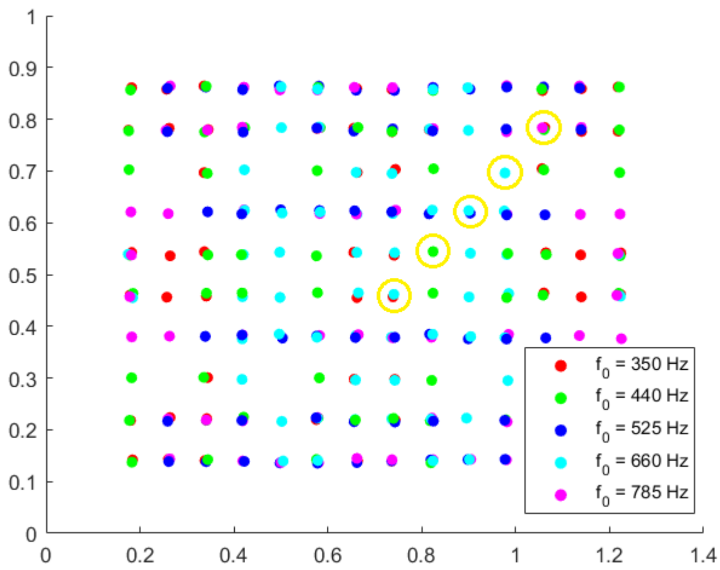


Figure 2: Bridge points grid location

It is clear that the model results oversimplified and barely reconnects to a real piano soundboard. This is mostly due to the fact that the chosen points depend on the discretization and therefore lead to an irregular shape for the bridge, not sinuous as a real one. This limitation could be bypassed considering an higher number of points, thus noticeably increasing the computational complexity of the study.We also plotted an impedance 2D pattern of the considered portion of the soundboard for each frequency value, and its behavior as function of the frequency, in correspondence of the five points we selected (fig.**??**-**??**).

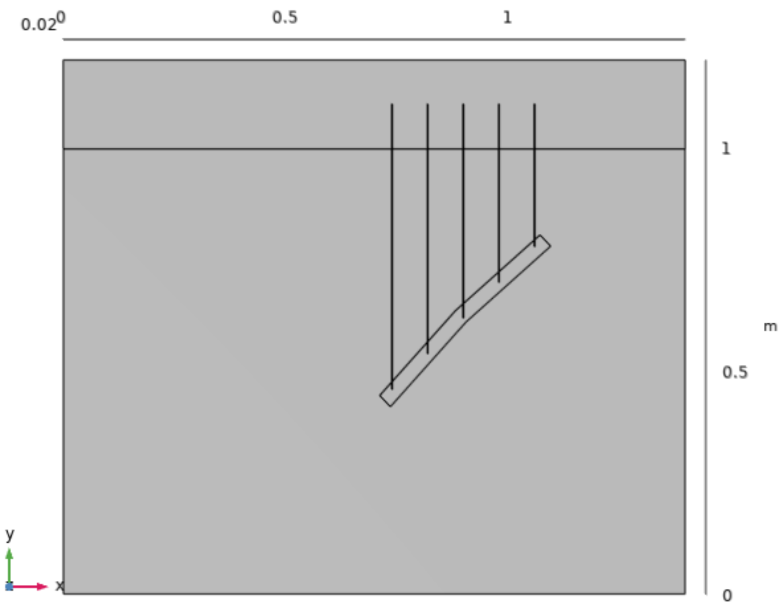
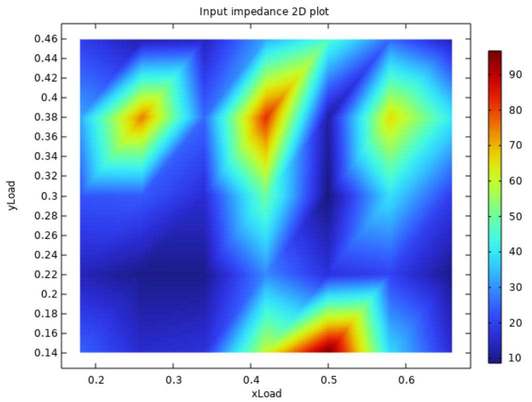


Figure 3: *Comsol* model of the bridge (xy plane)



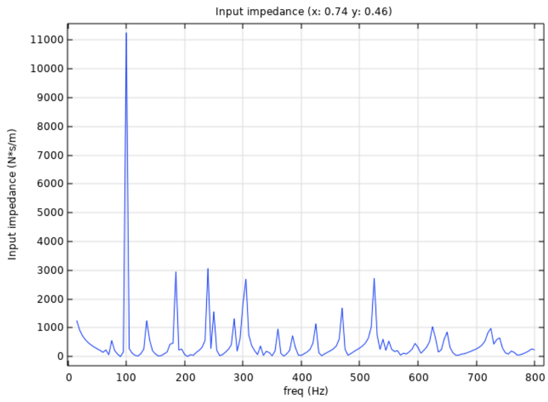
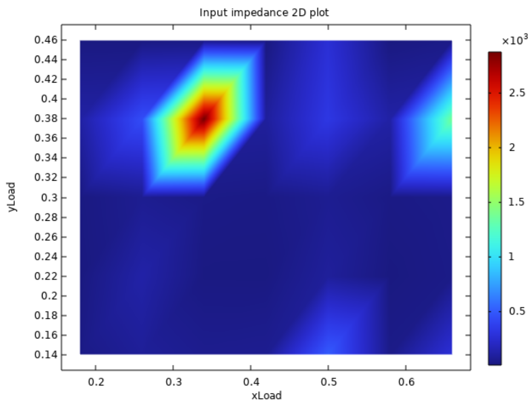


Figure 4: x: 0.74, y: 0.46, f=350 Hz



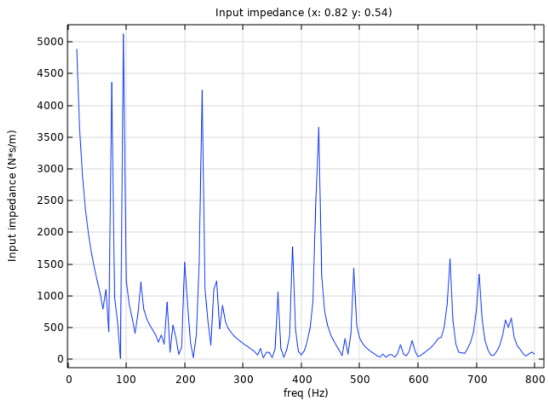


Figure 5: x: 0.82, y: 0.54, f=440 Hz

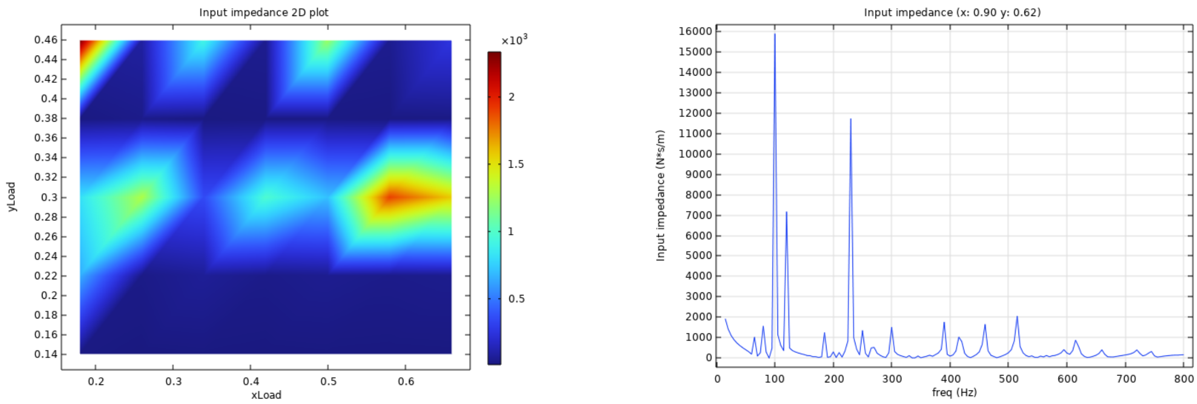


Figure 6: x: 0.90, y: 0.62, f=525 Hz

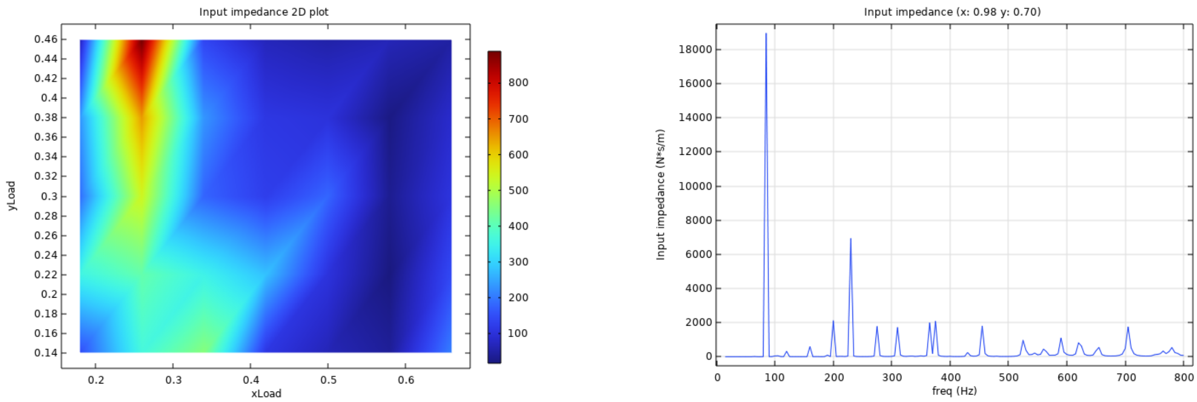


Figure 7: x: 0.98, y: 0.70, f=660 Hz

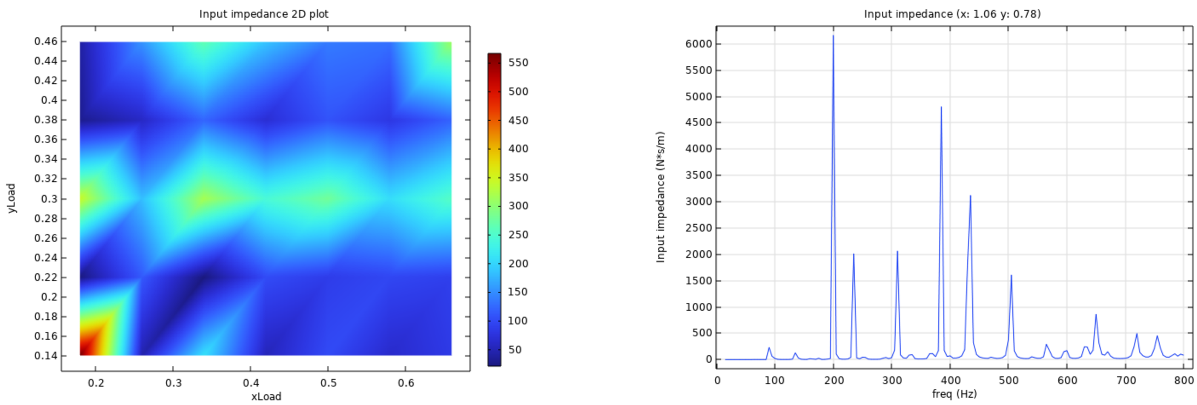


Figure 8: x: 1.06, y: 0.78, f=785 Hz

c) For all the strings assume that the weight per unit length is $10.8\,\mathrm{g/m}$. One of the strings in the pair is tuned to produce f_0 (i.e. $\omega_1 = 2\pi f_0$), while the second is detuned and $\omega_1 = 2\pi f_0(1 + 2\varepsilon)$. Compute the eigenfrequencies of the two strings in the pairs.

For the computation of the eigenfrequencies of each pair of strings, we made the following assumptions:

- The system is linear so that all the modes can be considered independently from each other;
- The strings of each pair have equal length, mass and cross-sectional area and only differ for the mistuning ϵ;
- The strings can only move perpendicularly to the soundboard, meaning that the lateral and longitudinal components of the motion have been neglected;
- The hammer action is neglected, the strings are freely vibrating;
- The dissipation associated to other mechanisms other than the bridge motion has been neglected;
- The two strings of each pair are mounted on the same position on the bridge;
- All strings are assumed to be disposed between the agraffe (to which they're rigidly fixed) and the bridge;

We started from the calculation of the strings' characteristic impedance Z_0, given by:

$$Z_0 = \sqrt{T \cdot \rho \cdot S} \tag{1}$$

where T is the tension of the strings and $\rho \cdot S = \mu_s = 10.8\,\mathrm{g/m}$. As regards the value of T, we considered the average tension range for an upright piano ($750\,\mathrm{N} < T < 900\,\mathrm{N}$) and consequently imposed $T = 900\,\mathrm{N}$. Having used the same tension for all the pairs is reasonable, since the corresponding notes lay in the central octaves of the keyboard, where the tension is constant.
Then, we derived the normalized admittance with respect to the characteristic impedance of the strings χ, starting from the values of Y_B previously computed, as:

$$Y_B = \frac{\pi}{jZ_0}\chi \Rightarrow \chi = j\frac{Z_0 Y_B}{\pi}$$

Finally, to compute the eigenfrequencies starting from the mistuning ϵ, we used the following equation, which derives from the description of the strings coupling, expressed in terms of forces.

$$a^2 - 2(\chi + \epsilon)a + 2\epsilon\chi = 0 \tag{2}$$

$a_\pm$ can be therefore written as:

$$a_\pm = \chi + \epsilon + \mu \tag{3}$$

with $\mu = (\epsilon^2 + \chi^2)^{1/2}$.
We considered a mistuning ϵ raging from -0.2 up to 0.2, so to avoid the presence of beats and to optimize the mistuning effect.
The charts below (fig. **??**) show the roots' real and imaginary part as function of the mistuning, for each pair of strings. At the increasing of $|\epsilon|$, the eigenfrequencies of the strings are close to those obtained in the uncoupled case. As ϵ tends to zero (unison), the bridge coupling is important and diverts the eigenfrequencies one from the other.
To determine the exact eigenfrequencies, we had to fix the ϵ value. After several attempts, we set $\epsilon = 0.0046$ for all the strings except for E_5, for which we set $\epsilon = 0.0017$: these values are a reasonable trade off, since they provide a low coupling effect together with

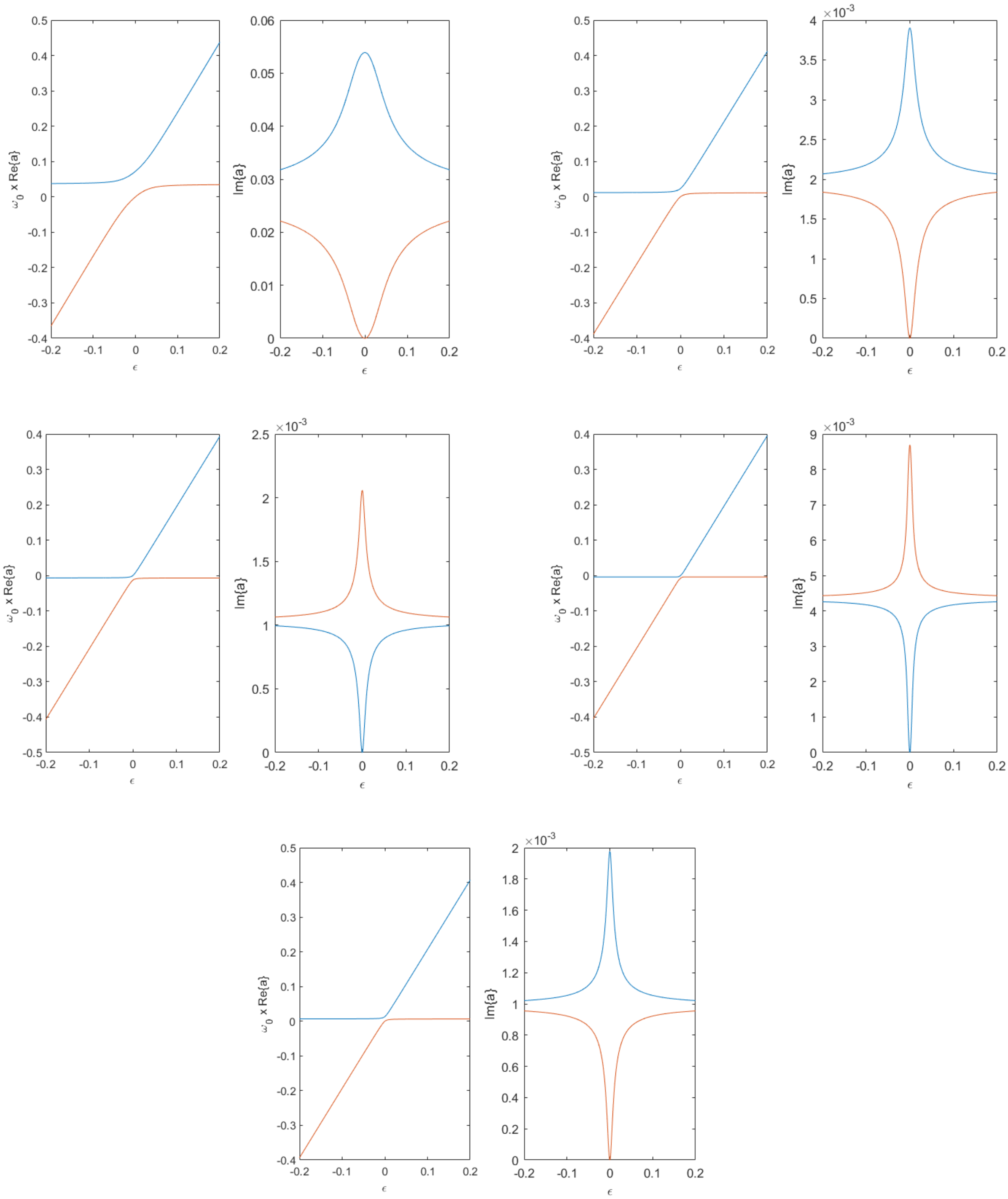


Figure 11: Eigenvalues of the system of strings coupled to the bridge

a reasonable decay time. We also tried with lower ϵ values, thus obtaining worse results. We computed the eigenfrequencies as:

$$f_{\pm} = f_0(1 + Re\{a_{\pm}\})$$

The results are computed in table **??**.
We also computed the strings' length, hypothesizing a vertical disposition of the strings together with the presence of a pin block overlapped to the soundboard, of length equal to $L = 20\,\mathrm{cm}$, to which the pins are fixed. The length of each string was therefore given

by the sum of two contributes, the first derived from the tension, the second given by $L/2$ (the portion of the string which lays on the pin block before wrapping around the pin).

f_0 [Hz]	f_+ [Hz]	f_- [Hz]
349.23	376.16	350.78
440	452.75	441.65
523.25	526.40	517.93
659.25	660.48	655.05
783.99	799.20	786.49

Table 2: Eigenfrequencies of each pair of strings

d) Compute the decay times for the two eigenfrequencies for all the pairs.

The temporal evolution of the system (meaning the decay time for each pair of strings) can be evaluated considering the real part of the expression of the bridge velocity. The latter is given by:

$$V_B = \frac{2\pi F_0 \chi}{\mu Z_0} e^{j(\epsilon+\chi)\omega_0 t}[\mu \cos \mu\omega_0 t + j\chi \sin \mu\omega_0 t]e^{j\omega_0 t}$$

Fig. **??** shows the absolute value of the bridge velocity, expressed in dB, as function of time. It's possible to notice that the above mentioned conditions for ϵ grant the double decay phenomenon (the presence of two successive parts with different time constants: shorter for the first section, longer for the second one).
We computed the decay time T_{60} for each frequency, which is defined as the time for the sound to decay by 60 dB. To do so, we hypothesized a linear decay, computed the slope of each curve and derived T_{60}, taking as reference $V_{B_1} = 0$ dB and $V_{B_2} = -60$ dB. Due to the presence of the double decay phenomenon, we found two values (${}_AT_{60}$ and ${}_BT_{60}$) for each frequency, corresponding to the decay times associated with the first and second decay rate respectively. The results are listed in table **??**.

note	${}_AT_{60}$ [s]	${}_BT_{60}$ [s]
F_4	0.058	22.500
A_4	0.563	19.056
C_5	0.765	11.654
E_5^*	0.188	9.309
G_5	0.477	8.149

Table 3: Decay rates ${}_AT_{60}$ and ${}_BT_{60}$

Comparing the values we got with the ones of a real upright piano, we can observe a good coherence of our results with respect to reality. The decay time ranges between around 0.06 s - 0.8 s for ${}_AT_{60}$ and 1.5 s - 22.5 s for ${}_BT_{60}$, progressively increasing at the decreasing of the frequency.

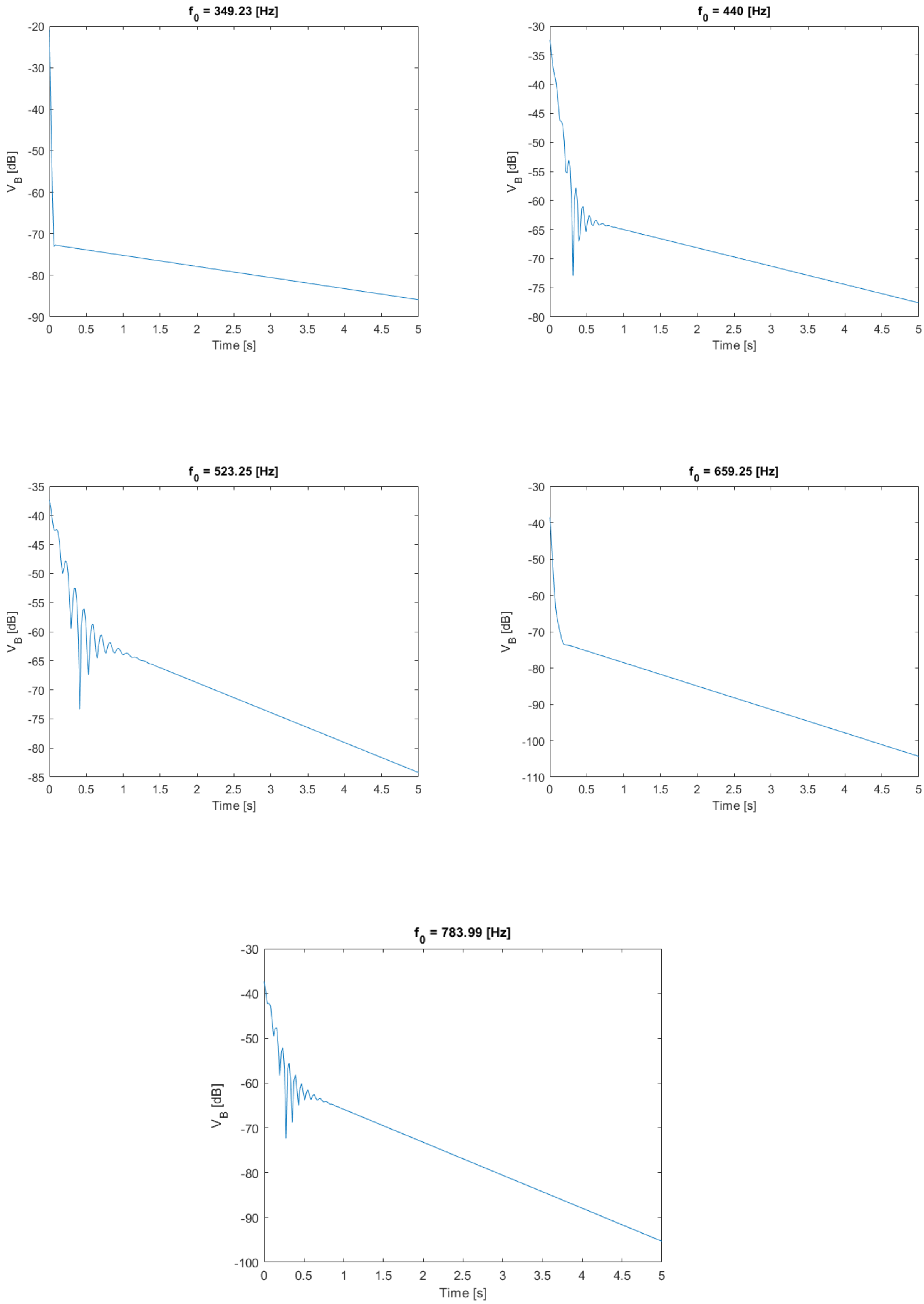


Figure 14: Temporal envelope of the bridge velocity V_B