Politecnico di Milano

Homework #3 - Horn design

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Course: Musical Acoustics – Professor: Fabio Antonacci
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Part 1: design of the exponential section

It is given an exponential horn. The equivalent radius (see diagram below) is $a = a_0 e^{mx}$, with $a_0 = 0.008\,\mathrm{m}$ and $m = 4.2\,\mathrm{m}^{-1}$. The length of the horn is $L = 0.35\,\mathrm{m}$. We intend to approximate the exponential horn as a concatenation of short sections of conical horns. In particular, we assume that all the sections are of the same length δ.

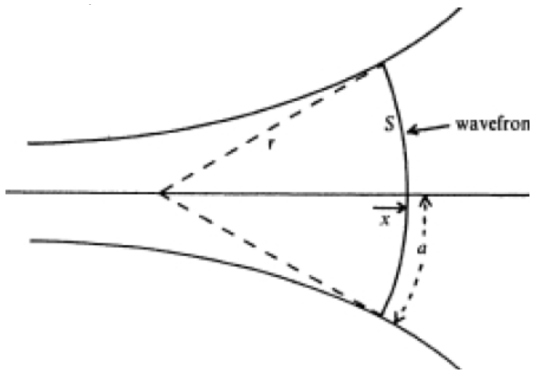

Figure 1: Exponential horn's cross section

We aim at computing, in Matlab, the input impedance of the concatenation and compare it, in the frequency range [0,2 kHz], with the analytical expression of the impedance of the exponential horn. In a first stage, we neglect the impedance radiation at the mouth of the horn. We measure the similarity between the impedances $Z_1(\omega)$ and $Z_2(\omega)$, which are the input impedance of the approximated and analytical model, respectively, through two metrics:

- $e_1 = \frac{1}{(\omega_{\max}-\omega_{\min})}\int_{\omega_{\min}}^{\omega_{\max}} |Z_1(\omega) - Z_2(\omega)|^2 d\omega$, which evaluates the mean squared error between the two impedances;
- $e_2 = \sum_1^5 \min|\arg\max_\omega Re(Z_1(\omega)) - \arg\max_\omega Re(Z_2(\omega))|$, which evaluates the difference in frequency between the real part of the first five maxima of the two impedances.

a) Evaluate the error e_1 as a function of the length δ of the conical sections and plot it in Matlab.
b) Evaluate the error e_2 as a function of the length δ of the conical sections and plot it in Matlab.

The input impedance of a conical horn is given by

$$Z_{IN} = \frac{\rho c}{S_1}\left[\frac{jZ_L[sin(kL-\theta_2)/sin\theta_2] + (\rho c/S_2)sin(kL)}{Z_L[sin(kL+\theta_1-\theta_2)/sin\theta_1 sin\theta_2] - (j\rho c/S_2)[sin(kL+\theta_1)/sin\theta_1]}\right] \tag{1}$$

where Z_L is the load impedance, meaning the the impedance radiation at the mouth of the horn; S_1 and S_2 are the areas of the throat and the mouth of the horn respectively; L is the length of the horn, and $\theta_{1/2} = tan^{-1}(kx_{1/2})$.

Under the assumption of planar wave-fronts, the cross-sectional areas of the mouth and throat of the horn can be expressed as $S_{1/2} = \pi a_{1/2}^2$, where $a_{1/2} = a_0 e^{mx_{1/2}}$ are the equivalent radius of the horn, computed in correspondence of x_1 and x_2.

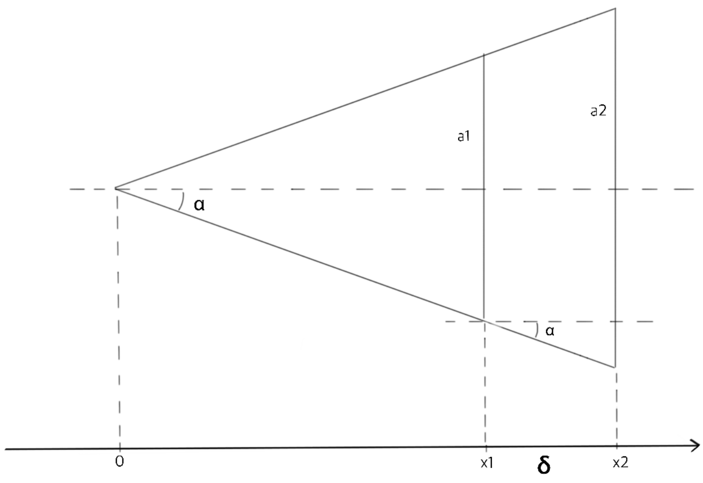

Figure 2: Section of the conical horn

In order to express the coordinates x_1 and x_2 as function of δ, we exploited the following trigonometrical relations:

$$tan(\alpha) = \frac{a_2 - a_1}{\delta} \tag{2}$$

$$x_1 = \frac{a_1}{tan(\alpha)} = \frac{a_1}{a_2 - a_1} \cdot \delta \tag{3}$$

$$x_2 = x_1 + \delta \tag{4}$$

where α is the flare angle of the conical horn (fig. **??**).

From these considerations, to compute the input impedance of the approximated model $Z_1(\omega)$, we started from the conical section at the right end of the horn, whose load impedance is equal to zero by assumption, and gradually moved towards the left.
In the approximated model, the input impedance of the n-th section acts as load impedance for the $(n-1)$-th one, until reaching the throat of the horn, where the impedance value is affected by the presence of all the other sections. The value of the input impedance of each section has been computed iterating along the whole horn's length the following Matlab function: `function [Z] = approximate_exponential_horn(l_exp_horn, a_0, m, delta, k, Z_l, rho, c)` which takes as input the horn's length, the parameters of the equivalent radius, the length of the sections, ρ, c and the load impedance (updated at each interaction, starting from `Z_l=0`).

In order to evaluate the errors e_1 and e_2, we divided the horn's length into j equivalent sections and we iterated the above written function. In particular we computed the radius of the two surfaces of each conical section as

$$a_1 = a_0 e^{m\delta(j-1)}$$

$$a_2 = a_0 e^{m\delta j}$$

with j ranging from 20 to 1.

The input impedance of a finite exponential horn is given by

$$Z_{IN} = \frac{\rho c}{S_1} \left[\frac{Z_L cos(bL + \theta) + j(\rho c/S_2) sin(bL)}{j Z_L sin bL + (\rho c/S_2) cos(bL - \theta)} \right] \tag{5}$$

where b and $theta$ are defined as

$$b^2 = k^2 - m^2$$
$$\theta = tan^{-1}(m/b)$$

Since once again we're neglecting the impedance radiation at the mouth of the horn, equation **??** can be simplified as

$$Z_{IN} = \frac{\rho c}{S_1}\left[\frac{jsin(bL)}{cos(bL-\theta)}\right] = Z_2(\omega)$$

The figures below show the errors e_1 and e_2 as function of the length δ of the conical sections. In order to better analyze the results, considering the impedances' high order of magnitude, we plotted e_1 adopting a dB scale.

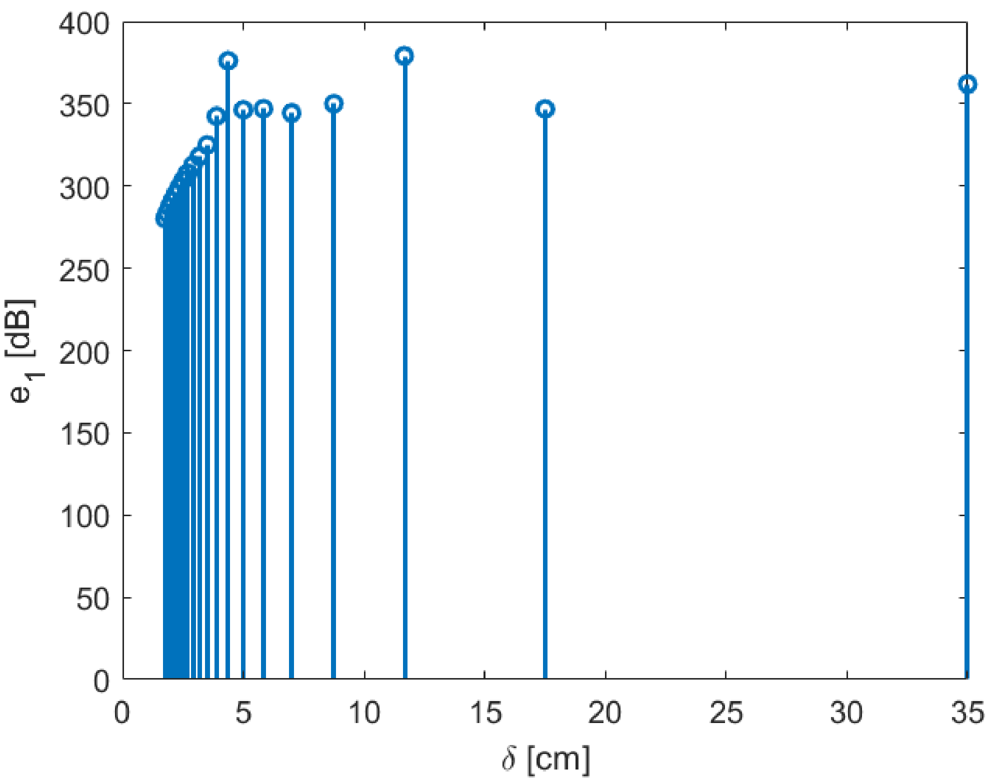

Figure 3: Error function e_1

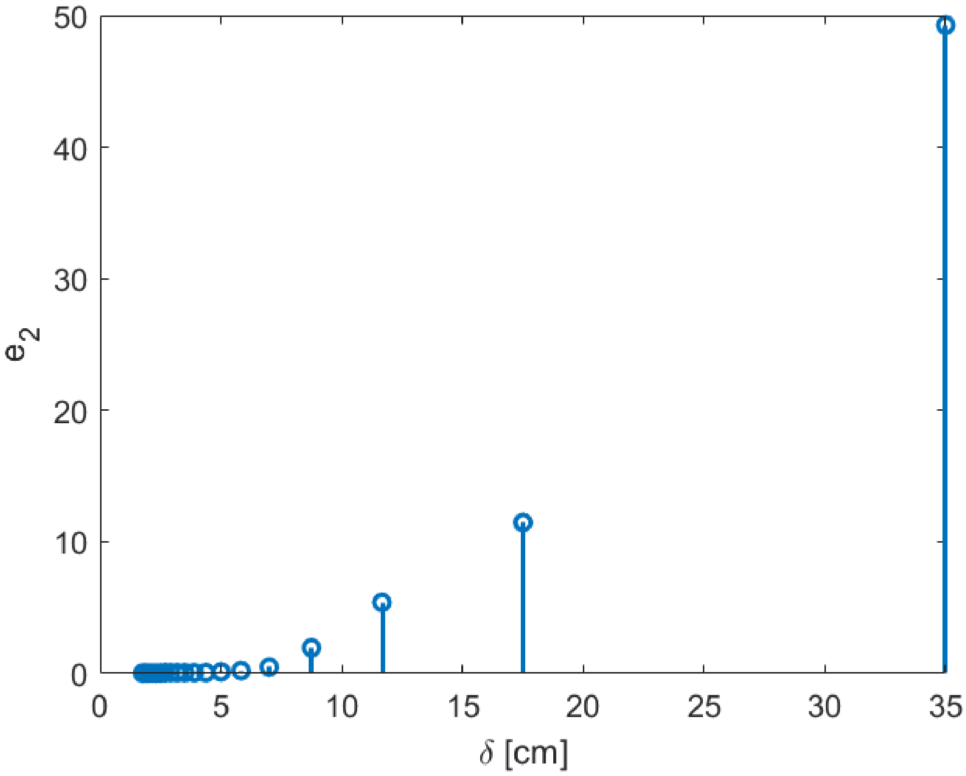

Figure 4: Error function e_2

By looking at figure **??**, we can see a significant decrease in the e_1 values by simply varying the value of δ. This means that the section's length deeply influences the values of e_1, i.e. the quality of the approximated model. In particular, we observed a significant increase of the error for δ higher than 5 cm.
Another parameter which influences both the behavior of e_1 and e_2 is the chosen frequency resolution N: the value of δ^* which minimizes e_2 becomes smaller as N increases.
In our analysis, we chose $N = 2^{16}$ and obtained $\delta^* = 2.5$ cm.

c) Compute the impedance of the approximated model when the radiation load is kept into account and evaluate the error brought by neglecting it in terms of the metric e_1.

The radiation load in the approximated horn is modeled through an additional section whose impedance is $Z_L(\omega) = Z_{L0}(\omega)\frac{S_p}{S_s}$, where $Z_{L0}(\omega)$ is the impedance of an unflanged cylindrical pipe of radius a, given by $Z_{L0}(\omega) = 0.25\frac{\omega^2\rho}{\pi c} + 0.61j\frac{\rho\omega}{\pi a}$, S_p is the cross-sectional area of the cylinder, and S_s the spherical wave front area at the open end of the cone. It can be approximated by $S_s = \frac{2S_p}{1+cos\theta}$, where θ is the flaring angle of the last conical section.

To include it, we used the previously described Matlab function, giving as input the load impedance of an unflanged cylindrical pipe. In particular, to define the flare angle θ of the last conical section, we applied the relations of equations **??**-**??**, with $a_1 = a_0e^{mL}$ and $a_2 = a_0e^{m(L-\delta^*)}$.

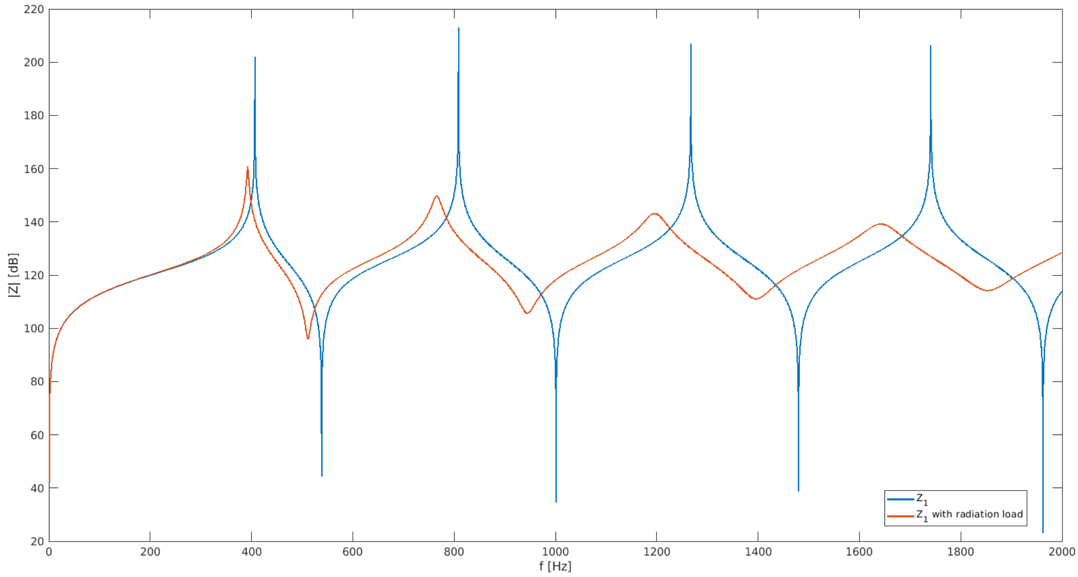

Figure 5: Impedance of the approximated model with and without the impedance load

From a physical point of view, the radiation impedance represents the impedance of the medium with which the horn interacts; a different modelization of Z_L allows to determine the termination's influence on the resonances aroused. For example, by imposing $Z_L = 0$ we are imposing a total negative reflection in correspondence of the mouth of the horn. Fig.**??** displays how the resonance frequencies of the horn (corresponding to the impedance peaks) show a strong dependence on the radiation impedance, both in height and location. The peaks are smoother and shifted towards the low end: this discrepancy becomes more evident for high frequencies, since the contribution given by the unflanged cylindrical pipe increases as the frequency rises.

The e_1 function, measured comparing the impedance of the approximated horns with and without the presence of the radiation impedance, is equal to 337 dB. As stated before, the MSE evaluation is based on the comparison between multiple measurements and this makes it difficult to singly evaluate this result. Despite of this, by comparing the e_1 value with the ones computed in $1.a$, we can say that radiation load has a non-negligible role in the determination of the impedance peaks. This could appear inconsistent with respect to what happens in case of a single conical horn (where the radiation impedance can be safely neglected, due to the low impedance mismatch between the air column inside and outside the horn) but can be explained recalling that our approximated model is much more complex and consists of multiple cones of smaller length, whose impedance adaptation is significantly diminished.

Part 2: design of the compound horn

Consider now a compound horn, composed by a cylindrical pipe, whose length is 0.85 m, followed by the exponential horn. For the exponential horn use the impedance obtained before, including the radiation load.

d) Compute the impedance of the compound horn and list in a table the frequencies of the first ten maxima of the impedance.

The input impedance of the compound horn under consideration (fig.??) corresponds to the one of a finite cylindrical pipe, whose load impedance Z_L equals the input impedance of the exponential horn analyzed up to now. The relation is:

$$Z_c = Z_0 \left[\frac{Z_L cos(kL) + jZ_0 sin(kL)}{jZ_L sin(kL) + Z_0 cos(kL)} \right] \tag{6}$$

where $Z_0 = \frac{\rho c}{S_1}$.

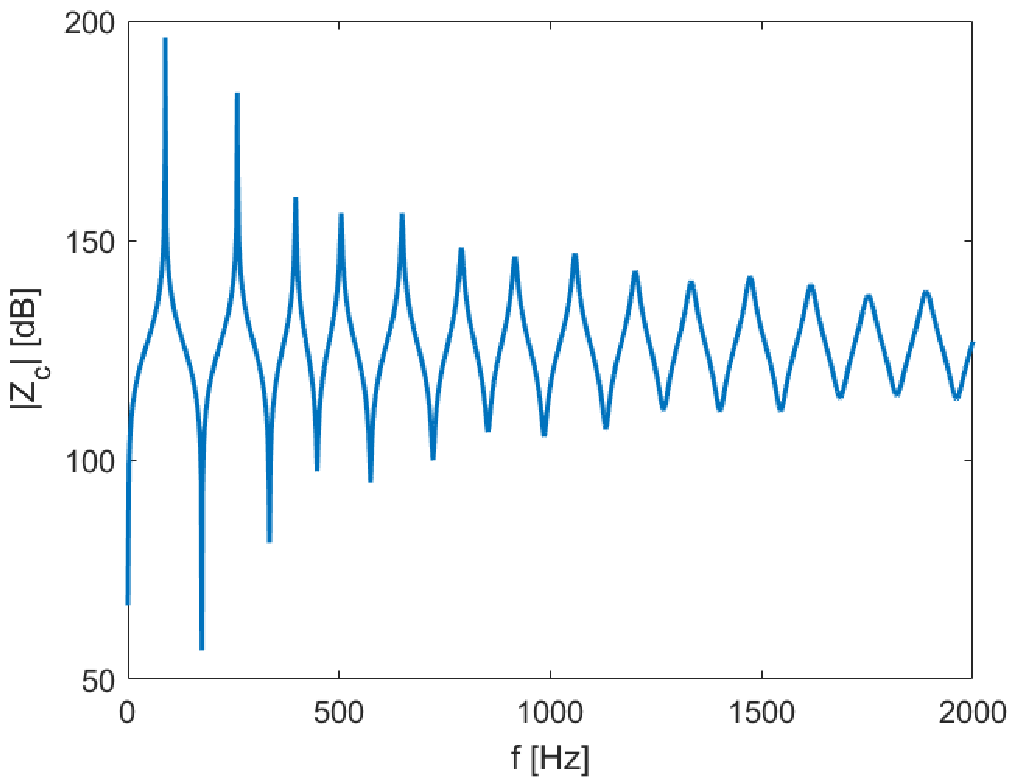

Figure 6: Impedance of the compound horn

Table 1 shows the frequencies of the first ten maxima of the impedance Z_C, found using the Matlab function `findpeaks`.

Frequency of the first 10 impedance peaks [Hz]
88.287
259.064
396.973
504.822
648.956
789.429
916.107
1057.922
1201.202
1333.405

Table 1: Frequencies of the first ten maxima of the compound horn's impedance

e) Evaluate the inharmonicity of the succession of impedance maxima for the different setups: exponential horn, exponential horn with impedance radiation and compound horn with impedance radiation.

We repeated the procedure described in point 2.*c* and applied it to find the impedance peaks of the impedance functions of the three desired configuration. The values of the impedance maxima for the exponential horn with and without the impedance radiation are reported in the first column of Table 2 and 3.
We evaluated the inharmonicity by quantifying the percentage deviation of the obtained peaks from the nearest whole multiples of the fundamental frequency (the one corresponding to the lowest peak).
The values of the correspondent whole multiples are listed in the second columns, while the third ones report the percentage inharmonicity.
As regards the compound horn not all the harmonics are present, therefore we computed the inharmonicity taking into account the nearest harmonic peak and highlighted the correspondent harmonics' number in an additional column (Table 4). By looking at the results we noticed that, for the compound horn, the majority of the peaks correspond to odd harmonics while all the harmonics are present for the other two horns. From a musical point of view, if we consider as approximation a fixed Just Noticeable Difference (JND) of 10 cents across all frequencies, we can say that for all three horns the inharmonicity will be perceived by the listener.

Impedance peak [Hz]	**Nearest harmonics [Hz]**	**Inharmonicity %**	**Inharmonicity [cst]**
405.975			
807.831	811.950	-0.51%	-9
1266.479	1217.925	+3.99%	+68
1740.234	1623.900	+7.16%	+120

Table 2: Impedance's maxima and inharmonicity of the exponential horn without impedance radiation

Impedance peak [Hz]	**Nearest harmonics [Hz]**	**Inharmonicity %**	**Inharmonicity [cst]**
391.876			
765.015	783.752	-2.39%	-42
1195.374	1175.628	+1.68%	+29
1642.303	1567.504	+4.77%	+81

Table 3: Impedance's maxima and inharmonicity of the exponential horn with impedance radiation

Impedance peak [Hz]	**Harmonics [Hz]**	**Harmonics' number**	**Inharmonicity %**	**Inharmonicity [cst]**
88.287				
259.064	264.861	3	-2.19%	-38
396.973	441.435	5	-10.07%	-184
504.822	529.722	6	-4.70%	-83
648.956	618.009	7	+5.01%	+85
789.429	794.583	9	-0.65%	-11
916.107	971.157	11	-5.67%	-101
1057.922	1059.444	12	-0.14%	-2
1201.202	1236.018	14	-2.82%	-49
1333.405	1324.305	15	+0.69%	+12

Table 4: Impedance's maxima and inharmonicity of the compound horn with impedance radiation