CPSC 406: Homework 1

1. **Backsolve** Here, we will explore the computational complexity of solving the system

$$Rx = b, \qquad R \in \mathbf{R}^{n\times n}$$

when R is either upper triangular ($R_{ij} = 0$ whenever $i > j$) or lower triangular ($R_{ij} = 0$ whenever $i < j$). If R were fully dense, then solving this system takes $O(n^3)$ flops. We will show that when R is upper or lower triangular, this system takes $O(n^2)$ flops. Assume that the diagonal elements $|R_{ii}| > \epsilon$ for ϵ suitably large in all cases.

   (a) Consider R lower triangular, e.g. we solve the system

$$\begin{bmatrix} R_{11} & 0 & \cdots & 0 \\ R_{21} & R_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{n,1} & R_{n,2} & \cdots & R_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

   Show how to find x_1. (This should take $O(1)$ flops.) Given $x_1, ..., x_i$, show how to find x_{i+1}. (This should take $O(i)$ flops.) Putting it all together, we get

$$O(1) + O(2) + \cdots + O(n-1) + O(n) = O(n^2) \text{ flops.}$$

   Ans.

$$x_1 = b_1/R_{11}, \quad x_{i+1} = \frac{b_{i+1} - \sum_{k=1}^{i} R_{i+1,k}x_k}{R_{i+1,i+1}}.$$

   (b) Now consider R upper triangular, e.g. we solve the system

$$\begin{bmatrix} R_{11} & \cdots & R_{1,n-1} & R_{1,n} \\ 0 & \cdots & R_{2,n-1} & R_{2,n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & R_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

   Show how to find x_n. (This should take $O(1)$ flops.) Given $x_{i+1}, ..., x_n$, show how to find x_i. (This should take $O(n-i)$ flops.) Putting it all together, we get

$$O(n) + O(n-1) + \cdots + O(2) + O(1) = O(n^2) \text{ flops.}$$

   Ans.

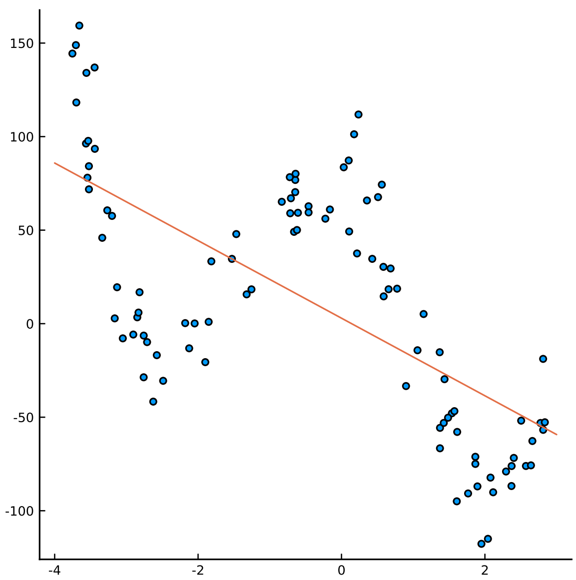
$$x_n = b_n/R_{nn}, \quad x_i = \frac{b_i - \sum_{k=i+1}^{n} R_{i,k}x_k}{R_{i,i}}.$$

2. **Linear data fit** Download data. Fit the best line

$$f(z) = x_1 + x_2 z$$

to the points $(z_1, y_1), ..., (z_n, y_n)$; that is, find the best approximation of the line $f(z)$ to y in the 2-norm sense. Plot the fit, and report $\|r\|_2$ the norm of the fit residual.

Ans.



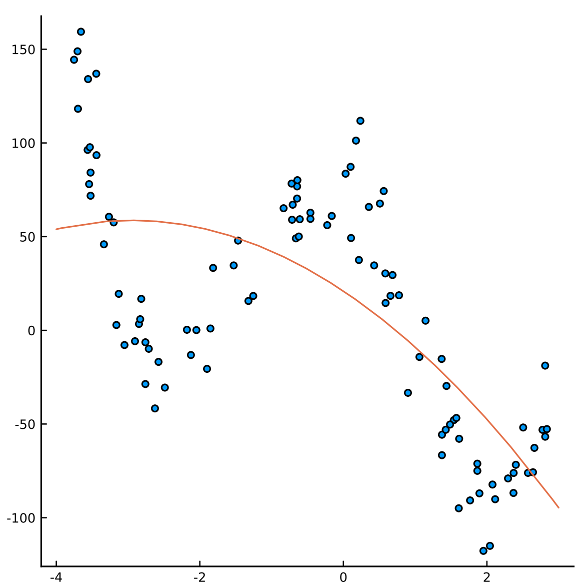
residual $\|b - Ax\|_2 = 498.56$

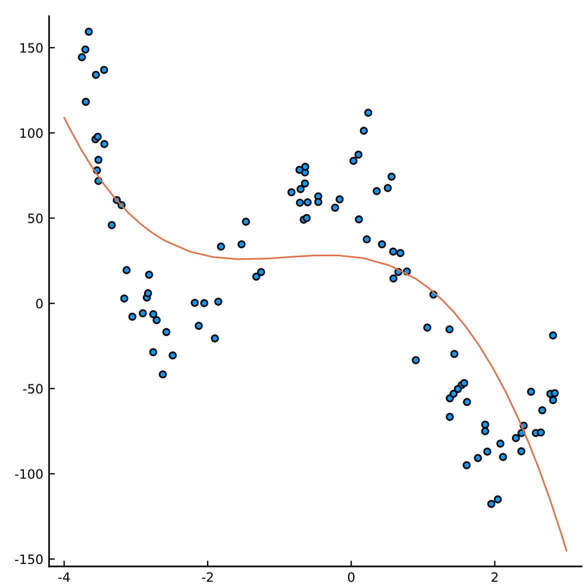
3. **Polynomial data fit** Using the same data as above, fit the best order-d polynomial to the points $(z_1, y_1), ..., (z_n, y_n)$, for $d = 2, 3, 4, 5$. That is, find $x_1, ..., x_{d+1}$ such that

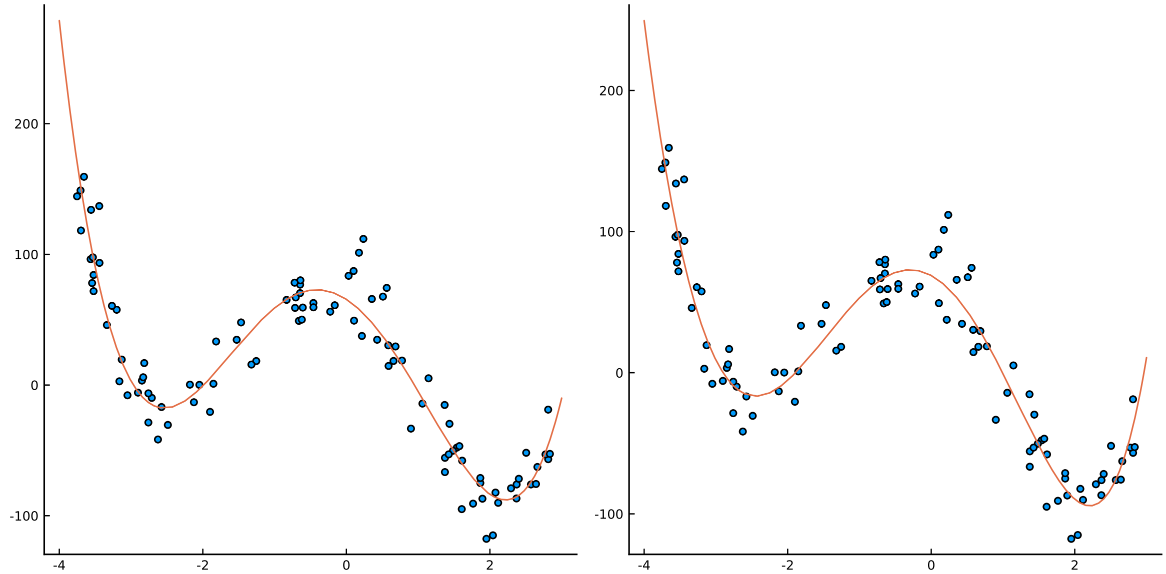
$$f(z) = x_1 + x_2 z + x_3 z^2 + \cdots + x_{d+1} z^d$$

best approximates the data in the 2-norm sense (minimizing $\sum_i (f(z_i) - y_i)^2$). Plot the fit, and report $\|r\|_2$ the norm of the fit residual. About how many degrees is needed for a reasonable fit?

Ans.







residuals:

- $d = 2$, $\|b - Ax\|_2 = 473.93$
- $d = 3$, $\|b - Ax\|_2 = 439.14$
- $d = 4$, $\|b - Ax\|_2 = 194.79$
- $d = 5$, $\|b - Ax\|_2 = 189.05$

You see a sharp improvement at $d = 4$, but $d = 5$ doesn't really add much, so $d = 4$ is needed for a reasonable fit.

4. Consider the full rank, underdetermined but consistent linear system $Ax = b$, where A is $m \times n$ with $m < n$.

   (a) Show how to use the QR factorization to obtain a soution of this system.
   Ans. There are two ways we can factor A. If we factor as $A = QR$, we get something like

$$\underbrace{\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \end{bmatrix}}_{R}$$

and we can solve

$$x = R^{-1}Q^T b.$$

The key advantage is that Q is only $m \times m$, and R is the same storage as A, so the only increase in storage is m^2. But, inverting R is tricky, as it is not exactly triangular.
We can also factor $A^T = QR$ to get something like

$$\underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{A^T} = \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}}_{R}.$$

Overall, we will solve this system in two steps:

$$R^T z = b, \quad Q^T x = z.$$

The first step is now much easier. When R is wide, it is tricky to figure out how to invert it. But when R is square, it is easy to "invert" through backsolving.

The second step is tricky, because Q^T is wide, and not easily left invertible. In fact, there are many solutions for x. One possible solution is $x = QQ^T z$, which is the least squares solution, and perhaps easiest to compute in this context.

In this regime, the solve system $z = R^{-T}b$ takes $O(m^2)$ flops, and $x = Qz$ requires $O(mn)$ flops, for a total of $O(mn + m^2)$ flops for the solve, and an extra $O(nm^2)$ flops for the original QR factorization.

(b) The following script can be used to generate random matrices in Julia, given dimensions m and n:

```
A = randn(m,n);
x = randn(n,1);
b = A*x;
```

Write Julia code for solving for x using the procedure outlined in the previous part of the question. Record the runtime using the Julia call `time`. (Make sure you are not running anything else or it will interfere with the timing results.) Record the runtimes for matrices of sizes $(m, n) = (10, 20)$, $(100, 200)$, $(100, 2000)$, $(100, 20000)$, and $(100, 200000)$. Compare the runtimes against finding x using `x = A\b`.

Ans. Here's some simple code:

```
~, t1 = @timed begin
F = qr(A')
x1 = F.Q*(F.R'\b)
end

~, t2 = @timed begin
x2 = A\b
end
```

On my personal laptop, in seconds

(m, n)	$(10, 20)$	$(100, 200)$	$(100, 2000)$	$(100, 20000)$	$(100, 200000)$
QR	0.00005	0.0012	0.0076	0.0654	0.7892
$A\backslash b$	0.00005	0.0020	0.0327	0.3377	4.8510

References

[1] A. Beck, *Introduction to nonlinear optimization: theory, algorithms, and applications with MATLAB*, vol. 19, Siam, 2014.