5. Nonlinear least squares

- nonlinear least-squares problem
- Gauss Newton method

Nonlinear least squares

Nonlinear least squares

- The NLLS (nonlinear least-squares) problem:

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad \frac{1}{2}\|r(x)\|_2^2, \quad r : \mathbf{R}^n \to \mathbf{R}^m \quad \text{(typically, } m > n\text{)}.$$

- "Residual" vector

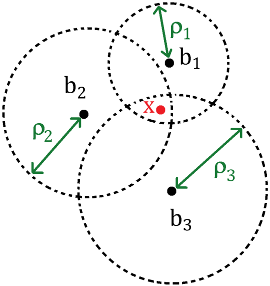
$$r(x) = \begin{bmatrix} r_1(x) \\ r_2(x) \\ \vdots \\ r_m(x) \end{bmatrix}, \quad r_i : \mathbf{R}^n \to \mathbf{R}$$

- Reduces to least-squares when r is **affine:**

$$r(x) = Ax - b$$

Example: position estimation from ranges

- Estimate $x \in \mathbf{R}^2$ from approximate distances to fixed beacons



- data: beacon positions

$$b_1, b_2, ..., b_m$$

- measurements

$$\rho_i = \|x - b_i\|_2 + v_i$$

- measurement error:

$$v_1, ..., v_m$$

- NLLS position estimate solves

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad \frac{1}{2} \sum_{i=1}^{m} r_i(x) = \sum_{i=1}^{m} (\rho - \|x - b_i\|_2)^2$$

- Must settle for locally optimal solution.

Gauss-Newton method for NLLS

given starting guess for $x^{(0)}$

repeat

1. linearize r near current guess for $\bar{x} = x^{(k)}$
2. solve a linear LS problem for next step

until converged

Linearization of residual

Linearization around a point $\bar{x} \in \mathbf{R}^n$:

$$r(x) = \begin{bmatrix} r_1(x) \\ \vdots \\ r_m(x) \end{bmatrix} \approx \begin{bmatrix} r_1(\bar{x}) + \nabla r_1(\bar{x})^T(x - \bar{x}) \\ \vdots \\ r_m(\bar{x}) + \nabla r_m(\bar{x})^T(x - \bar{x}) \end{bmatrix} = A(\bar{x})x - b(\bar{x})$$

where

$$A(\bar{x}) = \begin{bmatrix} \nabla r_1(x)^T \\ \vdots \\ \nabla r_m(x)^T \end{bmatrix} \in \mathbf{R}^{m \times n}, \qquad b(\bar{x}) = A(\bar{x})\bar{x} - r(\bar{x}) \in \mathbf{R}^m$$

and $A(\bar{x})$ is the **Jacobian** of mapping r at $\bar{x}$.

Linearized least-squares problem used to determine $x^{(k+1)}$

$$x^{(k+1)} = \operatorname*{argmin}_{x \in \mathbf{R}^n} \|A(x^{(k)})x - b(x^{(k)})\|_2^2$$

Dampening

Expand the linear least squares

$$\bar{A} = A(x^{(k)}) \quad \bar{b} = b(x^{(k)}), \quad \bar{r} = r(x^{(k)})$$

$$\begin{aligned} x^{(k+1)} &= \underset{x \in \mathbf{R}^n}{\operatorname{argmin}} \|\bar{A}x - \bar{b}\|_2^2 \\ &= (\bar{A}^T\bar{A})^{-1}\bar{A}^T\bar{b} \qquad \text{(if } \bar{A} \text{ has full rank)} \\ &= (\bar{A}^T\bar{A})^{-1}\bar{A}^T(\bar{A}x^{(k)} - \bar{r}) \\ &= x^{(k)} - \underbrace{(\bar{A}^T\bar{A})^{-1}\bar{A}^T\bar{r}}_{\text{step}} \end{aligned}$$

Dampened Gauss-Newton

$$x^{(k+1)} = x^{(k)} - \alpha z^{(k)}, \qquad z^{(k)} = \underset{x \in \mathbf{R}^n}{\operatorname{argmin}} \|A(x^{(k)})x - r(x^{(k)})\|^2$$

for $0 < \alpha \leq 1$.

Gauss-Newton method for NLLS

given starting guess for $x^{(0)}$

repeat

1. linearize r near current guess for $\bar{x} = x^{(k)}$

$$r(x) \approx A(\bar{x})(x - \bar{x}) - r(\bar{x})$$

2. solve a linear LS problem for next step

$$z^{(k)} = \underset{x \in \mathbf{R}^n}{\operatorname{argmin}} \|A(\bar{x})x - r(\bar{x})\|_2^2$$

3. take damped step

$$x^{(k+1)} = x^{(k)} - \alpha^{(k)} z^{(k)}, \quad 0 < \alpha^{(k)} \leq 1$$

until converged