

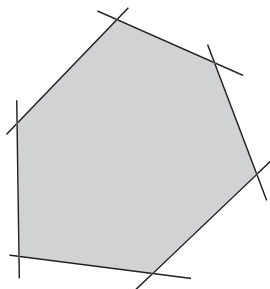
Geometry of Linear Programming

- extreme points
- vertices
- basic (feasible) solutions

Polyhedron (inequality form)

$A = [a_1 \ \cdots \ a_m]^T$ is $m \times n$, $b \in \mathbb{R}^m$

$$\mathcal{P} = \{ x \mid Ax \leq b \} = \{ x \mid a_i^T x \leq b_i, \ i = 1, \dots, m \}$$

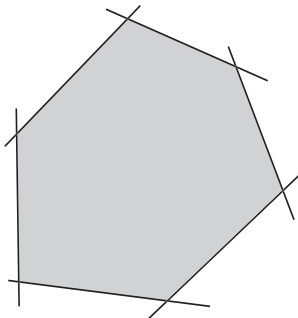


\mathcal{P} is convex because it's the intersection of halfspaces (the intersection of convex sets is convex)

Extreme points

$x \in \mathcal{P}$ is an **extreme point** of \mathcal{P} if there **does not exist** two vectors $y, z \in \mathcal{P}$ such that

$$x = \lambda y + (1 - \lambda)z \quad \text{for any } \lambda \in (0, 1)$$



Vertices

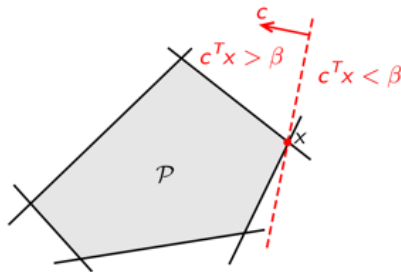
$x \in \mathcal{P}$ is a **vertex** of \mathcal{P} if there exists a vector $c \neq 0$ such that

$$c^T x < c^T y \quad \text{for all } y \in \mathcal{P}, y \neq x$$

two equivalent points of view:

- given a **vertex** x , find c such that $c^T x < c^T y$ for all $y \in \mathcal{P}, y \neq x$
- given a **vector** c , find x such that $c^T x < c^T y$ for all $y \in \mathcal{P}, y \neq x$, ie,

$$\underset{x}{\text{minimize}} \quad c^T x \quad \text{subject to} \quad x \in \mathcal{P}$$



Active constraints

define \mathcal{B} as the set of **active** or **binding** constraints (at x^*):

$$a_i^T x^* = b_i, \quad i \in \mathcal{B} \quad (\text{active constraints})$$

$$a_i^T x^* < b_i, \quad i \notin \mathcal{B} \quad (\text{inactive feasible constraints})$$

$$\text{or} \quad a_i^T x^* > b_i, \quad i \notin \mathcal{B} \quad (\text{inactive infeasible constraints})$$

define the subset of active constraints

$$\bar{A} = \begin{bmatrix} a_{i_1}^T \\ a_{i_2}^T \\ \vdots \\ a_{i_k}^T \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} b_{i_1} \\ b_{i_2} \\ \vdots \\ b_{i_k} \end{bmatrix}, \quad \mathcal{B} = \{i_1, i_2, \dots, i_k\}$$

Basic solutions

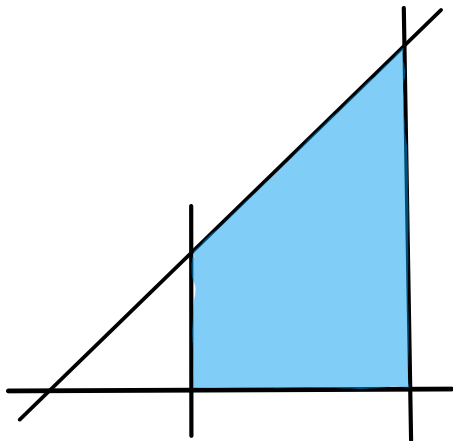
x^* is a **basic solution** if one of the following equivalent conditions hold:

- $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ are linearly independent
- $\bar{A}x^* = \bar{b}$ has a unique solution
- $\text{rank}(\bar{A}) = n$

basic feasible solution: x^* is a basic solution and $x^* \in \mathcal{P}$

Theorem: the following are equivalent

- x^* is a vertex
- x^* is an extreme point
- x^* is a basic feasible solution



Example

$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$$

- $(1, 1)$ is an extreme point
- $(1, 1)$ is a vertex: unique minimum of $c^T x$ with $c = (-1, -1)$
- $(1, 1)$ is a basic feasible solution: $B = \{2, 4\}$ and $\text{rank } \bar{A} = 2$, where

$$\bar{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Equivalence of definitions

Vertex \implies extreme point

Let x^* be a vertex of \mathcal{P} . Then there exists a $c \neq 0$ such that

$$c^T x^* < c^T x \quad \text{for all } x \in \mathcal{P} \text{ and } x \neq x^*.$$

Then for all $y, z \in \mathcal{P}$ with $y \neq x^*$ and $z \neq x^*$,

$$c^T x^* < c^T y \quad \text{and} \quad c^T x^* < c^T z.$$

If $\lambda \in [0, 1]$, then

$$c^T x^* < c^T(\lambda y + (1 - \lambda)z),$$

and $x^* \neq \lambda y + (1 - \lambda)z$. Therefore x^* is an extreme point.

Extreme point \implies basic feasible solution

Suppose $x^* \in \mathcal{P}$ is an extreme point with

$$a_i^T x^* = b_i \quad i \in \mathcal{B}, \quad \text{and} \quad a_i^T x^* < b_i \quad i \notin \mathcal{B}.$$

Proceed by contradiction. Suppose x^* is not a basic feasible solution. Thus, a_i for $i \in \mathcal{B}$ are not linearly independent. Then there exists a $d \neq 0$ with

$$a_i^T d = 0 \quad \text{for every } i \in \mathcal{B} \quad (\text{ie, } \bar{A}d = 0)$$

and for $\epsilon > 0$ small enough,

$$y = x^* + \epsilon d \in \mathcal{P}, \quad z = x^* - \epsilon d \in \mathcal{P}.$$

Summing, we have

$$x^* = \frac{1}{2}y + \frac{1}{2}z,$$

which contradicts the assumption that x^* is an extreme point.

Basic feasible solution \implies vertex

Suppose $x^* \in \mathcal{P}$ is a basic feasible solution and

$$a_i^T x^* = b_i \quad i \in \mathcal{B}, \quad \text{and} \quad a_i^T x^* < b_i \quad i \notin \mathcal{B}.$$

Take any $x \in \mathcal{P}$. For each $i \in \mathcal{B}$,

$$-a_i^T x \geq -b_i = -a_i^T x^*$$

Summing these all together:

$$c^T x = - \sum_{i \in \mathcal{B}} a_i^T x \geq - \sum_{i \in \mathcal{B}} b_i = c^T x^*, \quad c := - \sum_{i \in \mathcal{B}} a_i$$

with equality only if $a_i^T x = b_i, i \in \mathcal{B}$. Since $\{a_i \mid i \in \mathcal{B}\}$ are linearly independent, that holds only when $x = x^*$. Thus, $c^T x^* < c^T x$ for all $x \in \mathcal{P}, x \neq x^*$, so x^* is a vertex.

Unbounded directions

\mathcal{P} contains a **half-line** if there exists $d \neq 0$, x_0 such that

$$x_0 + \alpha d \in \mathcal{P} \quad \text{for all } \alpha \geq 0$$

equivalent conditions for $\mathcal{P} = \{x \mid Ax \leq b\}$:

$$Ax_0 \leq b, \quad Ad \leq 0$$

fact: \mathcal{P} unbounded $\iff \mathcal{P}$ contains a half line

\mathcal{P} contains a **line** if there exists $d \neq 0$, x_0 such that

$$x_0 + \alpha d \in \mathcal{P} \quad \text{for all } \alpha$$

equivalent conditions for $\mathcal{P} = \{x \mid Ax \leq b\}$:

$$Ax_0 \leq b, \quad Ad = 0$$

fact: \mathcal{P} has no extreme points $\iff \mathcal{P}$ contains a line

Optimal set of an LP

$$\text{minimize } c^T x \quad \text{subject to } Ax \leq b$$

- optimal value $p^* = \min \{ c^T x \mid Ax \leq b \}$ ($p^* = \pm\infty$ is possible)
- optimal point: x^* with $Ax^* \leq b$ and $c^T x^* = p^*$
- optimal set: $X^* = \{ x \mid Ax \leq b, c^T x = p^* \}$

example

$$\begin{array}{ll}\text{minimize} & c_1 x_1 + c_2 x_2 \\ \text{subject to} & -2x_1 + x_2 \leq 1, \quad x_1 \geq 0, \quad x_2 \geq 0\end{array}$$

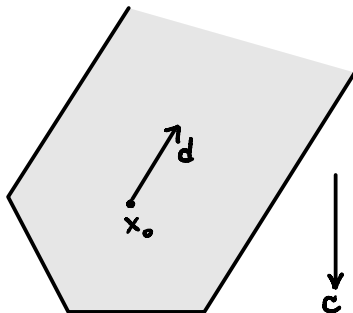
- $c = (1, 1)$: $X^* = \{ (0, 0) \}$, $p^* = 0$
- $c = (1, 0)$: $X^* = \{ (0, x_2) \mid 0 \leq x_2 \leq 1 \}$, $p^* = 0$
- $c = (-1, -1)$: $X^* = \emptyset$, $p^* = +\infty$

Existence of optimal points

- $p^* = -\infty$ if and only if there exists a feasible half line

$$\{x_0 + \alpha p \mid \alpha \geq 0\}$$

with $c^T p < 0$



- $p^* = +\infty$ if and only if $\mathcal{P} = \emptyset$
- p^* if finite if and only if $X^* \neq \emptyset$