Games and duality

- Two-person, zero-sum game
- Matrix games
- MiniMax Theorem
- Duality

A 2-person, zero-sum game

Player X			
Left	**Right**		
-10	20	**Left**	Player Y
10	-10	**Right**	

Elements of a game:

- Players
- Actions & strategies
- Payoffs

- **Pure strategy:** $x = (1, 0)$ and $y = (0, 1)$ Not optimal
- **Mixed strategy:** $x = (\frac{1}{2}, \frac{1}{2})$ and $y = (\frac{1}{2}, \frac{1}{2})$ Not optimal
  - **Y** expects $P_y = -10(\frac{1}{4}) + 20(\frac{1}{4}) + 10(\frac{1}{4}) - 10(\frac{1}{4}) = 2.5$
  - **X** expects $P_x = \ldots = 2.5$

Goal:

- Find strategies so that each player is happiest (not to deviate)

Saddle point:

- **X** has maximized her profit
- **Y** has minimized his loss

Matrix Games

Still two players **X** and **Y**. But now each has n and m strategies.

$$
\begin{array}{cccccccc}
 & & \boxed{\textbf{Player X}} & & & & & \\
x = (& x_1 & x_2 & \cdots & x_n) & & y & \\
 & \mathbf{1} & \mathbf{2} & \cdots & \mathbf{n} & & \shortparallel & \\
 & a_{11} & a_{12} & \cdots & a_{1n} & \mathbf{1} & \multirow{4}{*}{$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$} & \multirow{4}{*}{$\boxed{\textbf{Player Y}}$} \\
 & a_{21} & a_{22} & \cdots & a_{2n} & \mathbf{2} & & \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & & \\
 & a_{m1} & a_{m2} & \cdots & a_{mn} & \mathbf{m} & &
\end{array}
$$

Payoffs: a_{ij} = amount **Y** pays **X**

X Strategy: Choose x subj to $\sum_j x_j = 1, \quad x_j \geq 0$

Y Strategy: Choose y subj to $\sum_i y_i = 1, \quad y_i \geq 0$

Probability of outcome: (i, j) occurs w/ probability $x_j y_i$ and payoff a_{ij}

Total expected payoff: $\sum_i \sum_j a_{ij} x_j y_i$

Matrix Notation

Payoffs: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

X Strategy: Choose x subject to $e^T x = 1, \quad x \geq 0$
Y Strategy: Choose y subject to $e^T y = 1, \quad y \geq 0$

Total expected payoff:

$$y^T A x = [y_1, \ldots, y_m] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= a_{11} x_1 y_1 + \cdots + a_{mn} x_n y_m$$

Player Y's Analysis

Suppose **Y** chooses y as his strategy

- Then **X** will best defend by choosing x to

$$\max_x \quad y^T A x \qquad \text{(maximize expected payoff)}$$

- **Y** should then choose y to

$$\min_y \max_x \quad y^T A x$$

Player **Y** chooses his strategy to protect against the worst possible case:
When **X** knows what **Y** will do.

Solving for Y's Strategy

Y: $\min_y \max_x \; y^T A x$ subj to $e^T x = 1, \quad x \geq 0$
subj to $e^T y = 1, \quad y \geq 0$

The inner (**X**'s) problem: Given y, choose x to

$$\boxed{\begin{array}{ll} \underset{x}{\text{maximize}} & (y^T A)x \\ \text{subject to} & e^T x = 1, \quad x \geq 0 \end{array}} \iff \boxed{\underset{j}{\text{maximize}} \quad (y^T A)_j}$$

From LP theory, a **basic optimal solution** exists
$\Rightarrow x^*$ has only 1 nonzero component (equal to 1)

$$\text{Y}: \quad \text{Choose } y \text{ to} \quad \boxed{\begin{array}{ll} \underset{y}{\text{minimize}} & \underset{j}{\text{maximize}} \, (y^T A)_j \\ \text{subject to} & e^T y = 1, \quad y \geq 0 \end{array}}$$

Is **Y**'s problem an LP?

Solving for Y's Strategy: LP Formulation

Y: Choose y to

$$\boxed{\begin{array}{ll} \underset{y}{\text{minimize}} & \underset{j}{\text{maximize}}\ (y^T A)_j \\ \text{subject to} & e^T y = 1, \quad y \geq 0 \end{array}}$$

Reformulate as an LP:

$$\boxed{\begin{array}{ll} \underset{y,\nu}{\text{minimize}} & \nu \\ \text{subject to} & \nu e \geq A^T y \\ & e^T y = 1, \quad y \geq 0 \end{array}}$$

Player X's Analysis

(Player **X**'s analysis is symmetric to **Y**'s analysis)

- Suppose **X** chooses x as her strategy
- Then **Y** will best defend by choosing y to

$$\underset{y}{\text{minimize}} \quad y^T A x \qquad \text{(minimize expected payoff)}$$

- **X** should then choose x to

$$\boxed{\underset{x}{\text{maximize}} \; \underset{y}{\text{minimize}} \; y^T A x}$$

Player **X** choose her strategy to protect against the worst possible case:
When **Y** knows what **X** will do.

Solving for X's Strategy

X: $\underset{x}{\text{maximize}}\ \underset{y}{\text{minimize}}\ y^T A x$ subject to $e^T x = 1, \quad x \geq 0$

subject to $e^T y = 1, \quad y \geq 0$

The inner (**Y**'s) problem: Given x, choose y to

$$\boxed{\begin{array}{ll} \underset{y}{\text{minimize}} & y^T(Ax) \\ \text{subject to} & e^T y = 1, \quad y \geq 0 \end{array}} \iff \boxed{\underset{i}{\text{minimize}} \quad (Ax)_i}$$

From LP theory, a **basic optimal solution** exists

$\Rightarrow y^*$ has only 1 nonzero component (equal to 1)

X: Choose x to

$$\boxed{\begin{array}{ll} \underset{x}{\text{maximize}} & \underset{i}{\text{minimize}}\ (Ax)_i \\ \text{subject to} & e^T x = 1, \quad x \geq 0 \end{array}} \iff \boxed{\begin{array}{ll} \underset{x,\lambda}{\text{maximize}} & \lambda \\ \text{subject to} & \lambda e \leq Ax \\ & e^T x = 1, \quad x \geq 0 \end{array}}$$

Analysis Summary

Player **X** chooses her strategy to protect against worst possible case: **Y** knows what **X** will do.

$$\boxed{\underset{x}{\text{maximize}}\,\underset{y}{\text{minimize}}\, y^T A x} \quad \Longrightarrow \quad x^*$$

Player **Y** chooses his strategy to protect against worst possible case: **X** knows what **Y** will do.

$$\boxed{\underset{y}{\text{minimize}}\,\underset{x}{\text{maximize}}\, y^T A x} \quad \Longrightarrow \quad y^*$$

$$\boxed{\textbf{X}\text{'s worst-case expected win} \quad \begin{array}{c} \leq \\ = \\ \geq \end{array} \quad \textbf{Y}\text{'s worst-case expected loss}}$$

The **MiniMax Theorem**: Equality holds.

X and Y are Dual

X's Problem

$$\boxed{\underset{x}{\text{maximize}}\ \underset{y}{\text{minimize}}\ y^T A x}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{lll} \underset{x,\lambda}{\text{maximize}} & \lambda & \\ \text{subject to} & \lambda e \leq Ax & \\ & e^T x = 1, & x \geq 0 \end{array}}$$

$$P_x^*$$

(**X**'s worst-case expected win)

Dual pair of LPs

Y's Problem

$$\boxed{\underset{y}{\text{minimize}}\ \underset{x}{\text{maximize}}\ y^T A x}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{lll} \underset{y,\nu}{\text{minimize}} & \nu & \\ \text{subject to} & \nu e \geq A^T y & \\ & e^T y = 1, & y \geq 0 \end{array}}$$

$$P_y^*$$

(**Y**'s worst-case expected loss)

Weak duality: $P_x^* \quad \leq \quad P_y^*$

Strong duality: $P_x^* \quad = \quad P_y^*$

Proved the **MiniMax Theorem**