# Duality

- dual LP
- weak duality
- strong duality
- complementarity

# Duality

Consider the constrained problem

$$\begin{array}{ll} \underset{x_1, x_2}{\text{minimize}} & x_2^1 + x_2^2 \\ \text{subject to} & x_1 + x_2 = 1 \end{array}$$

and the unconstrained problem

$$\underset{x_1, x_2}{\text{minimize}} \quad \phi(x_1, x_2, y) \equiv x_1^2 + x_2^2 + y(1 - x_1 - x_2)$$

The scalar $y$ is the "price" for violating the constraint $x_1 + x_2 = 1$.
What price $y$ is enough to induce the optimal solution $x^* = (1/2, 1/2)$?

$$\left.\begin{array}{l} \dfrac{\partial \phi}{\partial x_1} = 2x_1 - y = 0 \\ \dfrac{\partial \phi}{\partial x_1} = 2x_2 - y = 0 \end{array}\right\} \implies \quad x_1 = x_2 = \frac{y}{2} \quad \implies \quad y^* = 1$$

# Dual function

**primal problem:** $\underset{x}{\text{minimize}} \quad c^T x \quad \text{subj to} \quad Ax = b,\ x \geq 0$

- $n$ variables, $m$ constraints
- optimal value $p^*$

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**relaxed problem:** $\underset{x}{\text{minimize}} \quad c^T x + y^T(b - Ax) \quad \text{subj to} \quad x \geq 0$

- relaxed problem is a lower bound for $p^*$:
- $g(y) := \min_{x \geq 0} \left\{ c^T x + y^T(b - Ax) \right\} \leq c^T x^* + y^T(b - Ax^*) = c^T x^* = p^*$

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**tightest lower bound:** find $y$ that solves

$$\underset{y}{\text{maximize}} \quad g(y)$$

# Dual of an LP

$$
\begin{aligned}
g(y) &= \min_{x \geq 0} \; c^T x + y^T(b - Ax) \\
&= b^T y + \min_{x \geq 0} \; x^T(c - A^T y) \\
&= \begin{cases} b^T y & \text{if} \quad c - A^T y \geq 0 \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

Because we want to **maximize** $g(y)$, we must have

$$
\begin{array}{ll} \underset{y}{\text{maximize}} & b^T y \\ \text{subject to} & c - A^T y \geq 0 \end{array}
\quad \Longleftrightarrow \quad
\begin{array}{ll} \underset{y,z}{\text{maximize}} & b^T y \\ \text{subject to} & A^T y + z = c \\ & z \geq 0 \end{array}
$$

this is the **dual** LP

# Weak duality

Suppose that $x$ is primal feasible:

$$Ax = b, \quad x \geq 0$$

Suppose that $(y, z)$ is dual feasible:

$$A^T y + z = c, \quad z \geq 0$$

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**Weak-duality theorem:** if $(x, y, z)$ is primal/dual feasible, then

- the primal value is an upper bound for the dual value
- the dual value is a lower bound for the primal value

# Complementarity

| **primal** | **dual** |
|---|---|
| $$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$ | $$\begin{aligned} \underset{y,z}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & A^T y + z = c \\ & z \geq 0 \end{aligned}$$ |

By weak duality:

$$(\text{primal value}) \equiv c^T x = b^T y + z^T x \geq b^T y \equiv (\text{dual value})$$

This bound is "tight" when $x$ and $z$ are **complementary**, ie, $x^T z = 0$:

$$\begin{aligned} x_j = 0 \quad & \text{and} \quad z_j \geq 0 \\ x_j \geq 0 \quad & \text{and} \quad z_j = 0 \end{aligned}$$

# Optimality conditions

Simplex maintains **primal feasibility** at every iteration:

$$Ax = b, \quad x \geq 0$$

It defines $y$ via $B^T y = c_B$ and $z = c - A^T y$, and maintains **complementarity**:

$$\begin{aligned} x_B \geq 0 \quad &\text{and} \quad z_B = 0 \qquad \text{(by construction)} \\ x_N = 0 \quad &\text{and} \quad z_N \lesseqqgtr 0 \end{aligned}$$

Simplex exits when $z \geq 0$, ie, $(y, z)$ is **dual feasible**, ie,

$$A^T y + z = c, \quad z \geq 0$$

**Strong duality theorem:** If an LP has an optimal solution, so does its dual, and the optimal values are equal, ie, $p^* = d^*$

# Sufficient conditions

Suppose that $(x, y, z)$ is primal/dual feasible.

By weak duality,

$$c^T x - b^T y = z^T x$$

By strong duality, if $(x, y, z)$ is primal-dual optimal,

$$z^T x = 0$$

Conversely, if $z^T x = 0$, then

- $c^T x$ achieves its upper bound
- $b^T y$ achieves its lower bound

therefore, $(x, y, z)$ is primal-dual optimal

**Theorem:** The primal-dual triple $(x, y, z)$ is optimal iff

$$Ax = b,\ x \geq 0, \quad A^T y + z = c,\ z \geq 0, \quad x^T z = 0$$

# Relationship between primal and dual LPs

| | finite opt | unbounded | infeasible |
|---|---|---|---|
| finite opt | ✓ | ✗ | ✗ |
| unbounded | ✗ | ✗ | ✓ |
| infeasible | ✗ | ✓ | ✓ |

# Interpretation of dual variables

**primal**

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

**dual**

$$\begin{array}{ll} \underset{y,z}{\text{maximize}} & b^T y \\ \text{subject to} & A^T y + z = c \\ & z \geq 0 \end{array}$$

Suppose that $x^*$ is optimal and nondegenerate, then

$$x^* = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} > 0 \qquad \text{and} \qquad x_B^*(\epsilon) = B^{-1}(b + \epsilon \Delta b) > 0 \quad \text{for small } \epsilon$$

Reduced costs $z^* = c - A^T y^*$ doesn't change. Thus $(x^*(\epsilon), y^*, z^*)$ is primal/dual feasible