1. Prove that the log-sum-exp function

$$f(x) = \log \sum_{i=1}^{m} \exp(a_i^T x)$$

is a convex function over $x \in \mathbf{R}^n$.

2. Suppose that the scalar random variable x takes values $\{a_1, a_2, \ldots, a_n\}$ with probability $\mathbf{prob}(x = a_i) = p_i$ for $i = 1, \ldots, n$. Is the variance

$$\mathbf{var}\ x = \mathbb{E}x^2 - (\mathbb{E}x)^2$$

a convex or concave function in the probabilities $p = (p_1, \ldots, p_n)$? Prove your answer.

3. Prove that the intersection of convex sets $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_n$ is a convex set.

4. Show that the *second-order cone*

$$\mathcal{S} = \{(x,t) \in \mathbf{R}^n \times \mathbf{R}_+ \mid \|x\|_2 \le t\}$$

is convex.

5. Consider the convex optimization problem

$$\underset{x}{\text{minimize}}\ \ f(x)\ \ \text{subject to}\ \ x \in \mathcal{C}, \tag{1}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is a convex and differentiable function, and $\mathcal{C} \subseteq \mathbf{R}^n$ is a convex set. A point $x^* \in \mathcal{C}$ is optimal for (1) if and only if

$$-\nabla f(x^*) \in \mathcal{N}_{\mathcal{C}}(x^*), \tag{2}$$

where $\mathcal{N}(x^*)$ is the normal cone of $\mathcal{C}$ at point x^*.

   (a) Prove that (2) is equivalent to the condition

$$x^* = \mathbf{proj}_{\mathcal{C}}(x^* - \gamma \nabla f(x^*)) \tag{3}$$

   for any constant $\gamma > 0$.

   (b) **Affine set.** Let $\mathcal{C} = \{x \mid Ax = b\}$.
      i. Derive the normal cone $\mathcal{N}_{\mathcal{C}}(x)$.
      ii. Use this expression for the normal cone to deduce that $\nabla f(x^*)$ is in the range of A^T.

   (c) **Nonnegative constraint.** Let $\mathcal{C} = \mathbf{R}^n_+$.
      i. Derive the normal cone $\mathcal{N}_{\mathcal{C}}(x)$.
      ii. Use this expression for the normal cone to deduce that $\nabla f(x^*) \ge 0$.