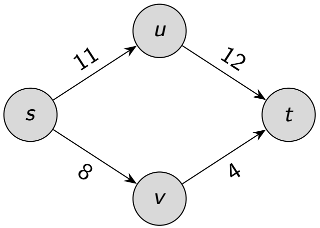
Application of duality: The max-flow min-cut theorem

Outline:

- Define network flow graph
- Maximizing the flow on a graph
- Define minimum cut of a graph
- Max-flow as a linear program (LP)
- Take the dual of the max-flow LP
- Prove max-flow min-cut theorem using strong duality

Network flow graph

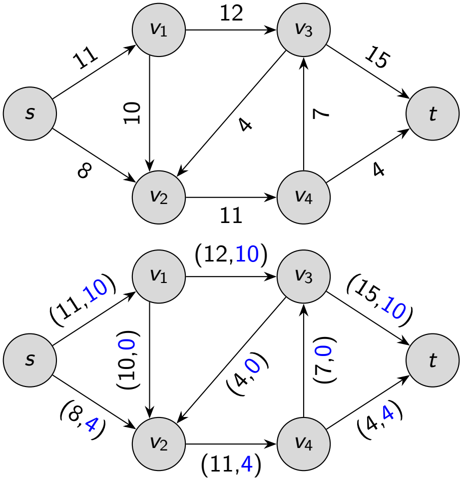
- Let $G = (V, A)$ be a directed graph, where V denotes a set of vertices and A denotes a set of arcs (directed eges).
- Example: $V = \{s, t, u, v\}$, $A = \{su, sv, ut, vt\}$.



- For every arc $uv \in A$, there is an associated capacity $c_{uv} \geq 0$.
- Designate special vertices $s, t \in V$ called *source* and *sink*, respectively.

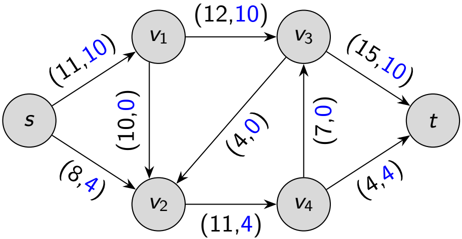
Flow on a graph

A flow is a nonnegative number assigned to every arc such that it doesn't exceed the capacity of the arc.



Definition of flow

- A valid s-t flow must obey the **flow in = flow out** rule.
- Denote the a flow by a vector $x \in \mathbb{R}^{|A|}$.
- x_{uv} = flow on arc uv.

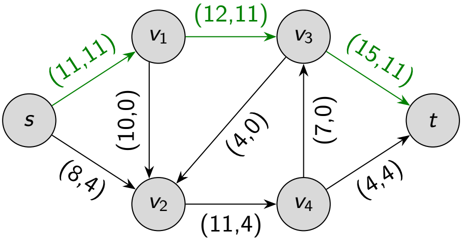


$$x = \begin{bmatrix} x_{sv_1} \\ x_{sv_2} \\ x_{v_1v_3} \\ x_{v_2v_4} \\ x_{v_1v_2} \\ x_{v_3v_2} \\ x_{v_4v_3} \\ x_{v_3t} \\ x_{v_4t} \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 10 \\ 4 \\ 0 \\ 0 \\ 0 \\ 10 \\ 4 \end{bmatrix}$$

- The flow **into** v is $f_x^+(v) = \sum_{(v,u)\in A} x_{vu}$.
- The flow **out of** v is $f_x^-(v) = \sum_{(u,v)\in A} x_{uv}$.

Increasing flow: example

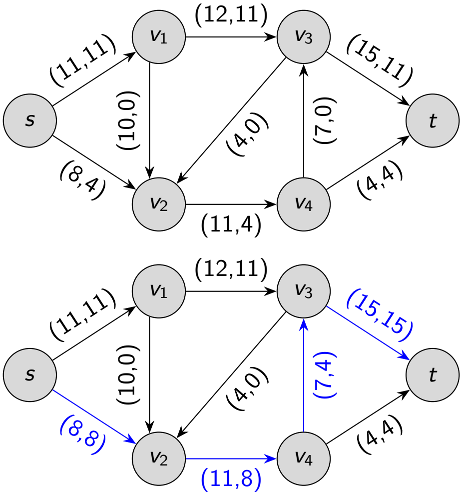
- If x is a valid flow, then $f_x^+(s) = f_x^-(t)$.
- We call the flow out of s, $f_x^+(s)$, the *total flow*.
- To increase the flow out of s, we must take into account the capacities of the subsequent edges.
- For example, we can increase the flow by 1.



- Total flow is now $f_x(s) = 11 + 4 = 15$.

Example

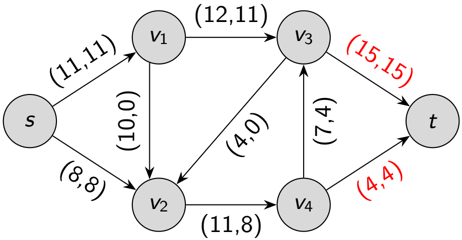
We can also increase the flow out of s by 4.



Total flow is now $f_x(s) = 11 + 8 = 19$.

Maximizing total flow

- How do we determine when we have reached the highest possible flow?
- How do we know that a different flow would give an even higher total flow?
- It is clear we cannot increase the current flow any further in this graph due to a bottleneck at t (and at s).



Bottleneck upper bounds the flow

- These bottlenecks can be defined more formally as *s*-*t* cuts.
- **Definition**: given a set of vertices $U \subset V$ such that $s \in U$ and $t \notin U$, the corresponding *s*-*t* cut is

$$\delta^+(U) := \{(u, v) \in A \colon u \in U,\ v \notin U\}.$$

- **The notation $\delta^+(U)$ means "edges leaving U".**
- **Definition**: the capacity of a cut $\delta^+(U)$ is given by the sum of capacities of all arcs in $\delta^+(U)$.

$$c(\delta^+(U)) := \sum_{uv \in \delta^+(U)} c_{uv}$$

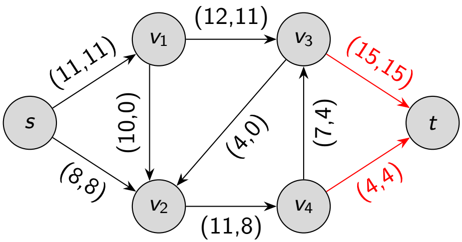
- Notice: the total flow on a graph cannot exceed the capacity of *any* *s*-*t* cut.

Cuts and bottlenecks: example

Let $U := \{s, v_1, v_2, v_3, v_4\} \subset V$. Then $\delta^+(U) = \{v_3 t, v_4 t\} \subset A$.

The capacity $c(\delta^+(U))$ of the cut is

$$c(\delta^+(U)) = c_{v_3 t} + c_{v_4 t} = 15 + 4 = 19.$$



Max-flow min-cut theorem

- The max-flow min-cut theorem: the max capacity s-t flow is **equal** to the minimum capacity s-t cut.
- A surprising fact: this can be proved using LP and strong duality!
- Start by writing the max flow LP as a linear program.

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & f_x^+(s) \\ \text{subject to} \quad & f_x^+(v) - f_x^-(v) = 0 \ \forall v \in V \setminus \{s, t\} \\ & 0 \leq x \leq c \end{aligned}$$

Vertex-arc incident matrix

Vertex-arc incident matrix. $M \in \mathbb{R}^{m \times n}$ where

$$m = |V| - 2 \quad n = |A|.$$

For all $i \in V \setminus \{s, t\}$ and $j \in A$.

$$M_{ij} = \begin{cases} +1 & \text{if vertex } i \text{ is the tail of arc } j \\ -1 & \text{if vertex } i \text{ is the head of arc } j \\ 0 & \text{if vertex } i \text{ is not an endpoint of arc } j \end{cases}$$

Recall x_j denotes the flow assigned to arc j.

Let's look at Mx row-by-row. Fix a vertex $v \in \{1, \ldots, m\}$.

$$\mathrm{row}_v(Mx) = \sum_{uv \in A} x_{uv} - \sum_{vu \in A} x_{vu} = f_x^+(v) - f_x^-(v)$$

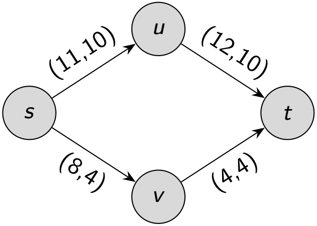
Vertex-arc incident matrix example

$$V = \{s, t, u, v\}$$

$$A = \{su, sv, ut, vt\}$$

rows of M: $i \in \{u, v\}$

columns of M: $j \in \{su, sv, ut, vt\}$



$$M = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \; x = \begin{bmatrix} x_{su} \\ x_{sv} \\ x_{ut} \\ x_{vt} \end{bmatrix}$$

$$Mx = \begin{bmatrix} x_{su} - x_{ut} \\ x_{sv} - x_{vs} \end{bmatrix} = \begin{bmatrix} 10 - 10 \\ 4 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Max-flow LP

Therefore we can write the constraint

$$f_x^+(v) - f_x^-(v) = 0 \;\forall v \in V \setminus \{s, t\} \qquad \text{as} \qquad Mx = 0.$$

For the objective, let $d \in \mathbb{R}^{|A|}$ be the indicator vector for arcs leaving s.

$$d^T x = f_x^+(s) \qquad d_a = \begin{cases} 1 & \text{if} \quad a \in \delta^+(s) \\ 0 & \text{otherwise} \end{cases}$$

We get the following LP:

$$\begin{array}{rl} \underset{x}{\text{maximize}} & d^T x \\ \text{subject to} & Mx = 0 \\ & 0 \leq x \leq c \end{array}$$

Taking the dual of the max-flow LP

Max-flow LP

$$
\text{(P)} \quad \begin{array}{ll} \underset{x}{\text{maximize}} & d^T x \\ \text{subject to} & Mx = 0 \\ & 0 \le x \le c \end{array}
$$

Dual LP

$$
\text{(D)} \quad \begin{array}{ll} \underset{y,z}{\text{minimize}} & c^T z \\ \text{subject to} & M^T y + z \ge d \\ & z \ge 0 \end{array}
$$

We can simplify the dual LP to the following.

$$
\text{(D)} \quad \begin{array}{ll} \underset{y,z}{\text{minimize}} & \sum_{uv \in A} c_{uv} z_{uv} \\ \text{subject to} & y_s = -1 \\ & y_t = 0 \\ & y_u - y_v + z_{uv} \ge 0 \ \forall uv \in A \\ & z \ge 0 \end{array}
$$

Max-flow min-cut theorem via Strong duality

- We can check that the primal and dual LPs both have feasible solutions.
- **Strong duality** tells us they both have *optimal solutions* and their *optimal values* are equal, i.e.

$$\mathrm{opt}(\mathrm{P}) = \mathrm{opt}(\mathrm{D}).$$

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- **Strong duality** tells us they both have *optimal solutions* and their *optimal values* are equal, i.e.

$$\mathrm{opt}(\mathrm{P}) = \mathrm{opt}(\mathrm{D}).$$

- The max-flow min-cut theorem: the max capacity s-t flow is **equal** to the minimum capacity s-t cut.
- All that's missing is to show that the optimal value of (D) equals the capacity of the minimum s-t cut, i.e.

$$(\mathrm{claim}) \quad \mathrm{opt}(\mathrm{D}) = c(\delta^+(W^*)),$$

where $\delta^+(W^*)$ is the minimum capacity cut.

Proof of claim

Let $W^* \subset V$ be such that $\delta^+(W^*)$ is a minimum capacity s-t cut.

We begin by showing $\mathrm{opt}(\mathrm{D}) \leq c(\delta^+(W^*))$.

Construct a feasible $(\hat{y}, \hat{z})$ by

$$\hat{y}_u = \begin{cases} -1 & \text{if} \quad u \in W^*, \\ \ \ 0 & \text{otherwise;} \end{cases} \qquad \hat{z}_{uv} = \begin{cases} 1 & \text{if} \quad uv \in \delta^+(W^*), \\ 0 & \text{otherwise.} \end{cases}$$

It's simple to check that $(\hat{y}, \hat{z})$ is feasible, and thus, $\mathrm{opt}(\mathrm{D}) \leq \mathrm{obj}(\hat{y}, \hat{z})$.

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The objective function of (D) evaluates to:

$$\underbrace{\mathrm{obj}(\hat{y}, \hat{z})}_{\geq \mathrm{opt}(\mathrm{D})} = \sum_{uv \in A} c_{uv} \hat{z}_{uv} = \sum_{uv \in \delta^+(W^*)} c_{uv} = c(\delta^+(W^*)).$$

Thus $\mathrm{opt}(\mathrm{D}) \leq c(\delta^+(W^*))$, as required.

Proof of claim

Now we show that $c(\delta^+(W^*)) \leq \mathrm{opt}(\mathrm{D})$.

Let (y^*, z^*) be an optimal solution to (D), i.e.,

$$\begin{cases} (y^*, z^*) \quad \text{is feasible for} \quad (\mathrm{D}), \\ \mathrm{obj}(y^*, z^*) = \mathrm{opt}(\mathrm{D}). \end{cases}$$

Let $\bar{W} := \{u \in V \ : \ y^*_u \leq -1\}$. This is a valid s-t cut since $s \in \bar{W}$, $t \notin \bar{W}$.

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We just need to show that $\mathrm{obj}(y^*, z^*) \ge c(\delta^+(\bar{W}))$.

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Why? We assumed $\delta^+(W^*)$ is a min capacity s-t. This gives the rightmost inequality

$$\underbrace{\mathrm{obj}(y^*, z^*)}_{=\mathrm{opt}(\mathrm{D})} \geq c(\delta^+(\bar{W})) \geq c(\delta^+(W^*)),$$

as required.

Last part of the proof!

To show that $\mathrm{obj}(y^*, z^*) \geq c(\delta^+(\bar{W}))$, we write:

$$\mathrm{obj}(y^*, z^*) = \sum_{uv \in \delta^+(\bar{W})} c_{uv} z_{uv} + \sum_{uv \notin \delta^+(\bar{W})} c_{uv} z_{uv}.$$

Assume that (y^*, z^*) is integral (we skip the reason why this is the case).

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Assume that (y^*, z^*) is integral (we skip the reason why this is the case).

Recall $\bar{W} := \{u \in V \ : \ y_u^* \leq -1\}$.

Thus $\forall uv \in \delta^+(\bar{W})$, $y_v^* > -1$, and since y_v^* is an integer, $y_v^* \geq 0$.

$$z_{uv}^* \geq \underbrace{y_v^*}_{\geq 0} - \underbrace{y_u^*}_{\leq -1} \geq 1 \quad \forall uv \in \delta^+(\bar{W})$$

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$$z_{uv}^* \geq \underbrace{y_v^*}_{\geq 0} - \underbrace{y_u^*}_{\leq -1} \geq 1 \quad \forall uv \in \delta^+(\bar{W})$$

Since (y^*, z^*) is feasible, $z_{uv}^* \geq 0 \; \forall uv \in A$. Therefore

$$\mathrm{obj}(y^*, z^*) = \underbrace{\sum_{uv \in \delta^+(\bar{W})} c_{uv} z_{uv}^*}_{\geq c(\delta^+(\bar{W}))} + \underbrace{\sum_{uv \notin \delta^+(\bar{W})} c_{uv} z_{uv}^*}_{\geq 0}.$$