Regret and Conservatism of Stochastic Model Predictive Control

Maik Pfefferkorn *maik.pfefferkorn@iat.tu-darmstadt.de*
Control and Cyber-Physical Systems Laboratory
Technical University of Darmstadt, Germany

Venkatraman Renganathan *venkatraman.renganathan@control.lth.se*
Department of Automatic Control - LTH
Lund University, Sweden

Rolf Findeisen *rolf.findeisen@iat.tu-darmstadt.de*
Control and Cyber-Physical Systems Laboratory
Technical University of Darmstadt, Germany

Abstract

We analyse conservatism and regret of stochastic model predictive control (SMPC) when using moment-based ambiguity sets for modeling unknown uncertainties. To quantify the conservatism, we compare the deterministic constraint tightening while taking a distributionally robust approach against the optimal tightening when the exact distributions of the stochastic uncertainties are known. Furthermore, we quantify the regret by comparing the performance when the distributions of the stochastic uncertainties are known and unknown. Analysing the accumulated sub-optimality of SMPC due to the lack of knowledge about the true distributions of the uncertainties marks the novel contribution of this work.

1 Introduction

Recently, there has been a surge of interest in analyzing control algorithms that operate subject to unknown entities through the lens of regret analysis. Entities of interest, also referred to as *quantities of interest* (QIs), are numerous, e.g., disturbance sequences, uncertainty distributions or cost function parameterizations. Lack of knowledge about a QI induces regret for the controller, i.e., a performance loss compared to when the QI was known. Analysing regret enables to better understand the performance trade-off of control algorithms due to the lack of knowledge about the entities. Notably, many robust control algorithms incorporate caution mechanisms into their decision making scheme in order to account for the lack of knowledge about the QIs. The cautiousness in the decision making can be quantified as the conservatism of the control algorithm. As a result of cautious decision making, robust control algorithms are known to incur regret. For instance, the regret associated with H_∞ control compared against an optimal hindsight controller that knows the disturbance sequences exactly beforehand was examined in Karapetyan et al. (2022). Further works investigated regret in different variants of the finite- and infinite-horizon linear-quadratic regulator problem, see e.g. Chen et al. (2023); Li et al. (2019) and references therein.

We are interested in comparing controllers that are robust with respect to a moment-based ambiguity set of distributions against a fully informed counterpart that knows the true uncertainty distribution. Specifically, we focus on stochastic model predictive control (SMPC) Rawlings et al. (2019). For SMPC, we analyze conservatism and regret of its distributionally robust formulation for unknown uncertainty distributions compared against its fully informed counterpart. We provide a detailed performance and constraint tightening analysis for SMPC. Our main contributions are as follows:

1. We define *constraint conservatism* and *distributional regret* of distributionally robust SMPC with uncertainties modeled using moment-based ambiguity sets.

2. We develop a framework for quantifying conservatism and regret and for analyzing their behaviors. Specifically, we derive analytic expressions for conservatism and regret to identify and analyze the effects that lead to regret. In doing so, we show how regret analysis can open the door for leveraging the power of dynamically setting state and input constraints in SMPC to achieve better performance.
3. We underline our theoretical findings in simulations.

The paper is organised as follows: The SMPC problem is introduced in Section 2. Regret and conservatism associated with SMPC are presented in Section 3, while some of its features are demonstrated using numerical simulation in Section 4 before concluding in Section 5.

Notation and Preliminaries. The set of real numbers, integers and the natural numbers are denoted by $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ respectively and the subset of real numbers greater than a given constant $a \in \mathbb{R}$ is denoted by $\mathbb{R}_{>a}$. The subset of natural numbers between two constants $a, b \in \mathbb{N}$ with $a < b$ is denoted by $[a : b]$. Given two sets $A \subset \mathbb{R}^n, B \subset \mathbb{R}^n$, their Pontryagin difference is denoted by $A \ominus B := \{a \in A \mid a + b \in A, \forall b \in B\} \subset \mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{n \times n}$, we denote its transpose and its trace by $A^\top$ and $\mathbf{Tr}(A)$ respectively. An identity matrix of dimension n is denoted by I_n. We denote by $\mathbb{S}^n$ the set of symmetric matrices in $\mathbb{R}^{n \times n}$. For $A \in \mathbb{S}^n$, we denote by $A \succ 0 (A \succeq 0)$ to mean that A is positive definite (positive semi-definite). Given $x \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$, the notation $\|x\|_A^2$ denotes $x^\top A x$.

2 Stochastic Model Predictive Control (SMPC) Problem Formulation

We consider SMPC with joint chance constraints similar to Paulson et al. (2020) using moment-based ambiguity set modeling to describe the stochastic system uncertainties. The detailed set-up is as follows.

2.1 System Dynamics & Constraints

We consider a stochastic discrete-time linear time-invariant system given by

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad \forall k \in \mathbb{N}, \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the system state and input at time step k, respectively. The matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ denote the dynamics matrix and the input matrix respectively. For the ease of exposition, we will assume that $x_0 \in \mathbb{R}^n$ is known. The process noise $w_k \in \mathbb{R}^n$ is a zero-mean random vector that is independent and identically distributed over time. The distribution of w_k, namely $\mathbb{P}^w$, is unknown. However, it is known to belong to a moment-based ambiguity set of distributions given by

$$\mathcal{P}^w = \left\{ \mathbb{P}^w \mid \mathbb{E}[w_k] = 0, \mathbb{E}[w_k w_k^\top] = \Sigma_w \right\}. \tag{2}$$

With $N \in \mathbb{N}$, we denote the prediction horizon considered in the predictive control problem to be defined shortly. The states, inputs and the disturbances over the prediction horizon are given by

$$\mathbf{x}_k := \begin{bmatrix} x_{0|k}^\top & x_{1|k}^\top & \cdots & x_{N|k}^\top \end{bmatrix}^\top \in \mathbb{R}^{(N+1)n}, \tag{3}$$

$$\mathbf{u}_k := \begin{bmatrix} u_{0|k}^\top & u_{1|k}^\top & \cdots & u_{N-1|k}^\top \end{bmatrix}^\top \in \mathbb{R}^{Nm}, \tag{4}$$

$$\mathbf{w}_k := \begin{bmatrix} w_{0|k}^\top & w_{1|k}^\top & \cdots & w_{N-1|k}^\top \end{bmatrix}^\top \in \mathbb{R}^{Nn}. \tag{5}$$

Here, the subscript $i|k$ indicates i time steps ahead of k and $x_{0|k} = x_k$. Starting at x_k, we compactly write the evolution of (1) over the prediction horizon as

$$\mathbf{x}_k = \mathbf{A}x_{0|k} + \mathbf{B}\mathbf{u}_k + \mathbf{D}\mathbf{w}_k, \tag{6}$$

with matrices $\mathbf{A}, \mathbf{B}$, and $\mathbf{D}$ being of appropriate dimension. The mean and covariance of (6) evolve as

$$\bar{\mathbf{x}}_k := \mathbb{E}[\mathbf{x}_k] = \mathbf{A}x_{0|k} + \mathbf{B}\mathbf{u}_k, \tag{7}$$

$$\mathbf{\Sigma_x} := \mathbb{E}[(\mathbf{x}_k - \bar{\mathbf{x}}_k)(\mathbf{x}_k - \bar{\mathbf{x}}_k)^\top] = \mathbf{D}\mathbf{\Sigma_w}\mathbf{D}^\top, \tag{8}$$

where $\mathbf{\Sigma_w}$ is a block diagonal matrix with N blocks and each block being equal to Σ_w. Since the support of $\mathbb{P}^w$ can be unbounded, we cannot guarantee the satisfaction of hard state constraints for all k. Hence, we enforce a distributionally robust joint risk constraint on the states as

$$\sup_{\mathbb{P}_k^{\mathbf{x}} \in \mathcal{P}_k^{\mathbf{x}}} \mathbb{P}_k^{\mathbf{x}} \left[\mathbf{x}_k \notin \mathcal{X} \right] \leq \Delta, \tag{9}$$

where $\mathcal{P}_k^{\mathbf{x}}$ denotes the moment-based ambiguity set for the compact state $\mathbf{x}_k$ and is given by

$$\mathcal{P}_k^{\mathbf{x}} = \left\{ \mathbb{P}_k^{\mathbf{x}} \mid \mathbb{E}[\mathbf{x}_k] = \bar{\mathbf{x}}_k, \mathbb{E}[(\mathbf{x}_k - \bar{\mathbf{x}}_k)(\mathbf{x}_k - \bar{\mathbf{x}}_k)^\top] = \mathbf{\Sigma_x} \right\}.$$

The set $\mathcal{X}$ is assumed to be a convex polytope defined by

$$\mathcal{X} := \bigcap_{i=1}^{n_x} \left\{ \mathbf{x} \mid f_i^\top \mathbf{x} \leq g_i \right\} = \left\{ \mathbf{x} \mid \mathbf{F}\mathbf{x} \leq \mathbf{g} \right\}, \tag{10}$$

where $f_i \in \mathbb{R}^{(N+1)n}, g_i \in \mathbb{R}, \mathbf{F} \in \mathbb{R}^{n_x \times (N+1)n}$ and $\mathbf{g} \in \mathbb{R}^{n_x}$. Further, $\Delta \in (0, 0.5]$ represents a user-prescribed total risk budget for the worst-case probability of constraint violation over the entire prediction horizon. A similar probabilistic treatment of control constraints can also be formulated. For the ease of exposition, we consider the hard input constraint formulation $u_k \in \mathcal{U}, \forall k$, where $\mathcal{U}$ is a convex polytope

$$\mathcal{U} := \bigcap_{j=1}^{n_u} \left\{ \mathbf{u} \mid c_j^\top \mathbf{u} \leq d_j \right\} = \left\{ \mathbf{u} \mid \mathbf{C}\mathbf{u} \leq \mathbf{d} \right\} \tag{11}$$

with $c_j \in \mathbb{R}^{Nm}, d_i \in \mathbb{R}, \mathbf{C} \in \mathbb{R}^{n_u \times Nm}$ and $\mathbf{d} \in \mathbb{R}^{n_u}$. The objective of the SMPC is to minimize the cost

$$J_{\text{SMPC}}(\mathbf{u}_k, x_{0|k}) := \mathbb{E}\left[\|\mathbf{x}_k\|_{\mathbf{Q}}^2 + \|\mathbf{u}_k\|_{\mathbf{R}}^2 \right]. \tag{12}$$

Note that $\mathbf{Q}$ is a block diagonal matrix with $N+1$ blocks and each block being equal to $Q \succeq 0$. Similarly, $\mathbf{R}$ is a block diagonal matrix with N blocks and each block being equal to $R \succ 0$. Then, $J_{\text{SMPC}}(\mathbf{u}_k, x_{0|k})$ can be rewritten as

$$J_{\text{SMPC}}(\mathbf{u}_k, x_{0|k}) = \|\bar{\mathbf{x}}_k\|_{\mathbf{Q}}^2 + \|\mathbf{u}_k\|_{\mathbf{R}}^2 + \mathbf{Tr}(\mathbf{Q}\mathbf{\Sigma_x}). \tag{13}$$

The following problem formally establishes the SMPC optimization problem along with the state and input constraints.

2.2 Stochastic Model Predictive Control (SMPC)

Problem 1. *Given an initial state $x_{0|k} = x_k \in \mathbb{R}^n$, process noise $w_{i|k} \sim \mathbb{P}^w \in \mathcal{P}^w, \forall i \in [0 : N-1]$, and the penalty matrices $Q \succeq 0, R \succ 0$, we seek to find an input sequence $\mathbf{u}_k$ that optimizes the following optimal control problem (OCP):*

$$\begin{aligned}
\min_{\mathbf{u}_k} \quad & J_{\text{SMPC}}(\mathbf{u}_k, x_{0|k}) && \text{(14a)}\\
\text{s.t.} \quad & \mathbf{x}_k = \mathbf{A}x_{0|k} + \mathbf{B}\mathbf{u}_k + \mathbf{D}\mathbf{w}_k, \quad x_{0|k} = x_k, && \text{(14b)}\\
& \mathbf{u}_k \in \mathcal{U}, w_{i|k} \sim \mathbb{P}^w \in \mathcal{P}^w, && \text{(14c)}\\
& \sup_{\mathbb{P}_k^{\mathbf{x}} \in \mathcal{P}_k^{\mathbf{x}}} \mathbb{P}_k^{\mathbf{x}} \left[\mathbf{x}_k \notin \mathcal{X} \right] \leq \Delta. && \text{(14d)}
\end{aligned}$$

Problem 1 is solved online in a receding horizon fashion at each time instance k. The first element $u_{0|k}^\star$ of the optimal input sequence $\mathbf{u}_k^\star$ is applied to the system given by (1) until the next sampling instant $k+1$[1].

[1] We refer the reader to Rawlings et al. (2019) for a comprehensive introduction to and overview of model predictive control.

Note that the OCP (14) is in general computationally intractable as (14d) is an infinite-dimensional distributionally robust joint risk constraint. Using Boole's inequality, (14d) can be decomposed into individual risk constraints across each time step along the prediction horizon. Suppose that $\sum_{i=1}^{n_x} \delta_i \leq \Delta$ and

$$\sup_{\mathbb{P}_k^{\mathbf{x}} \in \mathcal{P}_k^{\mathbf{x}}} \mathbb{P}_k^{\mathbf{x}} \left[f_i^\top \mathbf{x}_k > g_i \right] \leq \delta_i, \quad \forall i = 1, \ldots, n_x. \tag{15}$$

Then, (15) implies (14d). That is,

$$\begin{aligned}
\sup_{\mathbb{P}_k^{\mathbf{x}} \in \mathcal{P}_k^{\mathbf{x}}} \mathbb{P}_k^{\mathbf{x}} [\mathbf{x}_k \notin \mathcal{X}] &= \sup_{\mathbb{P}_k^{\mathbf{x}} \in \mathcal{P}_k^{\mathbf{x}}} \mathbb{P}_k^{\mathbf{x}} \left[\mathbf{x}_k \in \bigcup_{i=1}^{n_x} \left\{ \mathbf{x}_k \,|\, f_i^\top \mathbf{x}_k > g_i \right\} \right] \\
&\leq \sum_{i=1}^{n_x} \sup_{\mathbb{P}_k^{\mathbf{x}} \in \mathcal{P}_k^{\mathbf{x}}} \mathbb{P}_k^{\mathbf{x}} \left[f_i^\top \mathbf{x}_k > g_i \right] \\
&\leq \sum_{i=1}^{n_x} \delta_i \\
&\leq \Delta.
\end{aligned}$$

Though (15) is an infinite-dimensional risk constraint, it can be equivalently formulated as a second order cone constraint on the state mean through deterministic constraint tightening using Lemma 1 or Lemma 2 given below without proofs.

Lemma 1. *(From Ono & Williams (2008)) Let $\mathbf{x} \in \mathbb{R}^n$ be a random vector with known distribution $\mathbb{P}^{\mathbf{x}}$ defined using mean $\bar{\mathbf{x}}$ and covariance $\boldsymbol{\Sigma}^{\mathbf{x}}$. Then, $\forall \delta_i \in (0, 0.5]$, and $i \in [1:n_x]$, we have*

$$\mathbb{P}^{\mathbf{x}} \left[f_i^\top \mathbf{x} > g_i \right] \leq \delta_i \Leftrightarrow f_i^\top \bar{\mathbf{x}} \leq g_i - \Phi_{\mathbb{P}^{\mathbf{x}}}^{-1}(1 - \delta_i) \left\| \boldsymbol{\Sigma}_{\mathbf{x}}^{\frac{1}{2}} f_i \right\|_2,$$

where $\Phi_{\mathbb{P}^{\mathbf{x}}}$ denotes the Cumulative Distribution Function (CDF) of the known distribution ($\mathbb{P}^{\mathbf{x}}$) when normalized.

Lemma 2. *(From Calafiore & El Ghaoui (2006)) Let $\mathbf{x} \in \mathbb{R}^n$ be random with mean $\bar{\mathbf{x}}$ and covariance $\boldsymbol{\Sigma}_{\mathbf{x}}$. Then, $\forall \delta_i \in (0, 0.5], i \in [1:n_x]$,*

$$\sup_{\mathbb{P}^{\mathbf{x}} \in \mathcal{P}^{\mathbf{x}}} \mathbb{P}^{\mathbf{x}} \left[f_i^\top \mathbf{x} > g_i \right] \leq \delta_i \Leftrightarrow f_i^\top \bar{\mathbf{x}} \leq g_i - \sqrt{\frac{1 - \delta_i}{\delta_i}} \left\| \boldsymbol{\Sigma}_{\mathbf{x}}^{\frac{1}{2}} f_i \right\|_2.$$

The surrogate of problem 1, called the surrogate SMPC, can be written as

$$\begin{aligned}
\min_{\mathbf{u}_k} \quad & J_{\mathrm{SMPC}}(\mathbf{u}_k, x_{0|k}) && (16a) \\
\text{s.t.} \quad & \mathbf{x}_k = \mathbf{A} x_{0|k} + \mathbf{B} \mathbf{u}_k + \mathbf{D} \mathbf{w}_k, \quad x_{0|k} = x_k, && (16b) \\
& \mathbf{u}_k \in \mathcal{U}, w_k \sim \mathbb{P}^w \in \mathcal{P}^w, && (16c) \\
& f_i^\top \bar{\mathbf{x}}_k \leq g_i - \psi_i \left\| \boldsymbol{\Sigma}_{\mathbf{x}}^{\frac{1}{2}} f_i \right\|_2, i \in [1:n_x], && (16d)
\end{aligned}$$

Here, the tightening constant $\psi_i, \forall i = 1, \ldots, n_x$ is given by

$$\psi_i := \begin{cases} \Phi_{\mathbb{P}_k^{\mathbf{x}}}^{-1}(1 - \delta_i), & \text{when } \mathbb{P}_k^{\mathbf{x}} \text{ is known,} \\ \sqrt{\frac{1 - \delta_i}{\delta_i}}, & \text{when } \mathbb{P}_k^{\mathbf{x}} \text{ is unknown.} \end{cases} \tag{17}$$

Note that (16) has finite-dimensional constraints unlike its original counterpart in (14). Let the control input sequences that minimize (16) with exact constraint tightening and distributionally robust constraint tightening according to (17) be denoted by $\mathbf{u}_k^*$ and $\mathbf{u}_k^\dagger$ respectively. Their corresponding optimal value functions are given by

$$J_{\mathrm{SMPC}}^*(x_k) := J_{\mathrm{SMPC}}(\mathbf{u}_k^*, x_k). \tag{18}$$

$$J_{\mathrm{SMPC}}^\dagger(x_k) := J_{\mathrm{SMPC}}(\mathbf{u}_k^\dagger, x_k). \tag{19}$$

3 Conservatism & Regret Analyses

In this section, we introduce and define the concepts of conservatism and regret for the previously introduced SMPC algorithm.

3.1 Conservatism of SMPC

We would like to study the difference in constraint tightening when the true distributions of the stochastic uncertainties are known and when they are unknown. To do so, we define the constraint conservatism associated with the SMPC problem with joint chance constraint formulation as follows.

Definition 1. *The constraint conservatism, denoted by $\mathfrak{C}_{\mathrm{SMPC}} \in \mathbb{R}$, associated with the SMPC problem is defined as the difference in volume of the deterministically tightened state constraint set with and without the knowledge of $\mathbb{P}^w$ and $\mathbb{P}_k^{\mathbf{x}}$ respectively. That is,*

$$\mathfrak{C}_{\mathrm{SMPC}} := \int_{\underbrace{(\mathcal{X}_{\mathrm{True}} \ominus \mathcal{X}_{\mathrm{DR}})}_{:=\mathcal{X}_{\mathrm{Diff}}}} dx, \quad \textit{where}, \tag{20}$$

$$\mathcal{X}_{\mathrm{True}} := \bigcap_{i=1}^{n_x} \left\{ \mathbf{x} \mid f_i^\top \mathbf{x} \leq g_i - \Phi^{-1}_{\mathbb{P}_k^{\mathbf{x}}}(1-\delta_i) \left\| \boldsymbol{\Sigma}_{\mathbf{x}}^{\frac{1}{2}} f_i \right\|_2 \right\}, \tag{21}$$

$$\mathcal{X}_{\mathrm{DR}} := \bigcap_{i=1}^{n_x} \left\{ \mathbf{x} \mid f_i^\top \mathbf{x} \leq g_i - \sqrt{\frac{1-\delta_i}{\delta_i}} \left\| \boldsymbol{\Sigma}_{\mathbf{x}}^{\frac{1}{2}} f_i \right\|_2 \right\}. \tag{22}$$

Note that $\mathcal{X}_{\mathrm{True}}$ and $\mathcal{X}_{\mathrm{DR}}$ can be equivalently written as

$$\mathcal{X}_{\mathrm{True}} = \left\{ \mathbf{x} \mid \mathbf{F}\mathbf{x} \leq \mathbf{g}_{\mathrm{True}} \right\}, \tag{23}$$

$$\mathcal{X}_{\mathrm{DR}} = \left\{ \mathbf{x} \mid \mathbf{F}\mathbf{x} \leq \mathbf{g}_{\mathrm{DR}} \right\}, \quad \text{where} \quad \forall i \in [1:n_x] \tag{24}$$

$$\left[\mathbf{g}_{\mathrm{True}}\right]_i := \left[g_i - \Phi^{-1}_{\mathbb{P}_k^{\mathbf{x}}}(1-\delta_i) \left\| \boldsymbol{\Sigma}_{\mathbf{x}}^{\frac{1}{2}} f_i \right\|_2 \right], \quad \text{and}, \tag{25}$$

$$\left[\mathbf{g}_{\mathrm{DR}}\right]_i := \left[g_i - \sqrt{\tfrac{1-\delta_i}{\delta_i}} \left\| \boldsymbol{\Sigma}_{\mathbf{x}}^{\frac{1}{2}} f_i \right\|_2 \right]. \tag{26}$$

Let $\mathbf{z} := \left[\begin{array}{c|c} \mathbf{x}^\top & \mathbf{x}^\top \end{array} \right]^\top \in \mathbb{R}^{2n(N+1)}$ and

$$\mathbf{h} := \left[\begin{array}{c} \mathbf{g}_{\mathrm{True}} \\ \hline \mathbf{g}_{\mathrm{DR}} \end{array} \right] \in \mathbb{R}^{2n_x}, \quad \mathbf{H} := \left[\begin{array}{c|c} \mathbf{F} & \mathbf{F} \\ \hline \mathbf{0}_{n_x \times n(N+1)} & \mathbf{F} \end{array} \right]. \tag{27}$$

Then, using Theorem 3.3 in Gabidullina (2019), we can express $\mathcal{X}_{\mathrm{Diff}}$ as

$$\mathcal{X}_{\mathrm{Diff}} := \left\{ \mathbf{z} \mid \mathbf{H}\mathbf{z} \leq \mathbf{h} \right\}. \tag{28}$$

Although analyzing (28) is in general hard, its volume (i.e., conservatism) can be computed numerically, see e.g. Chevallier et al. (2022).

Remark 1: Note that (20) uses the Pontryagin difference of the two deterministically tightened convex polyhedrons $\mathcal{X}_{\mathrm{True}}$ and $\mathcal{X}_{\mathrm{DR}}$. The constraint tightening in both (21) and (22) depends upon the risk[2] δ_i corresponding to the i^{th} constraint. Only for $\delta_i, i = 1, \ldots, n_x$ close to 0, the distributionally robust constraint tightening will be significantly stricter than the exact tightening.

---

[2]When a non-uniform risk allocation is performed by solving an optimization problem as in Ono & Williams (2008), both $\mathcal{X}_{\mathrm{Diff}}$ and $\mathfrak{C}_{\mathrm{SMPC}}$ will have interesting observations. Analysing this aspect is out of the scope of this paper and is left for future.

3.2 Regret of SMPC

We start by noting that for SMPC formulation (16), regret is introduced by conservatism: Since the deterministic constraint tightening will be different for the case when the true distributions $\mathbb{P}_w$ and $\mathbb{P}_{\mathbf{x}}$ are known and unknown respectively, the resulting optimal cost from solving the Surrogate SMPC problem given by (16) will be different when tightened state constraints become active. This difference of the optimal costs is essentially referred to as regret.

Assumption 1. *Both the systems under the distributionally robust and fully informed SMPC respectively encounter the same disturbance realization w_k at all time steps $k \in \mathbb{N}$.*

Definition 2. *Given Assumption 1, the distributional regret associated with the SMPC problem at time step k denoted by $\mathfrak{R}_k^{\text{SMPC}} \in \mathbb{R}$ is defined as the difference in the cost associated while solving the Surrogate SMPC problem (16) with and without the knowledge of $\mathbb{P}^w$ and $\mathbb{P}_k^{\mathbf{x}}$ respectively. That is,*

$$\mathfrak{R}_k^{\text{SMPC}} := J_{\text{SMPC}}^{\dagger}(x_k^{\dagger}) - J_{\text{SMPC}}^{*}(x_k^{*}). \tag{29}$$

To analyze the regret, we derive a closed-form expression for the optimal input sequences $\mathbf{u}_k^*$ and $\mathbf{u}_k^{\dagger}$. We reformulate OCP (16) as a quadratic program (QP) of the form

$$\min_{\mathbf{u_k}} \quad \frac{1}{2}\left\|\mathbf{u}_k\right\|_H^2 + h_k^\top \mathbf{u}_k + r_k \tag{30a}$$

$$\text{s.t.} \quad \mathbf{M}\mathbf{u}_k - \mathbf{b}_k \leq \mathbf{0}. \tag{30b}$$

We start by reformulating the cost function (13) of OCP (16). Substituting (7) and (8) into (13), we obtain

$$J_{\text{SMPC}}(\mathbf{u}_k, x_k) = \frac{1}{2}\left\|\mathbf{u}_k\right\|_H^2 + h_k^\top \mathbf{u}_k + r_k, \tag{31}$$

where $H = 2(\mathbf{B}^\top\mathbf{Q}\mathbf{B} + \mathbf{R}) \succ 0, h_k^\top = 2x_{0|k}^\top \mathbf{A}^\top\mathbf{Q}\mathbf{B}$ and $r_k = \mathbf{Tr}\left(\mathbf{Q}\mathbf{D}\boldsymbol{\Sigma}_w\mathbf{D}^\top\right) + \left\|x_{0|k}\right\|_{\mathbf{A}^\top\mathbf{Q}\mathbf{A}}^2$. Next, we substitute (7) and (8) into (16d) to reformulate the tightened state constraints in terms of only the input as

$$\underbrace{f_i^\top \mathbf{B}}_{=:\bar{f}_i^\top}\mathbf{u}_k \leq \underbrace{g_i - f_i^\top \mathbf{A}x_{0|k}}_{=:\bar{g}_{k,i}} - \psi_i \underbrace{\left\|(\mathbf{D}\boldsymbol{\Sigma}_{\mathbf{w}}\mathbf{D}^\top)^{\frac{1}{2}} f_i\right\|_2}_{=:v_i}. \tag{32}$$

Then, $\forall i \in [1 : n_x]$, writing (32) equivalently in vectorized form as $\bar{\mathbf{F}}\mathbf{u}_k \leq \bar{\mathbf{g}}_k - \psi^\top \mathbf{v}$ enables us to define (30b) via

$$\mathbf{M} = \begin{bmatrix} \mathbf{C} \\ \bar{\mathbf{F}} \end{bmatrix}, \quad \mathbf{b}_k = \begin{bmatrix} \mathbf{d} \\ \bar{\mathbf{g}}_k - \operatorname{diag}(\boldsymbol{\psi})\mathbf{v} \end{bmatrix}. \tag{33}$$

Definition 3. *An inequality constraint is said to be active if $\mathbf{M}_{i:}\mathbf{u}_k - \mathbf{b}_{k,i} = 0$ and inactive if $\mathbf{M}_{i:}\mathbf{u}_k - \mathbf{b}_{k,i} < 0$, where $\mathbf{M}_{i:}$ denotes the i^{th} row of $\mathbf{M}$ and $\mathbf{b}_{k,i}$ the i^{th} entry of $\mathbf{b}_k$. The active set $\mathcal{A}_k \subseteq \mathcal{I} := [1 : n_x]$ is the index set of active inequality constraints.*

Assumption 2. *The active set $\mathcal{A}_k^\diamond$ of QP (30) is known $\forall k$.*

Assumption 3. *QP (30) satisfies the linear independence constraint qualification (LICQ) criterion, i.e., the gradients of the active inequality constraints are linearly independent.*

Proposition 1. *Under Assumptions 2 and 3, the unique and global solution to QP (30) at time step k is given by*

$$\mathbf{u}_k^\diamond = \left(V_k^\diamond \tilde{\mathbf{M}}_k^\diamond H^{-1} - H^{-1}\right) h_k + V_k^\diamond \tilde{\mathbf{b}}_k^\diamond, \tag{34}$$

where $V_k^\diamond = H^{-1}\tilde{\mathbf{M}}_k^{\diamond\top}\left(\tilde{\mathbf{M}}_k^\diamond H^{-1}\tilde{\mathbf{M}}_k^{\diamond\top}\right)^{-1}$, $\tilde{\mathbf{M}}_{\mathbf{k}}^\diamond = [\mathbf{M}_{i:}]_{i\in\mathcal{A}_k^\diamond}$, *and* $\tilde{\mathbf{b}}_k^\diamond = [\mathbf{b}_{k,i}]_{i\in\mathcal{A}_k^\diamond}$.

Proof. The result directly follows from applying the method of Lagrange multipliers Boyd & Vandenberghe (2004); Ghojogh et al. (2021) to QP (30), exploiting sufficiency of the active set Arnström (2023). We first formulate the Lagrangian as

$$\mathcal{L}(\mathbf{u}_k, \mu_k) = \frac{1}{2}\mathbf{u}_k^\top H\mathbf{u}_k + h_k^\top \mathbf{u}_k + r_k + \boldsymbol{\mu}_k^\top(\mathbf{M}\mathbf{u}_k - \mathbf{b}_k), \tag{35}$$

where μ_k is the vector of Lagrange multipliers. Applying the Karush-Kuhn-Tucker (KKT) conditions yields

$$H\mathbf{u}_k^{\diamond} + h_k + \mathbf{M}^\top \boldsymbol{\mu}_k = \mathbf{0} \tag{36a}$$
$$\mathbf{M}\mathbf{u}_k^{\diamond} - \mathbf{b}_k \leq \mathbf{0} \tag{36b}$$
$$\boldsymbol{\mu}_k \geq \mathbf{0} \tag{36c}$$
$$\mu_{k,i}(\mathbf{M}_{i:}\mathbf{u}_k^{\diamond} - \mathbf{b}_{k,i}) = 0, \quad \forall i \in [0:n_x]. \tag{36d}$$

According to the complementary slackness condition (36d), we have $\mu_{k,i} = 0, \forall i \in \mathcal{I} \setminus \mathcal{A}_k^{\diamond}$. This enables to remove inactive inequality constraints from the problem formulation (c.f., sufficiency of the active set Arnström (2023)). Hence, the KKT system (36) is reduced to a linear system of equations given by

$$\begin{bmatrix} H & \tilde{\mathbf{M}}_k^{\diamond\top} \\ \tilde{\mathbf{M}}_k^{\diamond} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_k^{\diamond} \\ \tilde{\boldsymbol{\mu}}_k \end{bmatrix} = \begin{bmatrix} -h_k \\ \tilde{\mathbf{b}}_k^{\diamond} \end{bmatrix}, \tag{37}$$

where $\tilde{\boldsymbol{\mu}}_k = [\mu_{k,i}]_{i \in \mathcal{A}_k^{\diamond}}, \tilde{\mathbf{M}}_{\mathbf{k}}^{\diamond} = [\mathbf{M}_{i:}]_{i \in \mathcal{A}_k^{\diamond}}$, $\tilde{\mathbf{b}}_k^{\diamond} = [\mathbf{b}_{k,i}]_{i \in \mathcal{A}_k^{\diamond}}$ and the dual feasibility condition (36c) is trivially fulfilled if $\mathcal{A}_k^{\diamond}$ is the correct active set. Exploiting invertibility of the coefficient matrix of (37) yields (34). Expression (34) is guaranteed to be the optimal solution to QP (30) as the KKT conditions are necessary and sufficient under Assumption 3. The solution is unique and global as $H \succ 0$. □

The optimal input sequences $\mathbf{u}_k^*$ and $\mathbf{u}_k^{\dagger}$ are obtained using (34) (Proposition 1) when exact and distributionally robust constraint tightening are employed, respectively.

Remark 2: The active set $\mathcal{A}_k^{\diamond}$ can be constructed in an iterative manner through additions and removals of constraints to a *working set* $\mathcal{A}_k$ until primal feasibility (36b) and dual feasibility (36c) are satisfied by solution (34). This idea is exploited in so-called *active-set methods* for solving QPs Arnström (2023). Hence, Assumption 2 is not restrictive and only used to avoid computations that are out of the scope of this work. As for Assumption 3, the LICQ criterion is commonly employed in practice as it is a rather weak condition. In the considered controller set-up, LICQ can be established during control design and is naturally given in many cases. However, other constraint qualifications might be used, see e.g. Bergmann & Herzog (2019).

Remark 3: Next, we investigate the regret as defined in (29). Given the optimal input sequences $\mathbf{u}_k^{\star}$ and $\mathbf{u}_k^{\dagger}$ according to Proposition 1 for appropriate constraint tightening, it can be seen that regret is induced through three main effects:

(i) The active sets $\mathcal{A}_k^{\star}$ and $\mathcal{A}_k^{\dagger}$ differ as the distributionally robust controller is more cautious and constraints might become active or inactive at earlier or later times when compared to the fully-informed controller.

(ii) The initial states $x_{0|k}^{\star}$ and $x_{0|k}^{\dagger}$ differ as the feasible set of states of the distributionally robust controller is smaller than that of the fully-informed controller.

(iii) The tightening factors (17) of the active state constraints differ between the distributionally robust and the fully-informed controller.

While regret can be easily computed using the previously stated results, analyzing its general behavior is challenging. Since the active sets enter the respective control sequences in a highly nonlinear way, their influence on the regret is hardly analyzable. Therefore, we focus in the following on a special case defined in the following assumption.

Assumption 4. *The active sets $\mathcal{A}_k^*$ and $\mathcal{A}_k^{\dagger}$ are identical at time step k, such that $\tilde{\mathbf{M}}_k^{\star} = \tilde{\mathbf{M}}_k^{\dagger} = \tilde{\mathbf{M}}_k$, $\mathbf{V}_k^{\star} = \mathbf{V}_k^{\dagger} = \mathbf{V}_k$ and the difference between $\tilde{\mathbf{b}}_{\mathbf{k}}^{\star}$ and $\tilde{\mathbf{b}}_{\mathbf{k}}^{\dagger}$ is only due to the different tightening factors $\tilde{\boldsymbol{\psi}}^{\star}$ and $\tilde{\boldsymbol{\psi}}^{\dagger}$ corresponding to the active state constraints.*

Theorem 1. *Let Assumption 4 hold and write* $\mathbf{V}_k = \begin{bmatrix}\mathbf{V}_{1,k} & \mathbf{V}_{2,k}\end{bmatrix}$ *as block-matrix composed of* $\mathbf{V}_{1,k}$ *and* $\mathbf{V}_{2,k}$. *Then, the regret at time step* k *is given by*

$$\begin{aligned}\mathfrak{R}_k^{\mathrm{SMPC}} =& -(x_{0|k}^\star - x_{0|k}^\dagger)^\top \Lambda_{1,k}(x_{0|k}^\star + x_{0|k}^\dagger)\\ &- \tilde{\mathbf{v}}_k^\top \mathrm{diag}(\tilde{\boldsymbol{\psi}}_k^\star - \tilde{\boldsymbol{\psi}}_k^\dagger)\Lambda_{2,k}\mathrm{diag}(\tilde{\boldsymbol{\psi}}_k^\star + \tilde{\boldsymbol{\psi}}_k^\dagger)\tilde{\mathbf{v}}_k\\ &+ (x_{0|k}^\star - x_{0|k}^\dagger)^\top \Lambda_{3,k}\mathrm{diag}(\tilde{\boldsymbol{\psi}}_k^\star + \tilde{\boldsymbol{\psi}}_k^\dagger)\tilde{\mathbf{v}}_k\\ &+ (x_{0|k}^\star + x_{0|k}^\dagger)^\top \Lambda_{3,k}\mathrm{diag}(\tilde{\boldsymbol{\psi}}_k^\star - \tilde{\boldsymbol{\psi}}_k^\dagger)\tilde{\mathbf{v}}_k\\ &- (x_{0|k}^\star - x_{0|k}^\dagger)^\top \Lambda_{4,k}\\ &+ \tilde{\mathbf{v}}_k^\top \mathrm{diag}(\tilde{\boldsymbol{\psi}}_k^\star - \tilde{\boldsymbol{\psi}}_k^\dagger)\Lambda_{5,k},\end{aligned} \tag{38}$$

where

$$\begin{aligned}\Lambda_{1,k} &= \frac{1}{2}\alpha_k^\top H\alpha_k + \frac{1}{2}(\tilde{h}^\top\alpha_k + \alpha_k^\top\tilde{h}) + \mathbf{A}^\top\mathbf{Q}\mathbf{A},\\ \Lambda_{2,k} &= \frac{1}{2}\mathbf{V}_{2,k}^\top H\mathbf{V}_{2,k}, \quad \Lambda_{3,k} = \frac{1}{2}(\tilde{h}^\top\mathbf{V}_{2,k} + \alpha_k^\top H\mathbf{V}_{2,k})\\ \Lambda_{4,k} &= \tilde{h}^\top\gamma_k + \alpha_k^\top H\gamma_k, \quad \Lambda_{5,k} = \mathbf{V}_{2,k}^\top H\gamma_k\end{aligned}$$

with $\alpha_k = (V_k\tilde{\mathbf{M}}_k H^{-1} - H^{-1})\tilde{h} - \mathbf{V}_{2,k}\tilde{\mathbf{F}}_k\mathbf{A}$, $\tilde{h} = 2\mathbf{B}^\top\mathbf{Q}\mathbf{A}$ *and* $\gamma_k = \mathbf{V}_{1,k}\tilde{\mathbf{d}}_k + \mathbf{V}_{2,k}\tilde{\mathbf{g}}_k$.

Proof. Expression (38) is obtained from (29) using the cost function representation (31), substituting the optimal input sequences $\mathbf{u}_k^*$ and $\mathbf{u}_k^\dagger$ from (34) and the definitions of $\tilde{\mathbf{M}}_k$ and $\tilde{\mathbf{b}}_k$ and performing a series of algebraic operations (not shown here for the brevity of presentation). □

Note that (38) is quadratic in the initial conditions $x_{0|k}^\star$, $x_{0|k}^\dagger$ and the tightening factors $\tilde{\boldsymbol{\psi}}_k^\star$, $\tilde{\boldsymbol{\psi}}_k^\dagger$ but not directly in their respective differences.

Corollary 1. *If no (state and input) constraints are active, then* (38) *simplifies to*

$$\begin{aligned}\mathfrak{R}_k^{\mathrm{SMPC}} &= -(x_{0|k}^\star - x_{0|k}^\dagger)^\top\Lambda_1(x_{0|k}^\star + x_{0|k}^\dagger),\\ \Lambda_1 &= \mathbf{A}^\top\mathbf{Q}\mathbf{A} - \frac{1}{2}\tilde{h}^\top H^{-1}\tilde{h}\end{aligned} \tag{39}$$

and the optimal input sequences are given by the linear feedback laws $\mathbf{u}_k^\star = -H^{-1}\tilde{h}x_{0|k}^\star$ *and* $\mathbf{u}_k^\dagger = -H^{-1}\tilde{h}x_{0|k}^\dagger$.

We continue analyzing the general behavior of regret when no constraints are active, given that the fully informed and the distributionally robust SMPC are stabilizing in the sense of the following definitions.

Definition 4. *(From Culbertson et al. (2023)) A random variable* $w \sim \mathbb{P}_w$ *belongs to* L^p, *denoted by* $w \in L^p$, *for* $p > 0$ *if* $\|w\|_{L^p} := \mathbb{E}\left[\|\xi\|^p\right]^{\frac{1}{p}} < \infty$, *where* $\|\cdot\|$ *defines a typical norm on* $\mathbb{R}^n$.

Definition 5. *(From Culbertson et al. (2023)) System* (1) *is input-to-state stable in probability (ISSp) with respect to* L^p *if* $\forall\epsilon \in (0,1)$, $K \in \mathbb{N}$ *and* $w_k \in L_p$, $\exists\beta \in \mathcal{KL}$ *and* $\exists\varrho \in \mathcal{K}$ *such that*

$$\mathbb{P}_x\left[\|x_{k+i}\| \leq \beta(\|x_k\|, k+i) + \varrho(\|w_{k+i}\|_{L^p}), \forall i \leq K\right] \geq 1 - \epsilon. \tag{40}$$

If this holds for $\beta(\|x_k\|, k+i) = M\nu^{k+i}\|x_k\|$ *with* $M > 0$ *and* $\nu \in (0,1)$, *system* (1) *is exponentially ISSp (eISSp).*

Corollary 2. *(From Culbertson et al. (2023)) The nominal (undisturbed) system is asymptotically stable according to*

$$\mathbb{P}[\|\bar{x}_{k+i}\| \leq \beta(\|x_k\|, k+i), \forall i \leq K)] = 1. \tag{41}$$

Definition 6. *(From Culbertson et al. (2023)) A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISSp Lyapunov function for system (1) if there exist functions $\kappa_1, \kappa_2, \kappa_3 \in \mathcal{K}_\infty$ and $\kappa_4 \in \mathcal{K}$ such that*

$$\kappa_1(\|x_k\|) \leq V(x_k) \leq \kappa_2(\|x_k\|)$$
$$\mathbb{E}[V(f(x_k, u_k) - V(x_k)] \leq \kappa_3(V(x_k)) + \kappa_4(\|w_k\|_{L^p})$$

hold for all $x_k \in \mathcal{X}$ and $w_k \in L_p$. If $\kappa_1(\|x_k\|) = a\|x_k\|^c$, $\kappa_2(\|x_k\|) = b\|x_k\|^c$ and $\kappa_3(V(x)) = \nu V(x)$ with $a, b, c \in \mathbb{R}_{>0}$ and $\nu \in (0,1)$, then V is an eISSp Lyapunov function.

Assumption 5. *System (1) is eISSp under SMPC for both the exact and the distributionally robust cases. That is, $\exists \beta^*, \beta^\dagger \in \mathcal{KL}$, given by $\beta^*(s,t) = M^* {\nu^*}^t s$, $\beta^\dagger(s,t) = M^\dagger {\nu^\dagger}^t s$ with $M^*, M^\dagger > 0$ and $\nu^*, \nu^\dagger \in (0,1)$ such that (40) holds for $w_k \in L_p, w_k \sim \mathbb{P}^w$ with $p > 0$, given $\varepsilon \in (0,1)$ and $\bar{K} = \min\{K^*, K^\dagger\}$. The optimal value functions $J^*_{\mathrm{SMPC}}(x_k), J^*_{\mathrm{SMPC}}(x_k)$ are eISSp Lyapunov functions.*

Note that there is an interplay between the probability ε and the horizon lengths $K^*, K^\dagger$. For fixed ε, the (maximum) horizon lengths $K^*, K^\dagger$ for which (40) holds true in the respective case is determined by the problem formulation and vice versa. We consider the common horizon length $\bar{K} = \min\{K^*, K^\dagger\}$ to compare the fully informed and the distributionally robust case as (40) remains valid for $\bar{K}$ in both cases.

Assumption 6. *There exists a set $\Phi := \{x \in \mathcal{X} \mid \|x\| \leq r\} \subset \mathcal{X}_{\mathrm{DR}} \subset \mathcal{X}_{\mathrm{True}} \subset \mathcal{X}$, where $r \in \mathbb{R}_{>0}$ is chosen such that Φ is the largest set such that no input or (tightened) state constraints are active for all $x \in \Phi$ for both the fully informed and the distributionally robust controller.*

Definition 7. *(From Culbertson et al. (2023)) For a bounded set $\mathcal{S} \in \mathcal{X}$, the hitting time is defined as $\tau_{\mathcal{S}}(x) := \inf\{k \in \mathbb{N} \mid x_k \in \mathcal{S}, x_0 = x\}$. The set $\mathcal{S}$ is recurrent if $\forall x \in \mathcal{X}, \mathbb{P}[\tau_{\mathcal{S}}(x) < \infty] = 1$.*

Lemma 3. *The set Φ is recurrent for system (1) under both the fully informed and the distributionally robust controller. Furthermore, if $x_k \in \Phi$, then system (1) satisfies*

$$\mathbb{P}[x_{k+i} \in \Phi, \forall i \leq \bar{K}] \geq 1 - \varepsilon. \tag{43}$$

Proof. If V is an eISSp Lyapunov functions for system (1), any sublevel set $\mathcal{V}_\gamma := \{x \in \mathcal{X} \mid V(x) \leq \gamma\}$ is recurrent, see Culbertson et al. (2023). As the optimal value functions are eISSp Lyapunov functions for the respective closed-loop systems according to Assumption 5 and $r > 0$ by Assumption 6, there exist $\gamma^*, \gamma^\dagger$ such that $\mathcal{V}_{\gamma^*} \subset \Phi, \mathcal{V}_{\gamma^\dagger} \subset \Phi$. Hence, Φ is recurrent under both the fully informed and the distributionally robust controller. Expression (43) follows from (40) as $\beta(\|x_k\|, k+i) + \rho(\|w_{k+i}\|_{L_p}) \leq \beta(\|x_k\|, k) + \rho(\|w_k\|_{L_p}) = \|x_k\| \leq r$ for $x_k \in \Phi$. □

By Lemma 3, Φ takes the role of a probabilistic invariant set.

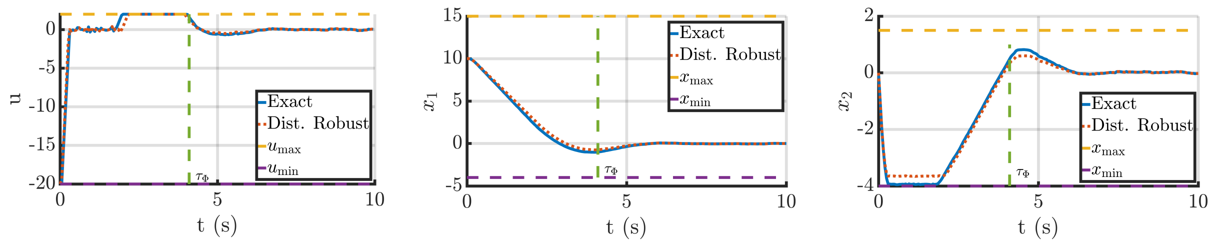
Lemma 4. *There exists $\tilde{\beta} \in \mathcal{KL}$ such that if $x^*_k, x^\dagger_k \in \Phi$,*

$$\mathbb{P}[\|x^*_{k+i} - x^\dagger_{k+i}\| \leq \tilde{\beta}(\|x^*_k\| + \|x^\dagger_k\|, k+i), \forall i \leq \bar{K}] \geq 1 - \varepsilon. \tag{44}$$

Proof. If $x^*_k, x^\dagger_k \in \Phi$, both systems remain in Φ for horizon $\bar{K}$ with probability at least $1 - \varepsilon$ according to (43). While $x^*_k, x^\dagger_k \in \Phi$, the unconstrained error dynamics between the fully informed and the distributionally robust closed-loop system is given by $e_{k+i} := x^*_{k+i} - x^\dagger_{k+i} = \bar{x}^*_{k+i} - \bar{x}^\dagger_{k+i}$ as both systems encounter the same disturbance w_{k+i} according to Assumption 1. Hence, exploiting (41), we find that $\forall i \leq \bar{K}$

$$\mathbb{P}[e_{k+i} = \bar{x}^*_{k+i} - \bar{x}^\dagger_{k+i}] \geq 1 - \varepsilon \tag{45}$$
$$\Leftrightarrow \; \mathbb{P}[\|e_{k+i}\| \leq \|x^*_{k+i}\| + \|x^\dagger_{k+i}\|] \geq 1 - \varepsilon \tag{46}$$
$$\Leftrightarrow \; \mathbb{P}[\|e_{k+i}\| \leq \beta^*(\|x^*_k\|, k+i) + \beta^\dagger(\|x^\dagger_k\|, k+i)] \geq 1 - \varepsilon \tag{47}$$

with $\beta^*, \beta^\dagger$ from Assumption 5. We can now construct $\tilde{\beta}(\|x^*_k\| + \|x^\dagger_k\|, k+i) = (M^* + M^\dagger)(\nu^* + \nu^\dagger)^{k+i}(\|x^*_k\| + \|x^\dagger_k\|) \geq \beta^*(\|x^*_k\|, k+i) + \beta^\dagger(\|x^\dagger_k\|, k+i)$, where clearly $\tilde{\beta} \in \mathcal{KL}$, which concludes the proof. □



(a) Control inputs $u_k^\star$ & $u_k^\dagger$

(b) System State: $[x]_1$

(c) System State: $[x]_2$

Figure 1: Simulation results for the distributionally robust and the fully informed case. The fully informed, distributionally robust, upper bound and lower bound quantities for all the respective parameters in all three subplots are plotted in blue, red, yellow and purple colors respectively. The green, dashed line indicates the time τ_Φ when both systems entered the set Φ. It is evident that the distributionally robust system makes cautious decisions and incurs regret.

Theorem 2. *If $x_k^*, x_k^\dagger \in \Phi$, then $\exists \sigma \in \mathcal{KL}$ such that*

$$\mathbb{P}[\|\mathfrak{R}_{k+i}^{\text{SMPC}}\| \le \sigma(\|x_k^*\| + \|x_k^\dagger\|, k+i), \forall i \le \bar{K}] \ge 1-\varepsilon. \tag{48}$$

Proof. If $x_k^*, x_k^\dagger \in \Phi$, both systems remain in Φ for horizon $\bar{K}$ with probability at least $1-\varepsilon$ according to (43). As no constraints are active for $x \in \Phi$, we find that $\forall i \le \bar{K}$

$$\begin{aligned} &\mathbb{P}[\mathfrak{R}_{k+i}^{\text{SMPC}} = (39)] \ge 1-\varepsilon \\ \Leftrightarrow\ &\mathbb{P}[\|\mathfrak{R}_{k+i}^{\text{SMPC}}\| \le \|x_{k+i}^* - x_{k+i}^\dagger\| \|\Lambda_1\| \|x_{k+i}^* + x_{k+i}^\dagger\|] \ge 1-\varepsilon \\ \Leftrightarrow\ &\mathbb{P}[\|\mathfrak{R}_{k+i}^{\text{SMPC}}\| \le \tilde{\beta}(\|x_k^*\| + \|x_k^\dagger\|, k+i) 2\|\Lambda_1\| r] \ge 1-\varepsilon, \end{aligned}$$

where $\|x_{k+i}^* + x_{k+i}^\dagger\| \le \|x_k^*\| + \|x_k^\dagger\| \le 2r$ as $x_{k+i}^*, x_{k+i}^\dagger \in \Phi$. Clearly, $2\|\Lambda_1\| r > 0$ and hence, $\sigma(\|x_k^*\| + \|x_k^\dagger\|, k+i) = 2\|\Lambda_1\| r \tilde{\beta}(\|x_k^*\| + \|x_k^\dagger\|, k+i)$ is a class $\mathcal{KL}$ function. $\square$

Theorem 2 implies that the cumulative regret $\mathfrak{c}\mathfrak{R}_k^{\text{SMPC}} = \sum_{i=0}^{k} \mathfrak{R}_i^{\text{SMPC}}$ will have sublinear growth over time. Furthermore, since regret is only observed when state constraints become active, those results indicate the potential of dynamically setting the constraints by leveraging the insights obtained from regret analysis.

4 Numerical Simulation

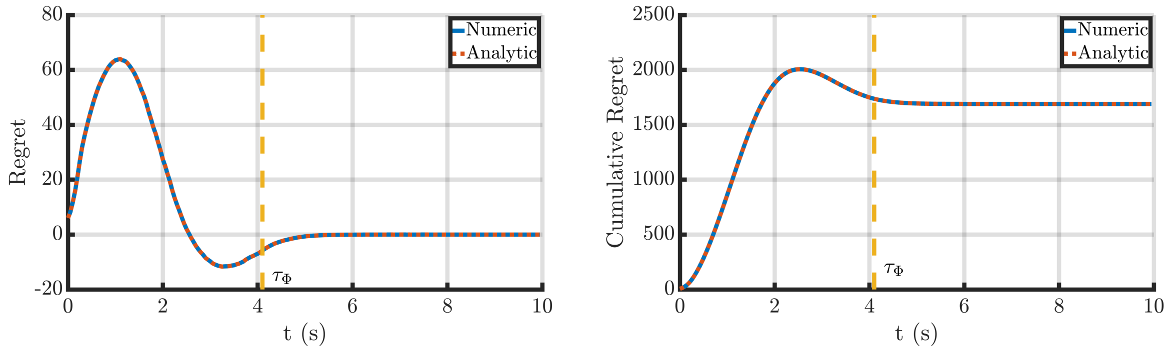
We consider a discretized double integrator with discretization time step $dt = 0.05$ given by

$$A = \begin{bmatrix} 1 & dt \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ dt \end{bmatrix}, C = I_2, D = 0_{2\times 1}. \tag{49}$$

The model was simulated for a total of $T = 200$ time steps with a total risk budget of $\Delta = 0.1$. The prediction horizon was set to be $N = 5$ time steps. The process noise was sampled from a multivariate Laplacian distribution with zero mean and covariance of $\mathbf{\Sigma^w} = 0.01^2 I_2$. The state and control penalty matrices are $Q = \text{diag}(1, 0.1)$ and $R = 0.1$ respectively. The control and the state constraints are given by $u \in [-20, 2]$ and $x \in [-4, 15] \times [-4, 1.5]$. The tightening constants for both the distributionally robust case and the fully informed case was obtained using (17). Further, the regret was computed using (29) for all time steps[3].

The results of simulating the system given by (49) using both the fully informed and the distributionally robust SMPC controller are shown in Figures 1 and 2. Specifically, the control inputs and the states are

[3]The code is available under: https://github.com/mpfefferk/Regret-and-Conservatism-of-SMPC.



(a) Instantaneous Distributional Regret

(b) Cumulative Distributional Regret

Figure 2: Distributional regret associated with the distributionally robust & the fully informed controller cases are shown here.

plotted in Figures 1(a) and 1(b-c) respectively. Similarly, the instantaneous and the cumulative distributional regret associated with the distributionally robust SMPC controller are plotted in Figures 2(a) and 2(b) respectively. It is evident that in general the distributionally robust system exhibits a cautious decision making and that we observe regret according to the three effects identified in Remark 3. Precisely, at $t = 4.1s$, all constraints become inactive for both the distributionally robust and the fully informed controller, indicating that both systems have entered the (here implicitly defined) set Φ. The distributionally robust controller being cautious to avoid constraint violation is at that time closer to the reference than the fully informed controller as seen in Figures 1(b-c). Since its remaining unconstrained cost-to-go is lower than its counterpart, this explains the observed, negative instantaneous regret in Figure 2(a).

5 Conclusion

A distributionally robust SMPC formulation exploiting moment-based ambiguity sets has been analysed in regard of conservatism and regret. We have performed an analysis of both conservatism and regret associated with distributionally robust SMPC for linear stochastic systems. We observed a sublinear growth of regret over time when state constraints became active. These findings were underlined by simulations. Future research will be dedicated towards applying the established framework to extended SMPC formulations that may include online estimation or learning of unknown quantities. Furthermore, the effect of non-uniform risk allocation on conservatism and regret is to be further investigated.

Acknowledgment

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program under grant agreement No 834142 (Scalable Control) and from the German Research Foundation (Research Training Group 2297). V. Renganathan is a member of the ELLIIT Strategic Research Area in Lund University.

References

Daniel Arnström. *Real-Time Certified MPC: Reliable Active-Set QP Solvers*. PhD thesis, Division of Automatic Control, Department of Electrical Engineering, Linköping University (Sweden), 2023.

Ronny Bergmann and Roland Herzog. Intrinsic Formulation of KKT Conditions and Constraint Qualifications on Smooth Manifolds. *SIAM Journal on Optimization*, 29(4):2423–2444, 2019.

S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

G. C. Calafiore and L. El Ghaoui. On distributionally robust chance-constrained linear programs. *Journal of Optimization Theory and Applications*, 130(1), December 2006.

Yitian Chen, Timothy L Molloy, Tyler Summers, and Iman Shames. Regret analysis of online lqr control via trajectory prediction and tracking. In *Proceedings of The 5th Annual Learning for Dynamics and Control Conference*, volume 211 of *Proceedings of Machine Learning Research*, pp. 248–258. PMLR, 15–16 Jun 2023.

Augustin Chevallier, Frédéric Cazals, and Paul Fearnhead. Efficient computation of the volume of a polytope in high-dimensions using piecewise deterministic Markov processes. In *25th International Conference on Artificial Intelligence and Statistics*, 2022.

Preston Culbertson, Ryan K. Cosner, Maegan Tucker, and Aaron D. Ames. Input-to-state stability in probability. *arXiv:2304.14578*, 2023.

ZR Gabidullina. The Minkowski difference for convex polyhedra and some its applications. *arXiv:1903.03590*, 2019.

Benyamin Ghojogh, Ali Ghodsi, Fakhri Karray, and Mark Crowley. KKT Contitions, First-Order and Second-Order Optimization, and Distributed Optimization: Tutorial and Survey. *arXiv:2110.01858*, 2021.

Aren Karapetyan, Andrea Iannelli, and John Lygeros. On the regret of H_∞ control. In *2022 IEEE 61st Conference on Decision and Control (CDC)*, pp. 6181–6186. IEEE, 2022.

Yingying Li, Xin Chen, and Na Li. Online Optimal Control with Linear Dynamics and Predictions: Algorithms and Regret Analysis. In *33rd Conference on Neural Information Processing Systems*, 2019.

Masahiro Ono and Brian C Williams. Iterative risk allocation: A new approach to robust model predictive control with a joint chance constraint. In *2008 47th IEEE Conference on Decision and Control*, pp. 3427–3432. IEEE, 2008.

Joel A Paulson, Edward A Buehler, Richard D Braatz, and Ali Mesbah. Stochastic model predictive control with joint chance constraints. *International Journal of Control*, 93(1):126–139, 2020.

James B. Rawlings, David Q. Mayne, and Moritz M. Diehl. *Model Predictive Control: Theory, Computation, and Design.* Nob Hill Publishing, 2 edition, 2019.