

# Elastic Full Procrustes Means for Sparse and Irregular Planar Curves

Masters Thesis

in partial fulfillment of the requirements for the degree

**M.Sc. Statistics**

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6<sup>th</sup> December, 2021, Berlin



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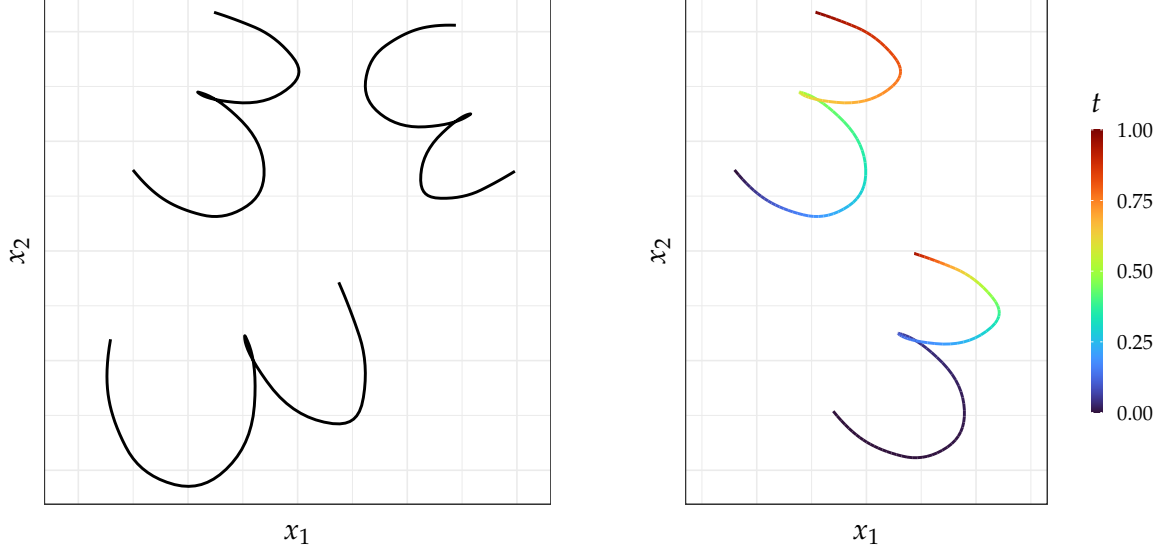
# 1. Introduction

sec:1

Statistical Shape Analysis (see e.g. DRYDEN and MARDIA 2016) is the branch of statistics concerned with modelling geometrical information. Such information might come in the form of outlines of bones or organs in a medical image, traced points along a handwritten digit, or data on the folding structure of a protein. This data is commonly captured using *landmarks*, which are characteristic points on the objects of interest that “match between and within populations”(DRYDEN and MARDIA 2016, p. 3). As an example, we might geometrically compare a set of mouse vertebrae by comparing the coordinates of prominent points along the bone outlines, which are common between all mouse vertebrae. More formally, we could say that each mouse vertebra’s geometrical information is then represented by a landmark configuration  $X \in \mathbb{R}^{k \times d}$ , which is the stacked matrix of the  $k$   $d$ -dimensional landmark coordinates, allowing for a multivariate treatment of shape or geometrical form.

A more flexible approach might be to treat e.g. the outline of an object as a whole, represented in the form of a continuous curve  $\beta : [0, 1] \rightarrow \mathbb{R}^d$ . Landmarks have the drawback that there is no clear way of choosing which points to include in the configuration, leaving the decision up to the subjectivity of the researcher. Furthermore, using landmarks leads to an inherently discrete treatment of the available data, which means modes of variation that lie in between landmarks may not be picked up by the analysis. By using curves, the analysis is not restricted to a fixed set of  $k$  discrete points, but instead uses all the available information. At the same time, the subjectivity in choosing the landmarks is eliminated. As each object then corresponds to one observation, this opens up a connection to the branch of statistics concerned with observations that are whole functions: Functional Data Analysis (see e.g. RAMSAY and SILVERMAN 2005).

When analyzing the geometry of objects, differences in location, rotation, and size are often not of interest. Instead, the focus lies purely on their differences in *shape*, a widely adapted definition of which was established by KENDALL 1977 and which might be



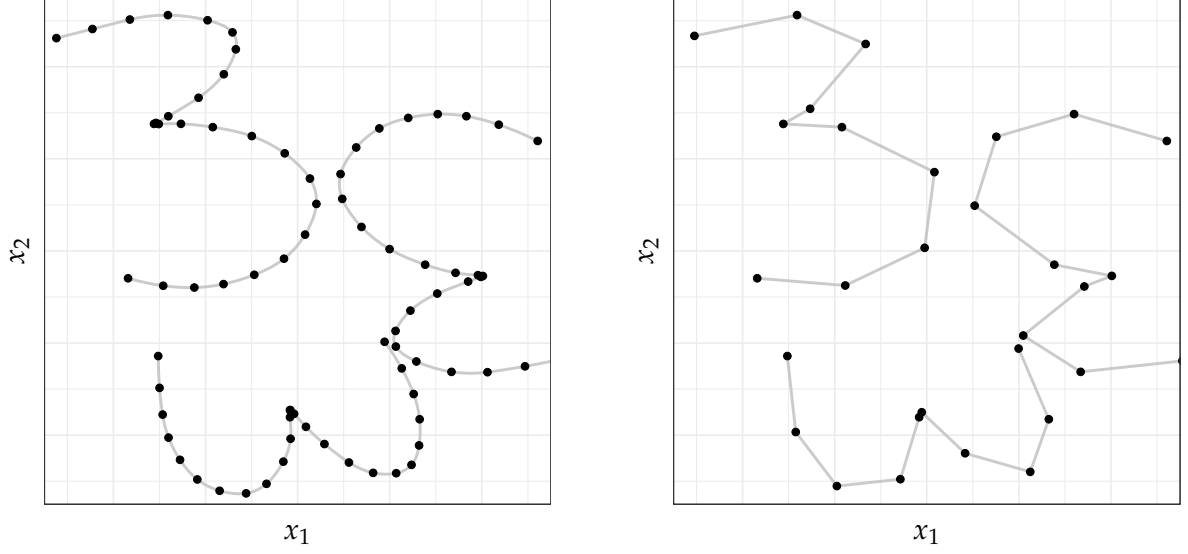
(a) The same digit with randomized rotation, scaling and translation applied three times.

(b) Original (left) and re-parameterized digit (right;  $t \mapsto t^5$ ). Color indicates the value of the parametrization  $t \in [0, 1]$  at point  $\beta(t)$ .

**Figure 1.1.:** Several representations of the same shape. Data: digits3.dat from the shapes package (DRYDEN 2019) for the R programming language (R CORE TEAM 2021) with smoothing applied using methods discussed in Appendix A.3. Original dataset collected by ANDERSON 1997.

formulated in the following way: “[A]ll the geometrical information that remains when location, scale and rotational effects are removed from an object” (DRYDEN and MARDIA 2016, p. 1). This is illustrated in Fig. 1.1a, where the same shape of a handwritten digit ‘3’ is plotted in three different orientations and sizes. When considering the shape of a curve  $\beta : [0, 1] \rightarrow \mathbb{R}^d$ , one has to additionally take into account effects relating to the parametrization  $t \in [0, 1]$ . As illustrated in Fig. 1.1b, curves  $\beta(t)$  and  $\beta(\gamma(t))$ , with some re-parametrization or *warping function*  $\gamma : [0, 1] \rightarrow [0, 1]$  monotonically increasing and differentiable, have the same image and therefore represent the same shape as well.

A pre-requisite for any statistical analysis of shape is the ability to calculate a distance between and to estimate a mean from observations, in a fashion that does not depend on location, rotation, scale and/or parametrization of the input. In this thesis two established approaches to shape analysis will be combined: Firstly, the *full Procrustes distance* and *mean* are widely used for translation-, rotation-, and scaling-invariant analysis of landmark data (see e.g. DRYDEN and MARDIA 2016, Chap. 4, 6). Secondly, SRIVASTAVA, KLASSEN, et al. 2011 introduced a mathematical framework for the *elastic*

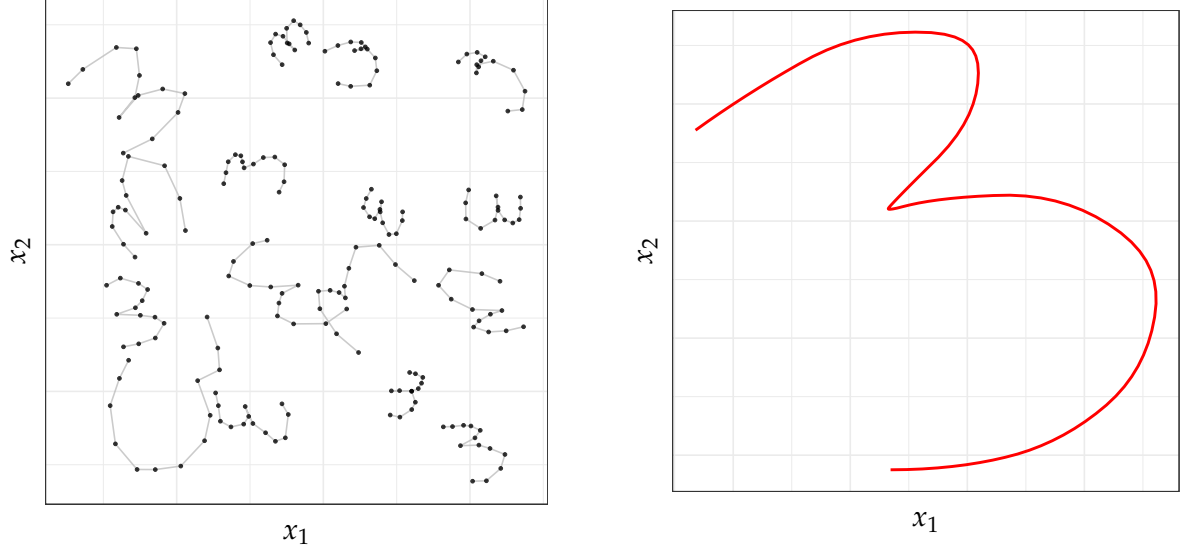


**Figure 1.2.:** Dense (left) and sparse (right) observations of the same three digits. Data: see Fig. 1.1, with the smooth curves sampled on a dense (left) and sparse (right) grid.

g:1-sparse

(re-parametrization invariant) shape analysis of curves, by using their square-root-velocity (SRV) representations. Taken together, both approaches allow for analyzing curves in a fashion that is invariant to all four shape-preserving transformations, leading to an *elastic full Procrustes distance and mean*. As the full Procrustes mean has particularly nice properties in two dimensions, when identifying  $\mathbb{R}^2$  with  $\mathbb{C}$  (see DRYDEN and MARDIA 2016, Chap. 8), this thesis will be restricted to the case of planar curves.

While we are interested in modelling a (planar) object's geometrical information as a continuous curve  $\beta : [0, 1] \rightarrow \mathbb{R}^2$ , the curve itself is usually only observed as a discrete set of points  $\beta(t_1), \beta(t_2), \dots, \beta(t_m)$ . As shown on the left side of Fig. 1.2 this is no problem when the number  $m$  of observed points is high and the whole length of the curve is densely observed, as we can easily interpolate  $\beta(t)$  for any  $t \in [0, 1]$ . However, in cases where  $\beta$  is only observed over a small number of points (right side) and where the density and position of observed points may even vary between different curves—a setting known as *sparse* and *irregular*—more sophisticated smoothing techniques have to be applied. While the SRV framework has been combined with a Procrustes distance before, to estimate elastic shape means that also include invariance under scaling, rotation and translation (see SRIVASTAVA, KLASSEN, et al. 2011), these approaches



**Figure 1.3.:** Elastic full Procrustes mean function (right) estimated from sparse and irregular observations (left) using a 13 knot linear B-spline basis, with a 2nd order roughness penalty on SRV level. Data: Original (un-smoothed) digits3.dat with additional random rotation, scaling and translation applied.

fig:1-mean

have mostly focused on *Riemannian* or *geodesic* mean concepts and are not specially designed with sparse or irregular observations in mind. On the other hand, as will be shown, the estimation of the *elastic full Procrustes mean* in two dimensions is related to an eigenfunction problem over the complex covariance surface of the observed curves. This offers an advantage, as we can then make use of established smoothing techniques for the estimation of covariance surfaces in the sparse and irregular setting. Here, in particular CEDERBAUM, SCHEIPL, and GREVEN 2018 offers a method for efficient covariance smoothing using *tensor product P-splines* (see e.g. FAHRMEIER et al. 2013, Chap. 8.2).

The aim of this thesis, as illustrated in Fig. 1.3, is to extend existing methods for elastic mean estimation of sparse and irregularly sampled curves, as proposed by STEYER, STÖCKER, and GREVEN 2021 and implemented in the package `elasdics` (STEYER 2021) for the R programming language (R CORE TEAM 2021), to also include invariance with respect to rotation and scaling. The later will be achieved by generalizing the concept of the full Procrustes mean from landmark to functional data and by iteratively applying full Procrustes mean estimation, rotation-alignment and parametrization-alignment, leading to the estimation of elastic full Procrustes means. Here, techniques

for Hermitian smoothing of the complex covariance surfaces as available in the R package `sparseFLMM` (CEDERBAUM, VOLKMANN, and STÖCKER 2021) will be used.

The thesis is organized as follows. In Chapter 2 relevant background material is reviewed and the elastic full Procrustes mean is derived, in the case where curves  $\beta : [0, 1] \rightarrow \mathbb{R}^2$  are fully observed. In Chapter 3 an estimation procedure for the setting of sparse and irregularly observed curves  $\beta(t_1), \dots, \beta(t_m)$  is proposed, concluding the theoretical part of this thesis. In Chapter 4 the proposed methods will be verified and applied using simulated and empirical datasets. Finally, all results will be summarized in Chapter 5. Appendix A and Appendix B offer additional considerations and some reproducibility guidelines.



## 2. Elastic Full Procrustes Means for Planar Curves

sec:2

As a starting point, it is important to establish a notational and mathematical framework for the treatment of planar shapes. While the restriction to the 2D case might seem a major one, it still covers all shape data extracted from e.g. imagery and is therefore very applicable in practice. The outline of a 2D object may be naturally represented by a planar curve  $\beta : [0, 1] \rightarrow \mathbb{R}^2$  with  $\beta(t) = (x_1(t), x_2(t))^\top$ , where  $x_1(t)$  and  $x_2(t)$  are the scalar-valued *coordinate functions*. Calculations in two dimensions, and in particular the derivation of the full Procrustes mean, are greatly simplified by using complex notation. We will therefore identify  $\mathbb{R}^2$  with  $\mathbb{C}$ , as shown in Fig. 2.1, and always use complex notation when representing a planar curve:

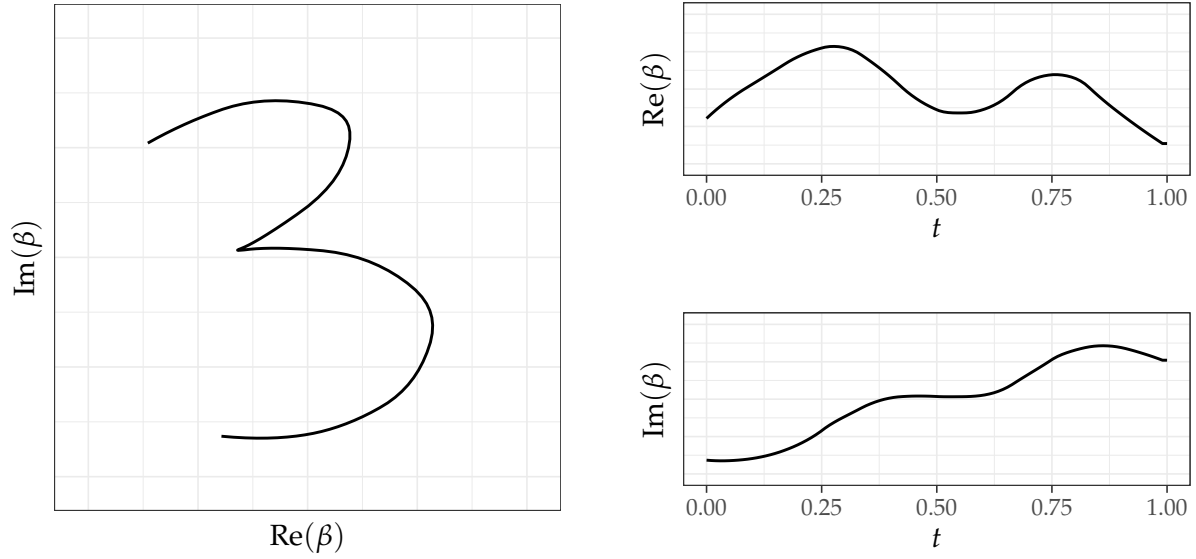
$$\beta : [0, 1] \rightarrow \mathbb{C}, \quad \beta(t) = x_1(t) + ix_2(t).$$

We will assume the curves to be absolutely continuous, denoted as  $\beta \in \mathcal{AC}([0, 1], \mathbb{C})$ , guaranteeing that  $\beta(t)$  has an integrable derivative. This is important when working in the square-root-velocity (SRV) framework, as will be discussed in Section 2.2. All considerations will be restricted to the case of open curves, with possible extensions to closed curves  $\beta \in \mathcal{AC}(\mathbb{S}^1, \mathbb{C})$  discussed in Appendix A.2 of the appendix.

### 2.1. Equivalence Classes and Shape

ec:2-shape

As mentioned in the introduction, shape is usually defined by its invariance under the transformations of scaling, translation, and rotation. When considering the shape of curves, we additionally have to take into account invariance with respect to re-parametrisation. This can be seen, by noting that the curves  $\beta(t)$  and  $\beta(\gamma(t))$ , with some re-parametrisation or *warping function*  $\gamma : [0, 1] \rightarrow [0, 1]$  monotonically increasing and differentiable, have the same image and therefore represent the same geometrical object (see Fig. 1.1b). We can say that the actions of translation, scaling, rotation, and



**Figure 2.1.:** Example of a planar curve (left) with respective coordinate functions (right) using complex notation, where  $\text{Re}(\beta) = x_1$  and  $\text{Im}(\beta) = x_2$  denote the *real* and *imaginary* parts of  $\beta$ . Data: see Fig. 1.1.

ig: 2-curve

re-parametrisation are *equivalence relations* with respect to shape, as each action leaves the shape of the curve untouched and only changes the way it is represented. The shape of a curve can then be defined as the respective *equivalence class*, i.e. the set of all possible shape preserving transformations of the curve. As two equivalence classes are necessarily either disjoint or identical, we can consider two curves as having the same shape, if they are elements of the same equivalence class (see SRIVASTAVA and KLASSEN 2016, p. 40).

When defining an equivalence class, one has to first consider how each individual transformation acts on a planar curve  $\beta : [0, 1] \rightarrow \mathbb{C}$ . This is usually done using the notion of *group actions* and *product groups*, with the later describing multiple transformations acting at once. A brief introduction to group actions may be found in SRIVASTAVA and KLASSEN 2016, Chap. 3.

1. The *translation* group  $\mathbb{C}$  acts on  $\beta$  by  $(\xi, \beta) \xrightarrow{\text{Trl}} \beta + \xi$ , for any  $\xi \in \mathbb{C}$ . We can consider two curves as equivalent with respect to translation  $\beta_1 \stackrel{\text{Trl}}{\sim} \beta_2$ , if there exists a complex scalar  $\tilde{\xi} \in \mathbb{C}$  such that  $\beta_1 = \beta_2 + \tilde{\xi}$ . Then, for some function  $\beta$ , the related equivalence class with respect to translation is given by  $[\beta]_{\text{Trl}} = \{\beta + \xi \mid \xi \in \mathbb{C}\}$ .
2. The *scaling* group  $\mathbb{R}^+$  acts on  $\beta$  by  $(\lambda, \beta) \xrightarrow{\text{Scl}} \lambda\beta$ , for any  $\lambda \in \mathbb{R}^+$ . We define  $\beta_1 \stackrel{\text{Scl}}{\sim} \beta_2$ , if there exists a scalar  $\tilde{\lambda} \in \mathbb{R}^+$  such that  $\beta_1 = \tilde{\lambda}\beta_2$ . An equivalence class

is  $[\beta]_{\text{Scl}} = \{\lambda\beta \mid \lambda \in \mathbb{R}^+\}$ .

3. The *rotation* group  $[0, 2\pi)$  acts on  $\beta$  by  $(\theta, \beta) \xrightarrow{\text{Rot}} e^{i\theta}\beta$ , for any  $\theta \in [0, 2\pi)$ . We define  $\beta_1 \stackrel{\text{Rot}}{\sim} \beta_2$ , if there exists a  $\tilde{\theta} \in [0, 2\pi)$  with  $\beta_1 = e^{i\tilde{\theta}}\beta_2$ . An equivalence class is  $[\beta]_{\text{Rot}} = \{e^{i\theta}\beta \mid \theta \in [0, 2\pi)\}$ .
4. The *warping* group  $\Gamma$  acts on  $\beta$  by  $(\gamma, \beta) \xrightarrow{\text{Wrp}} \beta \circ \gamma$ , for any  $\gamma \in \Gamma$  with  $\Gamma$  being the set of monotonically increasing and differentiable warping functions. We define  $\beta_1 \stackrel{\text{Wrp}}{\sim} \beta_2$ , if there exists a warping function  $\tilde{\gamma} \in \Gamma$  with  $\beta_1 = \beta_2 \circ \tilde{\gamma}$ . An equivalence class is  $[\beta]_{\text{Wrp}} = \{\beta \circ \gamma \mid \gamma \in \Gamma\}$ .

In a next step, we can consider how these transformations act in concert and whether they *commute*, i.e. whether the order of applying the transformations changes outcomes. Consider for example the actions of the *rotation and scaling* product group  $\mathbb{R}^+ \times [0, 2\pi)$  given by  $((\lambda, \theta), \beta) \xrightarrow{\text{Scl}+\text{Rot}} \lambda e^{i\theta}\beta$ , which clearly commutes as  $\lambda(e^{i\theta}\beta) = e^{i\theta}(\lambda\beta)$ . On the other hand, the joint actions of *scaling and translation* do not commute, as  $\lambda(\beta + \xi) \neq \lambda\beta + \xi$ , with the same holding for the joint actions of *rotation and translation*. As the order of translating and rotating or scaling matters, one usually takes the translation to act on the already scaled and rotated curve. The joint action defined using this ordering is called an *Euclidean similarity transformation* with  $((\xi, \lambda, \theta), \beta) \xrightarrow{\text{Eucl}} \lambda e^{i\theta}\beta + \xi$  (see DRYDEN and MARDIA 2016, p. 62). Considering the action of *warping* or re-parametrization, we can note that it necessarily commutes with all Euclidean similarity transformations as those only act on the image of  $\beta$ , while the former only acts on the parametrization. Putting everything together we can give a formal definition of the shape of a planar curve as the following equivalence class:

**Definition 2.1** (Shape). The *shape* of an absolutely continuous planar curve  $\beta \in \mathcal{AC}([0, 1], \mathbb{C})$  is given by its equivalence class  $[\beta]$  with respect to all Euclidean similarity transformations and re-parametrizations

$$[\beta] = \left\{ \lambda e^{i\theta}(\beta \circ \gamma) + \xi \mid \xi \in \mathbb{C}, \lambda \in \mathbb{R}^+, \theta \in [0, 2\pi), \gamma \in \Gamma \right\}.$$

The *shape space*  $\mathcal{S}$  is then given by  $\mathcal{S} = \{[\beta] \mid \beta \in \mathcal{AC}([0, 1], \mathbb{C})\}$ .

## 2.2. The Elastic Full Procrustes Distance for Planar Curves

Let us now turn to the construction of an appropriate *shape distance*  $d([\beta_1], [\beta_2])$  for two curves  $\beta_1, \beta_2$ . As the shapes  $[\beta_1]$  and  $[\beta_2]$  are elements of a non-Euclidean quotient space (the shape space  $\mathcal{S}$ ), calculating a distance between them is not straight-forward. A common approach is to map each equivalence class  $[\beta]$  to a suitable representative  $\tilde{\beta}$ , so that the distance calculation in shape space can be identified with a (much simpler) distance calculation over the representatives in an underlying functional space.

To illustrate this, let us first discuss each type of shape-preserving transformation individually, starting with the Euclidean similarity transformations. Consider two equivalence classes with respect to translation  $[\beta_1]_{\text{Trl}}, [\beta_2]_{\text{Trl}}$ . They might be uniquely mapped to their centered elements  $\tilde{\beta}_i^{\text{Trl}} = \beta_i - \bar{\beta}_i \in [\beta_i]_{\text{Trl}}$  for  $i = 1, 2$ . The distance between the centered elements then defines a distance that is invariant under translation  $d_{\text{Trl}}([\beta_1]_{\text{Trl}}, [\beta_2]_{\text{Trl}}) = \|\tilde{\beta}_1^{\text{Trl}} - \tilde{\beta}_2^{\text{Trl}}\|$ , which is equal to the minimal distance when optimizing over the translation group. Similarly, a distance that is invariant under scaling might be defined over the normalized elements  $\tilde{\beta}_i^{\text{Scl}} = \frac{\beta_i}{\|\beta_i\|} \in [\beta_i]_{\text{Scl}}$  for  $i = 1, 2$ , as  $d_{\text{Scl}}([\beta_1]_{\text{Scl}}, [\beta_2]_{\text{Scl}}) = \|\tilde{\beta}_1^{\text{Scl}} - \tilde{\beta}_2^{\text{Scl}}\|$ . When considering invariance under rotation, we can first note that no “standardization” procedure comparable to normalizing and centering exists for the case of rotation. Instead of mapping  $[\beta]_{\text{Rot}}$  to a fixed representative, we therefore have to identify an appropriate representative on a case-by-case basis. This can be achieved by defining the distance as the minimal distance  $d_{\text{Rot}}([\beta_1]_{\text{Rot}}, [\beta_2]_{\text{Rot}}) = \min_{\tilde{\beta}_2^{\text{Rot}} \in [\beta_2]_{\text{Rot}}} \|\beta_1 - \tilde{\beta}_2^{\text{Rot}}\| = \min_{\theta \in [0, 2\pi)} \|\beta_1 - e^{i\theta} \beta_2\|$ , when keeping one curve fixed and rotationally aligning the other curve (see e.g. STÖCKER and GREVEN 2021).

### The Full Procrustes Distance

The three approaches can be combined to formulate the family of *Procrustes* distances, which are invariant under Euclidean similarity transform. The *partial Procrustes distance* is defined as the minimizing distance  $d_{PP}([\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}}) = \min_{\theta \in [0, 2\pi)} \|\tilde{\beta}_1 - e^{i\theta} \tilde{\beta}_2\|$ , when rotationally aligning the centered and normalized curves  $\tilde{\beta}_i = \frac{\beta_i - \bar{\beta}_i}{\|\beta_i - \bar{\beta}_i\|}$ ,  $i = 1, 2$ . On the other hand, the *full Procrustes distance* (see Definition 2.2) includes an additional alignment over scaling, leading to a slightly different geometrical interpretation (see

DRYDEN and MARDIA 2016, pp. 77–78). Finally, the *Procrustes distance* given by  $d_P([\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}}) = \arccos |\langle \tilde{\beta}_1, \tilde{\beta}_2 \rangle|$  defines a geodesic distance in the space of curves modulo Euclidean similarity transforms. As earlier approaches to elastic shape mean estimation, such as SRIVASTAVA, KLASSEN, et al. 2011, have been focused on calculating *intrinsic* means, they mostly use the Procrustes distance. Although no distance definition is inherently better than the other, in the context of mean estimation for sparse and irregular curves the full Procrustes distance might be slightly more suitable, as the additional scaling alignment offers more flexibility in a setting where—as will become clear later on—calculating a norm  $\|\beta\| = \int_0^1 \|\beta(t)\| dt$  may already present a challenge. In this thesis, we will therefore only consider the full Procrustes distance and calculate *extrinsic* shape means. Note that in Definition 2.2 the optimization over scaling  $\lambda \in \mathbb{R}^+$  and rotation  $\theta \in [0, 2\pi)$  was combined into a single optimization over *rotation and scaling*  $\omega = \lambda e^{i\theta} \in \mathbb{C}$ .

def:2-fpdist

**Definition 2.2** (Full Procrustes distance). The *full Procrustes distance* for two equivalence classes  $[\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}}$  is defined as

$$d_{FP}([\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}}) = \min_{\omega \in \mathbb{C}} \|\tilde{\beta}_1 - \omega \tilde{\beta}_2\| \quad (2.1) \quad \{\text{eq:2-fpdist}\}$$

with centered and normalized representatives  $\tilde{\beta}_i = \frac{\beta_i - \bar{\beta}_i}{\|\beta_i - \bar{\beta}_i\|}$ .

By using a similar proof for complex-valued landmark data in DRYDEN and MARDIA 2016, Chap 8 as a blueprint, we can show that Eq. (2.1) has the following analytical solution.

m:2-fpdist

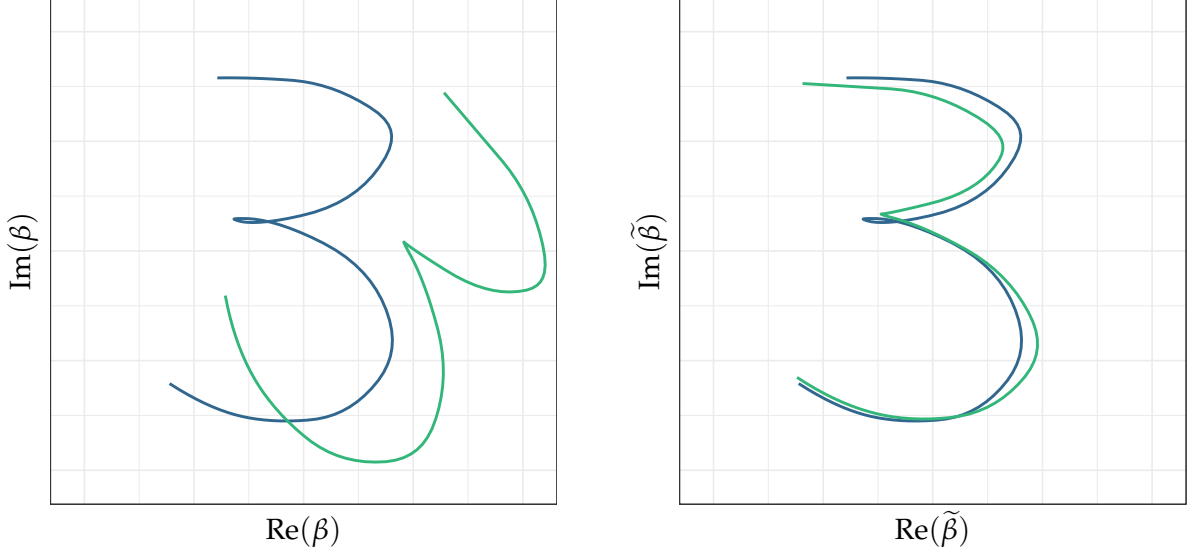
**Lemma 2.1.** Let  $\beta_1, \beta_2 : [0, 1] \rightarrow \mathbb{C}$  be two planar curves with corresponding equivalence classes  $[\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}}$  with respect to Euclidean similarity transforms and let  $\tilde{\beta}_i = \frac{\beta_i - \bar{\beta}_i}{\|\beta_i - \bar{\beta}_i\|}$ .

i.) The full Procrustes distance between  $[\beta_1]_{\text{Eucl}}$  and  $[\beta_2]_{\text{Eucl}}$  is given by

$$d_{FP}([\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}}) = \sqrt{1 - \langle \tilde{\beta}_1, \tilde{\beta}_2 \rangle \langle \tilde{\beta}_2, \tilde{\beta}_1 \rangle} \quad (2.2)$$

ii.) The optimal rotation and scaling alignment of  $\tilde{\beta}_2$  onto  $\tilde{\beta}_1$  is given by  $\omega^{\text{opt}} = \langle \tilde{\beta}_2, \tilde{\beta}_1 \rangle$ .

The aligned curve  $\tilde{\beta}_2^P = \langle \tilde{\beta}_2, \tilde{\beta}_1 \rangle \cdot \tilde{\beta}_2$  is then called the Procrustes fit of  $\tilde{\beta}_2$  onto  $\tilde{\beta}_1$ .



**Figure 2.2.:** Procrustes fit (right; normalized and centered) of two example curves (left). The Procrustes fit of  $\beta_2$  (green) onto  $\beta_1$  (blue) is given by  $\tilde{\beta}_2^p = \langle \tilde{\beta}_2, \tilde{\beta}_1 \rangle \tilde{\beta}_2$ . Data: See Fig. 1.1.

fig:2-pfit

| *Proof.* See Appendix A.1.1. □

Fig. 2.2 shows an example of two curves that were aligned by minimizing their full Procrustes distance using Lemma 2.1.

### The Elastic Distance

When considering warping, we would like to do something similar to rotation in trying to find an optimal warping alignment between two curves  $\beta_1, \beta_2$  by optimizing their distance over the space of warping functions  $\Gamma$ . A usual choice would be to optimize over the  $\mathbb{L}^2$ -distance by  $\inf_{\gamma \in \Gamma} \|\beta_1 - \beta_2 \circ \gamma\|_{\mathbb{L}^2}$ , where the  $\mathbb{L}^2$ -norm and scalar-product are given by  $\|f\|_{\mathbb{L}^2} = \sqrt{\int_0^1 \overline{f(t)} f(t) dt}$  and  $\langle f, g \rangle_{\mathbb{L}^2} = \int_0^1 \overline{f(t)} g(t) dt$  with  $\bar{z} = \text{Re}(z) - i\text{Im}(z)$  denoting the complex conjugate of  $z \in \mathbb{C}$ . However, as optimizing over re-parametrization using the  $\mathbb{L}^2$ -distance has problems relating to the so called *pinching effect* and *inverse-inconsistency*, this does not define a proper distance. Here, the latter means that aligning the parametrization of one curve to another by  $\inf_{\gamma \in \Gamma} \|\beta_1 - \beta_2 \circ \gamma\|_{\mathbb{L}^2}$  may yield different results than  $\inf_{\gamma \in \Gamma} \|\beta_2 - \beta_1 \circ \gamma\|_{\mathbb{L}^2}$  (see SRIVASTAVA and KLASSEN 2016, pp. 88–90).

A solution proposed in SRIVASTAVA, KLASSEN, et al. 2011 is to replace the  $\mathbb{L}^2$ -distance with an *elastic distance*, on which warping acts by isometry. Calculation of

this metric, the Fisher-Rao Riemannian metric (RAO 1945), can be greatly simplified by using the *square-root-velocity* (SRV) framework, as the Fisher-Rao metric of two curves can be equivalently calculated as the  $\mathbb{L}^2$ -distance of their respective SRV curves. As this SRV representation makes use of derivatives, any curve  $\beta$  that has a SRV curve must fulfill some kind of differentiability constraint. Here it is enough to consider only curves that are absolutely continuous  $\beta \in \mathcal{AC}([0, 1], \mathbb{C})$ , which in particular means that the original curves do not have to be smooth but might also be, e.g. piecewise linear (see SRIVASTAVA and KLASSEN 2016, p. 91). Note that, because of the use of derivatives, any elastic analysis of curves will automatically be translation invariant as well. See Fig. 2.3 for an example SRV curve of a digit ‘3’.

def:2-eldist

**Definition 2.3** (Elastic distance (SRIVASTAVA, KLASSEN, et al. 2011)). The *elastic distance* between equivalence classes  $[\beta_1]_{\text{WRP+Trl}}, [\beta_2]_{\text{WRP+Trl}}$  is defined as

$$d_E([\beta_1]_{\text{WRP+Trl}}, [\beta_2]_{\text{WRP+Trl}}) = \inf_{\gamma \in \Gamma} \|q_1 - (q_2 \circ \gamma) \cdot \sqrt{\dot{\gamma}}\|_{\mathbb{L}^2} \quad (2.3)$$

{eq:2-eldist}

with the respective *square-root-velocity* (SRV) representations  $q_i \in \mathbb{L}^2([0, 1], \mathbb{C})$  given by

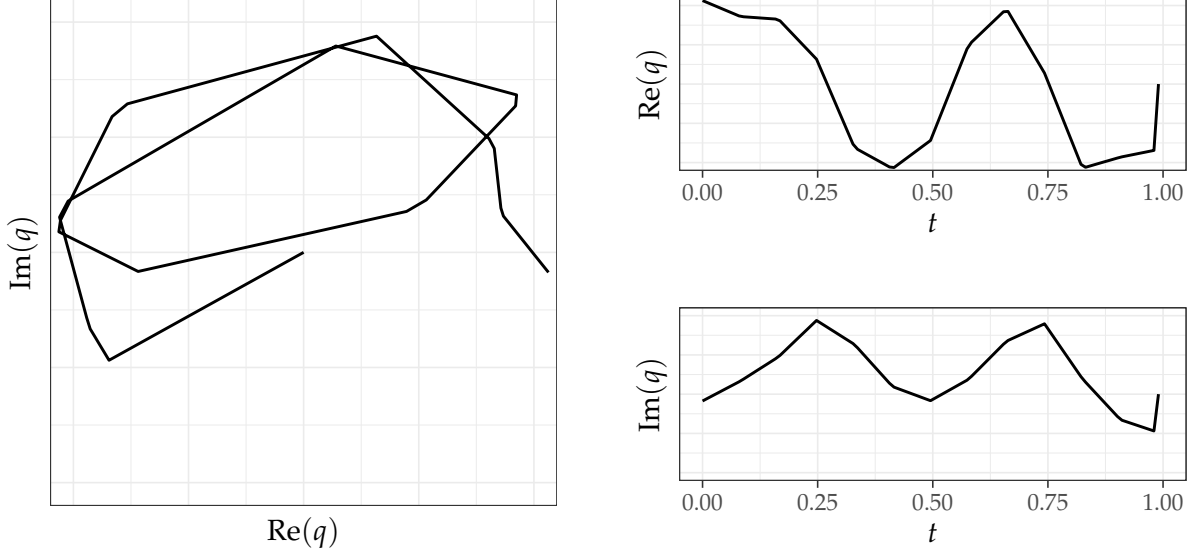
$$q_i(t) = \begin{cases} \frac{\dot{\beta}_i(t)}{\sqrt{\|\dot{\beta}_i(t)\|}} & \text{for } \dot{\beta}_i(t) \neq 0, \\ 0 & \text{for } \dot{\beta}_i(t) = 0, \end{cases} \quad (2.4)$$

where  $\beta_i \in \mathcal{AC}([0, 1], \mathbb{C})$  and  $\dot{\beta}_i(t) = \frac{d\beta_i(t)}{dt}$  for  $i = 1, 2$ .

Unlike the optimization over rotation in the definition of the full Procrustes distance, no analytical solution exists for the optimization over warping in Eq. (2.3). Instead, it is usually solved numerically, by minimizing a cost function  $H[\gamma] = \int_0^1 \|q_1(t) - q_2(\gamma(t)) \sqrt{\dot{\gamma}(t)}\| dt$  using a dynamic programming algorithm (see e.g. SRIVASTAVA and KLASSEN 2016, p. 152) or gradient based methods (see e.g. STEYER, STÖCKER, and GREVEN 2021).

## The Elastic Full Procrustes Distance

When the original curves  $\beta$  are absolutely continuous, the SRV curves are always ensured to be  $\mathbb{L}^2$ -integrable. As a consequence, we can re-construct the original



**Figure 2.3.:** SRV function (left) of the planar curve in Fig. 2.1 with respective SRV coordinate functions (right). Note that the polygon-like look of the SRV curve is an artifact of the (linear) smoothing applied to the original data on SRV level (see Appendix A.3). Data: see Fig. 1.1.

curve  $\beta$  up to translation from its respective SRV curve  $q$  by integration  $\beta(t) = \beta(0) + \int_0^t q(s) \|q(s)\| ds$ . Because the translation of the original curve is usually not of interest from the point of shape analysis, the SRV curve holds all relevant information about the shape of  $\beta$ . This means, in particular, that instead of analysing the shape of  $\beta$ , we can equivalently analyse the shape of  $q$ . The shape preserving transformations on original curve level translate to SRV curve level by actions laid out in Lemma 2.2.

**Lemma 2.2.** *The actions of the translation, scaling, rotation, and re-parametrization groups commute on SRV level. Furthermore, the individual transformations translate to SRV level by*

$$i.) (\xi, q) \xrightarrow{Trl} q, \quad ii.) (\lambda, q) \xrightarrow{Scl} \sqrt{\lambda} q, \quad iii.) (\theta, q) \xrightarrow{Rot} e^{i\theta} q, \quad iv.) (\gamma, q) \xrightarrow{Wrp} (q \circ \gamma) \sqrt{\dot{\gamma}}$$

(see e.g. SRIVASTAVA and KLASSEN 2016, p. 142).

*Proof.* The SRVF  $\tilde{q}(t)$  of  $\tilde{\beta}(t) = \lambda e^{i\theta} \beta(\gamma(t)) + \xi$  is given by

$$\tilde{q}(t) = \frac{\lambda e^{i\theta} \dot{\beta}(\gamma(t)) \dot{\gamma}(t)}{\sqrt{\|\lambda e^{i\theta} \dot{\beta}(\gamma(t)) \dot{\gamma}(t)\|}} = \sqrt{\lambda} e^{i\theta} \frac{\dot{\beta}(\gamma(t))}{\sqrt{\|\dot{\beta}(\gamma(t))\|}} \sqrt{\dot{\gamma}(t)} = \sqrt{\lambda} e^{i\theta} (q \circ \gamma) \sqrt{\dot{\gamma}(t)}.$$

The result is irrespective of the order of applying the transformations.  $\square$



We can note that the SRV curves are invariant under translation of the original curves, that the rotation is preserved on the SRV level, that scaling translates to SRV level by  $\sqrt{\cdot}$ . It is in particular noteworthy, that warping the original curve changes the image of the SRV curve.

Going forward, we will work in the SRV framework and combine the elastic distance with the full Procrustes distance. While the full Procrustes distance (see Definition 2.2) was defined over the normalized and centered curves, the SRV curves are already translation invariant so additional centering is not necessary (see Lemma 2.2 i.). We will therefore define the *elastic full Procrustes distance* as the minimal distance, when aligning the scaling, rotation, and warping of the normalized SRV curves  $\tilde{q} = \frac{q}{\|q\|}$ . Note that when the original curve  $\beta$  is of unit length  $L[\beta] = \int_0^1 |\dot{\beta}(t)| dt = 1$  the SRV curve  $q = \frac{\dot{\beta}}{\|\dot{\beta}\|}$  will be normalized, as

$$\|q\| = \sqrt{\int_0^1 |q(t)|^2 dt} = \sqrt{\int_0^1 |\dot{\beta}(t)|^2 dt} = \sqrt{L[\beta]}. \quad (2.5) \quad \text{\texttt{\{eq:2-norm\}}}$$

**Definition 2.4** (Elastic full Procrustes distance). The *elastic full Procrustes distance* between shapes  $[\beta_1], [\beta_2]$  of two continuously differentiable planar curves  $\beta_1, \beta_2 \in \mathcal{AC}([0, 1], \mathbb{C})$  is given by

$$d([\beta_1], [\beta_2]) = \inf_{\omega \in \mathbb{C}, \gamma \in \Gamma} \|\tilde{q}_1 - \omega(\tilde{q}_2 \circ \gamma) \sqrt{\dot{\gamma}}\|_{\mathbb{L}^2}, \quad (2.6) \quad \text{\texttt{\{eq:2-elfpdist\}}}$$

with normalized SRV representation  $\tilde{q}_i = \frac{q_i}{\|q_i\|} \in \mathbb{L}^2([0, 1], \mathbb{C})$ , where  $q_i$  is the SRV representation of  $\beta_i$ , for  $i = 1, 2$ .

To calculate the elastic full Procrustes distance, we need to solve the joint optimization problem over  $\mathbb{C} \times \Gamma$

$$(\omega^{\text{opt}}, \gamma^{\text{opt}}) = \underset{\omega \in \mathbb{C}, \gamma \in \Gamma}{\operatorname{argmin}} \|\tilde{q}_1 - \omega(\tilde{q}_2 \circ \gamma) \sqrt{\dot{\gamma}}\|_{\mathbb{L}^2}, \quad (2.7) \quad \text{\texttt{\{eq:2-elfpdis\}}}$$

so that the elastic full Procrustes distance is given as the  $\mathbb{L}^2$ -distance of the optimally aligned normalized SRV curves

$$d([\beta_1], [\beta_2]) = \|\tilde{q}_1 - \omega^{\text{opt}}(\tilde{q}_2 \circ \gamma^{\text{opt}}) \sqrt{\dot{\gamma}^{\text{opt}}}\|_{\mathbb{L}^2}. \quad (2.8)$$

Following SRIVASTAVA, KLASSEN, et al. 2011 we adapt an iterative procedure, where in each step  $k$  we optimize over the sets of parameters individually, iterating through both solutions until some form of convergence is reached. Let us first consider the optimization over  $\omega \in \mathbb{C}$  for a fixed  $\gamma^{(k)} \in \Gamma$ .

$$\omega^{(k)} = \underset{\omega \in \mathbb{C}}{\operatorname{argmin}} \|\tilde{q}_1 - \omega(\tilde{q}_2 \circ \gamma^{(k)})\sqrt{\dot{\gamma}^{(k)}}\|_{\mathbb{L}^2}, \quad (2.9)$$

$\Gamma$  nicht  
voll-  
ständig.

Eq. (2.9) is equivalent to the optimization problem of the full Procrustes distance defined in Definition 2.2. Following Lemma 2.1 ii.), the solution is given by the Procrustes fit of  $(\tilde{q}_2 \circ \gamma^{(k)})\sqrt{\dot{\gamma}^{(k)}}$  onto  $\tilde{q}_1$  with

$$\omega^{(k)} = \langle (\tilde{q}_2 \circ \gamma^{(k)})\sqrt{\dot{\gamma}^{(k)}}, \tilde{q}_1 \rangle \quad (2.10)$$

{eq:2-elfp

For fixed rotation and scaling  $\omega^{(k)} \in \mathbb{C}$  the optimization problem over  $\gamma \in \Gamma$  is given by

$$\gamma^{(k+1)} = \underset{\gamma \in \Gamma}{\operatorname{arginf}} \|\tilde{q}_1 - (\omega^{(k)}\tilde{q}_2 \circ \gamma)\sqrt{\dot{\gamma}}\|_{\mathbb{L}^2}. \quad (2.11)$$

{eq:2-elfp

Again, Eq. (2.11) is equivalent to the optimization problem of the elastic distance defined in Definition 2.3, when aligning the parametrization of the normalized SRV curve  $\tilde{q}_1$  and the rotation and scaling aligned, normalized SRV curve  $\omega^{(k)}\tilde{q}_2$ . A solution  $\gamma^{(k+1)}$  can be found by applying known optimization techniques such as a dynamical programming algorithm or a gradient based method. In this thesis we will use the methods laid out in STEYER, STÖCKER, and GREVEN 2021 and implemented in the R package `elastdics` (STEYER 2021) for solving Eq. (2.11) in the setting of sparse and irregularly sampled curves.

alg:2-dist

**Algorithm 2.1** (Elastic full Procrustes distance).  $\beta_1, \beta_2$  absolutely continuous planar curves with SRV curves  $q_1, q_2 \in \mathbb{L}^2([0, 1], \mathbb{C})$  and normalized SRV curves  $\tilde{q}_i = \frac{q_i}{\|q_i\|}$ . Set  $\gamma^{(0)}(t) = t$  as the initial parametrization alignment. Set  $k = 0$ .

1. Set  $\tilde{q}^{(k)} = (\tilde{q} \circ \gamma^{(k)})\sqrt{\dot{\gamma}^{(k)}}$
2. Calculate  $\omega^{(k)} = \langle \tilde{q}_2^{(k)}, \tilde{q}_1 \rangle$ . **Stop** if  $k > 1$  and  $\|\omega^{(k)}\tilde{q}^{(k)} - \omega^{(k-1)}\tilde{q}^{(k-1)}\| < \epsilon$
3. Solve  $\gamma^{(k+1)} = \underset{\gamma \in \Gamma}{\operatorname{argmin}} \|\tilde{q}_1 - (\omega^{(k)} \cdot \tilde{q}_2 \circ \gamma)\sqrt{\dot{\gamma}}\|_{\mathbb{L}^2}$ .
4. Set  $k = k + 1$  and return to Step 1.

The elastic full Procrustes distance is given as  $d([\beta_1], [\beta_2]) = \|\tilde{q}_1 - \omega^{(k)}(\tilde{q}_2 \circ \gamma^{(k)})\sqrt{\dot{\gamma}^{(k)}}\|_{\mathbb{L}^2}$ .

### 2.3. The Elastic Full Procrustes Mean for Planar Curves

sec:2-mean

We now want to use the elastic Full Procrustes distance to calculate shape means for sets of planar curves. Again, we assume all curves to be absolutely continuous  $\beta_i \in \mathcal{AC}([0, 1], \mathbb{C})$  with corresponding SRV curves  $q_i \in \mathbb{L}^2([0, 1], \mathbb{C})$ ,  $i = 1, \dots, N$ . We can take into account invariance with respect to shape-preserving transformations by defining the mean as a minimizer over the sum of squared elastic full Procrustes distances between the shape of each curve and a mean shape. If the resulting mean is a global minimum, it is usually called a “sample Fréchet mean” (FRÉCHET 1948), if it is a local minimum a “sample Karcher mean” (KARCHER 1977) (see DRYDEN and MARDIA 2016, p. 111).

def:2-mean

**Definition 2.5** (Elastic full Procrustes mean). For a set of curves  $\beta_i \in \mathcal{AC}([0, 1], \mathbb{C})$ ,  $i = 1, \dots, N$ , their *elastic full Procrustes mean* is given by a minimizing shape  $[\hat{\mu}]$  with

$$[\hat{\mu}] = \operatorname{arginf}_{[\mu] \in \mathcal{S}} \sum_{i=1}^N d_{EF}([\mu], [\beta_i])^2, \quad (2.12)$$

{eq:2-mean-d

where  $\mathcal{S} = \{[\beta] : \beta \in \mathcal{AC}([0, 1], \mathbb{C})\}$  is the shape space.

In practice, we will always solve Eq. (2.12) directly on SRV level, by using the definition of the elastic full Procrustes distance (see Definition 2.4) and writing it as an optimization problem over a normalized SRV mean function.

$$\hat{\mu}_q = \operatorname{argmin}_{\mu_q \in \mathbb{L}^2, \|\mu_q\|=1} \sum_{i=1}^N \left( \inf_{\omega_i \in \mathbb{C}, \gamma_i \in \Gamma} \|\mu_q - \omega_i(\tilde{q}_i \circ \gamma_i)\sqrt{\dot{\gamma}_i}\| \right)^2. \quad (2.13)$$

{eq:2-mean

The estimated normalized SRV mean  $\hat{\mu}_q$  then defines a representative unit-length mean  $\hat{\mu} \in [\hat{\mu}]$  by integration  $\hat{\mu}(t) = \hat{\mu}(0) + \int_0^t \hat{\mu}_q(s) \|\hat{\mu}_q(s)\| ds$  (compare Eq. (2.5)), which is unique up to translation  $\hat{\mu}(0)$ . When re-constructing  $\hat{\mu}$  from  $\hat{\mu}_q$ , one may decide to set  $\hat{\mu}(0)$  to a certain value depending on the application. In particular, setting  $\hat{\mu}(0) = 0$ , so that the mean curve starts at the origin, makes sense when the object represented by the mean curve has a ‘natural’ starting point shared across all objects of this type. An

example, explored in Section 4.4 are tongue shapes, which all connect to the back of the mouth on one end. Another possibility would be to choose a  $\hat{\mu}(0)$  that centers the mean curve, by setting  $\hat{\mu}(0) = \int_0^1 \int_0^t \hat{\mu}_q(s) \|\hat{\mu}_q(s)\| ds dt$ . From the point of shape analysis, the choice of representation does not make a difference, as both mean curves are elements of  $[\hat{\mu}]$  and therefore have the same shape. However, the distinction becomes important when the estimated mean curve  $\hat{\mu}$  is used in concert with other curves, for example in visualizing multiple curves or in comparing multiple class mean shapes, as those do not typically share the same center or starting point.

Turning back to the calculation of  $\hat{\mu}_1$ , we can simplify Eq. (2.13) by applying the following Lemma, which uses the analytical solution for the optimization over rotation in the full Procrustes distance (see Lemma 2.1 i.).

2-elfpdist

**Lemma 2.3.** *Let  $\beta_1, \beta_2$  be two absolutely continuous planar curves with corresponding shape  $[\beta_1], [\beta_2]$ . Let  $\tilde{q}_1, \tilde{q}_2$  be the respective normalized SRV curves. The elastic full Procrustes distance is given by*

$$d([\beta_1], [\beta_2]) = \inf_{\gamma \in \Gamma} \sqrt{1 - \langle \tilde{q}_1, (\tilde{q}_2 \circ \gamma) \sqrt{\dot{\gamma}} \rangle \langle (\tilde{q}_2 \circ \gamma) \sqrt{\dot{\gamma}}, \tilde{q}_1 \rangle} \quad (2.14)$$

| *Proof.* This follows from applying Lemma 2.1 i.) to Eq. (2.6), keeping  $\gamma$  fixed.  $\square$

Then Eq. (2.13) can be rewritten as

$$\hat{\mu}_q = \underset{\mu_q \in \mathbb{L}^2, \|\mu_q\|=1}{\operatorname{argmin}} \sum_{i=1}^N \inf_{\gamma_i \in \Gamma} \left( 1 - \langle \mu_q, (\tilde{q}_i \circ \gamma_i) \sqrt{\dot{\gamma}_i} \rangle \langle (\tilde{q}_i \circ \gamma_i) \sqrt{\dot{\gamma}_i}, \mu_q \rangle \right) \quad (2.15)$$

$$\hat{\mu}_q = \underset{\mu_q \in \mathbb{L}^2, \|\mu_q\|=1}{\operatorname{argmax}} \sum_{i=1}^N \sup_{\gamma_i \in \Gamma} \langle \mu_q, (\tilde{q}_i \circ \gamma_i) \sqrt{\dot{\gamma}_i} \rangle \langle (\tilde{q}_i \circ \gamma_i) \sqrt{\dot{\gamma}_i}, \mu_q \rangle \quad (2.16)$$

and we end up with a two step optimization problem consisting of an outer optimization over  $\mu_q$  and an inner optimization over the set  $\{\gamma_i\}_{i=1, \dots, N}$ . Similarly to the approaches discussed in SRIVASTAVA and KLASSEN 2016 and to STEYER, STÖCKER, and GREVEN 2021, we solve this iteratively by iterative *template based alignment* (see e.g. SRIVASTAVA and KLASSEN 2016, p. 271): In each step, the mean  $\hat{\mu}_q$  is estimated while keeping the parametrizations  $\gamma_i$  fixed, after which the  $\gamma_i$  are updated by calculating the warping alignment of the full Procrustes fit of each  $\tilde{q}_i$  onto  $\hat{\mu}_q$ .<sup>1</sup> Mean estimation

<sup>1</sup>Note that this is *not* the same as performing an elastic full Procrustes fit of  $\tilde{q}_i$  onto  $\hat{\mu}_q$ , because the

and warping alignment are iterated until the mean shape has converged.

Let us consider the outer optimization problem in step  $k$  for a fixed set of warping functions  $\{\gamma_i^{(k)}\}_{i=1,\dots,N}$  with corresponding warping aligned normalized SRV curves  $\tilde{q}_i^{(k)} = (\tilde{q}_i \circ \gamma_i^{(k)})\sqrt{\gamma_i^{(k)'}}$ . Note that if no warping alignment has happened yet, we can always set  $\gamma_i^{(0)}(t) = t$  for all  $i = 1, \dots, N$  as a starting value. The problem we have to solve is

$$\hat{\mu}_q^{(k)} = \underset{\mu_q \in \mathbb{L}^2, \|\mu_q\|=1}{\operatorname{argmax}} \sum_{i=1}^N \langle \mu_q, \tilde{q}_i^{(k)} \rangle \langle \tilde{q}_i^{(k)}, \mu_q \rangle. \quad (2.17)$$

We can reformulate this by writing out the complex functional scalar products  $\langle f, g \rangle = \int_0^1 \overline{f(t)} g(t) dt$  for functions  $f, g \in \mathbb{L}^2([0, 1], \mathbb{C})$ , where  $\overline{f(t)}$  denotes the complex conjugate of  $f(t)$ .

$$\hat{\mu}_q^{(k)} = \underset{\mu_q \in \mathbb{L}^2, \|\mu_q\|=1}{\operatorname{argmax}} \sum_{i=1}^N \int_0^1 \int_0^1 \overline{\mu_q(s)} \tilde{q}_i^{(k)}(s) \overline{\tilde{q}_i^{(k)}(t)} \mu_q(t) ds dt \quad (2.18)$$

$$\hat{\mu}_q^{(k)} = \underset{\mu_q \in \mathbb{L}^2, \|\mu_q\|=1}{\operatorname{argmax}} \int_0^1 \int_0^1 \overline{\mu_q(s)} \left( \sum_{i=1}^N \tilde{q}_i^{(k)}(s) \overline{\tilde{q}_i^{(k)}(t)} \right) \mu_q(t) ds dt \quad (2.19)$$

We can identify the inner term as proportional to a sample estimator  $\check{C}^{(k)}(s, t) = \frac{1}{N} \sum_{i=1}^N \tilde{q}_i^{(k)}(s) \overline{\tilde{q}_i^{(k)}(t)}$  of the population covariance surface of the normalized SRV curves  $C^{(k)}(s, t) = \mathbb{E}[\tilde{q}^{(k)}(s) \overline{\tilde{q}^{(k)}(t)}]$ , when noting that  $\mathbb{E}[\tilde{q}^{(k)}(t)] = 0$  for all  $t \in [0, 1]$  due to rotational symmetry.

$$\hat{\mu}_q^{(k)} = \underset{\mu_q \in \mathbb{L}^2, \|\mu_q\|=1}{\operatorname{argmax}} N \cdot \int_0^1 \int_0^1 \overline{\mu_q(s)} \check{C}^{(k)}(s, t) \mu_q(t) ds dt \quad (2.20)$$

By replacing  $\check{C}^{(k)}(s, t)$  by its expectation  $C^{(k)}(s, t)$ , we can analogously formulate an estimator on the population level.

$$\mathbb{E}[\hat{\mu}_q^{(k)}] = \underset{\mu_q \in \mathbb{L}^2: \|\mu_q\|=1}{\operatorname{argmax}} \int_0^1 \int_0^1 \overline{\mu_q(s)} C^{(k)}(s, t) \mu_q(t) ds dt \quad (2.21)$$

We can rewrite this again as a functional scalar product by considering the *covariance operator*  $C$  with  $(C\mu_q)(s) = \int_0^1 C(s, t) \mu_q(t) dt$  (see RAMSAY and SILVERMAN 2005,

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warping and Procrustes alignments are themselves not iterated over.

p. 153).

$$\mathbb{E}[\mu_q^{(k)}] = \underset{\mu_q \in \mathbb{L}^2, \|\mu_q\|=1}{\operatorname{argmax}} \left\langle \mu_q, C^{(k)} \mu_q \right\rangle \quad (2.22)$$

{eq:quadr\_}

This is a well known problem in the context of functional principal component analysis (FPCA), but typically only for real-valued covariance operators. From  $\overline{C(s,t)} = \overline{\mathbb{E}[\tilde{q}(s)\tilde{q}(t)]} = \mathbb{E}[\tilde{q}(t)\tilde{q}(s)] = C(t,s)$  it follows that  $\langle \mu_q, C\mu_q \rangle = \langle C\mu_q, \mu_q \rangle$  and therefore that  $C$  is a *self-adjoint* operator. The optimization problem then reduces to an eigenfunction problem

$$C^{(k)} u^{(k)} = \lambda^{(k)} u^{(k)} \Leftrightarrow \int_0^1 C^{(k)}(s,t) u^{(k)}(t) dt = \lambda^{(k)} u^{(k)}(s), \quad (2.23)$$

{eq:funcei}

where  $\lambda^{(k)} = \langle \mu_q, C^{(k)} \mu_q \rangle$  is the target function to maximize. For normalized eigenfunctions  $u_1^{(k)}, u_2^{(k)}, \dots$  and corresponding eigenvalues  $\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \dots$  of  $C^{(k)}(s,t)$ , the expectation  $\mathbb{E}[\hat{\mu}_q^{(k)}(t)]$  is given by the leading normalized eigenfunction  $u_1^{(k)}(t)$  of  $C^{(k)}(s,t)$  (see RAMSAY and SILVERMAN 2005, pp. 153, 397).

We will estimate  $C^{(k)}(s,t)$  using the methods for covariance estimation from sparse and irregular observations laid out in Chapter 3. Given such an estimate  $\hat{C}^{(k)}(s,t)$ , we can calculate the elastic full Procrustes mean by the following algorithm.

algo:mean

**Algorithm 2.2** (Elastic full Procrustes mean). *Let  $\{\beta_i\}_{i=1,\dots,N}$  be a set of planar curves with corresponding SRV curves  $\{q_i\}_{i=1,\dots,N}$ . Let  $\tilde{q}_i = \frac{q_i}{\|q_i\|}$ . Set  $\gamma_i^0(t) = t$  for all  $i = 1, \dots, N$  as the initial parametrisation alignment. Set  $k = 0$ .*

1. For  $i = 1, \dots, N$ : Set  $\tilde{q}_i^{(k)} = (\tilde{q}_i \circ \gamma_i^{(k)}) \cdot \sqrt{\dot{\gamma}_i^{(k)}}$ .
2. Estimate  $\hat{C}^{(k)}(s,t)$  from  $\{\tilde{q}_i^{(k)}\}_{i=1,\dots,N}$ .
3. Estimate  $\hat{u}_1^{(k)}$  by eigendecomposition of  $\hat{C}^{(k)}(s,t)$ .
4. Set  $\hat{\mu}_q^{(k)}$  as  $\hat{u}_1^{(k)}$ . **Stop** if  $k > 1$  and  $\|\hat{\mu}_q^{(k)} - \hat{\mu}_q^{(k-1)}\| < \epsilon$ .
5. For  $i = 1, \dots, N$ : Calculate  $\omega_i^{(k)} = \langle \tilde{q}_i^{(k)}, \hat{\mu}_q^{(k)} \rangle$ .
6. For  $i = 1, \dots, N$ : Solve  $\gamma_i^{(k+1)} = \operatorname{argmin}_{\gamma \in \Gamma} \|\hat{\mu}_q^{(k)} - \omega_i^{(k)}(\tilde{q}_i \circ \gamma)\sqrt{\dot{\gamma}}\|$ .
7. Set  $k = k + 1$  and return to Step 1.

### 3. Mean Estimation for Sparse and Irregular Observations

sec:3

So far, we have considered estimation of the elastic full Procrustes mean in a setting, where each curve  $\beta_i$  is assumed fully observed. This is usually not the case in practice, as each observation  $\beta_i$  may itself only be observed at a finite number of discrete points  $\beta_i(t_{i1}), \dots, \beta_i(t_{in_i})$ . Additionally, the number of observed points per curve  $n_i$  might be quite small and the points do not need to follow a common sampling scheme across all curves, a setting which is respectively known as *sparse* and *irregular*.

Following the steps laid out in Algorithm 2.2, this chapter proposes a mean estimation strategy for dealing with sparse and irregular observations. In a first step, the construction of SRV and warped SRV curves from discrete (and possibly sparse) observations will be shown in Section 3.1. Section 3.2 discusses efficient estimation of the complex covariance surface  $C^{(k)}(s, t)$  from sparse observations. In Section 3.3, calculation of the leading eigenfunction  $\hat{u}_1^{(k)}$  of  $C^{(k)}(s, t)$  in a fixed basis will be shown. Section 3.4 deals with the estimation of the optimal rotation and scaling alignment  $\omega_i^{(k)} = \langle \tilde{q}_i^{(k)}, \hat{\mu}_q^{(k)} \rangle$ , where  $\tilde{q}_i^{(k)}$  is a sparsely observed normalized SRV curve, while  $\hat{\mu}_q^{(k)}$  is a smooth SRV mean function. Note that the final warping alignment step in Algorithm 2.2 is solved by using methods for warping alignment of sparse and irregular curves provided in STEYER, STÖCKER, and GREVEN 2021.

#### 3.1. Discrete Treatment of SRV Curves

3-discrete

As a first step, we need to calculate the *normalized SRV curves*  $\tilde{q}_i = \frac{q}{\|q\|}$  from sparse observations. As the SRV curve of  $\beta \in \mathcal{AC}([0, 1], \mathbb{C})$  is defined as  $q = \dot{\beta} / \sqrt{\|\dot{\beta}\|}$  (for  $\dot{\beta} \neq 0$ ), we have to be able to calculate a derivate of  $\beta$ . However, as we never observe the whole function  $\beta$  but only a discrete set of points  $\beta(t_1), \dots, \beta(t_n)$ , as seen in Fig. 3.1, we cannot simply calculate a pointwise derivative. Following STEYER, STÖCKER, and

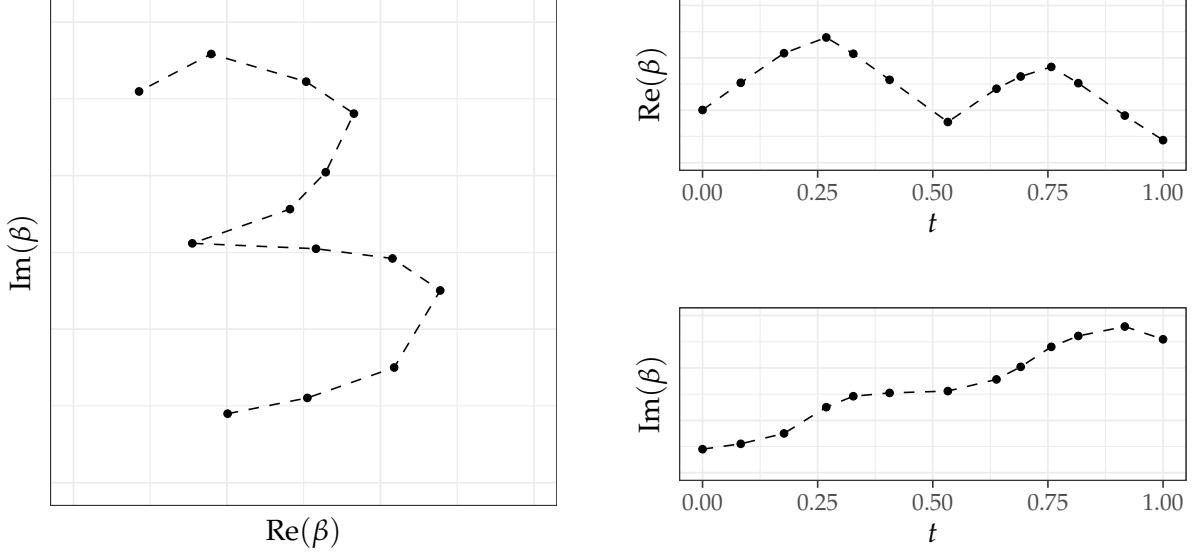


fig:3-disc

**Figure 3.1.:** Example of a sparse and irregularly observed digit ‘3’. Data: digits3.dat.

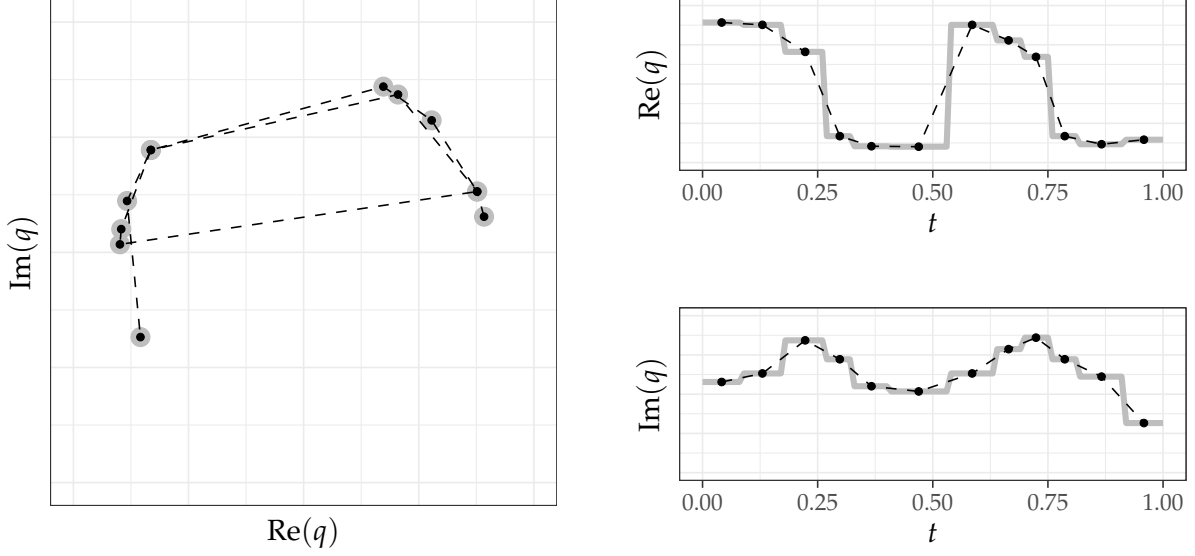
GREVEN 2021, we may treat a discretely observed curve  $\beta$  as piecewise linear between its observed corners  $\beta(t_1), \dots, \beta(t_n)$ , which allows us to calculate a piecewise constant derivative on the intervalls  $[t_j, t_{j+1}]$  for  $j = 1, \dots, n-1$ . As usually only the image  $\beta(t_1), \dots, \beta(t_n)$  but not the parametrisation  $t_1, \dots, t_n$  is observed, it is first necessary to construct an initial parameterisation. A common choice is an *arc-length-parametrisation*, where we set  $t_j = l_j/l$  with  $l_j = \sum_{k=1}^{j-1} |\beta(t_{k+1}) - \beta(t_k)|$  the polygon-length up to point  $j$  for  $j \leq 2$  with  $l_1 = 0$  and  $l = l_n$ .

Consider the discrete derivative  $\Delta\beta|_{[t_j, t_{j+1}]} = \frac{\beta(t_{j+1}) - \beta(t_j)}{t_{j+1} - t_j}$ , which assumes that  $\beta$  is linear between its observed corners. The corresponding SRV curve  $q$  can then be treated as piecewise constant  $q|_{[t_j, t_{j+1}]} = q_j$  with

$$q_j = \Delta\beta|_{[t_j, t_{j+1}]} / \sqrt{\|\Delta\beta|_{[t_j, t_{j+1}]}\|} = \frac{\beta(t_{j+1}) - \beta(t_j)}{\sqrt{t_{j+1} - t_j} \cdot \sqrt{\|\beta(t_{j+1}) - \beta(t_j)\|}} \quad (3.1)$$

the constant *square-root-velocity* of  $\beta$  between its corners  $\beta(t_j)$  and  $\beta(t_{j+1})$ . As shown in STEYER, STÖCKER, and GREVEN 2021, Fig. 3, treating the SRV curves as piecewise-constant functions can lead to overfitting, where the mean shape is estimated too polygon-like. As an alternative they propose to approximate the derivative, by assuming that it attains the value of the discrete derivative  $\Delta\beta|_{[t_j, t_{j+1}]}$  at the center  $s_j = \frac{t_{j+1} - t_j}{2}$  of the interval  $[t_j, t_{j+1}]$ . While STEYER, STÖCKER, and GREVEN use this to approximate





**Figure 3.2.:** Discretely approximated (black, dashed) and piecewise constant (gray) SRV curve of the digit ‘3’ in Fig. 3.1. Data: digits3.dat.

3-disc-srv

an integral, in this thesis it is used for approximating observations  $q(s_j) \approx q_j$  of the SRV curve  $q$  for the covariance estimation step. See Fig. 3.2 for a visualization of both approaches. Finally, we can approximate the normalized SRV curve  $\tilde{q} = q/\|q\|$  using the polygon-length  $l$  of  $\beta$  by  $\tilde{q}_j = q_j/\sqrt{l}$  (see Eq. (2.5)).

When considering the warped normalized SRV curve  $(q \circ \gamma)\sqrt{\gamma}$ , the warped discrete derivative is given by  $\Delta(\beta \circ \gamma)|_{[\gamma^{-1}(t_j), \gamma^{-1}(t_{j+1})]} = \frac{\beta(t_{j+1}) - \beta(t_j)}{\gamma^{-1}(t_{j+1}) - \gamma^{-1}(t_j)}$ . The corresponding warped SRV curve is then given by  $(q \circ \gamma)\sqrt{\gamma}|_{[\gamma^{-1}(t_j), \gamma^{-1}(t_{j+1})]} = \frac{1}{\sqrt{\gamma^{-1}(t_{j+1}) - \gamma^{-1}(t_j)}} \cdot \frac{\beta(t_{j+1}) - \beta(t_j)}{\sqrt{\|\beta(t_{j+1}) - \beta(t_j)\|}}$ . Note that this does not change the normalization as re-parametrization is norm-preserving on SRV level.

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### 3.2. Efficient Estimation using Hermitian Covariance Smoothing

sec:3-cov

Given approximate observations of the warped normalized SRV curves  $\tilde{q}_i^{(k)}(s_{ij})$  for  $j = 1, \dots, n_i - 1$  and  $i = 1, \dots, N$ , where  $n_i$  denotes the number of observed points per curve, we want to estimate the warping aligned complex covariance surface  $C^{(k)}(s, t) = \mathbb{E}[\tilde{q}^{(k)}(s)\overline{\tilde{q}^{(k)}(t)}]$ . We can treat this estimation as a smoothing problem, by constructing responses  $y_{ilm}^{(k)} = \tilde{q}_i^{(k)}(s_{il})\overline{\tilde{q}_i^{(k)}(s_{im})}$  and treating the pairs  $s_{il}, s_{im}$  as covariates  $s$  and  $t$ . Smoothing the responses  $y_{ilm}^{(k)}$  gives an estimate  $\hat{C}^{(k)}(s, t)$  of  $C^{(k)}(s, t)$ , as each response has expectation  $\mathbb{E}[y_{ilm}^{(k)}|s_{il}, s_{im}] = C^{(k)}(s_{il}, s_{im})$ . We carry out the

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smoothing in a flexible *penalized tensor product spline* basis

$$C^{(k)}(s, t) = b(s)^\top \Xi^{(k)} b(t) \quad (3.2)$$

where  $b(s) = (b_1(s), \dots, b_K(s))$  denotes the vector of a spline basis and  $\Xi^{(k)}$  is a  $K \times K$  coefficient matrix to be estimated under a roughness penalty to prevent overfitting. As  $C^{(k)}(s, t)$  is complex, we choose the spline basis to be real-valued with  $b_j : [0, 1] \rightarrow \mathbb{R}$  for  $j = 1, \dots, K$  and the coefficient matrix to be complex-valued with  $\Xi^{(k)} \in \mathbb{C}^{K \times K}$  without loss of generality. The exact choice of basis and penalty will be discussed in Section 3.3.

Taking into account the symmetry properties of the covariance surface by considering every unique pair  $(s_{il}, s_{im})$  only once allows for more efficient estimation, as shown in CEDERBAUM, SCHEIPL, and GREVEN 2018. In the complex case the covariance surface is Hermitian with  $C^{(k)}(s, t) = \overline{C^{(k)}(t, s)}$ , which means we can decompose the estimation into two separate regression problems over the symmetric real and skew-symmetric imaginary parts of  $C^{(k)}(s, t)$ . We estimate the two models

$$\mathbb{E}[\text{Re}(y^{(k)})|s, t] = b(s)^\top \Xi_{\text{Re}}^{(k)} b(t) \quad (3.3)$$

$$\mathbb{E}[\text{Im}(y^{(k)})|s, t] = b(s)^\top \Xi_{\text{Im}}^{(k)} b(t), \quad (3.4)$$

with  $\Xi_{\text{Re}}^{(k)}, \Xi_{\text{Im}}^{(k)} \in \mathbb{R}^{K \times K}$  and  $\Xi^{(k)} = \Xi_{\text{Re}}^{(k)} + i\Xi_{\text{Im}}^{(k)}$ , under the constraints that  $(\Xi_{\text{Re}}^{(k)})^\top = \Xi_{\text{Re}}^{(k)}$  and  $(\Xi_{\text{Im}}^{(k)})^\top = -\Xi_{\text{Im}}^{(k)}$ .

In this thesis  $\Xi_{\text{Re}}^{(k)}$  and  $\Xi_{\text{Im}}^{(k)}$  are estimated using the `gam` function from the R package `mgcv` (WOOD 2017), where the smoothing parameters are selected via restricted maximum likelihood (REML) estimation. Two `mgcv` smooths provided in the package `sparseFLMM` (CEDERBAUM, VOLKMANN, and STÖCKER 2021) are used for efficient Hermitian smoothing, which generalize the approach proposed by CEDERBAUM, SCHEIPL, and GREVEN 2018 for symmetric tensor product P-splines to the skew-symmetric case. Note that `mgcv` automatically adds a sum-to-zero constraint to a specified basis (see WOOD 2017, p. 175), which makes it is necessary to transform the coefficient matrices  $\tilde{\Xi}_{\text{Re}}^{(k)}, \tilde{\Xi}_{\text{Im}}^{(k)}$  recovered from `gam` with an appropriate transformation matrix  $D$ . The coefficient matrices in the specified basis are given by  $\Xi_{\text{Re/Im}}^{(k)} = D \cdot \tilde{\Xi}_{\text{Re/Im}}^{(k)}$ , where  $D$  may be calculated from the constrained and unconstrained design matrices via

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$$X = D \cdot \tilde{X}.$$

### 3.3. Estimating the Elastic Full Procrustes Mean in a Fixed Basis

sec:3-mean

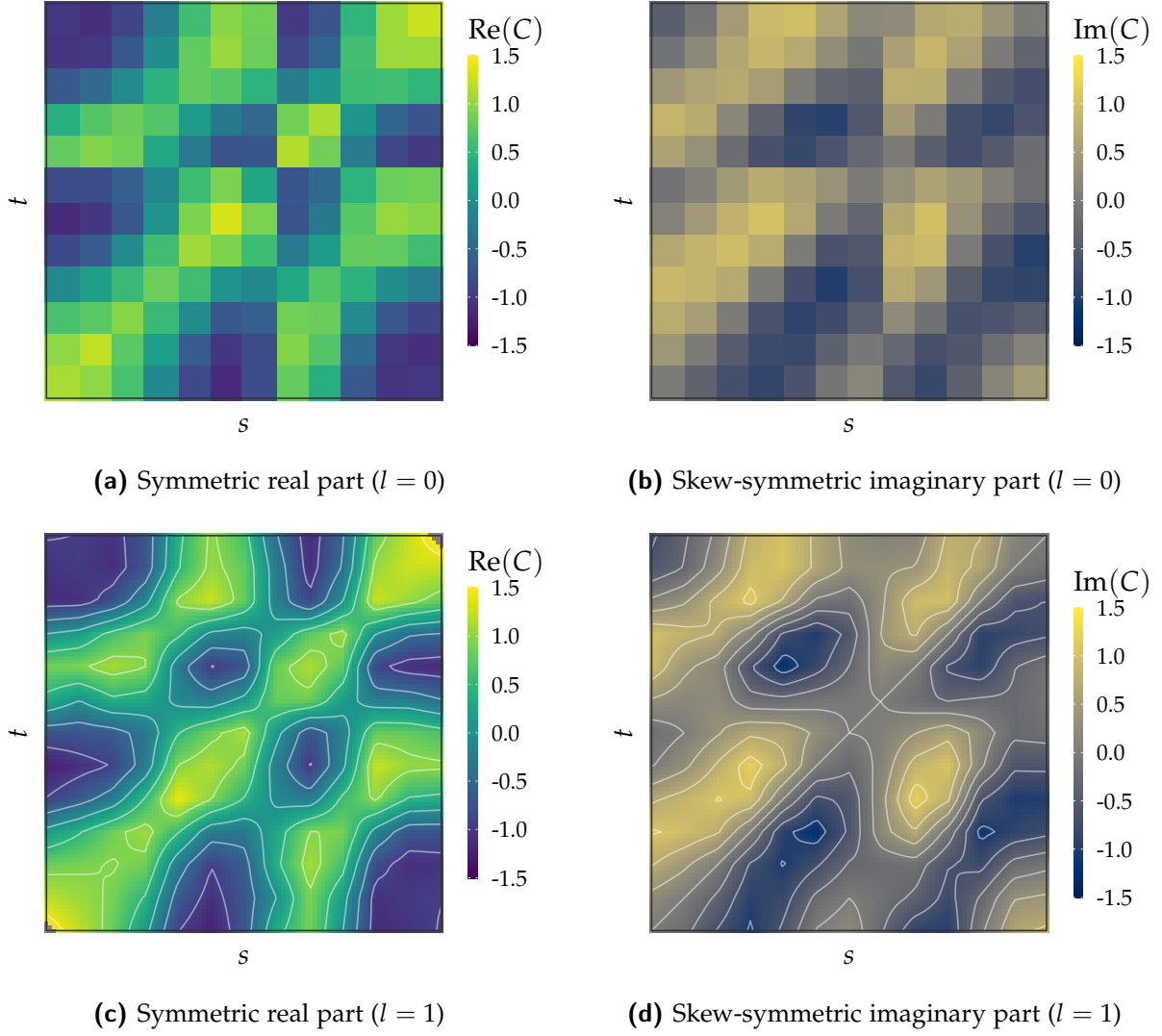
Our goal is to estimate a smooth mean function, which might be constructed with the same univariate basis  $b(s)$  used in the tensor product basis of the covariance surface, so that the mean is given by  $\mu_q(s) = b(s)^\top \theta$ . We can choose an appropriate basis  $b(s)$  by considering which, e.g., smoothness properties we want the estimated mean to have. In this thesis we will use penalized B-spline basis functions (P-splines), which are piecewise polynomials of degree  $l$ , fused at  $m$  knots  $\{\kappa_j\}_{j=1,\dots,m}$ , where the  $p$ -th order differences between coefficients of neighbouring splines are penalized in the covariance estimation (see FAHRMEIER et al. 2013, Chap. 8.1). It should be noted that by using a penalty, the number and location of knots do not have an influence on the estimated function when their number is high enough and they are evenly distributed. Furthermore, because of results relating to the identifiability of spline SRV curves modulo warping and translation, only piecewise linear ( $l = 1$ ) and piecewise constant ( $l = 0$ ) B-splines will be considered in the estimation (see STEYER, STÖCKER, and GREVEN 2021). See Fig. 3.3 for examples of a covariance surface estimated using piecewise constant and piecewise linear P-splines. This still allows us the flexibility of estimating either polygonal or smooth means, as we estimate the mean function on original curve level by integration (see Fig. 3.4).

To calculate the elastic full Procrustes Mean, we had to solve a functional eigenvalue problem on the estimated covariance surface  $\hat{C}(s, t) = b(s)^\top \hat{\Xi} b(t)$  (omitting  $\cdot^{(k)}$  in this section). Remember that the elastic full Procrustes mean (for fixed warping) is estimated by the solution to the optimization problem

$$\hat{\mu}_q = \operatorname{argmax}_{\mu_q \in \mathbb{L}^2, \|\mu_q\|=1} \int_0^1 \int_0^1 \overline{\mu_q(s)} \hat{C}(s, t) \mu_q(t) \, ds \, dt . \quad (3.5)$$

Then, estimating the mean  $\hat{\mu}_q(s) = b(s)^\top \theta$  reduces to estimating the vector of coefficients  $\theta = (\theta_1, \dots, \theta_K) \in \mathbb{C}^K$  with

$$\hat{\theta} = \operatorname{argmax}_{\theta \in \mathbb{C}^K, \|b^\top \theta\|=1} \int_0^1 \int_0^1 \theta^H b(s) b(s)^\top \hat{\Xi} b(t) b(t)^\top \theta \, ds \, dt \quad (3.6)$$



**Figure 3.3.:** Complex covariance surface on SRV curve level. Estimated using 13 equidistant knots, a 2nd order penalty and piecewise constant (top) or piecewise linear (bottom) B-splines. Data: digits3.dat

$$= \underset{\theta \in \mathbb{C}^K, \|\theta\|=1}{\operatorname{argmax}} \theta^H \left( \int_0^1 b(s)b(s)^\top ds \right) \hat{\Sigma} \left( \int_0^1 b(t)b(t)^\top dt \right) \theta \quad (3.7)$$

$$= \underset{\theta \in \mathbb{C}^K, \theta^H G \theta = 1}{\operatorname{argmax}} \theta^H G \hat{\Sigma} G \theta \quad (3.8)$$

{eq:3-targ}

where  $(\cdot)^H = \overline{(\cdot)}^\top$  denotes the conjugate transpose and  $G$  is the  $K \times K$  Gram matrix with entries given by the basis products  $g_{ij} = \langle b_i, b_j \rangle$ . For an orthonormal basis the Gram matrix is an identity matrix as  $\langle b_i, b_j \rangle = \delta_{ij}$ , however, this is not the case for many basis representations such as, e.g., the B-spline basis. In thesis the R package `orthogonalsplinebasis` (REDD 2015) is used to calculate Gram matrices for B-splines analytically, using the methods laid out in REDD 2012.

Having reduced the functional eigenvalue problem to a multivariate eigenvalue problem over the covariance coefficient matrix, we may solve it using Lagrange optimization with the following Langrangian:

$$\mathcal{L}(\theta, \lambda) = \theta^H G \hat{\Xi} G \theta - \lambda(\theta^H G \theta - 1) \quad (3.9)$$

Taking into account that we identified  $\mathbb{R}^2$  with  $\mathbb{C}$  we can split everything into real and imaginary parts and optimize with respect to  $\text{Re}(\theta)$  and  $\text{Im}(\theta)$  seperately, which avoids having to take complex derivatives. Using  $\theta = \theta_{\text{Re}} + i\theta_{\text{Im}}$  and  $\hat{\Xi} = \hat{\Xi}_{\text{Re}} + i\hat{\Xi}_{\text{Im}}$  we can write

$$\begin{aligned} \mathcal{L}(\theta_{\text{Re}}, \theta_{\text{Im}}, \lambda) &= (\theta_{\text{Re}}^\top - i\theta_{\text{Im}}^\top) G (\hat{\Xi}_{\text{Re}} + i\hat{\Xi}_{\text{Im}}) G (\theta_{\text{Re}} + i\theta_{\text{Im}}) \\ &\quad - \lambda \left( (\theta_{\text{Re}}^\top - i\theta_{\text{Im}}^\top) G (\theta_{\text{Re}} + i\theta_{\text{Im}}) - 1 \right). \end{aligned}$$

By multiplying everything out and using  $\hat{\Xi}_{\text{Re}}^\top = \hat{\Xi}_{\text{Re}}$  and  $\hat{\Xi}_{\text{Im}}^\top = -\hat{\Xi}_{\text{Im}}$  we get

$$\begin{aligned} \mathcal{L}(\theta_{\text{Re}}, \theta_{\text{Im}}, \lambda) &= \theta_{\text{Re}}^\top G \hat{\Xi}_{\text{Re}} G \theta_{\text{Re}} + i\theta_{\text{Re}}^\top G \hat{\Xi}_{\text{Im}} G \theta_{\text{Re}} + \theta_{\text{Im}}^\top G \hat{\Xi}_{\text{Im}} G \theta_{\text{Re}} - \theta_{\text{Re}}^\top G \hat{\Xi}_{\text{Im}} G \theta_{\text{Im}} \\ &\quad + \theta_{\text{Im}}^\top G \hat{\Xi}_{\text{Re}} G \theta_{\text{Im}} + i\theta_{\text{Im}}^\top G \hat{\Xi}_{\text{Im}} G \theta_{\text{Im}} + \lambda\theta_{\text{Re}}^\top G \theta_{\text{Re}} + \lambda\theta_{\text{Im}}^\top G \theta_{\text{Im}} - \lambda. \end{aligned}$$

Differentiation with respect to  $\theta_{\text{Re}}$  and  $\theta_{\text{Im}}$  yield

$$\frac{\partial \mathcal{L}}{\partial \theta_{\text{Re}}} = 2G\hat{\Xi}_{\text{Re}}G\theta_{\text{Re}} - 2G\hat{\Xi}_{\text{Im}}G\theta_{\text{Im}} - 2\lambda G\theta_{\text{Re}} \stackrel{!}{=} 0 \quad (3.10) \quad \boxed{\text{eq:lagrRe}}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_{\text{Im}}} = 2G\hat{\Xi}_{\text{Re}}G\theta_{\text{Im}} + 2G\hat{\Xi}_{\text{Im}}G\theta_{\text{Re}} - 2\lambda G\theta_{\text{Im}} \stackrel{!}{=} 0 \quad (3.11) \quad \boxed{\text{eq:lagrIm}}$$

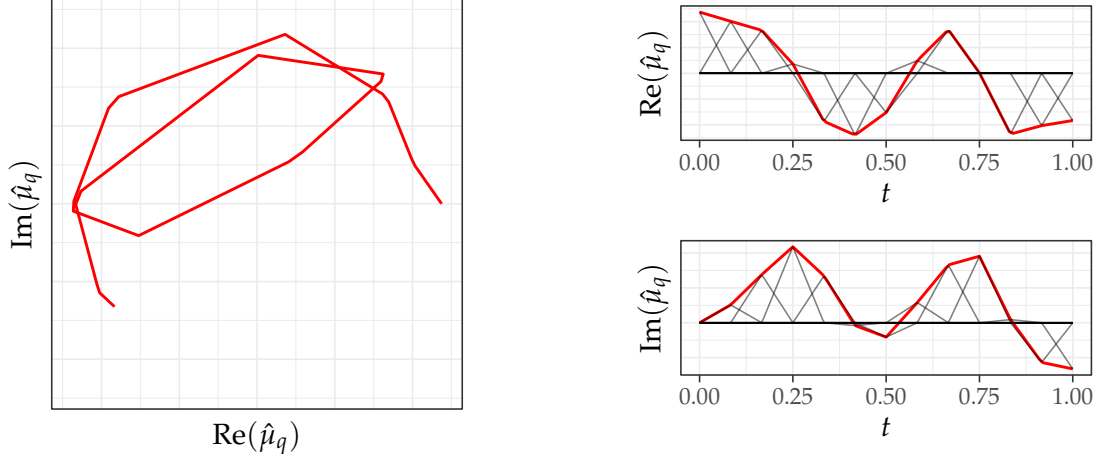
with the additional constraint  $\theta_{\text{Re}}^\top G \theta_{\text{Re}} + \theta_{\text{Im}}^\top G \theta_{\text{Im}} = 1$ . We can simplify this further and multiply Eq. (3.11) by  $i$ , leading to

$$\hat{\Xi}_{\text{Re}}G\theta_{\text{Re}} - \hat{\Xi}_{\text{Im}}G\theta_{\text{Im}} = \lambda\theta_{\text{Re}} \quad (3.12)$$

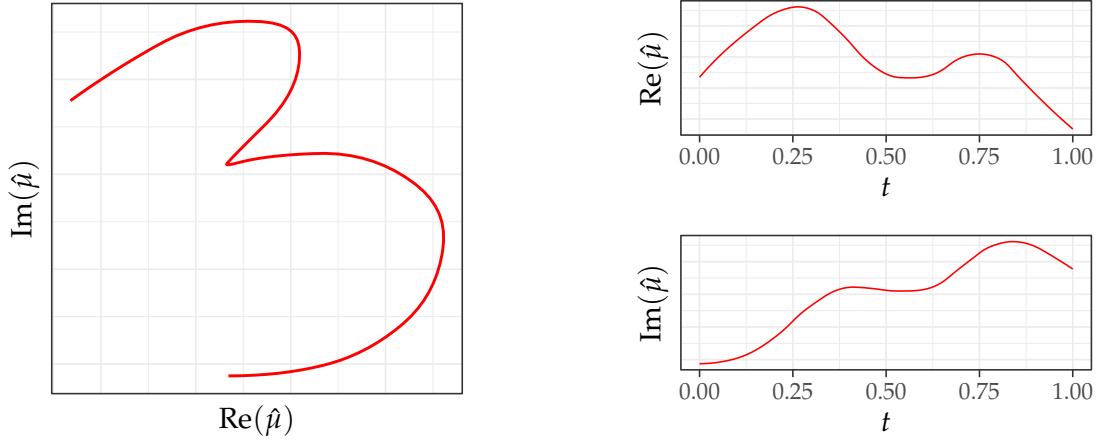
$$i\hat{\Xi}_{\text{Re}}G\theta_{\text{Im}} + i\hat{\Xi}_{\text{Im}}G\theta_{\text{Re}} = i\lambda\theta_{\text{Im}}. \quad (3.13)$$

Adding both equations finally leads to

$$(\hat{\Xi}_{\text{Re}} + i\hat{\Xi}_{\text{Im}})G(\theta_{\text{Re}} + i\theta_{\text{Im}}) = \lambda(\theta_{\text{Re}} + i\theta_{\text{Im}}) \quad (3.14)$$



(a) Mean function  $\hat{\mu}_q(t) = b(t)^\top \hat{\theta}$  on SRV level with scaled linear B-spline basis functions (right).



(b) Mean function  $\hat{\mu}(t) = \int_0^t \hat{\mu}_q(s) \|\hat{\mu}_q(s)\| ds$  on (unit-length) original curve level.

**Figure 3.4.:** Elastic full Procrustes mean of handwritten digits ‘3’. Estimated using 13 equidistant knots, a 2nd order penalty and piecewise linear B-splines. Data: digits3.dat

fig:3-mean

$$\hat{\Xi}G\theta = \lambda\theta \quad (3.15)$$

which is an eigenvalue problem on the product of the complex coefficient matrix and the Gram matrix. Note that this mirrors the result by REISS and XU 2020 for FPCA on real-valued tensor product spline coefficient matrices. Multiplying by  $\theta^H G$  from the left and using  $\theta^H G \theta = 1$  yields  $\lambda = \theta^H G \hat{\Xi} G \theta$ , i.e. the eigenvalues  $\lambda$  correspond to the target function to maximize (see Eq. (3.8)). It follows that the estimate for the coefficient vector of the elastic full Procrustes mean  $\hat{\theta}$  is given by the eigenvector of the leading eigenvalue of  $\hat{\Xi}G$  or likewise of  $\hat{\Xi}^{(k)}G$ , when taking into account the warping alignment in step  $k$  of Algorithm 2.2.

### 3.4. Numerical Integration of the Procrustes Fits

Given an estimated mean function  $\hat{\mu}_q^{(k)} = b(s)^\top \hat{\theta}^{(k)}$ , we need to calculate the Procrustes alignment  $\omega_i^{(k)} = \langle \tilde{q}_i^{(k)}, \hat{\mu}_q^{(k)} \rangle$ ,  $i = 1, \dots, N$ , of each curve onto the current mean, before we can update the warping alignment (see Algorithm 2.2). In the sparse and irregular setting, calculating a scalar product such as  $\langle \tilde{q}_i^{(k)}, \hat{\mu}_q^{(k)} \rangle$  can be a challenge, as we have to evaluate  $\tilde{q}_i^{(k)}(t)$  at every  $t \in [0, 1]$ .

$$\langle \tilde{q}_i^{(k)}, \hat{\mu}_q^{(k)} \rangle = \int_0^1 \langle \tilde{q}_i^{(k)}(t), \hat{\mu}_q^{(k)}(t) \rangle dt \quad (3.16)$$

Following STEYER, STÖCKER, and GREVEN 2021 and as discussed in Section 3.1, this may be achieved by treating  $\tilde{q}_i^{(k)}$  as piecewise constant between the warping-aligned corners of the original curve  $\beta_i$ , with the normalized and warping aligned values given by

$$\tilde{q}_i^{(k)} \Big|_{\left[ \left( \gamma_i^{(k)} \right)^{-1}(t_{ij}), \left( \gamma_i^{(k)} \right)^{-1}(t_{ij+1}) \right]} = \tilde{q}_{ij}^{(k)} \quad (3.17)$$

with  $j = 1, \dots, m_i - 1$  for  $m_i$  the number of observed points of the original curve  $\beta_i$  and where

$$\tilde{q}_{ij}^{(k)} = \frac{1}{\sqrt{L[\beta_i]}} \cdot \frac{1}{\sqrt{\left( \gamma_i^{(k)} \right)^{-1}(t_{ij+1}) - \left( \gamma_i^{(k)} \right)^{-1}(t_{ij})}} \cdot \frac{\beta_i(t_{ij+1}) - \beta_i(t_{ij})}{\sqrt{\|\beta_i(t_{ij+1}) - \beta_i(t_{ij})\|}}. \quad (3.18)$$

Given a fully observed smooth mean function  $\hat{\mu}_q^{(k)}$  we can then analytically calculate the scalar product as

$$\langle \tilde{q}_i^{(k)}, \hat{\mu}_q^{(k)} \rangle = \int_0^1 \langle \tilde{q}_i^{(k)}(t), \hat{\mu}_q^{(k)}(t) \rangle dt = \sum_{j=0}^{m_i-1} \int_{\left( \gamma_i^{(k)} \right)^{-1}(t_{ij})}^{\left( \gamma_i^{(k)} \right)^{-1}(t_{ij+1})} \langle \tilde{q}_{ij}^{(k)}, \hat{\mu}_q^{(k)}(t) \rangle dt. \quad (3.19)$$

Noch ein paragraph zu curve-wise interpolation mit der penalty auf cov matrix.  
Verweis auf appendix.

## 4. Application to Simulated and Empirical Datasets

sec:4

In this chapter the proposed methods will be applied and verified. Section 4.1 offers a comparison between the elastic full Procrustes mean, the elastic mean and the full Procrustes mean over two datasets, with the goal of illustrating some properties of the elastic full Procrustes mean. Section 4.3 will briefly discuss the effect of the penalty parameter on the estimation. Section 4.2 discusses differences between the elastic full Procrustes fits and the warping aligned Procrustes fits, and how they might matter practice. Finally, Section 4.4 applies the proposed methods to an empirical dataset of tounge shapes, which was kindly provided by ....

### 4.1. Comparison to the Elastic and the Full Procrustes Mean

ec:4-means

In this section we will estimate and compare the elastic mean, the full Procrustes mean and the elastic full Procrustes mean for two datasets. The first dataset is the `digits3.dat` dataset provided in the `shapes` package (DRYDEN 2019) and originally collected by ANDERSON 1997, which has already been used throughout this thesis for illustrative purposes. It consist of 30 handwritten digits '3', each of which was sampled in a regular fashion at 13 points along the digit, leading to sparse but somewhat regular observations. The second dataset consists of ten simulated spirals, each of which is a sample of the curve  $\beta(t) = t \cos(13t) + i \cdot t \sin(13t)$ , evaluated  $m_i \in [10, 15]$  times over a noisy grid, with additional noise applied to the output, leading to sparse and irregular observations. The code simulating these spirals was adapted from the `compute_elastic_mean` function's documenation in the `elasdics` package (STEYER 2021). Because the spirals "speed up" w.r.t.  $t$  towards the end ( $t = 1$ ), they start out quite densly sampled but become sparser towards the end, making the mean estimation, especially close to  $t = 1$ , a challenge. The datasets will be considered in two settings: In the first setting, each dataset is considered as is, which means that all curves are centered and similarly aligned. In the second setting, for each

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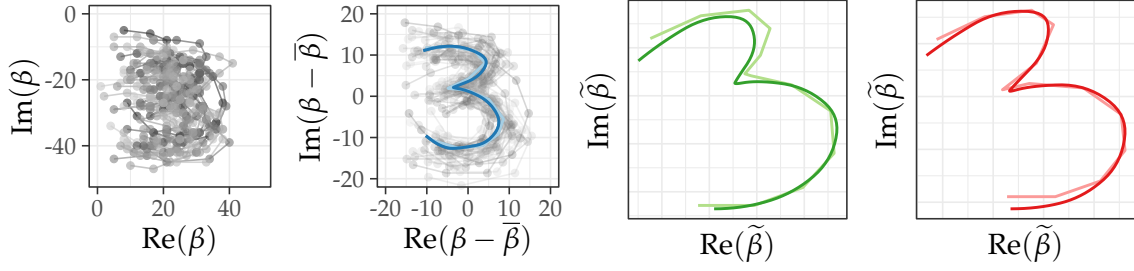


curve  $\beta_i$  a random Euclidean similarity transform with translation  $\xi_i \sim \mathcal{U}([\xi_{\min}, \xi_{\max}])$ , rotation  $\theta_i \sim \mathcal{U}([0, 2\pi))$  and scaling  $\lambda_i \sim \mathcal{U}([0.5, 1.5])$  is drawn, where  $\xi_{\min}, \xi_{\max}$  are respectively set to  $\pm 60$  and  $\pm 2$  for the digits and spirals. The curves are then transformed by  $\lambda_i e^{i\theta_i} \beta_i + \xi_i$ .

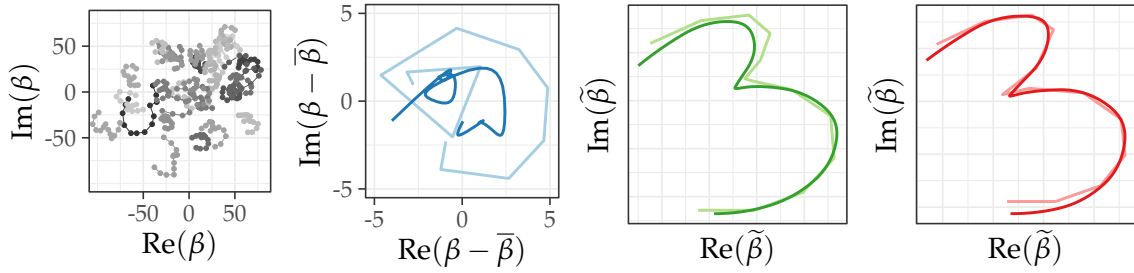
The four sets of data curves and means, are show in Fig. 4.1. Here, the elastic mean (blue) is estimated using `compute_elastic_mean` from the `elasdics` package. The full Procrustes (green) and the elastic full Procrustes mean (red) are estimated using the methods proposed in Chapters 2 and 3, where for the full Procrustes mean the estimation is stopped before the warping alignment step during the first iteration. Note that the full Procrustes mean calculated in this way is not exactly a minimizer of the sum of squared full Procrustes distance defined in Definition 2.2, but instead is a minimizer of the sum of squared full Procrustes distances on SRV level. When comparing the different mean types in Fig. 4.1 we can see that, unlike the elastic mean, both Procrustes means are invariant with respect to all Euclidean similarity transforms, as the estimated mean is the same for the transformed and original datasets. However for this very same reason, both Procrustes means hold no information about the scale or rotation of the original curves, as they are of unit-length and have a rotation dependent on the eigendecomposition of the covariance surface. The elastic mean is only invariant with respect to re-parametrization and translation so that its scale and rotation match the original data curves. Unlike the Procrustes means, it can therefore be meaningfully plotted together with the original curves, when they are centered.

Apart from illustrating these properties, Fig. 4.1 provides two important validation checks for the estimation procedure proposed in Chapters 2 and 3. Firstly, the estimated elastic full Procrustes mean is invariant to all Euclidean similarity transforms, as the mean shapes do not change with transformations of the input curves. Secondly, when considering untransformed curves, the estimated elastic full Procrustes mean shapes are very comparable to elastic mean shapes estimated with the method proposed STEYER, STÖCKER, and GREVEN 2021. This is especially notable when comparing the means in Fig. 4.1 (a) where the prominent “notch” in the center of the mean shape is similarly pronounced for the elastic and the elastic full Procrustes means. Taken together this shows that the proposed mean estimation method works as intended in the setting of sparse and irregular curves.

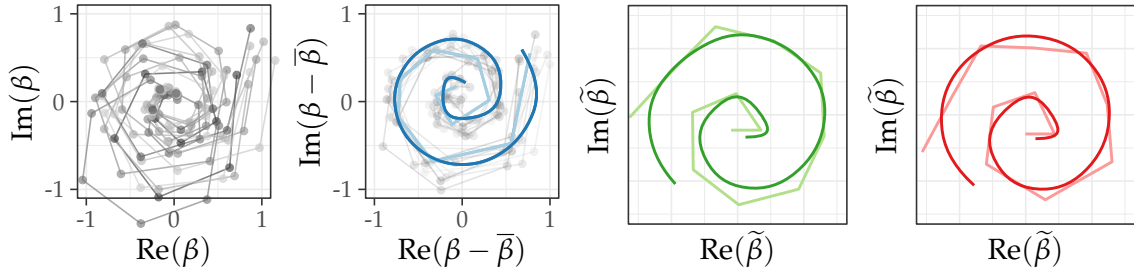
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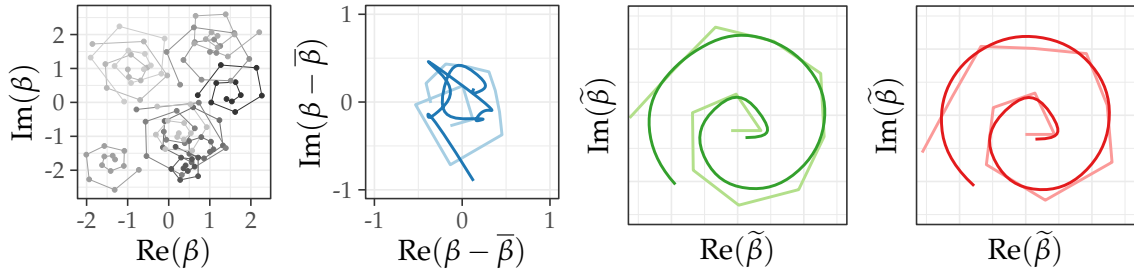
(a) Means for digits3.dat.



(b) Means for digits3.dat with random Euclidean similarity transform applied.

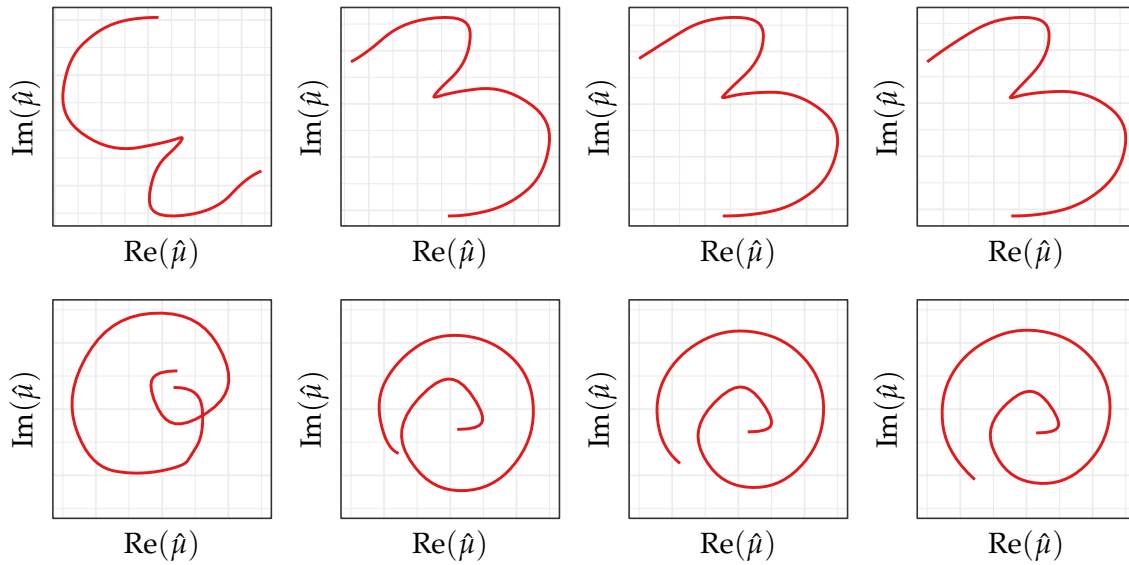


(c) Means for simulated sparse spirals.



(d) Means for simulated sparse spirals with random Euclidean similarity transform applied.

**Figure 4.1.:** Comparison of three mean types: Elastic mean (blue), full Procrustes mean (green) and elastic full Procrustes mean (red), estimated over four sets of data curves (grey). Each mean is estimated as polygonal (light, 16 knots) and smooth (dark, 13 knots), where in the estimation of the two Procrustes means a 2nd order penalty was applied.



**Figure 4.2.:** Elastic full Procrustes mean under different penalties. Estimated using no penalty (left) and order 0/1/2 penalties (center-left/center-right/right) respectively, as well as 13 equidistant knots and linear B-splines on SRV level. Data: See Figs. 4.1b and 4.1d

## 4.2. Problems Relating to Outliers

- Discuss elastic Procrustes fits.
- Discuss warping aligned Procrustes fits in last iteration
- Discuss plotting together with mean
- Discuss outliers that get shrunk away.
- Discuss normalization problem when keeping scaling fixed.

## 4.3. Effect of the Penalty on Mean Estimation

The elastic full Procrustes mean is given by the leading eigenfunction of the complex covariance surface, which was estimated using tensor product P-splines. As a consequence, the order of the roughness penalty applied in the estimation of the covariance surface directly influences the shape of the estimated mean function, as shown in Fig. 4.2. No penalty, ridge penalty, order 1 penalty -> constant, order 2 penalty -> linear?

#### 4.4. Mean Differences of Tounge Shapes in a Phonetics Dataset

:4-tounges

## 5. Summary and Outlook

sec:5

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## A. Appendix

app:a

### A.1. Additional Proofs and Derivations

pp:a-deriv

#### A.1.1. Derivation of Lemma 2.1

riv-fpdist

$$\begin{aligned}
 d_{FP}([\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}})^2 &= \min_{\omega \in \mathbb{C}} \|\tilde{\beta}_1 - \omega \tilde{\beta}_2\|^2 \\
 &= \min_{\lambda \in \mathbb{R}^+, \theta \in [0, 2\pi)} \|\tilde{\beta}_1 - \lambda e^{i\theta} \tilde{\beta}_2\|^2 \\
 &= \min_{\lambda \in \mathbb{R}^+, \theta \in [0, 2\pi)} \langle \tilde{\beta}_1 - \lambda e^{i\theta} \tilde{\beta}_2, \tilde{\beta}_1 - \lambda e^{i\theta} \tilde{\beta}_2 \rangle \\
 &= \min_{\lambda \in \mathbb{R}^+, \theta \in [0, 2\pi)} \|\tilde{\beta}_1\|^2 + \lambda^2 \|\tilde{\beta}_2\|^2 - \lambda(e^{i\theta} \langle \tilde{\beta}_1, \tilde{\beta}_2 \rangle + e^{-i\theta} \langle \tilde{\beta}_2, \tilde{\beta}_1 \rangle)
 \end{aligned}$$

Define  $\langle \tilde{\beta}_1, \tilde{\beta}_2 \rangle = \kappa e^{i\phi} \in \mathbb{C}$  with  $\kappa \in \mathbb{R}^+$ ,  $\phi \in [0, 2\pi)$  and use  $\|\tilde{\beta}_1\| = \|\tilde{\beta}_2\| = 1$ .

$$\begin{aligned}
 d_{FP}([\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}})^2 &= \min_{\lambda \in \mathbb{R}^+, \theta \in [0, 2\pi)} 1 + \lambda^2 - \lambda(e^{i\theta} \kappa e^{i\phi} + e^{-i\theta} \kappa e^{-i\phi}) \\
 &= \min_{\lambda \in \mathbb{R}^+, \theta \in [0, 2\pi)} 1 + \lambda^2 - \lambda \kappa (e^{i(\theta+\phi)} + e^{-i(\theta+\phi)}) \\
 &= \min_{\lambda \in \mathbb{R}^+} 1 + \lambda^2 - \max_{\theta \in [0, 2\pi)} 2\lambda \kappa \cos(\theta + \phi) \\
 &\stackrel{\theta^{\text{opt}} = -\phi}{=} \min_{\lambda \in \mathbb{R}^+} 1 + \lambda^2 - 2\lambda \kappa
 \end{aligned}$$

From  $\frac{\partial}{\partial \lambda} (1 + \lambda^2 - 2\lambda \kappa) = 2\lambda - 2\kappa \stackrel{!}{=} 0$  it follows that  $\lambda^{\text{opt}} = \kappa$ .

$$d_{FP}([\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}})^2 = (1 + \kappa^2 - 2\kappa^2) = (1 - \kappa^2)$$

Lemma 2.1 i.) follows by considering  $\kappa^2 = |\langle \tilde{\beta}_1, \tilde{\beta}_2 \rangle|^2 = \langle \tilde{\beta}_1, \tilde{\beta}_2 \rangle \langle \tilde{\beta}_2, \tilde{\beta}_1 \rangle$ . Then

$$d_{FP}([\beta_1]_{\text{Eucl}}, [\beta_2]_{\text{Eucl}}) = \sqrt{1 - \langle \tilde{\beta}_1, \tilde{\beta}_2 \rangle \langle \tilde{\beta}_2, \tilde{\beta}_1 \rangle}.$$

Lemma 2.1 ii.) follows by  $\omega^{\text{opt}} = \lambda^{\text{opt}} e^{i\theta^{\text{opt}}} = \kappa e^{-i\phi} = \overline{\langle \tilde{\beta}_1, \tilde{\beta}_2 \rangle} = \langle \tilde{\beta}_2, \tilde{\beta}_1 \rangle$ .



p:a-closed

## A.2. Discussion of Possible Extensions to Closed Curves

p:a-smooth

## A.3. Shape-Smoothing Using the Estimated Covariance-Surface

Idea: Smooth  $q$  in mean basis  $b(t)$ , so that  $\hat{q}(t) = b(t)^T \hat{\theta}_q$ . Then the scalar products simplify to

$$\langle q, \hat{p} \rangle \approx \hat{\theta}_q^H G \hat{\theta}$$

$$\langle q, q \rangle \approx \hat{\theta}_q^H G \hat{\theta}_q$$

As  $q$  can be sparse, we may want to use the estimated covariance matrix  $\hat{\Xi}$  when estimating  $\theta_q$  in the form of a Normal prior  $\theta_q \sim \mathcal{N}_{\mathbb{C}^k}(0, \hat{\Xi})$  leading to

$$\hat{\theta}_q = (B^T B + \Xi^{-1})^{-1} B^T q$$

or equivalently using eigendecomposition  $\Xi = V \Lambda V^{-1}$  with  $V^H V = \mathcal{I}$  and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_k)$

$$\hat{\theta}_q = V(V^H B^T B V + \Lambda^{-1})^{-1} V^H B^T q$$

In general how well this works tends to depend on how close the curve is to the estimated mean.

Maybe using a penalty parameter - controlling the strength of the regularization - is also thinkable? Something like this:

$$\hat{\theta}_q(\lambda) = V(V^H B^T B V + \lambda \cdot \Lambda^{-1})^{-1} V^H B^T q$$

**\*\*Thought\*\*:** A normal prior is probably not appropriate, because while its true that  $\mathbb{E}[\theta] = 0$  (due to rotational symmetry) I would think that in general  $\mathbb{E}[\|\theta\|] \neq 0$ . A distribution more like a Normal distribution lying on a ring around 0 with radius  $r = \|\theta\|$  is probably better? Then estimates wouldn't be pressed to zero so strongly, and instead would be pressed to curve of size equal to the mean.

## B. Implementation Notes

app:b