Problem Set 4: Quantum Circuit and Deutsch's Algorithm

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1 Quantum Circuit

1.1 Write down the quantum states denoted point P, Q, R

(a) $P : |1\rangle\,, \quad Q : |-\rangle\,, \quad R : -|1\rangle$

(b) $P : \alpha|0\rangle - \beta|1\rangle\,, \quad Q : \alpha|1\rangle - \beta|0\rangle\,, \quad R : \frac{\alpha-\beta}{\sqrt{2}}|0\rangle + \frac{\alpha+\beta}{\sqrt{2}}|1\rangle$

(c) $P : \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$
$Q : \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|11\rangle + \beta\delta|10\rangle$
$R : \alpha\gamma|10\rangle + \alpha\delta|11\rangle + \beta\gamma|00\rangle + \beta\delta|01\rangle$

(d) $P : \frac{1}{2}\big(|00\rangle - |01\rangle + |10\rangle - |11\rangle\big)$
$Q : \frac{1}{2}\big(|00\rangle - |01\rangle + |11\rangle - |10\rangle\big)$
$R : |11\rangle$

1.2 Simplifying Quantum Circuit

(a) I

(b) Z

(c) iX

(d) $-Z$

(e) $-iX$

1.3 Quantum Oracle

Recall that the oracle action is defined as $U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$
For example, if $x = |1\rangle, y = |0\rangle, f(x) = x \wedge x$, then

$$U_f|1\rangle|0\rangle = |1\rangle|0 \oplus (1 \wedge 1)\rangle = \underline{|1\rangle|1\rangle}$$

<u>**Exercises: Calculate the output of the circuit after the oracle**</u>

(a) $x = |0\rangle, y = |1\rangle, f(x) = x \wedge x \wedge 1$

(b) $x = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle, y = |1\rangle, f(x) = (x \wedge 1) \vee x$

(c) $x = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, y = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, f(x) = \overline{(x \vee 0)}$

2 Deutsch's Algorithm

2.1 Quantum Oracle: Calculate the output of the circuit after the oracle

(a) Note that $f(x) = x \wedge x \wedge 1 = x$

$$U_f |0\rangle |1\rangle = |0\rangle |1 \oplus 0\rangle = |0\rangle |1\rangle$$

(b) Note that $f(x) = (x \wedge 1) \vee x$

$$|x\rangle |y\rangle = \frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{aligned} U_f |x\rangle |y\rangle &= \frac{1}{\sqrt{2}} |0\rangle |1\rangle - \frac{1}{\sqrt{2}} |1\rangle |1\rangle \\ &= \frac{1}{\sqrt{2}} |0\rangle |1 \oplus 0\rangle - \frac{1}{\sqrt{2}} |1\rangle |1\rangle \\ &= \frac{1}{\sqrt{2}} |0\rangle |1\rangle - \frac{1}{\sqrt{2}} |1\rangle |0\rangle \end{aligned}$$

(c) Note that $f(x) = \overline{(x \vee 0)} = \overline{x}$

$$|x\rangle |y\rangle = |+\rangle |+\rangle = \frac{1}{2}\big(|00\rangle + |01\rangle + |10\rangle + |11\rangle \big)$$

$$\begin{aligned} U_f |x\rangle |y\rangle &= \frac{1}{2}\big(U_f |00\rangle + U_f |01\rangle + U_f |10\rangle + U_f |11\rangle\big) \\ &= \frac{1}{2}\big(|0\rangle |0 \oplus 1\rangle + |0\rangle |1 \oplus 1\rangle + |1\rangle |0 \oplus 0\rangle + |1\rangle |1 \oplus 0\rangle \big) \\ &= \frac{1}{2}\big(|0\rangle |1\rangle + |0\rangle |0\rangle + |1\rangle |0\rangle + |1\rangle |1\rangle \big) \end{aligned}$$

2.2 Deutsch's Oracle: Design an oracle for the negation function

The negation oracle has the action $f(x) = \overline{x}$. Equivalently,

$$U_f |x\rangle |y\rangle = |x\rangle |y \oplus \overline{x}\rangle$$

To perform $|y \oplus \overline{x}\rangle$, we need to add an X gate before the control bit of the CX gate. To get the $|x\rangle$ back, we apply another X gate after the control bit.