Problem Set 4: Quantum Circuit and Deutsch's Algorithm

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1 Quantum Circuit

For this problem set, we are working with a standard gates set that is widely used throughout the quantum literature. This includes single qubit and double qubit gates.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

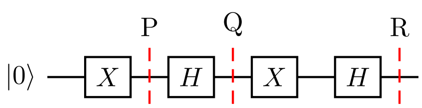
$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}, \quad P(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

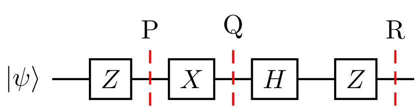
$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

To recap from previous lectures, gates in the quantum circuit are applied in opposite order to that of matrix multiplications.

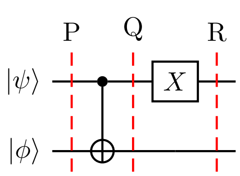
$$|0\rangle \;—\boxed{A}—\boxed{B}—\boxed{C}—\; = CBA\,|0\rangle$$

1.1 Write down the quantum states denoted point P, Q, R in bra-ket notation

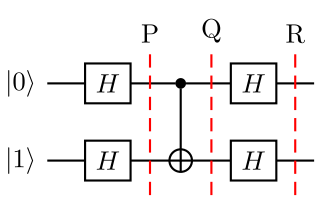
(a) 

(b) 

, where $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

(c) 

$\left.\begin{matrix} |\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \\ |\phi\rangle = \gamma\,|0\rangle + \delta\,|1\rangle \end{matrix}\right.$

(d) 

1.2 Simplifying Quantum Circuit

For these exercises, we can simplify the circuit by following some basic identities.

Gate-Inverse Identity

$$-\boxed{M}-\boxed{M^\dagger}- \;=\; -\boxed{M^\dagger}-\boxed{M}- \;=\; -\boxed{I}-$$

Phase Addition Identity

This tells us that phase gates when apply in series, combines constructively, and so they are commutative.

$$-\boxed{P(\alpha)}-\boxed{P(\beta)}- \;=\; -\boxed{P(\beta)}-\boxed{P(\alpha)}- \;=\; -\boxed{P(\alpha+\beta)}-$$

$X - Z$ Transform Identity

$$-\boxed{H}-\boxed{Z}-\boxed{H}- \;=\; -\boxed{X}-$$

$$-\boxed{H}-\boxed{X}-\boxed{H}- \;=\; -\boxed{Z}-$$

Inverse Phase Identity

$$-\boxed{X}-\boxed{P(\theta)}-\boxed{X}- \;=\; -\boxed{e^{i\theta}}-\boxed{P(-\theta)}-$$

Note that $e^{i\theta}$ is a global phase.

Nested Gate-Inverse Gate Identity

This is a particularly powerful identity which can help us decompose many non-intuitive circuits. Since this is more of a mathematical identity, it will be presented in terms of matrix multiplication.

$$U(AB)U^\dagger = (UAU^\dagger)(UBU^\dagger)$$

We can extend this to a nested product of an arbitrary number of matrices. For a set L containing many matrices, we have the identity

$$U\left(\prod_{A\in L} A\right)U^\dagger = \prod_{A\in L}\left(UAU^\dagger\right)$$

Note that the $\prod$ denotes matrix multiplication; just as $\sum$ denotes a sum.

The reason behind this is quite easy to understand, if we expand the product out to a few terms, we have

$$\begin{aligned}(UAU^\dagger)(UBU^\dagger)(UCU^\dagger)...(UVU^\dagger)(UWU^\dagger) &= UA(U^\dagger U)B(U^\dagger U)C(U^\dagger ...U)V(U^\dagger U)WU^\dagger \\ &= UA(I)B(I)C...V(I)WU^\dagger \\ &= U(ABC...VW)U^\dagger\end{aligned}$$

We use this identity when the nested product of the individual matrix inside is easier to expand the the original product. For example,

$$H(ZX)H = (HZH)(HXH) = XZ$$

We know from $\underline{X - Z \text{ Transform Identity}}$ that each part of the product multiplies to a simpler matrix, X and Z respectively.

<u>Exercises</u>

Decompose the circuit into a single simple quantum gate. Include global phase if there are any. Use circuit identities as listed above whenever possible. In the worst case, perform normal matrix multiplication. Note that $S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}$, and $S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{2}} \end{pmatrix}$

(a) $-\boxed{H}-\boxed{T}-\boxed{T^\dagger}-\boxed{T}-\boxed{T^\dagger}-\boxed{H}-$

(b) $-\boxed{H}-\boxed{T}-\boxed{S^\dagger}-\boxed{T^\dagger}-\boxed{S}-\boxed{X}-\boxed{H}-$

(c) $-\boxed{H}-\boxed{S}-\boxed{X}-\boxed{S^\dagger}-\boxed{X}-\boxed{T}-\boxed{T^\dagger}-\boxed{H}-$

(d) $-\boxed{T}-\boxed{X}-\boxed{T^\dagger}-\boxed{H}-\boxed{S}-\boxed{S^\dagger}-\boxed{X}-\boxed{H}-\boxed{T}-\boxed{X}-\boxed{T^\dagger}-$

(e) $-\boxed{H}-\boxed{T}-\boxed{X}-\boxed{T^\dagger}-\boxed{X}-\boxed{T}-\boxed{X}-\boxed{T^\dagger}-\boxed{X}-\boxed{H}-$

2 Deutsch's Algorithm

2.1 Quantum Oracle

Recall that the oracle action is defined as $U_f \left|x\right\rangle \left|y\right\rangle = \left|x\right\rangle \left|y \oplus f(x)\right\rangle$

For example, if $x = \left|1\right\rangle, y = \left|0\right\rangle, f(x) = x \wedge x$, then

$U_f \left|1\right\rangle \left|0\right\rangle = \left|1\right\rangle \left|0 \oplus (1 \wedge 1)\right\rangle = \underline{\left|1\right\rangle \left|1\right\rangle}$

<u>Exercises: Calculate the output of the circuit after the oracle</u>

(a) $x = \left|0\right\rangle, y = \left|1\right\rangle, f(x) = x \wedge x \wedge 1$

(b) $x = \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle, y = \left|1\right\rangle, f(x) = (x \wedge 1) \vee x$

(c) $x = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle, y = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle, f(x) = \overline{(x \vee 0)}$

2.2 Deutsch's Oracle

The negation oracle has the action $f(x) = \overline{x}$. Equivalently,

$$U_f \left|x\right\rangle \left|y\right\rangle = \left|x\right\rangle \left|y \oplus \overline{x}\right\rangle$$

<u>Exercises: Design an oracle for the negation function</u>

The oracle goes between the two red lines