Chapter 3 Draft

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Chapter 1

Matrices

1.1 Matrices

A matrix is a rectangular array of numbers.

For example,

$$M = \begin{pmatrix} 3 & 9 & -6 \\ -5i & 6.2 & -\sqrt{\pi} \end{pmatrix}$$

The shape of the matrix is [# rows] $\times$ [# columns]. The shape of M is (2×3)

M_{kj} describes the element in the in row j & column k.

Note that the index starts with 0.

$$M = \begin{pmatrix} M_{0,0} & M_{0,1} & M_{0,2} \\ M_{1,0} & M_{1,1} & M_{1,2} \end{pmatrix}$$

For example,

$$M_{1,2} = -\sqrt{\pi}$$

1.2 Matrix Multiplication

Matrix multiplication is not commutative which means that

$$AB \neq BA$$

Multiplication is valid when the shapes of both matrices are compatible.

$$(n \times p)(p \times m) = (n \times m)$$

However,

$$(p \times m)(n \times p) \text{ is not possible}$$

1.3 Row Vector x Column Vector

This is the special case where multiplying two matrices (vectors) together results in a scalar. For $\mathbb{R}^n$, this is equivalence to inner product.

$$(1 \times p)(p \times 1) = (1 \times 1) \text{ (scalar)}$$

1.4 Definition of Matrix Multiplication

The mechanics of matrix multiplication can be seen as taking all the different combinations of row-column multiplication.

For matrix A with shape $(n \times p)$ & matrix B with shape $(p \times m)$,

$$C_{j,k} = (AB)_{j,k} \equiv \sum_{\ell=0}^{p-1} A_{j,\ell} \cdot B_{\ell,k}$$

where $A_{j,\ell}$ is the jth row vector and $B_{\ell,k}$ is the kth column vector

For example,

$$A = \begin{pmatrix} -5+6i & 9-2i & 4 \\ 4+i & -2.5 & 6i \end{pmatrix}, \quad B = \begin{pmatrix} 6-8i & 5+2i \\ -5i & 7-7i \\ 15 & 2i \end{pmatrix}$$

$$C_{2,1} = (AB)_{2,1} = \begin{pmatrix} -5+6i & 9-2i & 4 \end{pmatrix} \begin{pmatrix} 5+2i \\ 7-7i \\ 2i \end{pmatrix} = 12 - 49i$$

Exercise: Calculate in the rest of matrix C

Assuming that the shape is compatible, matrix multiplication is associative.

$$(AB)C = A(BC)$$

1.5 Matrix Addition

Matrix addition requires both matrices to have the same shape.

This is because matrix addition adds elements with the same index on different matrix together.

For matrix A with shape $(n \times m)$ & matrix B with shape $(n \times m)$,

$$C_{j,k} = A_{j,k} + B_{j,k}$$

Matrix multiplication is commutative

$$A + B = B + A$$

And associative

$$A + (B + C) = (A + B) + C$$

1.6 Scalar Multiplication

Scalar multiplication with a matrix is defined as

$$C_{j,k} = cA_{j,k}$$

Scalar multiplication with a matrix is distributive.

$$c(A + B) = cA + cB$$

1.7 Zero Matrix (Additive Identity)

Every matrix has an additive identity with the same shape associated with it.

$$M + (0) = (0) + M = M$$

For M with shape $(n \times m)$,

$$(0)_{j,k} = 0$$

Matrices with all 0s are called zero matrix.

All matrix multiplication with a compatible zero matrix results in a zero matrix.

$$M(0) = (0)M = (0)$$

Note that the shape of the resulting zero matrix maybe different depending on the shape of its zero matrix factor.

1.8 Identity Matrix (Multiplicative Identity)

Every matrix has an multiplicative identity with compatible shape associated with it.

$$\mathbb{I}\ M = M\ \mathbb{I} = M$$

Identity matrices are square matrices with shape $(p \times p)$.

$$\mathbb{I}_{j,k} = \delta_{j,k}$$

This means that $\mathbb{I}$ has 1s along its main diagonal (top left to bottom right) and 0s everywhere else.

When M is a non-square matrix, the shape of $\mathbb{I}$ is different depending on its position in the multiplication. However, product matrix is still the same.

1.9 Matrix Transpose

The transpose of a matrix is its reflection the main diagonal.

For A with shape $(n \times m)$, its tranposed A^T has shape $(m \times n)$.

For each element in the matrix,

$$(A_{j,k})^T = A_{k,j}$$

For example,

$$\begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} \\ A_{1,0} & A_{1,1} & A_{1,2} \end{pmatrix}^T = \begin{pmatrix} A_{0,0} & A_{1,0} \\ A_{0,1} & A_{1,1} \\ A_{0,2} & A_{2,2} \end{pmatrix}$$

One application of this is proving orthonormality of basis.

For a proposed set orthonormal basis $\mathcal{A} = \{\vec{v}_k\}_{k=0}^{n-1}$, place the vectors side by side into a matrix.

$$\mathcal{A} = \{\vec{v}_k\}_{k=0}^{n-1} \longrightarrow A = \begin{pmatrix} \vec{v}_0 & \vec{v}_1 & \vec{v}_2 & ... \, \vec{v}_k & \vec{v}_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

We know from **2.8** that the number of elements in the basis set is a well-defined number for the vector space. If this number is minimal, the matrix built from these vectors should be a square matrix.

In this case if the product of A and its transpose A^T equals the identity matrix with the same size, the basis set $\mathcal{A}$ is orthonormal.

$$\underbrace{A^T A}_{1} = \underbrace{A A^T}_{2} = \mathbb{I} \rightarrow \mathcal{A} \text{ is orthonormal}$$

Proof that 1. $A^T A = \mathbb{I}$

Recall that matrix multiplication is simply taking multiple row-column vector product.

From the definition of orthonormality **2.10**,

$$\vec{v}_j \cdot \vec{v}_k = \delta_{jk}$$

This means that for our $A^T A$, the product is 1 only when the ℓth row vector is dotted the corresponding ℓth column vector.

From the definition of identity matrix **3.8**, this shows that

$$(A^T A)_{j,k} = \delta_{j,k} = \mathbb{I}_{j,k}$$

Proof that 2. $AA^T = \mathbb{I}$

From the above proof, we know that

$$A^T A = \mathbb{I}$$

Multiply the whole expressions with matrix A gives

$$A(A^T A) = A(\mathbb{I})$$

Using the associative property and the definition of the identity matrix,

$$A(A^T A) = (AA^T)A = (\mathbb{I})A$$

From this, we can deduce that

$$AA^T = \mathbb{I}$$

1.10 Matrix Product of Vectors

Vectors are matrices with shape $(1 \times m)$, or $(m \times 1)$.

Because of this, vectors has all the properties of matrix addition and multiplication from above.

One that is especially useful is the distributive property:

$$M(\vec{v} + \vec{w}) = M\vec{v} + M\vec{w}$$

Matrix product of vectors are sometimes called linear transformations.

This is because multiplying a vector with a matrix returns a vector.

$$(m \times n)(n \times 1) = (m \times 1)$$

or

$$(1 \times n)(n \times m) = (1 \times m)$$

Horizontal vectors with shape $(1 \times m)$ are called row vectors.

For example,

$$\vec{v} = \begin{pmatrix} 1 & -2i & 5\pi & 4\sqrt{3} \end{pmatrix}$$

Vertical vectors with shape $(m \times 1)$ are called column vectors.

$$\vec{u} = \begin{pmatrix} 1 \\ -2i \\ 5\pi \\ 4\sqrt{3} \end{pmatrix}$$

Notice that a row vector can be transposed to form to column vector, and a column vector can be transposed to form to row vector.

1.11 Determinants of a 2 x 2 Matrix

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

The determinant of A can be calculated by

$$det\ A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

1.12 Determinants of a 3 x 3 Matrix

For $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$,

$$det\ B = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

= a (minor of a) - b(minor of b) + c(minor of c)

Minor is the determinant of a smaller matrix made by crossing out the element's row & column.

1.13 Determinants of an n x n Matrix

$det(A) = |A| = \sum_{k=1}^{n} (-1)^{k+j} A_{jk}$ (minor of $A_j k$)

1.14 Determinants of Products

$det(AB) = det(A)\, det(B)$

1.15 Matrix Inverses

$A^{-1}A = AA^{-1} = I$

If A has an inverse, it's invertible or non-singular.

Little Inverse Theorem M is singular if M$\vec{v} = 0$, for $\vec{v} \neq 0$

Big Inverse Theorem M is singular $\iff$ det(M) = 0

1.16 System of Linear Equations

A system of n unknowns is only solvable if there are n independent equations.

1.17 Matrix Equations

M$\vec{v}$ = c, where

M is the matrix of the linear combination

$\vec{v}$ is the vectors of the unknown

c is the vectors of the constants

To solve, $M^{-1}M\vec{v} = \vec{v} = M^{-1}c$

1. Determine if matrix is invertible
2. If yes, compute inverse

1.18 Cramer's Rule

For M$\vec{v}$ = c, $x_k = \frac{detM_k}{detM}$

where M_k is the matrix M with the kth element column replaced by the constant vector c,

To find inverse, split inverse into column & solve for individual variables with Cramer's Rule.

1.19 Complex Vector Space, $\mathbb{C}^n$

$$\mathbb{C}^n = \{ \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix} c_k \in C,\ k = 0, ..., n-1\} \text{ (complex scalar)}$$

1.20 Complex Inner Product

$a \cdot b = \langle a, b\rangle = \langle a|b\rangle = \sum_{k=0}^{n-1} \bar{a}_k b_k = \sum_{k=0}^{n-1} (a_k^*) b_k$

* Non-commutative: $\langle a|b\rangle \neq \langle b|a\rangle$

However, $\langle a|b\rangle^* = \langle b|a\rangle$

* Physicists conjugate the left vector of the inner product.

- Distributive: $\langle a|b + b'\rangle = \langle a|b\rangle + \langle a|b'\rangle$ and $\langle a + a'|b\rangle = \langle a|b\rangle + \langle a'|b\rangle$
- Anti-linear in the 1st position: $c\langle a|b\rangle = \langle c^* a|b\rangle$
- Linear in the 2nd position: $c\langle a|b\rangle = \langle a|c\, b\rangle$ For both cases, c $\in$ C

1.21 Norm

$||\vec{a}|| = \sqrt{\sum_{k=0}^{n-1} |\vec{a}_k|^2} = \sqrt{\sum_{k=0}^{n-1} (\vec{a}_k)^* \, \vec{a}_k}$

1.22 Distance

$\mathrm{dist}(\vec{a}, \vec{b}) = ||b - a|| = \sqrt{\sum_{k=0}^{n-1} |\vec{b}_k - \vec{a}_k|^2}$

These all results in $||b - a|| \geq 0$

1.23 Expansion Coefficients

$\langle b_k | \vec{v} \rangle = \sum_{j=0}^{n-1} \beta_j \delta_{kj}$ with $\vec{v}$ on the right side of inner product.

This only works for orthonormal basis.