Chapter 2 Draft

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Contents

Chapter 1

Real and Complex Vector Spaces

All of quantum computing occurs in one vector space or another. In fact, the qubit, the fundamental unit, of quantum computing is a vector quantity.

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

The definition of a vector space is called its axioms. Axioms defines the objects and the rules (operations on the objects) of the vector space.

1.1 Objects of the Vector Space

Every vector space is made up of two fundamental sets: scalars and vectors.

Scalars The underlying field (number system) of the vector space

Examples of scalars sets: $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers).

Examples of scalars: $a = -\pi,\ b = e^{i\frac{\pi}{4}} = \sqrt{2} + \sqrt{2}i$

Vectors The objects (ordered n-tuples) of the vector space.

Examples of vector set (space): $\mathbb{R}^2$ (2-d real vectors), $\mathbb{C}^3$ (3-d complex vectors).

Examples of vectors: $\vec{v} = \begin{pmatrix} -e \\ \sqrt{\pi} \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 5-9i \\ -6 \\ 4i \end{pmatrix}$

1.2 Rules

These are the valid operations with scalars and vectors, assuming that the objects of the vector space are already defined.

Vector Addition: $\vec{v} + \vec{w} \Rightarrow \vec{u}$

$$\begin{pmatrix} 16+20i \\ -6+7i \end{pmatrix} + \begin{pmatrix} 5-20i \\ -9-10i \end{pmatrix} = \begin{pmatrix} 21 \\ -15-3i \end{pmatrix}$$

Some properties of vector addition:

- Zero Vector: $\vec{v} + 0 = 0 + \vec{v} = \vec{v}$

  By definition, every vector space contains a zero vector. This vector is an n-tuples made up of all 0.

$$\begin{pmatrix} -13-20i \\ 3+14i \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -13-20i \\ 3+14i \end{pmatrix} = \begin{pmatrix} -13-20i \\ 3+14i \end{pmatrix}$$

- Additive Inverses: $\vec{v} + (-\vec{v}) = 0$

$$\begin{pmatrix} 15-8i \\ -2+7i \end{pmatrix} + - \begin{pmatrix} 15-8i \\ -2+7i \end{pmatrix} = \begin{pmatrix} 15-8i \\ -2+7i \end{pmatrix} + \begin{pmatrix} -15+8i \\ 2-7i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Commutativity: $\vec{v} + \vec{t} = \vec{t} + \vec{v}$

$$\begin{pmatrix} 9-2i \\ -19+i \end{pmatrix} + \begin{pmatrix} 8i \\ 15-6i \end{pmatrix} = \begin{pmatrix} 8i \\ 15-6i \end{pmatrix} + \begin{pmatrix} 9-2i \\ -19+i \end{pmatrix} = \begin{pmatrix} 9+6i \\ -4-5i \end{pmatrix}$$

- Associativity: $(\vec{v} + \vec{t}) + \vec{w} = \vec{v} + (\vec{t} + \vec{w})$

$$\left[\begin{pmatrix} 2+9i \\ -2+3i \end{pmatrix} + \begin{pmatrix} 1-12i \\ -2+3i \end{pmatrix}\right] + \begin{pmatrix} 4+6i \\ -5-2i \end{pmatrix} = \begin{pmatrix} 2+9i \\ -2+3i \end{pmatrix} + \left[\begin{pmatrix} 1-12i \\ -2+3i \end{pmatrix} + \begin{pmatrix} 4+6i \\ -5-2i \end{pmatrix}\right]$$

  Exercise: Evaluate

Scalar Multiplication: $c * \vec{v} \Rightarrow \vec{w}$ ($*$ represents multiplication)

$$(6+5i) * \begin{pmatrix} 8-3i \\ -6+5i \end{pmatrix} = \begin{pmatrix} 63+22i \\ -61 \end{pmatrix}$$

Some properties of scalar multiplication:

- Scalar Identity: $1 * \vec{v} = \vec{v}, \quad \forall\, \vec{v}$

$$1 * \begin{pmatrix} -19+16i \\ 10-11i \end{pmatrix} = \begin{pmatrix} -19+16i \\ 10-11i \end{pmatrix}$$

- Associativity: $c * \vec{v} = \vec{v} * c$

$$2 * \begin{pmatrix} 6-3i \\ -8+2i \end{pmatrix} = \begin{pmatrix} 6-3i \\ -8+2i \end{pmatrix} * 2 = \begin{pmatrix} 12-6i \\ -16+4i \end{pmatrix}$$

- Distributivity: $c * (\vec{v} + \vec{t}) = (c * \vec{v}) + (c * \vec{t})$

$$(-15-2i) * \left[\begin{pmatrix} 5 \\ 2-10i \end{pmatrix} + \begin{pmatrix} -6+i \\ -17i \end{pmatrix}\right]$$

Exercise: Expand the above expression by distributivity and evaluate

Inner Product: $\vec{v} \cdot \vec{w} \Rightarrow c$

This is also called dot product.

To see how to calculate real and complex inner (dot) product, see Chapter 1, Complex Inner Product

$$\begin{pmatrix} 2i \\ 9-5i \end{pmatrix} \cdot \begin{pmatrix} -12-6i \\ 5+6i \end{pmatrix} = -106-60i$$

The complex inner product can be either real or complex.

Some properties of inner products:

- Commutativity: $\vec{v} \cdot \vec{t} = \vec{t} \cdot \vec{v}$

  Recall that for complex inner product,

$$\vec{v} \cdot \vec{t} = (\vec{v})^{\dagger} \cdot \vec{t}$$

  Commutativity only holds if the Hermitian conjugate is the same vector

$$(\vec{v})^{\dagger} \cdot \vec{t} = \vec{t} \cdot (\vec{v})^{\dagger}$$

$$\begin{pmatrix} -2+i \\ 1+5i \end{pmatrix} \cdot \begin{pmatrix} -2-6i \\ 15+16i \end{pmatrix} = \begin{pmatrix} -2-6i \\ 15+16i \end{pmatrix} \cdot \begin{pmatrix} -2+i \\ 1+5i \end{pmatrix} = -123+47i$$

- Distributivity: $\vec{v} \cdot (\vec{t} + \vec{w}) = \vec{v} \cdot \vec{t} + \vec{v} \cdot \vec{w}$

$$\begin{pmatrix} 8-9i \\ 12-13i \end{pmatrix} \cdot \left[\begin{pmatrix} 19-i \\ -5i \end{pmatrix} + \begin{pmatrix} 16 \\ -12-7i \end{pmatrix} \right]$$

  Exercise: Let the left vector be the Hermitian conjugate, expand the above expression by distributivity and evaluate

- Associativity with Scalar Multiplication: $c * (\vec{v} \cdot \vec{t}) = (c * \vec{v}) \cdot \vec{t} = \vec{v} \cdot (c * \vec{t})$

  However, for complex inner product, there are some minor modifications:

$$c * (\vec{v} \cdot \vec{t}) = (c^{\dagger} * \vec{v}) \cdot \vec{t} = \vec{v} \cdot (c * \vec{t})$$

  Exercise: Explain why the $(c^{\dagger} * \vec{v})$ in the middle expression is necessary

Length (Modulus or Norm): $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$, $||\vec{v}|| \geq 0$

$$\left\|\begin{matrix} 5-8i \\ 6+3i \end{matrix}\right\| = \sqrt{\begin{pmatrix} 5-8i \\ 6+3i \end{pmatrix} \cdot \begin{pmatrix} 5-8i \\ 6+3i \end{pmatrix}} = \sqrt{134}$$

Orthogonality: $\vec{v} \cdot \vec{t} = 0$

Any two vectors that are not multiples of one another will form a plane. On that plane, if the two vectors are perpendicular with one another, they are orthogonal.

This can be represented by their dot product being 0.

$$\begin{pmatrix} 5+6i \\ 9-3i \end{pmatrix} \cdot \begin{pmatrix} -15-15i \\ 16-7i \end{pmatrix} = 0 \therefore \text{ orthogonal}$$

If a set of vectors are all orthogonal to one another and all has length of 1, they are called an orthonormal.

1.3 Positive Definiteness of Inner Product

More generally, the inner product of a space is any operation that satisfies positive definiteness. In theory, a vector space can have more than one inner products or it can have none.

- The inner product of any vectors with itself is non-negative scalar value.

$$\vec{v} \cdot \vec{v} \geq 0$$

- The length of any vector that is not the zero vector is non-zero.

$$\vec{v} \neq 0 \Rightarrow ||\vec{v}|| > 0$$

If an operation does not satisfies positive definiteness, it's called a paring.

1.4 Unit Vector

A unit vector has length of 1 unit. To turn a vector into a unit vector, simply scale the magnitude of the vector. This is done to conserve the direction of the vector.

$$\vec{v} \longrightarrow \frac{\vec{v}}{||\vec{v}||}$$

1.5 Linear Combination

The sums of the scalar products of vectors are called linear combinations in maths & superpositions in physics.

$$\vec{t} = \alpha\vec{v} + \beta\vec{w} + \gamma\vec{u} + ...$$

Or in a more concise notation, where c_k are coefficients and $\vec{v}_k$ are vectors

$$\vec{t} = \sum_{k=0}^{n-1} c_k\vec{v}_k$$

1.6 Linear Independence

Set $\mathcal{S}$ is linearly independent if the span of set $\mathcal{S}$ with all non-zero coefficients does not contain the zero vector.

$$\sum_{k=0}^{n-1} c_k\vec{v}_k \neq \vec{0} \text{ if } c_k \text{ are non-zero}$$

The consequences of this are

1. The vectors in set $\mathcal{S}$ are not scaled version of one another.

$$\vec{v}_{k1} \neq \alpha\vec{v}_{k2}$$

2. One vector cannot be represented as the linear combination of another.

$$\vec{v}_t \neq \sum_{\substack{\text{all } k \\ \text{except } t}} -\frac{c_k}{c_t}\vec{v}_k = \sum_{\substack{\text{all } k \\ \text{except } t}} d_k\vec{v}_k,\ t \in \{0, ..., n-1\}$$

If this is not the case, then set $\mathcal{S}$ linearly dependent.

For example,

$$\text{Let } \mathcal{L} = \left\{ \begin{pmatrix} 8-5i \\ 6+3i \end{pmatrix}, \begin{pmatrix} -34-i \\ -12-21i \end{pmatrix}, \begin{pmatrix} 9+7i \\ 2i \end{pmatrix} \right\}$$

After doing some arithmetic, we can see that the second vector is $-3-2i$ times the first vector.

This simple can tell us right off the bat that set $\mathcal{L}$ is linearly dependent. However, if we remove either one of these vectors, $\mathcal{L}$ will be linearly independent.

The process of determining the linear independence of a set is by solving for the constant c_k in the simultaneous equations. If at least one c_k is 0, then the set is linearly independent.

$$\sum_{k=0}^{n-1} c_k \vec{v}_k = \vec{0}$$

1.7 Span

Let there be a set of vectors $\mathcal{S} = \{\vec{v}_0, \vec{v}_1, \vec{v}_2, ..., \vec{v}_{n-1}\}$

The span of set $\mathcal{S}$ is another set of vectors $\mathcal{T}$ that can be created by taking all the possible linear combinations of the vectors in set $\mathcal{S}$.

If linear combination was a function f with domain $\mathcal{S}$, then its range (span) would be $\mathcal{T}$.

$$f(\mathcal{S}) \longrightarrow \mathcal{T}$$

In other words, the span of a set $\mathcal{S}$ is all the vectors linearly dependent to it.

1.8 Bases of a Vector Space

The basis of a space is the minimal set of vectors that completely span the vector space.

Conjecture: All elements of the basis set are linearly independent vectors.

To proof this conjecture, we have to show that these vectors satisfies:

1. Completeness (Spanning)
2. Set Minimality

1. Completeness (Spanning)

Let $\mathcal{A} = \{\vec{v}_k\}_{k=0}^{z-1}$ be a set of z vectors in the vector space $\mathcal{V}$.

To repeat, the span of $\mathcal{A}$ is all the possible linear combinations of the vectors in $\mathcal{A}$.

$$\text{span}(\mathcal{A}) = \sum_{k=0}^{z-1} c_k \vec{v}_k,\ c_k \in \text{field of } \mathcal{A}$$

We want to make the set $\mathcal{A}$ spans completely over $\mathcal{V}$.

$$\text{span}(\mathcal{A}) = \mathcal{V}$$

Let there be an arbitrary vector $\vec{w} \notin \text{span}(\mathcal{A})$,

From 1.7, this implies linearly independence with $\mathcal{A}$.

$$\vec{w} \neq \sum_{k=0}^{z-1} c_k \vec{v}_k$$

Add $\vec{w}$ to set $\mathcal{A}$ to extend the span of $\mathcal{A}$.

Repeat the process until $\mathcal{A}$ spans completely over $\mathcal{V}$.

$$\mathcal{A}_f = \{\vec{w}_0, \vec{w}_1, ..., \vec{v}_k, \}_{k=0}^{z-1}$$

2. Set Minimality

The cardinality of a set is a measure of its size, or the number of elements in that set.

For examples, set $\mathcal{B} = \{0, 1, 2, 3\}$ has cardinality 4.

$$|\mathcal{B}| = 4$$

A minimal set contains a finite and is bounded away from zero number of elements that satisfies certain conditions. In this case, it's the smallest set that satisfies completeness.

$$0 < \# \textbf{ elements in a basis set} < \infty$$

A vector space can have more than one set of basis. However, the number of elements of each of them is the same.

Proof of Lower Bound > 0

Assume set $\mathcal{A} = \{\vec{v}_k\}_{k=0}^{n-1}$ from before is a random collection of vectors that span $\mathcal{V}$.

From 1.7, if a vector from set $\mathcal{A}$ is linearly dependent to the set, this implies

$$\sum_{k=0}^{n-1} c_k \vec{v}_k = \vec{0} \text{ if } c_k \text{ are non-zero}$$

This means that at least one vector is made up of linear combinations of other vectors.

$$\vec{v}_t = \sum_{\substack{\text{all } k \\ \text{except } t}} -\frac{c_k}{c_t}\vec{v}_k = \sum_{\substack{\text{all } k \\ \text{except } t}} d_k \vec{v}_k,\ t \in \{0, ..., n-1\}$$

Substituting $\vec{v}_t$ back into the first equation, we get

$$\sum_{\substack{\text{all } k \\ \text{excluding } t}} c_k \vec{v}_k + \sum_{\substack{\text{all } k \\ \text{excluding } t}} d_k \vec{v}_k = \sum_{\substack{\text{all } k \\ \text{excluding } t}} (c_k + d_k)\vec{v}_k = \vec{0}$$

The initial sum has more elements than the final sum.

$$\sum_{k=0}^{n-1} c_k \vec{v}_k = \vec{0} = \sum_{\substack{\text{all } k \\ \text{excluding } t}} (c_k + d_k)\vec{v}_k$$

Follow this process, we can reduce the number of elements in the set until we have at least only one element, which give us the lower bound of 1.

$$\underline{1} < |\mathcal{A}| < \infty$$

Because we eliminated all the dependent vectors, we have have left are linearly independent vectors.

Now we only have to confirm the upper bound.

Proof of Upper Bound $< \infty$

The vector space $\mathcal{V}$, in essence, is a set of n-tuples vectors in the field $\mathbb{C}$.

The famous mathematician Georg Cantor has proven three results that are relevant to our proof:

1. The cardinality of real numbers $\mathbb{R}$, called $\beth_1$ (beth-one), is bigger than the cardinality of natural numbers $\mathbb{N}$, called $\aleph_0$ (aleph-null) or $\beth_0$ (beth-null).

   $$|\mathbb{R}| > |\mathbb{N}| \text{ or } \beth_1 > \beth_0$$

   In other words, what this means is that there are more real numbers than natural numbers.

   Specifically, $|\mathbb{R}|$, or $\beth_1$, is equal to the cardinality of the power set of $|\mathbb{N}|$, or $\beth_0$.

   $$\beth_1 = P(\beth_0)$$

   The power set of a set $\mathcal{B}$ is the set of all subsets of $\mathcal{B}$, including the empty set and $\mathcal{B}$ itself.

   For example, let $\mathcal{B} = \{a, b, c\}$

   $$P(\mathcal{B}) = \{\varnothing\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$$

   The cardinality of the power set of a set with size n is 2^n.

   $$|\mathcal{S}| = n \to |P(\mathcal{S})| = \sum_{k=0}^{n-1} \binom{n}{k} = 2^n$$

   The cardinality of the power set of is larger than that of the set.

   $$P(\mathcal{S}) = 2^{|\mathcal{S}|} > |\mathcal{S}|$$

   For the case of real numbers, we can conclude that the following

   $$\beth_1 = P(\beth_0) = 2^{\beth_0} > \beth_0$$

2. The cardinality of the real numbers $\mathbb{R}$ is equal to that of any n-dimensional Euclidean space $\mathbb{R}^n$ (see space-filling curve).

   $$|\mathbb{R}| = |\mathbb{R}^n| = \beth_1$$

3. The cardinality of the real numbers $\mathbb{R}$ is equal to that of complex numbers $\mathbb{C}$.

$$|\mathbb{R}| = |\mathbb{C}| = \beth_1$$

From these three results, we can reason that the cardinality of any n-dimensional Euclidean space $\mathbb{R}^n$ is equal to that of any n-dimensional complex Euclidean space $\mathbb{C}^n$.

$$|\mathbb{R}^n| = |\mathbb{C}^n| = \beth_1$$

Now, we have all the necessary tools to proceed with our proof.

Returning to $\mathcal{A} = \{\vec{v}_k\}_{k=0}^{z-1}$ after we have eliminated all the linearly dependent vectors, repeat the process of adding linearly independent vectors not in the span of $\mathcal{A}$ to it.

The power set, $P(\mathcal{A})$, has 2^z elements. Each element in this $P(\mathcal{A})$ is a subset of $\mathcal{A}$, containing a different combination of $\vec{v}_k$.

Now take the sum of of each subset.

$$\text{sum}(\{\vec{v}_\alpha, \vec{v}_\beta, \vec{v}_\gamma\}) = \vec{v}_\alpha + \vec{v}_\beta + \vec{v}_\gamma$$

If we don't consider the coefficients, c_k, we can produce at least 2^z linear combinations from a set of z linearly independent vectors

The results of these linear combinations are vectors that are linearly dependent to $\mathcal{A}$

Notice that if we pick a random vector $\vec{u}$ out of $\mathcal{V}$, there are only two possibilities, $\vec{u}$ is linearly independent to $\mathcal{A}$ or not.

If $\vec{u}$ is linearly independent to $\mathcal{A}$, we add it to $\mathcal{A}$.

$\mathcal{A}$ now has $z+1$ elements. Follow the same process as above, we can show that the vectors in $\mathcal{A}$ can produce at least 2^{z+1} vectors linearly dependent to $\mathcal{A}$.

Do this long enough, we should get a $\mathcal{A}$ that spans completely over $\mathcal{V}$, and we called this $\mathcal{A}_f$.

Supposition: The cardinality $\mathcal{A}_f$ is infinite.

More rigorously, the cardinality of $\mathcal{A}_f$ is equal to that of $\mathcal{V}$.

$$|\mathcal{A}_f| = |\mathcal{V}|$$

From $\mathcal{A}_f$, we can produce at least $2^{|\mathcal{A}_f|}$ linearly combinations that are vectors linearly dependent to $\mathcal{A}_f$.

The set of these vectors, call it $\mathcal{D}$, has the following cardinality

$$|\mathcal{D}| = 2^{|\mathcal{A}_f|} = P(\mathcal{A}_f)$$

We have already shown that the cardinality of a power set is larger than that of the set. Therefore,

$$|\mathcal{D}| = P(\mathcal{A}_f) > |\mathcal{A}_f|$$

This means that the cardinality of the linearly dependent vectors set is larger than that of the vector space itself!

$$|\mathcal{A}_f| = |\mathcal{V}| \longrightarrow |\mathcal{D}| > |\mathcal{V}|$$

This obviously cannot be true, thus the supposition is refuted.

This means that $\mathcal{A}_f$ is a finite set.

$$1 < |\mathcal{A}| < \underline{\infty}$$

Because all the vectors independent to $\mathcal{A}$ in $\mathcal{V}$ is already in $\mathcal{A}$, we know that all other vectors in $\mathcal{V}$ are linearly dependent to $\mathcal{A}$. Therefore,

$$\text{span}(\mathcal{A}) = \mathcal{V}$$

This implies that all elements in $\mathcal{A}$ is linearly independent.

Finally, we have to proof that $|\mathcal{A}|$ well-defined for all $\mathbb{C}^n$ vector space.

Can someone help with this?

To summarize, the basis of a space is a minimal set of linearly independent vectors that span the vector space completely.

1.9 Natural (Standard) Basis

The natural basis set are the set of vectors in $\mathbb{C}^n$ is $\mathcal{S} = \{\hat{e}_j\}_{j=0}^{n-1}$, where

$$\hat{e}_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ n \text{ elements} \end{pmatrix} \text{at } j\text{th index}$$

*hat denotes unit length

Notice that this is a unit vector because it is 1 at the jth index and 0 everywhere.

$$\sqrt{0^2 + 0^2 + 1^2 + ... + 0^2} = 1$$

Also, because the position of 1 is different for all $\hat{e}_j \in \mathcal{S}$, we know that they are orthogonal to each other.

$$\hat{e}_a \cdot \hat{e}_b = 0$$

Natural basis set is a special set of orthonormal bases.

For example, the natural basis of $\mathbb{C}^2$ is

$$\mathcal{S} = \{\hat{x}, \hat{y}\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Exercise: Prove Orthonormality (Unit length and Orthogonality)

From this we can build any vectors in $\mathbb{C}^2$ by taking linear combinations of the natural basis vectors.

$$\begin{pmatrix} 8+\pi i \\ -\sqrt{3}-5i \end{pmatrix} = (8+\pi i)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-\sqrt{3}-5i)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

1.10 Orthonormal Bases

To iterate, orthonormal bases are set of basis vectors that are of unit length and are mutually orthogonal to each other.

$$\text{The basis set } \mathcal{B} = \{\vec{b}_k\} \text{ is orthonormal if } \vec{b}_k \cdot \vec{b}_j = \delta_{kj},\ \forall\, \vec{b}_j, \vec{b}_k \in \mathcal{B}$$

This means that $\vec{b}_k$ is orthogonal with all other basis vectors except itself.

Turning an arbitrary vector into a unit vector requires normalization,

$$\vec{v} \to \frac{\vec{v}}{||\vec{v}||}$$

The Gram-Schmidt process turns a random set of basis vectors in to and orthonormal basis.

$$\{\vec{v}_j\}_{j=0}^{n-1} \longrightarrow \left\{ \frac{\vec{v}_j - \sum_{i=1}^{j-1}(\vec{v}_j \cdot \hat{e}_j)\hat{e}_i}{\left\| \vec{v}_j - \sum_{i=1}^{j-1}(\vec{v}_j \cdot \hat{e}_j)\hat{e}_i \right\|} \right\}_{j=0}^{n-1} \text{ where } \hat{e}_0 = \frac{\vec{v}_0}{||\vec{v}_0||}$$

1.11 Expansion Coefficients Of Orthonormal Basis

For a linear combination of an arbitrary set of basis vectors $\{\vec{v_k}\}_{k=0}^{n-1}$,

$$v_f = c_0\vec{v_0} + c_1\vec{v_1} + c_2\vec{v_2} + ...c_k\vec{v_k}$$

The scalar constants c_k are called expansion coefficients.

Finding these coefficients for an arbitrary set of basis vectors, requires solving systems of equations,

$$x + 3y = 5$$

$$6x - 3y = -7$$

However, the set of basis vectors is orthonormal, we can simply take the dot product of the resultant vectors with each of the basis to get the corresponding coefficient.

For $\mathbf{v} = \sum_{k=1}^{n} \alpha_k \mathbf{b_k}$, if we want to extract the coefficient α_j,

$$\mathbf{b_j} \cdot \mathbf{v} = \mathbf{b_j} \cdot \sum_{k=0}^{n-1} (\alpha_k \mathbf{b_k}) = \sum_{k=0}^{n-1} \mathbf{b_j} \cdot (\alpha_k \mathbf{b_k}) = \sum_{k=0}^{n-1} \alpha_k (\mathbf{b_j} \cdot \mathbf{b_k})$$

From the definition of orthonormal basis above,

$$\sum_{k=0}^{n-1} \alpha_k (\mathbf{b_j} \cdot \mathbf{b_k}) = \sum_{k=1}^{n} \alpha_k \delta_{kj} = \alpha_j$$

When the basis set is orthonormal, the dot product with one of the basis vector is the coordinate representation expanded along that basis.

Can someone help visualize this?

Let $\vec{t}$ be the linear combination of vectors in the orthonormal basis set $\mathcal{A} = \{\vec{v}_k\}_{k=0}^{n-1}$,

$$\vec{t} = \sum_{k=0}^{n-1} c_k \vec{v}_k$$

$\vec{t}$ can be represented a vector of coefficients c_k under the basis $\mathcal{A}$

$$\vec{t} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{pmatrix}_{\mathcal{A}}$$

More about this in the next chapter.

1.12 Subspace

A subspace of a (parent) vector space is a subset of the parent vectors that is closed under the vector/scalar operations.

For example, let $\mathcal{A}$ the basis of the space $\mathcal{V}$, the subset of $\mathcal{A}$ spans a subspace of $\mathcal{V}$

Take an arbitrary basis in $\mathbb{R}^3$

$$\mathcal{A} = \left\{ \begin{pmatrix} 5 \\ 8 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} \right\}$$

The span of these vectors is the 3-D real space.

Remove one of the vectors so that $\mathcal{B} \subset \mathcal{A}$,

$$\mathcal{B} = \left\{ \begin{pmatrix} 5 \\ 8 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix} \right\}$$

Now, the linear combination of vectors in $\mathcal{B}$ only span a 2-D plane.

Can someone help draw this? A diagram of the 3-d space and the resulting 2-d subspace