Q Notes

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Chapter 1

Complex Arithmetic

1.1 Complex Dot Product

$\vec{a} \cdot \vec{b} = \sum_i \alpha_i \beta_i = \alpha_T \beta;\ \alpha, \beta \in \mathbb{R}$

$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} \geq 0, \in \mathbb{R}$

For complex vectors, $\sqrt{\vec{a} \cdot \vec{a}}$ could be negative and $|\vec{a}| \in \mathbb{C}$.

1.2 Complex Conjugate

For $\vec{\gamma} = \begin{pmatrix} a+ib \\ c+id \end{pmatrix}$, $\vec{\gamma}* = \begin{pmatrix} a-ib \\ c-id \end{pmatrix}$

Conjugation is distributive across sums and products.

$(\alpha + \beta)^* = \alpha^* + \beta^*$

$(\alpha \times \beta)^* = \alpha^* \times \beta^*$

$|z|^2 = z^* z = z z^* = a^2 + b^2$

Now, the dot product defintition changes:

$\vec{\alpha} \cdot \vec{\beta} = \vec{\alpha}_T \cdot \vec{\beta}$, where $\vec{\alpha} \cdot \vec{\alpha} = \vec{\alpha^*_T} \vec{\alpha} \in \mathbb{R}^+$

α^*_T is also called the Hermitian conjugate $\equiv \alpha^\dagger$ (alpha-dagger).

$zw =$ rs $e^{i\theta + \phi}$; $\frac{z}{w} =$ rs $e^{i\theta + \phi}$; $z^* =$ re$^- i\theta$ (Complex Polar Form)

1.3 Roots of Unity

$\omega_n = {}^n\sqrt{1} = cos(\frac{2\pi \cdot k}{n}) + i\ sin(\frac{2\pi \cdot k}{n}),\ k \in \{1, ..., n\}$

1.4 Summing Roots of Unity

$\sum_{k=0}^{n-1} \omega_n^k = A \Rightarrow A = 0$ because $A \times \omega_n^k = A$

Substitute $\omega_n =$ z and multiply by z-1, we get

$$\prod_{k=0}^{n-1} z - \omega_n^k = 0$$

1.5 Kronecker Delta

$\delta_{kj} = \{ {}^{1,\, if\ k=j}_{0,\, otherwise} \}$

$\sum_{k=0}^{N-1} \omega^{(j-m)k} = N\delta_j m$ for sum of roots of unity.

Chapter 2

Real Vector Spaces

2.1 Linear Algebra

Axioms defines the objects of a vector space and the rules.

2.2 The Objects

Scalars The scalars of the vector space are the underlying field.

Vectors The object of the vector space.

2.3 The Rules

Vector Addition: $\vec{v} + \vec{w} \Rightarrow \vec{u}$

- Zero Operator: $\vec{v} + 0 = 0 + \vec{v} = \vec{v}$
- Vector Opposites (additive inverses): $\vec{v} + (-\vec{v}) = 0$
- Commutativity & Associativity

Scalar Multiplication: $c\vec{v} \Rightarrow \vec{w}$

- Scalar Identity: $I\vec{v} = \vec{v} \,\forall\, \vec{v}$
- Associativity & Distributivity

Inner Product $\vec{v} \cdot \vec{w} \Rightarrow c$

- Commutativity & Distributivity
- Associativity with Scalar Multiplication

Length (Modulus or Norm): $|\vec{v}| \geq 0$

- $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$

Orthogonality

- $\vec{v} \cdot \vec{w} = 0$

A set of orthogonal vectors with norm 1 is called orthonormal.

2.4 Positive Definite Property

$\vec{v} \cdot \vec{v} \geq 0$ & $\vec{v} \neq 0 \Rightarrow ||\vec{v}|| > 0$

An operation that satisfies positive definiteness is an inner product, else it's a paring.

2.5 Linear Combination

$\vec{u} = \sum_{k=0}^{n-1} c_k \vec{v}_k$

This is called linear combination in maths & superposition in physics.

2.6 Bases of a Vector Space

If a minimal (linear independent) subset of vectors span the vector space, they are called the basis of the space.

2.7 Natural (Standard) Basis

$A = \{\hat{x}, \hat{y}\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$

*hat denotes unit length

2.8 Properties of a Basis

- Linear Independece: $\vec{u} \neq \vec{v}+\vec{w} \, \forall \, \vec{u}, \vec{v}, \vec{w}, \in A$where A is the subset of the basis of the space
- Completeness (Spanning): A is closed under linear combination

2.9 Orthonormal Bases

B is orthonormal if $\vec{b}_k \cdot \vec{b}_j = \delta_k j$

2.10 Expansion Coefficients

For $\vec{v} = \sum_{k=1}^{n} a_k b_k$, $\vec{v} \cdot \vec{b}_j = \sum_{k=1}^{n} \vec{a}_k \delta_{kj} = \vec{a}_j$

*Inner product is dependent on bases.

2.11 Subspace $\{a\vec{v} \mid a \in R\}$

A subset of vectors closed under vector/ scalar operations.

Chapter 3

Matrices

3.1 Matrices

A matrix is a rectangular arrays of number.

The size is expressed as [# rows] × [# columns].

A_{kj} describes the element in the row k & column j.

3.2 Matrix Multiplication

AB $\neq$ BA where (n × p)(p × q) = (n × q)

3.3 Row Vector x Column Vector

Special case where (1 × l)(l × 1) = 1 (scalar)

3.4 Definition of Matrix Multiplication

For A as (n × p) & B as (p×m),

$$C_{kl} = (AB)_{kl} \equiv \sum_{j=1}^{p} A_{kj} B_{jl}$$

where $k = 1, ..., n$; $l = 1, ..., m$

Notice that $(AB)C = A(BC)$

3.5 Matrix Product of Vectors

- $Av, v^T A$ is compatible
- Av is a linear transformation

3.6 Matrix Transpose

$(A_{kl})^T = A_{lk}$

3.7 Matrix Addition and Scalar Multiplication

Both are commutative, associative, and distributive.

3.8 Zero Matrix (Additive Identity)

$$(0)A = (0)\&(0)\vec{v} = 0$$

$$A + (0) = (0) + A = A$$

3.9 Identity Matrix (Multiplicative Identity)

$$IM = MI = M$$

$$\vec{v} = \vec{v}\&v^T I = v^T$$

3.10 Determinants of a 2 x 2 Matrix

For A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, detA = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ad-bc

3.11 Determinants of a 3 x 3 Matrix

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \equiv a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

= a (minor of a) - b(minor of b) + c(minor of c)

Minor is the determinant of a smaller matrix made by crossing out the element's row & column.

3.12 Determinants of an n x n Matrix

$det(A) = |A| = \sum_{k=1}^{n} (-1)^{k+j} A_{jk}$ (minor of $A_j k$)

3.13 Determinants of Products

$det(AB) = det(A)\, det(B)$

3.14 Matrix Inverses

$A^{-1}A = AA^{-1} = I$

If A has an inverse, it's invertible or non-singular.

Little Inverse Theorem M is singular if M$\vec{v}$ = 0, for $\vec{v} \neq 0$

Big Inverse Theorem M is singular $\iff$ det(M) = 0

3.15 System of Linear Equations

A system of n unknowns is only solvable if there are n independent equations.

3.16 Matrix Equations

M$\vec{v}$ = c, where

M is the matrix of the linear combination

$\vec{v}$ is the vectors of the unknown

c is the vectors of the constants

To solve, $M^{-1}M\vec{v} = \vec{v} = M^{-1}c$

1. Determine if matrix is invertible
2. If yes, compute inverse

3.17 Cramer's Rule

For M$\vec{v}$ = c, $x_k = \frac{detM_k}{detM}$

where M_k is the matrix M with the kth element column replaced by the constant vector c,

To find inverse, split inverse into column & solve for individual variables with Cramer's Rule.

3.18 Complex Vector Space, $\mathbb{C}^n$

$$\mathbb{C}^n = \{ \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix} c_k \in C,\ k = 0, ..., n-1\} \text{ (complex scalar)}$$

3.19 Complex Inner Product

$a \cdot b = \langle a, b\rangle = \langle a|b\rangle = \sum_{k=0}^{n-1} \bar{a}_k b_k = \sum_{k=0}^{n-1} (a_k^*) b_k$

* Non-commutative: $\langle a|b\rangle \neq \langle b|a\rangle$

However, $\langle a|b\rangle^* = \langle b|a\rangle$

* Physicists conjugate the left vector of the inner product.

- Distributive: $\langle a|b+b'\rangle = \langle a|b\rangle + \langle a|b'\rangle$ and $\langle a+a'|b\rangle = \langle a|b\rangle + \langle a'|b\rangle$
- Anti-linear in the 1st position: $c\langle a|b\rangle = \langle c^* a|b\rangle$
- Linear in the 2nd position: $c\langle a|b\rangle = \langle a|c\,b\rangle$ For both cases, c ∈ C

3.20 Norm

$||\vec{a}|| = \sqrt{\sum_{k=0}^{n-1} |\vec{a}_k|^2} = \sqrt{\sum_{k=0}^{n-1} (\vec{a}_k)^* \, \vec{a}_k}$

3.21 Distance

$\mathrm{dist}(\vec{a}, \vec{b}) = ||b - a|| = \sqrt{\sum_{k=0}^{n-1} |\vec{b}_k - \vec{a}_k|^2}$

These all results in $||b - a|| \geq 0$

3.22 Expansion Coefficients

$\langle b_k|\vec{v}\rangle = \sum_{j=0}^{n-1} \beta_j \delta_{kj}$ with $\vec{v}$ on the right side of inner product.

This only works for orthonormal basis.

Chapter 4

Hilbert Space, $\mathcal{H}$

4.1 Definitions

A Hilbert Space is a real or complex vector space that has:

1. Inner Product
2. Completeness

4.2 Finite Dimensional Hilbert Spaces

$\mathbb{R}^n$ and $\mathbb{C}^n$ are valid Hilbert spaces.

4.3 Infinite Dimensional Hilbert Spaces

This usually referes to function spaces. The vectors of these vector space consist of well-behaved functions.

4.4 The Space L^2 [a, b]

All complex-values functions defined over the real interval [a, b] and which are square-integrable.

$$\int_a^b |f(x)|^2 dx < \infty$$

The inner product is defined as

$$\langle f|g\rangle = \int_a^b f(x)^* \, g(x) dx < \infty$$

4.5 Properties of Hilbert Spaces

- Triangle Inequality: $\forall\, x, y, z$, $\text{dist}(\vec{x}, \vec{z}) = \text{dist}(\vec{x}, \vec{y}) + \text{dist}(\vec{y}, \vec{z})$
- Cauchy-Schwartz Inequality: $|\langle x|y\rangle|^2 \leq ||x||^2||y||^2$

Equality exist only if $x\&y$ are linearly dependent.

4.6 Modeling Quantum Systems

Most quantum computations take place in $\mathbb{C}^n$.

4.7 Ray

A ray is a set of all scalar multiples of some non-zero vector that passes through the origin.

Ray of $a \neq 0 \equiv$ [a] $= \{\alpha\, a \mid \alpha \in C\}$ (α is the global phase)

Every possible quantum state can be represented as a unit vector in $\mathcal{H}$.

For each $\vec{v} \in \mathcal{H}$, $\exists$ an infinite set of $e^{i\theta}\vec{v}$ rotated vectors. ($||e^{i\theta}\vec{v}|| = 1$)

4.8 0 in $\mathcal{H}$

- 0 (the zero vector) is not a quantum state because it's is not normalizable.
- Every other vectors correspond to a quantum state.
- Scalar multiples of a vector represents the same state.
- The collection of rays $\{[a]\}$ form a complex projective sphere with one-to-one quantum correspondence.
- However, this complex projective sphere is not a vector space.

4.9 Interacting with these Vectors

1. Identify a unit vector $\hat{v} \in \mathcal{H}$ of a quantum states
2. Apply unitary transformation
3. Renormalize vectors

*The projective sphere collapses entire ray onto a complex point

Chapter 5

Linear Transformation

5.1 Linear Transformation

Linear transformations map vectors from one vector space to another

Linearity $T(c\vec{v}) = c\,T(\vec{v})$ & $T(v_1 + v_2) = T(v_1) + T(v_2)$ where c is the domain scalar and $\vec{v}$ is the domain vector.

Identity $I\vec{v} = \vec{v}$

Zero $0(\vec{v}) = 0$

Scale $S_c(\vec{v}) = c\vec{v}$

Projection on to $\hat{x}_k$ $P_k(\vec{v}) = \vec{v}_k \hat{x}_k$

Projection on to $\hat{n}$ $P_{\hat{n}}(\vec{v}) = (\vec{v} \cdot \hat{n})\,\hat{n}$

Differentiation $D(\psi) \equiv \psi'$

Anti-differentiation $\int^x (f) \equiv \int^x f(x')dx'$

Multiplication by matrix of constants $T_A(\vec{v}) \equiv A\vec{v}$

5.2 Role of Bases

For $\vec{v} = \sum_{k=1}^{n} \beta_k b_k$, $T\vec{v} = \sum_{k=1}^{n} \beta_k T(b_k)$

To write T as a matrix, write $T(b_k)$ in a row of vectors and expand vertically

$$T\vec{v} = (T(b_0),\, T(b_1), \ldots) \begin{pmatrix} b_0 \\ b_1 \\ \vdots \end{pmatrix})$$

5.3 Dependence of Matrix on Basis

For $T\vec{v}$, T & $\vec{v}$ have to be in the same basis ($w = T\vec{v}$; $w_{|A} = T_{|A}(\vec{v}_{|A})$; $w_{|B} = T_{|B}(\vec{v}_{|B})$)

5.4 Matrix M_T in a Non-Standard Basis

$T_{|B} = (T(b_0)_{|B}, T(b_1)_{|B}, ...)$ with $b_{k|B} = b_k$

5.5 Transformation in an Orthonormal Basis

For an orthonormal basis B, $T_{jk|B} = \langle \hat{b}_j \mid T(\hat{b}_k) \rangle$

5.6 The Adjoint of a Matrix

$M^\dagger \equiv (M^T)^*$ for which $(M^T)_{jk} = M^*_{kj}$

5.7 Unitary Operators

Unitary operators are associated with quantum gates.

U is unitary if it preserves inner products for all vectors $\vec{v}, \vec{w}$.

This implies $||A\vec{v}|| = ||\vec{v}||$ and its matrix has orthonormal rows and columns.

1. For basis B $= \{b_k\}_{k=1}^n$, $\langle\, U(b_j)|U(b_k)\,\rangle = \langle b_j|b_k\rangle = \delta_{jk}$
2. $U^\dagger U = UU^\dagger = I$

Ex. $R_{\frac{\pi}{2}}$ and phase change gates.

In essence, unitary operators map one set of orthonormal basis to another.

5.8 Non-Unitary Operators

- Scaling
- Projection Operator

5.9 Hermitian Operators

Hermitian operators are associated with measurements.

For all bases, $M^\dagger = M$

Chapter 6

The Experimental Basis of Quantum Computing

6.1 Spin 1/2 Quantum Mechanics

The vector space for spin $\frac{1}{2}$ fermions are 2-dimensional with linear combinations as sums.

Spin $\frac{1}{2}$ vectors can represents classical 0 & 1 as well as superpositions.

6.2 Naive Electron Spin Definition

An electron has

1. Angular Momentum, S, scalar
2. Oritentation, $\hat{S}$, vectors with $S = (\frac{\sqrt{3}}{2}\hbar)\hat{n}_s$, where $\hat{n}_s = \frac{s}{|s|}$

6.3 Spherical Representation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} ; \; \hat{n} = (1, \theta, \phi)_{Sph}$$

6.4 Electron Spin z-Projection

Measuring the z- projection of a soup of randomly oriented e^- yields 50 % $+\frac{\hbar}{2}$&50%$-\frac{\hbar}{2}$ instead of a continuum between $\pm\frac{\sqrt{3}}{2}\hbar$. The post-measurement states remain for future measurements of the same orientation.

This is because of $\Delta x \Delta p \leq \frac{\hbar}{2}$ (Heisenberg's Uncertainty Principle).

Similar results are found for S_x & S_y

6.5 Follow up $|+\rangle_z$ in S_x

Measuring $|+\rangle_z$ in S_x yield the same result, 50-50. The post-measurement states remain for future measurements of the same orientation.

6.6 Follow up $|+\rangle_x$ in S_z

Measuring this yields 50 % $|+\rangle_z$ & $|-\rangle_z$, even though $|+\rangle_z$ was selected in the above measurements.

A S_a measurement collapses into 1 of 2 allowable orientation (eigenvalues) of the observable S_a.

In fact, any pairs of directions are incompatible observables.

6.7 Construction of Hilbert Space

Measurements show that $|-\rangle_x$ contains a portion of the original $|+\rangle$ & $|-\rangle$.

$|+\rangle_x = \alpha|+\rangle + \beta|-\rangle$.

These vectors will be normalized on a projective sphere.

6.8 Representing Spin

The (r, θ, ϕ) real vector is now represented by (α, β) complex vector with basis $|+\rangle \& |-\rangle$.

1. $|+\rangle$ & $|-\rangle$ are linearly independent of one another, instead of ± (multiples) of the same basis vector.
2. The observable of one basis can be written as linear combination of the two observables of another basis, instead of being linearly independent. What this means is that for basis A and B so that if $A \neq B, |\pm\rangle_A = \alpha\,|+\rangle_B + \beta\,|-\rangle_B$

6.9 Measuring Spin θ from the Standard Axis

Measuring spin θ away from S_a yields $|+\rangle$ with probability of $\cos^2(\frac{\theta}{2})$ and $|-\rangle$ with probability of $\sin^2(\frac{\theta}{2})$.

Non-zero azimuthal ϕ will not change the result.

*ϕ is the angle of the vector on the plane orthogonal to S_a.

*Two polar opposite direction in $\mathbb{R}^3$ correspond to two orthonormal vectors in $\mathbb{C}^2$. This is the basis of the spin state.

Chapter 7

Time Independent Quantum Mechanics

7.1 Particle Energy and Position

E_k takes on discrete values. This helps determine the probability curve of the location of the particle.

7.2 Stern-Gerlach Apparatus

This shoots a silver atom through a magnetic field. A plate measures deflection. This is the physical system of quantum mechanics.

7.3 First Postulate of Quantum Mechanics

7.4 Trait #1: The State Space

All system $\mathscr{S}$ has an associated Hilbert space $\mathcal{H}$. Each physical state in $\mathscr{S}$ correspond to some ray in $\mathcal{H}$ (a point on the projective sphere of $\mathcal{H}$).

$$\text{physical state} \in \mathscr{S} \leftrightarrow \vec{v} \in \mathcal{H}$$

These vectors in the space is represented as $|\psi\rangle$.

7.5 Fundamental State Space for Quantum Computing

$|+\rangle_z$ & $|-\rangle_z$ as the natural basis kets of a 2-dimensional complex Hilbert space form a the spin $\frac{1}{2}$ state space of complex ordered pairs with the orthogonal inner product.

$\mathcal{H} \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\alpha, \beta \in \mathbb{C}$ with $|+\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Any physical states can be described as

$|\psi\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \alpha|+\rangle + \beta|-\rangle$, where

$||\alpha||^2 + ||\beta||^2 = 1$

7.6 Orthonormality Expression

$\langle +|+\rangle = \langle -|-\rangle = 1$

$\langle +|-\rangle = \langle -|+\rangle = 0$

7.7 The x-Basis for $\mathcal{H}$

$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ 1\end{pmatrix}$

$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ -1\end{pmatrix}$

7.8 Reasoning for Complex Vector Space

Measurements dictates that $|+\rangle$ electrons has 50% being up & spin down in right-angled basis. However, all 3 basis must be able to be represented as linear combination of the other. Real vector space cannot represent this, therefore complex vector space are used. If there is 1 more basis, quaternions would be used.

7.9 Second Postulate of Quantum Mechanics

7.10 Trait #2: The Operator for an Observable

An observable quantity A correspond to an operator (linear transformation) in $\mathcal{H}$. This matrix is always related to a Hermitian.

For Observable $\mathcal{A} \in \mathscr{S}$,

T_A: $\mathcal{H}$ linear & $T_A^\dagger = T_A$ (Hermitian condition)

7.11 Completeness of the Eigenbasis

Eigenvectors of an observable span the state space

7.12 The Observable S_z

$S_z = \sigma_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$ represents

1. the observable "spin projected onto the z-aixs"
2. the associate linear operator
3. the matrix of the operator

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

These are Pauli spin matrices (eigenvalues diagonal matrix).

7.13 Third Postulate of Quantum Mechanics

7.14 Trait #3: Eigenvalues of an Observables

The only possible measurements of an observable quantity $\mathcal{A}$ are eigenvalues of the operator's matrix.

7.15 Eigenvectors and Eigenvalues

For real or complex M, with $\vec{u} \neq 0, M\vec{u} = a\vec{u}$

$\vec{u}$ is the eigenvector, a is the eigenvalues.

$\{\vec{u}_{ak} \leftrightarrow a_k\}_{k=1}^{n\ or\ \infty}$

Uniqueness Degenerate eigenvalues have non-unique eigenvectors.

Diagonality The eigenvectors are the basis of the space (eigenbasis) only when the matrix is diagonal.

7.16 Trait #3': Eigenvectors and Eigenvalues of S_z

The only possible outcome of $\mathcal{A}$ are the solutions to the eigenvector-value equation

$T_A|\vec{v}_k\rangle = a_k|\vec{v}_k\rangle$

Eigenvalues with non-unique eigen-kets are degenerate observables.

Eigenvectors of S_z are $(1,0)^T$ & $(0,1)^T$.

7.17 Computing Eigenvectors and Eigenvalues

7.18 Eigenvalue Theorem

For matrix M, the eigenvalues are solution to the system of simultaneous equations of the unknown λ

$$det(M - \lambda I) = 0$$

To solve for the eigenvector:

1. Plug in 1 to solve
2. For complex vectors, split the variable into a+bi
3. If solution is a contradiction, plug in 0 instead

7.19 Summary of Eigenvectors and Eigenvalues for spin 1/2 Observables

The eigenvalues and eigenvectors for S_z, S_x, & S_y are:

$$\begin{aligned} S_z: \quad +\frac{\hbar}{2} &\leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}, \quad -\frac{\hbar}{2} \leftrightarrow \begin{pmatrix}0\\1\end{pmatrix} \\ S_x: \quad +\frac{\hbar}{2} &\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad -\frac{\hbar}{2} \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \\ S_y: \quad +\frac{\hbar}{2} &\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \quad -\frac{\hbar}{2} \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \end{aligned}$$

Expressed explicitly in terms of z-basis vectors, we find

$$\begin{aligned} S_z: +\frac{\hbar}{2} &\leftrightarrow |+\rangle, \quad -\frac{\hbar}{2} \leftrightarrow |-\rangle \\ S_x: +\frac{\hbar}{2} &\leftrightarrow |+\rangle_x = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \quad -\frac{\hbar}{2} \leftrightarrow |-\rangle_x = \frac{|+\rangle - |-\rangle}{\sqrt{2}} \\ S_y: +\frac{\hbar}{2} &\leftrightarrow |+\rangle_y = \frac{|+\rangle + i|-\rangle}{\sqrt{2}}, \quad -\frac{\hbar}{2} \leftrightarrow |-\rangle_y = \frac{|+\rangle - i|-\rangle}{\sqrt{2}} \end{aligned}$$

7.20 Trait #4: Real Eigenvalues and Orthonormal Eigenvectors

An observable $\mathcal{A}$ in $\mathscr{S}$ will always correspond to an operator

1. with real eigenvalues
2. and with unique eigenvectors that form an orthonormal basis for $\mathcal{H}$.

7.21 General States Expressed in Alternate Bases

$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle = \gamma|+\rangle_{|c} + \delta|-\rangle_{|c}$

To get $\psi\rangle$ in c-basis,

$|\psi\rangle = \begin{pmatrix} {}_c\langle+|\psi\rangle \\ {}_c\langle-|\psi\rangle \end{pmatrix}$

7.22 Orthonormal Basis in Higher Dimensions

For an n-dimensional state space, the orthonormal basis is $|\vec{v}_k\rangle_{k=1}^n$

7.23 Trait #5: Closure (Completeness) Relation

Any $\{|\vec{v}_k\rangle\}$ for the Hilbert space $\mathcal{H}$ satisfies the relation

$|\psi\rangle = \sum_{k=1}^n \langle\vec{v}_k|\psi\rangle\, |\psi\rangle = \sum_{k=1}^n (|\vec{v}_k\rangle\langle\vec{v}_k|)\, |\psi\rangle = I|\psi\rangle$

*The eigenvectors of any observable will satisfy the closure relation.

7.24 Fourth Postulate of Quantum Mechanics

7.25 Trait #6: Probability of Outcome

The normalized state $|\psi\rangle$ can be expanded along the eigenbasis $\{\vec{v}_k\}$ of some observable $\mathcal{A}$

$|\psi\rangle = \sum_{k=1}^{n} c_k |\psi\rangle_k$, where c_k is the amplitude

then the probability that a measurement of $\mathcal{A}$ yielding a non-degenerate eigenvalue and its associated eigenvectors is $|c_k|^2$.

$P(a_k)_{|\psi\rangle} = |\langle a_k|\psi\rangle|^2 = c_k^* c_k = |c_k|^2$, where a_k is the measurement, and ψ the state of the system

7.26 Fifth Postulate of Quantum Mechanics

7.27 Trait #7: Post-Measurement Collapse (complex inner product)

If the measurement of an observable of system $\mathscr{S}$ results in eigenvalue a_k , then the system collapse probabilistically into an eigenvectors associated with a_k. Further measurement collapses back to a_k with 100% certainty.

7.28 Dirac's Bra-ket Notation

$\langle\, bra \mid ket\, \rangle$, where bra is the Hermitian conjugate

$|\psi\rangle = \langle\psi|^\dagger$ and $\langle\psi| = |\psi\rangle^\dagger$

7.29 Transforming Ket to Bra

1. Turn all kets (left vectors into bras)
2. Take complex conjugate of any scalars
3. Form inner products
4. Use distributive property to combine bra & ket

7.30 The Bra Space (Adjoint of a ket)

The bra space is a different vector space from the ket space. It is an isomorphism in finite dimensions.

7.31 The Adjoint of an Operator

For $\mathcal{A}$ in the ket space, $\exists\, \mathcal{A}^\dagger$, where

$\langle\psi|A^\dagger \rightarrow \langle\phi|$. Notice that, $\langle\varnothing| = |\phi\rangle^\dagger$

7.32 Trait #8: Adjoint Conversion Rules

The bra space & the ket space can be mapped onto one another using the adjoint operator (†).

1. $\langle\psi| \rightarrow \langle\psi|^\dagger = |\psi\rangle$
2. $|\psi\rangle \rightarrow |\psi\rangle^\dagger = \langle\psi|$
3. $c \rightarrow c^\dagger = c^*$
4. $A \rightarrow A^\dagger$
5. $(AB)^\dagger \rightarrow B^\dagger A^\dagger$

7.33 Expectation Values

The mean of multiple measurements: $\bar{m} = \frac{1}{N}\sum_{j=1} m_j$

The law of large numbers dictates that $lim_{N\rightarrow\infty}\bar{m} = \mu$

The probability of collapse $|c_k|^2$ is seen by accumulate multiple instances.

$$\mu = \langle\mathcal{A}\rangle_{|\psi\rangle} \equiv \sum_k |c_k|^2\, a_k$$

" The expectation value of state ψ when measured under observable $\mathcal{A}$ is μ. "

7.34 Trait #9: Expectation Value Theorem

$$\langle\mathcal{A}\rangle_{|\psi\rangle} = \langle\psi|\mathcal{A}|\psi\rangle$$

where $\mathcal{A}$ is the Hermitian observable

Chapter 8

Time Dependent Quantum Mechanics

Time evolution mechanics represents noise or predictable changes in the system.

8.1 The Hamiltonian

The Hamiltonian describes the total energy of the system.

8.2 Trait #10: Constructing the Hamiltonian

1. Put $\mathscr{H}$ (LHS), classical energy in terms of classical concepts (RHS)
2. Change $\mathscr{H}$ to H, and classical variables to quantum operators.

For example, classical (x, y, z) → quantum (X, Y, Z).

8.3 Classical Hamiltonian (1/2 Spin)

A stationary e^- in a constant magnetic field has

$$\mathscr{H} = -\gamma \hat{B} \cdot S = -\gamma \hat{B} \cdot S_z$$

γ is the gyromagnetic ratio

$\hat{B} = B\hat{z} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$ is the magnetic field vector in the +z direction

$S = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$ is the intrinsic angular momentum (spin vector)

8.4 Quantum Hamiltonian

Replace classical variables with quantum operators, we get

$$H = -\gamma B S_z = -\gamma B\,\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

8.5 Energy Eigenkets

The eigenvectors of H is also known as energy eigenkets.

Rearrange & substitute from above the eigenvectors-eigenvalues relation

$H|+\rangle = (-\frac{\gamma B\hbar}{2})|+\rangle; H|\rangle = (-\frac{\gamma B\hbar}{2})|-\rangle$

$\therefore$ When measuring the energy of the system, we get $-\frac{\gamma B\hbar}{2}$ (minimum PE) for $|+\rangle$ and $\frac{\gamma B\hbar}{2}$ (maximum energy) for $|-\rangle$.

8.6 Trait #11: Quantization of Energy

The only allowable energies of a quantum system are the eigenvalues of the Hamiltonian.

8.7 Sixth Postulate of Quantum Mechanics

Time-dependent state: $|\psi\rangle \rightarrow |\psi(t)\rangle$

For which $|\psi(t)\rangle = \sum_{k=0}^{n-1} c_k(t)|\vec{v}_k\rangle$

At fixed time, $c'_k \equiv c_k(t')$

8.8 Trait #12: Time-Dependent Schrodinger Equation

The time evolution of a state vector is governed by the Schrodinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle \quad (1)$$

However, H is time-independent for our purpose.

The time-independent Schrodinger equation set A = H. This is in the eigenvectors-eigenvalues form.

$$H|\vec{v}_k\rangle = a_k|\vec{v}_k\rangle(2)$$

Solving this gives the eigenkets and the associated eigenvalues for the time-dependent Schrodinger equation.

8.9 Solving the Schrodinger Equation

1. Solve (2) for $\{E\}_k$ and the associated $\{|E\rangle\}$.

2. Expand $|\psi\rangle$ along the energy basis.

   $|\psi\rangle = \sum_k c_k |E_k\rangle$

3. Solve the differential equation (1) for time-dependent amplitude, $c_k(t)$.

$$c_k(t) = c_k e^{\frac{-itE_k}{\hbar}}$$

4. Put c_k back into the sum in the second step

$$|\psi(t)\rangle = \sum_k c_k e^{\frac{-itE_k}{\hbar}} |E_k\rangle$$

8.10 Trait #13: Stationary States

An eigenstate of the Hamiltonian operator evolves in a way that its measurement outcome does not change. It remains the same.

For spin $\frac{1}{2}$ system, the stationary states are $|+\rangle$ & $|-\rangle$.

$$|\psi(t)\rangle = e^{i\phi_t}|a\rangle \cong |a\rangle = |\psi\rangle$$

8.11 General Technique for Computing Time-Evolved States

8.12 Trait #14: Evolution of any Observable

1. Solve the Schrodinger equation
2. Take the inner product with the desired eigenket $|\vec{v}_j\rangle$ of $\mathcal{A}$

$$\alpha_j(t) = \langle \vec{v}_j | \psi(t) \rangle$$

3. Square to get the probability of $\mathcal{A}$ producing α_k at time t.

$$P(\alpha(t))_A = |\alpha_j(t)|^2 = \alpha_j(t)^* \alpha_j(t)$$

8.13 Larmor Precession

Combine time evolution & expectation values to relate 3-D real classical angular momentum vector to 2-D complex quantum state vector.

$$|\psi(t)\rangle = c_1 e^{it(\frac{\gamma B}{2})}|+\rangle + c_2 e^{-it(\frac{\gamma B}{2})}$$

8.14 Rewriting $|\psi\rangle$

Converting to polar form, rearranging & substituting, we get $|\psi(t)\rangle = \begin{pmatrix} ce^{i\phi(t)} \\ se^{-i\phi(t)} \end{pmatrix}$, where $\phi(t) = \frac{1}{2}(\omega + \phi_0)$

with $\omega = \gamma B$ and $\phi = \phi_1 - \phi_2$ (relative phase)

8.15 Convenient Angle

$$cs = \frac{sin(\theta)}{2}$$

and

$$c^2 - s^2 = cos(\theta)$$

8.16 Expectation Value at Time t

$$\langle S_z \rangle_{\psi(t)} = \frac{\hbar}{2} cos(\theta)$$

$$\langle S_y \rangle_{\psi(t)} = -\frac{\hbar}{2} sin(\theta)\ sin(\omega t + \phi_0)$$

$$\langle S_x \rangle_{\psi(t)} = \frac{\hbar}{2} sin(\theta)\ cos(\omega t + \phi_0)$$

$$s(t) = \begin{pmatrix} \langle S_x \rangle_{\psi(t)} \\ \langle S_y \rangle_{\psi(t)} \\ \langle S_z \rangle_{\psi(t)} \end{pmatrix} = \begin{pmatrix} sin\theta\ cos\phi(t) \\ -sin\theta\ sin\phi(t) \\ cos\theta \end{pmatrix}$$

- θ is the real angle between s(t) & the z-axis in $\mathbb{R}^3$. $\frac{\theta}{2}$ expresses $|\psi\rangle$ in $\mathbb{C}^2$. Domain: $(0 \leq \frac{\theta}{2} \leq \frac{\pi}{2})$. Range: $(0 \leq \theta \leq \pi)$
- ω (Larmor frequency) = γ (gyromagnetic ratio) B (magnetic field magnitude)
- $\phi_0 = \phi_1 - \phi_2$ (relative phase)

Chapter 9

The Qubit

9.1 Vector Space $\mathcal{B} = \mathbb{B}^2$ ($\mathbb{B} \equiv \{0, 1\}$)

$\mathcal{B} = \{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\}$

$\oplus$ is mod-2 addition (XOR).

Mod-2 Inner Product (pairing as inner product is not positive definite)

$$\begin{pmatrix}x_1\\y_1\end{pmatrix} \odot \begin{pmatrix}x_2\\y_2\end{pmatrix} = x_1 \cdot x_2 \oplus y_1 \cdot y_2$$

$||x|| = |x| = \sqrt{x \odot x}$

Note that $|| \begin{pmatrix}0\\1\end{pmatrix} || = || \begin{pmatrix}1\\0\end{pmatrix} || = 1$ and $|| \begin{pmatrix}0\\0\end{pmatrix} || = || \begin{pmatrix}1\\1\end{pmatrix} || = 0$

$(1, 0)^T$ & $(0, 1)^T$ are the orthonormal basis.

9.2 Definition of a Classical Bit

A bit is the vector space $\mathcal{B}$.

9.3 Definition of Bit Value

A bit's value is any unit vector in $\mathcal{B}$, which includes $(1, 0)^T$ & $(0, 1)^T$.

9.4 Alternate Definition of a Bit

A bit is a variable superposition of [0] & [1] of $\mathcal{B}$.

$x = \alpha[0] + \beta[1]$, where $\alpha^2 \oplus \beta^2 = 1$

9.5 Definition of a Classical Logical Operator

A logical operator is a linear transformation of $\mathcal{B}$ that maps one unit vector to another.

Unary gate with 1 input

Binary gate with 2 inputs

Constant-[0] A(x) = [0]

Constant-[1] A(x) = [1]

Negation (NOT) A(x) = ¬x (reversible)

Identity A(x) = Ix = x (reversible)

* An operator is reversible $\iff$ its matrix is unitary.

9.6 Unitary Properties

1. Preserve length: $||A\vec{v}|| = ||\vec{v}||$
2. Preserve inner product: $\langle A\vec{v} \mid A\vec{w}\rangle = \langle \vec{v}|\,|\vec{w}\rangle$
3. Rows or columns are orthonormal

9.7 Definition of a Qubit

A qubit is the vector space $\mathcal{H}$

9.8 Alternate Definition of a Qubit

A qubit is a variable superposition of $|0\rangle \& |1\rangle$ in $\mathcal{H}$.

$|\psi\rangle = \alpha|0\rangle + |1|\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$

9.9 Definition of Qubit Value

The value of a qubit is any unit vector in $\mathcal{H}$

9.10 Computational Basis State (CBS)

$$|+\rangle_a = |0\rangle_a;\ |-\rangle_a = |1\rangle_a$$

9.11 Global Phase Factors

$$e^{i\theta}|\psi\rangle = |\psi\rangle$$

9.12 Definition of a Quantum Logical Operator

A logical operator is a linear transformation of $\mathcal{H}$ that maps one unit vector to another.

Any quantum unary operator can be represented a 2 × 2 matrix.

*All quantum operators are unitary.

9.13 Bit Flip QNOT (X)

$$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$\alpha\,|0\rangle + \beta\,|1\rangle \;-\!\boxed{X}\!-\; \beta\,|0\rangle + \alpha\,|1\rangle$$

9.14 Phase Flip (Z)

$$Z|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

Z changes the relative phase of the two by π

$$\alpha\,|0\rangle + \beta\,|1\rangle \;-\!\boxed{Z}\!-\; \alpha\,|0\rangle - \beta\,|1\rangle$$

9.15 Bit and Phase Flip (Y)

$$Y|\psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -i \begin{pmatrix} \beta \\ -\alpha \end{pmatrix} \cong \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}$$

$$\alpha\,|0\rangle + \beta\,|1\rangle \;-\!\boxed{Y}\!-\; \text{-i } (\beta\,|0\rangle - \alpha\,|1\rangle)$$

* All unary operator takes the form $(\vec{\sigma} \cdot \hat{n})$, where

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} ; \quad \hat{n} = \begin{pmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{pmatrix} \text{ real spin vector}$$

9.16 Hadamard Gate (H)

$$H|\psi\rangle = \tfrac{1}{\sqrt{2}}(\alpha + \beta)|0\rangle + \tfrac{1}{\sqrt{2}}(\alpha - \beta)|1\rangle$$

$$H|\psi\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle)$$

9.17 Measurement

The probability of collapse for $|\psi\rangle$ into state $|c\rangle$

$$P(|\psi\rangle) \searrow |c\rangle)$$

9.18 Phase-Shift Gates (R_θ, S and T)

$$R_\theta|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} e^{i\theta}\alpha \\ \beta \end{pmatrix}$$

$S = R_{\frac{\pi}{2}}$ and $T = R_{\frac{\pi}{4}}$

9.19 Basis Conversion Theorem

Quantum gates map one orthonormal CBS to another.

For unitary U, $\langle x| U^{\dagger}U |y\rangle = \langle x|y\rangle = \delta_{xy}$

9.20 Combining Gates

For $|a\rangle_x, HXH = Z$

$X^2 = Y^2 = Z^2 = -iXYZ = iZYX = I$

$\therefore X = iYZ,\ Y = iZX,\ Z = iXY$

Non-identical Pauli matrices anti-commute: $\sigma_i\sigma_j = -\sigma_j\sigma_i$

*Gates operate left to right, whereas operator algebra operates right to left.

9.21 The Bloch Sphere

$|\psi\rangle$ can be written in polar form.

$|\psi\rangle = \begin{pmatrix} ce^{i\phi(t)} \\ se^{-i\phi(t)} \end{pmatrix}$, where $c^2 + s^2 = 1$

$c = cos\frac{\theta}{2};\ s = sin\frac{\theta}{2}$

9.22 Definition of the Bloch Sphere

The sphere in $\mathbb{R}^3\{|\hat{n}| = 1\}$ with coordinates

$$\hat{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \langle S_x\rangle_{\psi(t)} \\ \langle S_y\rangle_{\psi(t)} \\ \langle S_z\rangle_{\psi(t)} \end{pmatrix} = \begin{pmatrix} sin\theta cos\phi(t) \\ -sin\theta sin\phi(t) \\ cos\theta \end{pmatrix}$$

For spherical coordinate,

$$\hat{n} = \begin{pmatrix} 1 \\ \theta \\ \phi \end{pmatrix}_{Sph} \in \text{Bloch sphere} \leftrightarrow |\psi\rangle = \begin{pmatrix} e^{i\phi(t)}cos\frac{\theta}{2} \\ e^{-i\phi(t)}sin\frac{\theta}{2} \end{pmatrix} \in \mathcal{H}$$

Chapter 10

Tensor Products

10.1 Tensor Products

To construct a new vector space

1. Specifying scalars & vectors
2. Defining vector addition & scalar multiplication
3. Confirming all the required properties
4. Defining inner product
5. Establishing the preferred basis

The tensor product of V (dim=ℓ) and W (dim=m) is a new vector space with dim=ℓm

10.2 Scalars of V $\otimes$ W

Both V & W must have a common scalar set to form an inner product and that will be the scalar set for V $\otimes$ W.

10.3 Vectors of V $\otimes$ W

All of the separable tensor product of V $\otimes$ W are in the form v $\otimes$ w. The general vector is the finite sum of this.

$\sum_k \vec{v}_k \otimes \vec{w}_k$, with $\vec{v}_k \in V$ and $\vec{w}_k \in W$

This span the full space V $\otimes$ W.

10.4 Terms and Definition

Product Space The tensor propduct of 2 vector space is called the tensor space

Tensors Vectors in the product state

Separable Tensors Vectors in space V $\otimes$ W

Tensor Product Tensor product of spaces or separable vectors

10.5 Vector Addition

For any two tensors,

$\zeta + \zeta' \equiv \sum_k \vec{v}_k \otimes \vec{w}_k + \sum_j \vec{v}'_j \otimes \vec{w}'_j$

Tensor product distributes over sums in the component space.

$(\vec{v} + \vec{v}') \otimes \vec{w} = \vec{v} \otimes \vec{w} + \vec{v}' \otimes \vec{w}'$ and $\vec{v} \otimes (\vec{w} + \vec{w}') = \vec{v} \otimes \vec{w} + \vec{v} \otimes \vec{w}'$

Commutativity is implied: $\zeta + \zeta' = \zeta' + \zeta$

10.6 Scalar Multiplication

Separable Tensor

$c(\vec{v} \otimes \vec{w}) \equiv (c\vec{v}) \otimes \vec{w} \equiv \vec{v} \otimes (c\vec{w})$

$c(\zeta + \zeta') = c\zeta + c\zeta'$

10.7 Inner Product in V ⊗ W

If V & W have an inner product, the inner product of V⊗W is

$\langle \vec{v} \otimes \vec{w} | \vec{v}' \otimes \vec{w}' \rangle \equiv \langle \vec{v} | \vec{v}' \rangle \cdot \langle \vec{w} | \vec{w}' \rangle$

Dot product distributes as usual

10.8 Natural Basis for V ⊗ W

10.9 Tensor Product Basis Theorem

If V(dim=ℓ) has basis $\{\vec{v}_k\}_{k=0}^{l-1}$ and W(dim=m) has basis $\{\vec{w}_j\}_{j=0}^{m-1}$,

then V ⊗ W (dim=ℓm) inherits a natural orthonormal basis.

10.10 Proof of Basis Theorem

Spanning: any tensors can be expressed as linear combinations of v ⊗ w.

For $\vec{v} \in V = \sum \alpha_k \vec{v}_k$ and $\vec{w} \in W = \sum \beta_j \vec{w}_j$,

$\vec{v} \otimes \vec{w} = \sum \alpha_k \beta_j (\vec{v}_k \otimes \vec{w}_j)$

Linear Independence & Orthonormality

$\langle \vec{v}_k \otimes \vec{w}_j | \vec{v}'_k \otimes \vec{w}'_j \rangle = \langle \vec{v}_k | \vec{v}'_k \rangle \langle \vec{w}_j | \vec{w}'_j \rangle = \delta_{kk' \,\&\, jj'}$

∴ Any tensors in the product space can be expressed as

$$\vec{u} = \sum_{(k=0)(j=0)}^{(n-1)(m-1)} c_{kj} (\vec{v}_k \otimes \vec{w}_j)$$

10.11 Conventional Order of Tensor Basis

For $\vec{v}_{ij}$, i increments slowly & j quickly (row major).

10.12 Tensor Coordinates from Component-Space Coordinates

Natural Coordinates of Separable Tensors

For $\vec{v}$ (dim=ℓ) and $\vec{w}$ (dim=m),

$$\vec{v} \otimes \vec{w} = \begin{pmatrix} \vec{v}_0\vec{w}_0 \\ \vec{v}_0\vec{w}_1 \\ \vdots \\ \vec{v}_{l-1}\vec{w}_{m-1} \end{pmatrix}$$

Natural Coordinates of Basis Vectors

For $b_k \in V \otimes W, b_{k|B} = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

$\exists$ ℓm basis vectors for the product space.

Any natural basis vectors are made up of 2 separable natural bases.

Natural Coordinates of General Tensor

For tensor space with dim=ℓm,

$\zeta = \begin{pmatrix} \zeta_0 \\ \vdots \\ \zeta_{l-1} \end{pmatrix}$

Tensor as Matrices

The basis can be put into a matrix

$\begin{pmatrix} \zeta_{00} & \zeta_{01} & \\ \vdots & \ddots & \\ \zeta_{(l-1)0} & & a_{(l-1)(m-1)} \end{pmatrix}$

10.13 Linear Operators on the Tensor Product Space

Separable Operator

For A: V $\rightarrow V'$ & $B : W \rightarrow W', A \otimes B : V \otimes W \rightarrow V' \otimes W'$

On vectors, $[A \otimes B](\vec{v} \otimes \vec{w}) \equiv A\vec{v} \otimes B\vec{w}$

*Matrix tensor product follows the A-major format.

10.14 Matrix of General Operator

Any operator on the product space can be represented as sum of $\leq (\ell m)^2$ separable operators in the form P_{pq} with 1 at (p, q) and 0 everywhere, where $0 \leq p, q \leq \ell m$

10.15 Matrix Tensor Product

$$AB \otimes CD = (A \otimes C)(B \otimes D)$$

$$(a \times b\, matrix) \otimes (c \times d\, matrix) = (ac \times bd\, matrix)$$

- The product of unitary operators is unitary in the product space.

  Using the unitary inner product, assume

  $\langle (V \otimes W)(v \otimes w) | (V \otimes W)(v \otimes w) \rangle = \langle v \otimes w | v \otimes w \rangle$

  From the above expansion, this gives $\langle Vv \otimes Ww | Vv \otimes Ww \rangle$

  Using the tensor product inner product, expand with the unitary definition

  $\langle Vv | Vv \rangle \cdot \langle Ww | Ww \rangle = \langle v | v \rangle \cdot \langle w | w \rangle = \langle v \otimes w | v \otimes w \rangle$

- The product of Hermitian operators is Hermitian in the product space.

- The product of invertible operators is invertible in the product space.

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

Chapter 11

Two Qubits and Binary Quantum Gates

11.1 Definition of Two Qubits ($\mathcal{H}_A \otimes \mathcal{H}_B$)

A bipartite system is the product space $\mathcal{H} \otimes \mathcal{H}$. This is also referred to as an order-2 system. The need for tensor product came from entanglement.

11.2 Definition of a Two-Qubit Value

The value of a bipartite system is any unit vector in $\mathcal{H} \otimes \mathcal{H}$.

11.3 Preferred Bipartite CBS

Bipartite systme inherits separable products of component space.

$|0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle = |0\rangle^2$ (coordinate representation)

$\exists$ no alternative for general separable tensors.

*When expanding along a-basis, the b-basis kets have equal numbers of + & - terms except for the 0th CBS kets, which have all +.

11.4 Alternate Bipartite Bases

CBS can inherit from other orthonormal bases.

For example, $|0\rangle_x|0\rangle_x = |+\rangle|+\rangle = |++\rangle = |0\rangle^2_\pm$

11.5 Inherit Second Order Mixed CBS

It is possible to create $|0\rangle_a|0\rangle_b$ orthonormal product basis as long as a & b are orthonormal bases.

11.6 Non-Standard Second Order CBS in Natural Basis

For $|0\rangle_a = \alpha_a|0\rangle + \beta_a|1\rangle$ & $|0\rangle_b = \alpha_b|0\rangle + \beta_b|1\rangle$

Using the distributive property of tensor product

$|0\rangle_\alpha|0\rangle_\beta = \alpha_a\alpha_b|00\rangle + \beta_a\alpha_b|01\rangle + \alpha_b\beta_a|10\rangle + \beta_a\beta_b|11\rangle$

When expanded along the z-basis, the x-basis kets have equal number of + and - terms except for the zeroth CBS ket, $|00\rangle_x$, whose coordinates are all $+1$.

11.7 Alternate Definition of Two Qubits

Two qubits are represented by a variable superposition of the four tensors basis vectors of $\mathcal{H} \otimes \mathcal{H}$.

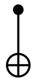
$|\psi\rangle^2 = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$, where

$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$

11.8 Binary Quantum Operator Definition

A binary quantum operator is a unitary transformation U, on the qubit system $\mathcal{H} \otimes \mathcal{H}$.

11.9 Complete Description of Binary Quantum Operator

 show symbol of the gate

$|x\rangle|y\rangle$ define gate's action on CBS

M construct matrix of the gate

$|\psi\rangle^2$ gate's action on a general state

$\searrow$ measurement probabilities of the output registers.

11.10 Measurement of Separable Outputs

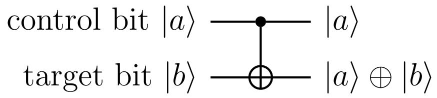
Probability of collapse is local and independent.

11.11 Quantum Entanglement (Non-separable outputs)

An entangled state in a product space is one that is not separable.

$\therefore$ Measurements on entangled state are non-local, which means that it affects the state not in the local system.

11.12 Controlled-NOT (CNOT Gate)



$$|y\rangle \to \begin{cases} |y\rangle, \text{ if } x = 0 \\ \neg y\rangle, \text{ if } x = 1 \end{cases}$$

$M_{CNOT} = (|00\rangle, |01\rangle, |10\rangle, |11\rangle)$

$CNOT(|\psi\rangle^2) = \alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle$

Notice that γ and δ swap position.

11.13 Quantum Entanglement for CNOT

A separable bipartite state going into CNOT does not usually result in a separable state going out of CNOT.

Separable operators allow separated components whereas non-separable state does not.

Control Register as CBS: separable output

Control Register as superposition: non-separable output

11.14 CNOT in Different CBS

For x-basis, B-register is the control & A is the target

11.15 Second Order Hadamard Gate

2nd-order H-gate is the tensor product of 2 Hadamard gates.

$$H^{\otimes 2} = H \otimes H$$

CBS states always map to separable states.

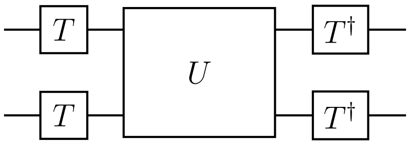
11.16 Condensed Form

$H^{\otimes 2} = \frac{1}{2}\sum_{y=0}^{3}(-1)^{x \odot y}|y\rangle^2$, where $\odot$ is mod-2 dot product.

$H^{\otimes 2}$ transform between z-CBS & x-CBS.

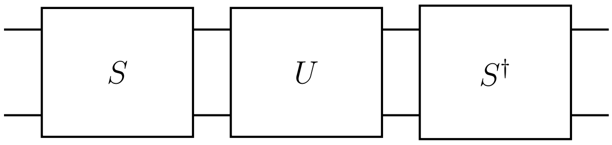
11.17 Circuits in Separable Basis

To operate circuit in alternate CBS, with operator in z-basis. First, apply T to transform from a-basis to z-basis & then $T^\dagger$ after operator to convert back to a-basis:



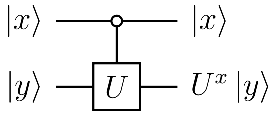
11.18 Circuits in Non-Separable Basis

To measure along a non-separable basis, define the circuit of z-basis, sandwich it between binary a-basis converter.



11.19 Controlled-U Gate

Apply U to target if CBS in control is 1.



where $U^0 = I$; $M_{CU} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & U \end{array}\right)$

11.20 Separable Operators on Separable States

Separable operators map separable states to separable states. They preserve locality and are called local operators.

11.21 Separable Operators on Entangled States

Separable operators modify both qubits of the entangled state.

11.22 Trait #15: Born Rule for Bipartite States

If a bipartite state is factored relative to the A-register.

$|\psi\rangle^2 = |0\rangle(\alpha|0\rangle + \beta|1\rangle) + |1\rangle(\gamma|0\rangle + \delta|1\rangle)$

The measurement of the A-register will collapse according to

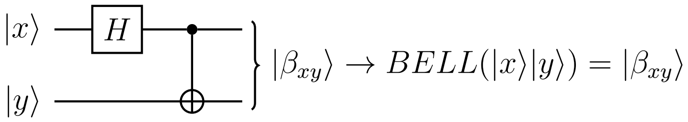
$A \searrow 0 \Rightarrow B \searrow \frac{\alpha|0\rangle+\beta|1\rangle}{\sqrt{|\alpha|^2+|\beta|^2}}$

$A \searrow 1 \Rightarrow B \searrow \frac{\gamma|0\rangle+\delta|1\rangle}{\sqrt{|\gamma|^2+|\delta|^2}}$

This solves the problem of non-normality.

11.23 Bell States Circuit

$BELL = (CNOT)(H \otimes I)$



$|x\rangle$
H
$|y\rangle$
$|\beta_{xy}\rangle \rightarrow BELL(|x\rangle|y\rangle) = |\beta_{xy}\rangle$

11.24 Binary Operators on Bell States

The Bell operator is unitary, $\therefore$ Bell states form orthonormal basis

$(I \otimes I)|\beta_{00}\rangle = |\beta_{00}\rangle$

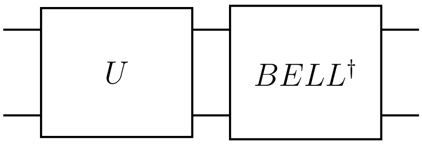
$(X \otimes I)|\beta_{00}\rangle = |\beta_{01}\rangle$

$(Z \otimes I)|\beta_{00}\rangle = |\beta_{10}\rangle$

$(iY \otimes I)|\beta_{00}\rangle = |\beta_{11}\rangle$

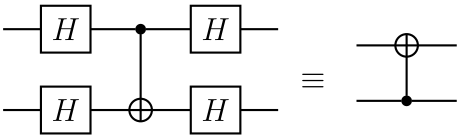
11.25 BELL as Basis Transform Operator

Measuring along Bell basis,



where $BELL^{\dagger} = (CNOT)^{\dagger}(H \otimes I)^{\dagger} = (H \otimes I)(CNOT)$

11.26 Upside Down CNOT Circuit



This works by transforming to x-basis, then perform CNOT (where the effect is opposite), then transform back to z-basis.

11.27 Order-3 Tensor Product

The product space can rely on second order construction.

$W = (A \otimes B) \otimes C = A \otimes (B \otimes C)$,

where separable tensors $\vec{w} = a \otimes b \otimes c$

The dimension is the product of 3-dimensions

$dim(W) = dim(A)dim(B)dim(C)$

W has basis $\{\vec{w}_{jkl} \equiv a_j \otimes b_k \otimes c_l\}$, where $\{a_j\}$, $\{b_k\}$, $\{c_l\}$ are bases of component space.

General tensors can be written in the form of linear combination.

$\vec{w} = \sum_{j,k,l} c_{j,k,l}(a_j \otimes b_k \otimes c_l)$

Separable operator distributes over tensors

$[T_A \otimes T_B \otimes T_C](a \otimes b \otimes c) \equiv T_A(a) \otimes T_B(b) \otimes T_C(c)$

11.28 Tripartite System Defintion ($\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C = \mathcal{H}_{(3)}$)

Three qubits are the tensor product of $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ and the value is any tensor with unit length in the product space

CBS in tripartite system can be represented in the same way

11.29 Trait #15': Born Rule for Tripartite States

If a tripartite system can be expressed as sum of 4 terms,

$|\psi\rangle^3 = \sum_{k=0}^{3} |k\rangle^2_{AB} |\psi\rangle_C$, where C is normalized

$A \otimes B \searrow |k\rangle^2 \Rightarrow C \searrow \frac{|\psi_k\rangle}{\sqrt{\langle\psi_k|\psi_k\rangle}}$

Chapter 12

First Quantum Algorithms

12.1 Superdense Coding

This technique sends two classical bits through one classical bit.

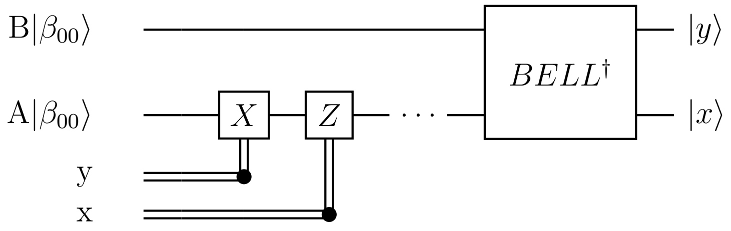
First, a Bell state (β_{00}) is prepared & separated. Then, depending on the desired number from 0-3, a unique set of gate is applied to the sender's qubit $\mathscr{A}$. After which, $\mathscr{A}$'s qubit is sent to $\mathscr{B}$. $\mathscr{B}$ measures both qubit in the Bell basis with $BELL^\dagger$.

The table below referes to the gate applied.

$\mathscr{A}$ to Send	$\mathscr{A}$ Applies	Equivalent Binary Gate	New Bipartite State
"00"	(nothing)	$\mathbb{I} \otimes \mathbb{I}$	$\lvert\beta_{00}\rangle$
"01"	X	$\mathbb{I} \otimes X$	$\lvert\beta_{01}\rangle$
"10"	Z	$\mathbb{I} \otimes Z$	$\lvert\beta_{10}\rangle$
"00"	iY	$\mathbb{I} \otimes Y$	$\lvert\beta_{11}\rangle$

Note, iY = ZX

x, y is the encoded binary message



12.2 Quantum Teleportation

Quantum teleportation transport 1 quantum bit using 2 classical bits.

12.3 Application of Born Rule

As long as all register is in the same orthonormal basis, Born's rule will work.

12.4 Expanding Initial States along $BELL_{AC}$ basis

$|\psi\rangle|\beta_{00}\rangle_{AB} = |\beta_{00}\rangle_{AC} \frac{\alpha|0\rangle_B\beta|1\rangle}{2}...$

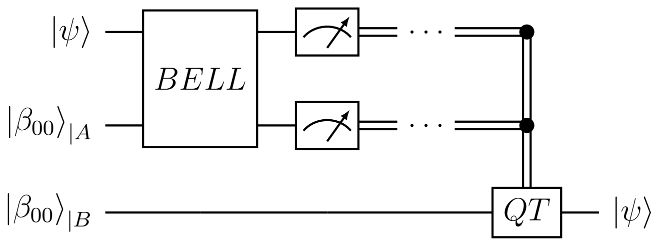
$\mathscr{A}$ measures this state through $BELL^{\dagger}$, then send the result to $\mathscr{B}$.

12.5 B's Action from A's Message

The table shows that gate that B applies on A's message, as shown by the gate QT, to get the desired result.

$\mathscr{B}$ Receives	$\mathscr{B}$ Applies	$\mathscr{B}$ Recovers
"00"	(nothing)	$\|\psi\rangle$
"01"	X	$\|\psi\rangle$
"10"	Z	$\|\psi\rangle$
"11"	iY	$\|\psi\rangle$

Note that iY = ZX



12.6 Boolean Functions and Reversibility

Boolean function is a function with one or more binary digits as input and one or binary output.

Most boolean functions are irreversible

12.7 Unary Gates as Boolean Function

$f : \{0, 1\} \to \{0, 1\}$ or $f : B \to B$

Example of reversible unary include NOT and Identity

Example of irreversible unary inlcude [0] and [1]

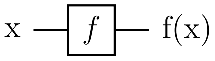
12.8 Binary Gates as Boolean Functions

$f : \mathbb{B}^2 \to B$

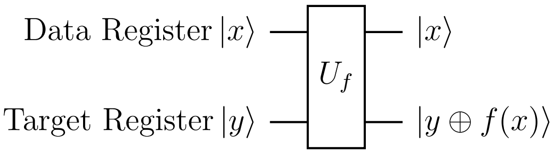
All two-bit cases are irreversible

12.9 Quantum Oracle for Boolean Functions

Oracle for Unary Functions

For  , a quantum analog must have

- two bits input
- two bits output
- unitary ∴ reversible
- computes f with the proper inputs
- equally efficient

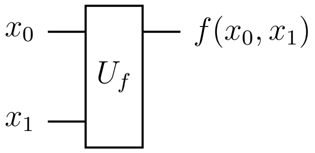


U_f for a CBS is always separable

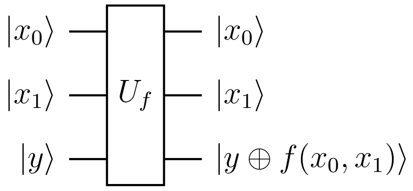
*y is almost always set to 0 so $y \oplus$ f(x) would be always return f(x)

Oracle for Binary Functions

Extending from



A three-in three-out oracle is defined by



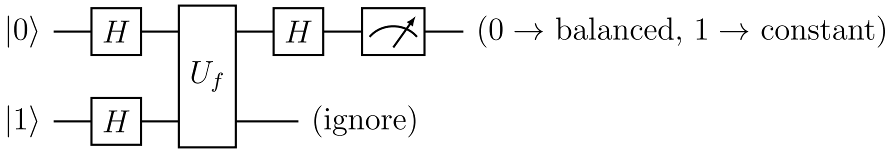
- x_0 & x_1 is usually shortened as $|x\rangle^2$
- U_f is its own inverse
- $U_f = f(x)$ with y = 0
- $U_f(|x\rangle\,|0\rangle) = |x\rangle^2\,|f(x)\rangle$

12.10 Deutsch's Algorithm

This determines whether a unary function is constant or balanced with a single query.

This takes advantage of two techniques:

1. Quantum Parallelism: Non-trivial superposition can explore both CBS at the same time. Bringing calculations out of the z-basis has this effect. Quantum entanglement also plays a big part
2. Phase Kick-back: Information of f(x) from B is transferred to A. The process includes transforming both A & B to the x-basis, where phase-kickback could occur. When this is done, A will respond differently to constant v.s balanced function. This effect is visible when brought back to the z-basis.



Changing A invert the outcome

Changing B leads to inconclusive result, due to loss of phase kickback

Chapter 13

Multi-Qubit Systems and Algorithms

13.1 Higher Order Tensor Products

Notation $\bigotimes$ and $\prod$ can be used to shorten expression

Product Space W $= \prod_{k=0}^{n-1} A_k$

Dimension dim(W) $= \prod_{k=0}^{n-1} d_k$, where $d_k = dim(A_k)$

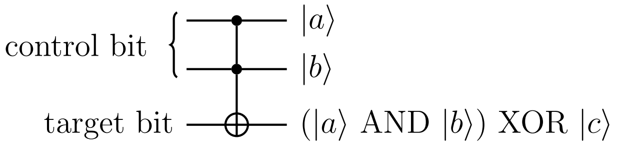
Separable Tensors $\{\prod_{j=0}^{n-1} a_j k_j\}_{k_0,\ldots,k_{n-1}=0,\ldots,0}^{d_0-1,\ldots,d_{n-1}-1}$

Linear Combination $w = \sum c_{k0,\ldots kn-1}\{\prod_{j=0}^{n-1} a_j k_j\}$

Separable Operators $\prod_{k=0}^{n-1} T_k \prod_{k=0}^{n-1} \vec{v}_k = \prod_{k=0}^{n-1} T_k(\vec{v}_k)$

13.2 Toffoli Gate (CCX)

The CCX gate flip the last bit if both a & b = 1



For general tensors, the CCX gate flip the last two amplitudes.

13.3 Definition of n Qubits

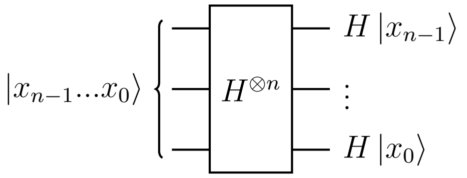
n qubits are the vector space $\mathcal{H}_{(n)}$, and the value is any unit tensors in the product space.

$$\mathcal{H}_{(n)} = \prod_{k=0}^{n-1} \mathcal{H}$$

13.4 n-qubit CBS

$|x\rangle^n = |x_{n-1}, ..., x_0\rangle$

13.5 nth Order Hadamard Gate



$H^{\otimes n}|x\rangle^n = (\frac{1}{\sqrt{2}})^n \sum_{y=0}^{2^{n-1}} (-1)^{x\odot y}|y\rangle^n$, where $x \odot y = x_{n-1} \cdot y_{n-1} \oplus ... \oplus x_0 \cdot y_0$

13.6 Higher Order Basis conversion

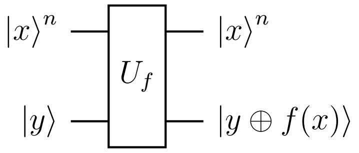
$H^{\otimes n}$ converts between x & z-basis in $\mathcal{H}_{(n)}$

x-basis CBS can be represented as $|x\rangle^n_{\pm}$

In z-basis, $|x\rangle^n_{\pm}$ has the same number of ± except for $|0\rangle^n_{\pm}$

13.7 Oracle for n-qubit Functions

Quantum oracle for n-qubit is the quantum analog of n-bit boolean function.

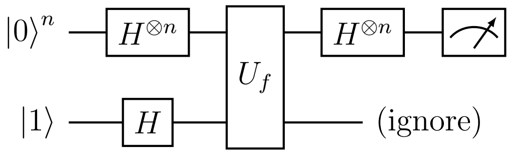


- U_f is its own inverse
- $U_f(|x\rangle|0\rangle) = |x\rangle^n|f(x)\rangle)$
- Both has the same spatial circuit complexity

13.8 Deutsch-Jozsa Problem

Deutsch-Jozsa algorithm determines if an n-bit boolean function is constant or balanced. The quantum worst case n is an improvement from the 2^{n-1} classical worst case.

$$f\{0,1\}^n \rightarrow \{0,1\}$$



The algorithm uses quantum parallelism and phase kickback.

$|0\rangle$ for constant; anything else for balanced

13.9 Quantum vs. Classical Time Complexity

The classical time complexity is $2^{n-1}+1$

The quantum time complexity is $\frac{n}{2}+1$

With $N=2^n$, classical is exponential to n & linear to N. Quantum is linear to n & logarithmic to N

These complexity are for deterministic algorithms.

13.10 Non-Deterministic Algorithm with small error $\epsilon << 1$

M-guess algorithm measures from a sample of registers. If all is 0 → constant, else balanced. The error of predicting constant when everything else is balanced is small for constant large sample, even as n increases.

13.11 Deutsch-Jozsa Measurements

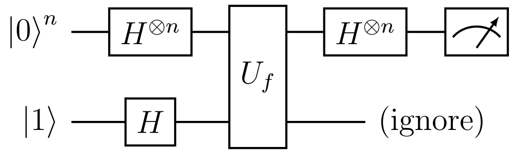
Not all balanced f has a corresponding x-CBS. However, the probability of measuring $|0\rangle^n_{\pm}$ is 0 if f is balanced

13.12 Bernstein-Vazirani Problem

For $f(x)=\alpha\odot x_n$, this algorithm returns a in one try instead of n in classical computing.

a is an n-bit binary number & $\odot$ is mod-2 dot product.

The circuit is the same as the Deutsch-Jozsa algorithm.



Compile the results from the $|0\rangle$ registers, we get α.

The classical complexity is linear in n and logarithmic in N. The quantum complexity is constant for both. N = 2^n (encoded integer size)

$|0\rangle^n = \frac{1}{2^n}\sum_{z=0}^{2^n-1}\left(\sum_{y=0}^{2^n-1}(-1)^{y\odot(a\oplus z)}\right)|z\rangle^n$

This problem is non-deterministic because for n-1 guess, the last registers has 50% probability of being 0 or 1.

13.13 Trait #15": Generalized Born Rule

For (n+m)th order state $|\psi^{n+m}\rangle$ in the product space $A \otimes B = \mathcal{H}_{(n)} \otimes \mathcal{H}_{(n)}$, $|\psi^{n+m}\rangle$ can be written as separable product of A CBS ket & general tensors in B.

$|\psi\rangle^{n+m} = \sum_{k=0}^{2^{n-1}} |k\rangle_A^n |\psi_k\rangle_B^m$

The probability of collapse for B given A observable is

$A \searrow |k\rangle^n \Rightarrow B \searrow \frac{|\psi_k\rangle^m}{\sqrt{(\langle\psi_k|\psi_k\rangle)}}$ for $k = 0, ..., 2^{n-1}$

Chapter 14

Probability Theory

14.1 Outcomes

An outcome is the most basic result. A partition of legal outcomes must

1. be mutually exclusive
2. collectively represents every possible results of the experiment.

14.2 Events

An event is a subset of outcomes

Simple event is an event that contains exactly one outcome

Compound event is an event that contatins more than one outcome

14.3 Sample Space (ω)

The sample space is the set of all possible outcomes

14.4 Null Event ($\varnothing$)

The event consisting of no outcomes, empty set

14.5 Probability $P(\mathcal{A})$

Probability is the likelihood of certain events represented by positive numbers from 0 to 1.

14.6 Set Operations

Union ($\cup$) $\Omega \equiv U_{x_k \in \{0,1\}} \{(x_0, .., x_{n-1})\}$

Intersections ($\cap$) Overlapping of two subset

Differences (- or /) $A - B = A \cap B^c$

Complement ($'$, $^{-}$, ^{c}, $\neg$) All elements NOT in subset

*Outcomes can also be represented as vectors on the $(Z_k)^n$ vector space or integers from 0 to $k^n - 1$

14.7 Linear Independence

In a vector space, a set of vectors $\vec{v}$ is linearly independent

$$c_0\vec{v}_0 + ... + c_{n-1}\vec{v}_{n-1} \Rightarrow c_k = 0$$

14.8 Span

The span of $\vec{v} = \{c_0\vec{v}_0 + ... + c_{m-1}\vec{v}_{m-1} | c_k \text{ are scalars } \}$

For mod-2 vectors $\{\vec{v}_{l0} + ... + \vec{v}_{ls} | \vec{v}_{lk} \in \mathscr{S}\}$

14.9 Fundamental Probability Theory

14.10 The Axioms

For $\{\mathscr{E}\}$, the probability measure, P(), satisfies

1. All P() values are non-negative

$$P(\mathscr{E}) \geq 0$$

2. The probability of something happening is certain

$$P(\Omega) = 1$$

3. The probability of mutually exclusive events can be added

$$P(\bigcup_{k=0}^{n-1} \mathscr{E}) = \sum_{k=0}^{n-1} P(\mathscr{E}_k)$$

However, for mutually non-exclusive events,

$$P(U_{k=0}^{n-1}\mathscr{E}) \leq \sum_{k=0}^{n-1} P(\mathscr{E}_k)(\text{accounting for intersections})$$

These axioms naturally leads to

- For any events, $P(\mathscr{E}) \leq 1$
- $P(\varnothing) = 0$
- If $\mathscr{E} \subseteq \mathscr{F}, P(\mathscr{E}) \leq P(\mathscr{F})$

14.11 Definitions for Finite Equiprobable Sample Space

Size of an Event

$$|\mathscr{E}| \equiv \#\,\text{outcomes} \in \mathscr{E}$$

Probability of an Event

$$P(\mathscr{E}) = \frac{|\mathscr{E}|}{|\omega|}$$

14.12 Conditional Probability

For $\mathscr{E}|\mathscr{F}$, $\mathscr{E}$ given $\mathscr{F}$ is true,

$P(\mathscr{E}|\mathscr{F}) = \frac{|\mathscr{E}\cap\mathscr{F}|}{|\mathscr{F}|}$, whenever $\mathscr{F} \neq 0$

14.13 Bayes' Law

Dividing top & bottom by $|\omega|$

$P(\mathscr{E}|\mathscr{F}) = \frac{P(\mathscr{E}\cap\mathscr{F})}{P(\mathscr{F})}$, whenever $P(\mathscr{F}) \neq 0$

14.14 Statistical Independece

Mutually exclusive means that two events cannot happen at the same time:

$$P(A \cap B) = 0$$

Independence means that the probability of seeing one won't affect the proability of seeing others: $P(A|B) = P(A)$

14.15 Two Independent Events

Substituting: $P(\mathscr{E}|\mathscr{F}) = P(\mathscr{E}))$ into Bayes' Law

$$P(\mathscr{E} \cap \mathscr{F}) = P(\mathscr{E})\, P(\mathscr{F})$$

*Note $\cap$ here has different meaning than mutually exclusive

14.16 Multiple Independent Events

n events are independent if:

$$P(\bigcap_{1<\ldots<n} \mathscr{E}_{ki}) = \prod_{1<\ldots<n} P(\mathscr{E}_{ki}))$$

14.17 Other Formulas

Events follows identity of Boolean Algebra

Probabilities:

$$P(\mathscr{E}) = P(\mathscr{E} \cap \mathscr{F}) + P(\mathscr{E} \cap \mathscr{F}')$$

$$P(\mathscr{E} \cap \mathscr{F}) = P(\mathscr{E}|\mathscr{F})\, P(\mathscr{F})$$

14.18 Wedge and Vee Notations

$\wedge \to \cap$; $\vee \to \cup$; $\neg \to \mathsf{c}$

Sets notations used for non-events sets

14.19 Applications to Deutsch-Jozsa

M and Guess sample M $< 2^n$ random events and test if $f(x') \neq f(x'') \Rightarrow$ balanced, else constant

14.20 Sampling with Replacement

$$P(\mathscr{S} \wedge \mathscr{B}) = P(\mathscr{S}|\mathscr{B})\, P(\mathscr{B}) = \frac{1}{2^{M-1}}\, P(\mathscr{B})$$

where $P(\mathscr{S} \wedge \mathscr{B})$ is the probability of errors (f is balanced yet all guesses yields constant) and $P(\mathscr{B})$ is the probability of f balanced

14.21 Sampling without Replacement

$$P(\theta) = P(\theta_{M-1} \wedge ... \wedge \theta_0) = \prod_{k=0}^{M-1} \frac{2^{n-1} - k}{2^n - k}$$

where M is independent of n, number of inputs.

The numerator represents the #outcomes of 0 and the denominator represents #outcomes in the sample space.

$$P(W \text{ without replacement}) = P(\mathscr{B}) \times 2 \prod_{k=0}^{M-1} \frac{2^{n-1} - k}{2^n - k}$$

This error rate of this is smaller than with replacement

14.22 Probability Algorithm

$\mathscr{A}$ is probabilistic with error tolerance ϵ

14.23 Looping Algorithm

For looping algorithm, failure only occurs if all loop fails

14.24 Constant Time Complexity for Looping Algorithm

Assume $\mathcal{A}$ is a probabilistic, looping algorithm with size N. If P(success) in a single loop is bounded away from 0.

$$P(\mathscr{S}) \geq p > 0$$

with p independent of the size N, then $\mathscr{A}$ is a constant-time algorithm.

For P($\mathscr{S}_k$) $\geq p$, for all $k \geq 1$,

$$P(\mathscr{S}_{tot}) = P(\neg\mathscr{S}_1) \ldots P(\neg\mathscr{S}_T) = (1-p)^T < \epsilon$$

where $P(\mathscr{S}_{tot})$ is the probability of a total failure

Because p & T is independent of size N, algorithm is constant time complexity (CTC)

To solve for integer T from $(1-p)^T = \epsilon$,

$$T = \lfloor log(\epsilon)/log(1-p) \rfloor + 1$$

Chapter 15

Computational Complexity

Computational complexity refers to the growth rate of an algorithm with respect to the size of the input.

Time complexity refers to the growth rate of running time of an algorithm whereas space complexity refers to the growth rate of hardware.

Space & time complexity together are referred to as computational complexity.

15.1 Big-O Growth

$$T_Q(N) = O(f(N))$$

$$\iff$$

$$\exists\, n_0, c \,\text{s.t}\, T_Q(N) \leq c|f(N)|, \forall\, N \geq n_0$$

where $T_Q(N) \equiv$ time required by the algorithm Q to process N elements

This means that above a certain point, T_Q grows no faster than c$|f(N)|$. This is the upper bound of the growth rate for algorithm Q.

- Constant factor K can be ignored
- For polynomial time complexity, ignore all but the highest degree term

15.2 Ω Growth

$$T_Q(N) = \Omega(f(N))$$

$$\iff$$

$$\exists\, n_0, c \,\text{s.t}\, T_Q(N) \geq c|f(N)|, \forall\, N \geq n_0$$

This means that above a certain point, T_Q grows no slower than c$|f(N)|$. This is the lower bound of the growth rate for algorithm Q.

15.3 Θ Growth

$$T_Q(N) = \Theta(f(N))$$

$$\iff$$

$$\text{both}$$

$$T_Q(N) = O(f(N)) \text{ and } T_Q(N) = \Omega(f(N))$$

15.4 Little-o Growth

$$T_Q(N) = o(f(N))$$

$$\iff$$

$$\text{both}$$

$$T_Q(N) = O(f(N)), \text{ but } T_Q(N) \neq \Omega(f(N))$$

15.5 Easy vs. Hard

Easy problems are those whose algorithms is polynomial time complexity.

Hard problems are those whose algorithms is exponential time complexity.

Quantum parallelism & entanglement offer exponential speed up to hard problems.

Two examples are Simon's algorithm and Shor's algorithm.

Chapter 16

Computational Basis States and Modular Arithmetic

16.1 Single Qubit Hilbert Space

One-qubit Hilbert Space, $\mathcal{H}$ consists of the 2-D complex vector space. The typical states is a superposition of the two CBS.

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

16.2 Multi Qubit Hilbert Spaces

Multi qubit states operate in 2^n-dimensional Hilbert space $\mathcal{H}_{(n)}$

$$|x\rangle^n = \bigotimes_{k=0}^{n-1} |x_k\rangle$$

where $|x_k\rangle$ is either $|0\rangle$ or $|1\rangle$.

The index is in decreasing order because the right-most bit correspond ot the least significant bit of the binary number $x_{n-1}...x_0$

16.3 $\mathbb{Z}_N$, or mod N Arithmetic

$\mathbb{Z}_N$ is the finite group consisting of N integers from 0 to N-1.

$$\mathbb{Z}_N \equiv \{0, 1, 2, ..., N-1\}$$

The addition modulo N is the remainder after dividing by N

- $x + y$ (mod N) $\equiv (x + y)\%N$
- $-x$ (mod N) $\equiv (N - x)$
- $x - y$ (mod N) $\equiv (x + -y)\%N$

For mod-2, the addition operator is the XOR operation ($\oplus$).

$\mathbb{Z}_2$ correspond to the CBS of $\mathcal{H}_1$

16.4 $(\mathbb{Z}_2)^n$ with $\oplus$ Arithmetic

$(\mathbb{Z}_2)^n$ is a vector space because it satisfies all the conditions of a vector space.

The set $(\mathbb{Z}_2)^n$ is the n-tuples that have 0 or 1 as their coordinates.

$(\mathbb{Z}_2)^n$ is a vector space with vectors as objects

$(\mathbb{Z}_2)^n$ correspond to the CBS of $\mathcal{H}_{(n)}$

Ex: $\vec{x} = (1, 0, 0, 0)^T$

16.5 The group $\mathbb{Z}_{2n}$ (encoded binary form)

$\mathbb{Z}_{2n}$ has 2^n elements

Each x in $(\mathbb{Z}_{2n})$ can be represented as n binary digits

$$x = \sum_{k=0}^{n-1} x_k 2^k$$

The addition operation is the bitwise $\oplus$ operator.

$$x \oplus y \equiv \sum_{k=0}^{n-1} (x_k \oplus y_k) 2^k$$

For any $x \in (\mathbb{Z}_{2n}, \oplus)$

$$x \oplus x = 0 \ \therefore \ x = -x$$

Ex: $x = (0110)$

16.6 Connection Between $(\mathbb{Z}_2)^n$ and $(\mathbb{Z}_{2n})$

$(\mathbb{Z}_2)^n$ and $(\mathbb{Z}_{2n})$ is the same group. They are isomorphic

$$(\mathbb{Z}_2)^n \cong (\mathbb{Z}_{2n})$$

16.7 General Notations

For $|x\rangle^n$, n represents the number of registers in the state, not the dimension of the space. The dimension would actually be 2^n. This means that $|x\rangle^n \otimes |y\rangle^m$ will result in in $|xy\rangle^{n+m}$

For x, y in $\mathbb{Z}_{2n}$, the mod-2 sum can be taken inside a ket $(x \oplus y)$

Chapter 17

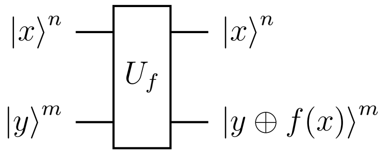
Quantum Oracle

Oracles are called black boxes, and they have to meet certain requirements:

- U_f's actions on CBS is unitary
- multi-qubit registers takes in CBS in of the form $|x\rangle^n$ & $|y\rangle^m$
- f have domain and range $\in \mathbb{Z}_{2n}$
- f is an easy function, which means that it can be computed in polynomial time

17.1 General Oracle (maps from $\mathbb{Z}_{2n}$ to $\mathbb{Z}_{2m}$)

For a general oracle,



The U_f matrix has shape $(2^{m+n})^2$ and is made up of $(2^n)^2$ sub-matrices of shape $(2^m)^2$.

The submatrices is x-major, so columns represent the output of one x and all y.

The submatrix for a particular x is the tensor product of the corresponding output in binary form, replace 0 with $\mathbb{I}$ and 0 with σ_x, where

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The submatrix M_x is expanded by $|x\rangle \otimes M_x$.

$f(x)$ maps from $\mathbb{Z}_{2n}$ of x to $\mathbb{Z}_{2m}$ of y.

17.2 Complexity of Quantum Oracle

Relativized Time Complexity This is the time complexity without the knowledge of the oracle's design (circuit + algorithm)

Absolute Time Complexity This includes knowledge of the oracle's design

Chapter 18

Simon's Algorithm

18.1 Ordinary Periodicity

$$f : \begin{Bmatrix} \mathbb{R} \\ \mathbb{C} \\ \mathbb{Z} \end{Bmatrix} \rightarrow \text{S is periodic if } \exists \text{ smallest } \alpha < 0, \text{where}$$

$$f(x+\alpha) = f(x) \, \forall \, x \in \text{dom}(f)$$

18.2 $(\mathbb{Z}_2)^n$ Periodicty

$$f : (\mathbb{Z}_2)^n \rightarrow \text{S is periodic if} \, \exists \, \alpha \in (\mathbb{Z}_2)^n, \alpha \neq 0 \, s.t$$

$$\forall \, x \neq y \in (\mathbb{Z}_2)^n, f(x) = f(y) \Leftrightarrow y = x \oplus \alpha$$

This means that the cycle has two elements in the progression.

$$x \rightarrow x \oplus \alpha \rightarrow x \oplus (\alpha \oplus \alpha) = x$$

Notice that $\alpha \oplus \alpha = 0$, which means that $\alpha = x \oplus y$ (the period can be recovered from one pair of elements).

This periodicity is isomorphically equivalent in $\mathbb{Z}_{2n}$

18.3 Collapsing 1 bit Periodicity

$f = \sum_{k=0}^{m} \begin{Bmatrix} 0 \text{ or } 1 \text{ if } k = c \\ x_k \end{Bmatrix} 2^k$

$f \oplus 2^k$ performs a bit flip on the kth element.

The period of f is 2^c.

18.4 Collapsing More Than 1 Bit → Not Periodic

Domain and Range

The domain of f is dom(f), the range of f is ran(f).

For $\omega \in$ ran(f), the pre-image of ω is $\{x|f(x) = \omega\}$

One-to-one

$x \neq y \Rightarrow f(x) \neq f(y) \, \forall \, x, y \in \text{dom}(f)$

For $S \subseteq$ dom(f) if f is not 1-to-1, S can still be 1-to-1.

Two-to-one

$\omega \in$ ran(f) has two pre-images in dom(f).

n-to-one

$\omega \in$ ran(f) has n pre-images in dom(f).

The domain of all periodic functions in $\mathbb{Z}_{2n}$ can be partitioned into 2 cosets

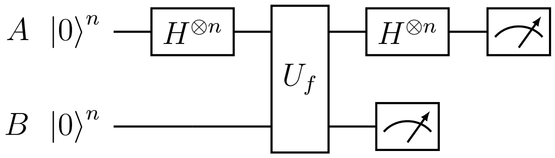
$$\mathbb{Z}_{2n} = R \cup Q$$
$$= \{..., x, ...\} \cup \{..., x \oplus \alpha, ...\}$$

Every periodic function in $\mathbb{Z}_{2n}$ is 2-to 1 in f and 1-to-1 in their respective coset.

18.5 Simon's Problem

Let f: $\{(\mathbb{Z}_2)^n \rightarrow (\mathbb{Z}_2)^n\}$ is periodic. Find α (the period)

18.6 The Circuit



- H-gate sets up quantum parallelism
- For B-channel, $|0\rangle$ is used for generalized Born's rule & $|1\rangle$ for phase kickback

18.7 The Strategy

Find an n-1 linear independent ω_k that is orthogonal to α.

Classical determine α.

18.8 Circuit Breakdown

After the oracle is applied, the state becomes

$$\left(\tfrac{1}{2}\right)^n \sum_{x=0}^{2^n-1} |x\rangle^n \, |f(x)\rangle^n$$

The superposition $|x\rangle^n \frac{1}{\sqrt{2}^n} |f(x)\rangle^n$ is separable, A-measurement will collapse B to normalized $|f(x)\rangle^n$

If B is measured, the collapsing state will be a superposition with individual probability of $\frac{1}{2^n}$

*The previous sum has 2^{2n} basis vectors but the amplitude for most is 0.

Partitioning the domain into 2 cosets, we can group the sum

$$|\psi\rangle = (\frac{1}{\sqrt{2}})^{n-1} \sum_{x \in \mathbb{R}} (\frac{|x\rangle^n + |x \oplus \alpha\rangle^n}{\sqrt{2}}) \, |f(x)\rangle^n$$

This shows that A collapses into 1 of 2 states.

18.9 Hypothetical B-Measurement

$$|\psi\rangle \searrow (\frac{|x\rangle^n + |x \oplus \alpha\rangle^n}{\sqrt{2}}) \, |f(x)\rangle^n$$

Which means that B measurement will collapse to

$$|\psi_{x_0}\rangle \searrow (\frac{|x_{x_0}\rangle^n + |x_{x_0} \oplus \alpha\rangle^n}{\sqrt{2}}) \, |f(x)\rangle^n$$

18.10 Final Hadamard Gate

The final states become

$$\left(\tfrac{1}{\sqrt{2}}\right)^{n+1} \sum_{y=0}^{2^n-1} (-1)^{y \cdot x_0} (1 + (-1)^{y \cdot \alpha}) \, |y\rangle^n$$

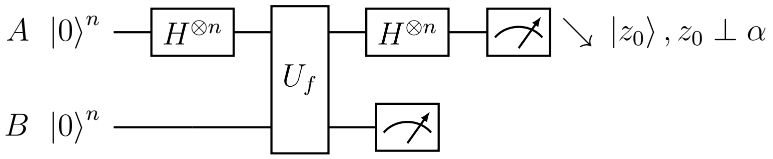
The inside expression reduces to

$$1 + (-1)^{y \cdot x} = \begin{cases} 0, \text{if}\, y \cdot \alpha = 1 \text{ (mod 2)}, \\ 2 \text{ if}\, y \cdot \alpha = 0 \text{ (mod 2)} \end{cases}$$

which leaves

$$\left(\tfrac{1}{\sqrt{2}}\right)^{n-1} \sum_{y \cdot \alpha = 0 \text{ (mod 2)}} (-1)^{y \cdot x_0} \, |y\rangle^n$$

The set of vectors in the sum is now halved (2^{n-1})



All output is orthogonal to α

z = 0 is orthogonal to everything and ∴ trivial

18.11 The Algorithm

1. Getting $n-1$ linear independent vectors in polynomial time with arbitrary good confidence
2. Checking linear independence & solving n equations in polynomial time

The complexity of the algorithm is O(n^3)

The Steps

- Select an integer T with P(failure) $= \frac{1}{2^T}$
- Initialize an empty set W for vectors $\in (\mathbb{Z}_2)^N$
- Repeat the loop at most $n+T$ times
    1. Apply Simon's circuit
    2. Measure the A register to get z
    3. Classically determine z's linear independence on W

       -If yes, name it ω_j & add ω_j to W

       * If $j = n-2$, $\exists\ n-1$ vectors, break loop (success)

       -If not (including $z = 0$), start new pass
- If loop end naturally after $n+T$, failure
- Else, add non-orthogonal ω_{n-1} independent to W. The result is n independent vectors

$$\omega_k \cdot \alpha = \begin{Bmatrix} 0, k = 0, ..., n-2 \\ 1, k = n-1 \end{Bmatrix}$$

- Classically solve for α

The first n-1 vectors form a basis in $\mathbb{Z}_2^n$. However, as the $n-1$th vector is not orthogonal to α, this basis cannot be an orthonormal basis

$$c_k \neq v \cdot w_k$$

18.12 Producing $n-1$ Linear Independent w_k in Polynomial time

Theorem 1 Select $m+T$ samples, P($\mathscr{S}(m)) > 1-(\frac{1}{2})^T$

$\mathscr{S}(m) \equiv$ event that $c_0, ..., c_{j-1}$ are linearly independent

$$P(\mathscr{S}(m) = \prod_{i=1}^{m} P(\mathscr{S}(j)|\mathscr{S}(j-1))$$

With $P(\mathscr{S}(j)|\mathscr{S}(j-1)) = (1-\left(\frac{1}{2}\right)^{m+T-j+1})$,

$$P(\mathscr{S}(m)) = \prod_{i=1}^{m}(1-\left(\frac{1}{2}\right)^i) \,\&\, \prod_{i=1}^{p}(1-\alpha_i) \geq 1-\sum_{i=1}^{p}\alpha_i$$

$$P(\mathscr{S}(m)) = 1-\left(\frac{1}{2}\right)^T [\sum_{i=1}^{m} 1-\left(\frac{1}{2}\right)^i] > 1-\left(\frac{1}{2}\right)^T$$

Theorem 2 Select mT samples, $P(\mathscr{S}(m)) > \frac{1}{4}$

$$P(\mathscr{S}(m)) \geq \sum_{i=1}^{\infty}(1-\left(\frac{1}{2}\right)^T) > \frac{1}{4}$$

This means that P(failure) $< (\frac{3}{4})^T$

Hidden Classical Algorithm

18.13 Gaussian Elimination (GE)

GE changes a matrix into row echelon form or reduced row echelon form. All-zero rows are at the bottom

RHS constant vector should also be included in the GE process

Three Operations of GE

1. Swap two rows
2. Multiply a row by a non-zero value
3. Add a multiple of one row to another

The cost for decimal-based GE is $O(n^3(log\, m)^2)$, whereas for mod-2 GE is O(n^3)

18.14 Back Substitution

In RREF, $x_{n-1} = b_{n-1}$ can be read off. Substitute the value into the above equation to solve for unknown.

The cost for decimal-based is $O(n^2(log\, m)^2)$, whereas for mod-2 back substitution is O(n^2)

18.15 GE in Determining Linear Independence

For $m \times n$ matrix, with $m < (n-1)$, there will be gap in the diagonal with leading 1s.

Add z to the matrix and reapply GE.

If z is linearly independent, one of the hole is filled. If not an all 0 row will appear at the bottom.

Once there are $n-1$ vectors, $W - \alpha = 0$

The cost of this is $O(n^4)$ (n quantum, n^3 classical)

18.16 Completing the Basis with Non-Orthogonal nth Vector

1. Find ω_k where there is a gap in the all-1 diagonal
   - Place $\omega_k \equiv 2^{n-2-k}$ below ω_k
   - If diagonal is all 1s place ω_{n-1} at the bottom
2. On the RHS vectors of all 0, insert 1 in the row of ω_{n-1}

The cost of this is $O(n)$ in series.

18.17 Back Substitution to Solve for α

W is already in RREF $\therefore$ back substitution is in series $O(n^2)$

18.18 New Linear Independence Step

The classical adding z & apply GE is replaced by

1. Loop until either z is added to W or $z = 0$ produced

   A $m \leftarrow$ MSB position of z

   B Search W for a ω_k row with the same MSB position

   - If not found, insert z in between. Success
   - If found, z $\leftarrow z \oplus \omega$. Continue next pass

This works because the addition process move the MSB to the right after each loop. In other words, if $z \in$ span (W).

$$z \oplus \sum_{\omega_0}^{\cdots} = 0$$

The cost of this step is $O(n^2)$. This brings the overall CTC to $O(n^3)$ ($O(n)$ for quantum).

18.19 New w_{n-1} Step

Because the new linear independence step produces and $(n-1) \times n$ RREF, simply add nth linear independent vector to form $n \times$ n RREF.

$$\omega_k \cdot \alpha = \begin{Bmatrix} 0, k = 0, ..., n-2 \\ 1, k = n-1 \end{Bmatrix}$$

18.20 Cost of the Circuit Summary

$O(n)$: Quantum with $O(n^2)$ GE nested → $O(n^3)$

In series: $O(n)$ inserting ω_{n-1}, $O(n^2)$ back-substitution

The CTC of the circuit is therefore $O(n^3)$

18.21 Classical Deterministic Cost

$2^{n-1}+1$ samples have to be tested in the worst case $\therefore O(2^n)$

18.22 Classical Probabilistic Cost

The upper bound of getting a repeat in m samples is

$$P(\mathscr{E}_{ij}) \leq \frac{m(m-1)}{2} \frac{1}{2^n - 1}$$

where $\mathscr{E}_{ij}$ is the event that any $f(x_i) = f(x_j)$.

With $lim_{n \to \infty} P = 0$, the problem is hard even probabilistically

Chapter 19

Real & Complex Fourier Series

19.1 Periodic Functions over $\mathbb{R}$

$f : \mathbb{R} \to \mathcal{S}$ is periodic if $\exists$ smallest $a > 0$, where

$$f(x+a) = f(x)\,\forall\, x \in \mathrm{dom}(f)$$

19.2 Functions with Bounded Domain or Compact Support

$f : x \in \mathbb{R}$ (defined over bounded interval)

A function has compact support when it is defined over an interval but all 0 outside the interval.

The support of a function is the closure of the domain where $f \neq 0$

19.3 Periodicity & Bounded Domain

A periodic function is a repeat of the bounded itnerval.

If the same action is applied to the bounded domain, we have an induced periodic function

19.4 Real Fourier Series (of 2π periodic function)

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

19.5 Interpretations

The amplitude determines the weighting of each term

When small-n coefficients are large, the function has low frequency characteristics. When large-n coefficients are large, the function has high frequency characteristics.

$\{\sin\, nx,\ \cos\, nx\}$ are called Fourier basis functions, or normal modes, or Fourier eigenfunctions

19.6 Finite Approximation

Practical use of the series requires approximation by taking the partial sum of $n = N < \infty$ terms.

19.7 The Spectrum

The Fourier coefficients list is called the spectrum of f. They represents a new function, $F(n)$.

19.8 Fourier Series as Function Mapping Operator

$$\mathcal{FS}[f(x)] = F(n) \longleftrightarrow \{a_n, b_n\}$$

$$f : \mathbb{R} \to \mathbb{R}; F : Z_{\geq 0} \to \mathbb{R}$$

$$\mathcal{FS} : f \to F$$

19.9 Computing Fourier Coefficients

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, n = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, n > 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, n > 0$$

This works for function with period 2π or bounded domain $[-\pi, \pi)$. For general period T, the form is as follow

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx, n = 0$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos nx \, dx, n > 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin nx \, dx, n > 0$$

The domain of the series is the entire real line, so for bounded functions, only one period would be necessary.

Compact support has no Fourier Series unless it is broken up

19.10 Complex Fourier Series

From Euler's formula

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\ ;\ \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Substituting into the real Fourier Series, we get

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n - ib_n)e^{inx} + \sum_{n=1}^{\infty}(a_n + ib_n)e^{-inx}$$

The Complex Fourier Series of 2π period function is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n \equiv \begin{cases} \frac{1}{2}(a_n - ib_n),\ n > 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}),\ n > 0 \\ \frac{1}{2}a_0,\ n = 0 \end{cases}$

The coefficient c_n is complex, but the sum is still real.

19.11 Computing Complex Fourier Coefficients

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$$

Note that the 2 in the denominator can be dropped depending on the format.

c_n is the weighted sum of e^{inx} in the summation, where c is a function of n and e^{-inx} is the weighted sum of $f(x)$ in the integral, with f as a function of x

The correspondence $f(x) : \mathbb{R} \to c(n) : \mathbb{Z}$ is the complex Fourier series

19.12 Ordinary Frequency

The frequency, f, is the reciprocal of the period

$$f = \frac{1}{T}$$

This is the number of period in a 1 unit interval

$$f \cdot T = 1$$

19.13 Angular Frequency

This expresses the number of period per cycle or revolution

$$\omega = \frac{2\pi}{T} \; ; \; \omega = 2\pi f \; ; \; \omega \cdot T = 2\pi$$

For frequency of arbitrary interval l,

$$f = \frac{l}{T} \; ; \; f \cdot T = l$$

Chapter 20

Continuous Fourier Transform

To represents unbounded periodic functions as sums of exponentials,

- The integer index n becomes a real number s
- The sum becomes an integral
- The limit of integration changes from $\pm\pi$ to $\pm\infty$
- Normalization constant $\left(\frac{1}{\sqrt{2\pi}}\right)$

20.1 Fourier Transform

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx} dx$$

$$F = \mathscr{F}[f]\,;\; f \underset{\rightarrow}{\mathscr{F}} F$$

20.2 Inverse Fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{isx} ds$$

20.3 Real vs. Complex, Even vs. Odd

If f is even, F is real-valued. If f is odd, F is imaginary-values. Else, F is mixed.

20.4 Conditions for Fourier Transform

A function that can be Fourier transformed must be absolutely integrable

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

20.5 The Delta Function

A delta function is 0 everywhere and spike to infinity at a point

$$\delta(x) = \begin{cases} \infty, \text{if x = 0} \\ 0, \text{otherwise} \end{cases} \quad \text{and} \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

20.6 Delta Functions as Limits of Rectangles

$$\delta_n(x) = \begin{cases} \frac{1}{n}, \text{if } x \in [\frac{-1}{2}n, \frac{1}{2}n] \\ 0, \text{otherwise} \end{cases}$$

This is true because the area of the rectangle is 1.

$$\delta(x) = lim_{n\to\infty}\delta_n(x)$$

20.7 Delta Functions as Limits of Exponentials

$$\delta_a(x) = \sqrt{\frac{a}{\pi}}e^{-ax^2}$$

With the area of the Gaussian function being 1, this is smoother than the rectangular approximation

$$\delta(x) = lim_{n\to\infty}\delta_n(x)$$

20.8 The Sifting Property

Using the rectangular sum approximation, we can prove

$$f(d) = \int_{-\infty}^{\infty} f(t)\delta(t-d)\,dt$$

or equivalently $= \int_{x_0-\in}^{x_0+\in} f(t)\delta(t-d)\,dt$

The integral returns the value of f at point d

By applying Fourier transform and substituting the whole expression, we get

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}ds$$

The delta function represent the different frequencies.

By definition of even function, $\delta(x) = \delta(-x)$, which implies

$$\delta(x-d) = \delta d - x \;\therefore\; \text{the delta function is even}$$

20.9 Fourier Transform with $\delta(x)$

Using the Delta function, the Fourier Transform of

- $f(x) = a \leftrightarrow F(s) = a\sqrt{2\pi}\delta(s)$
- $f(x) = \cos(\text{x}) \leftrightarrow F(x) = \sqrt{\frac{\pi}{2}}(\delta(s+1) + \delta(s-1))$
- $f(x) = \sin(\text{x}) \leftrightarrow F(x) = -\sqrt{\frac{\pi}{2}}(\delta(s+1) - \delta(s-1))$

20.10 Translation Invariance

$$f(x+a)\xrightarrow{\mathscr{F}}e^{ias}F(s)\ \&\ F(s-\gamma)\xrightarrow{\mathscr{F}^{-1}}e^{i\gamma x}f(x)\ \forall\, a,\gamma \in \mathbb{R}$$

A translation in the time or spatial domain causes a phase shift in the frequency domain. The multiple is a unit vector in $\mathbb{C}$ so there is no effect on the magnitude of the output function.

Because probabilities of collapse is amplitude absolute squared,

$$|e^{ias}F(s)|^2 = |e^{ias}|^2|F(s)|^2 = |F(s)|^2$$

translation doesn't affect collapse probabilities.

20.11 Plancherel's Theorem

For any Fourier transform pair, F and f, we have

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

If the function are amplitudes of quantum states, the squared absolute values are probabilities density of the function. This will always sum up to 1.

20.12 Convolution

Convolution is a binary operator on two functions that produces a third function

$$[f * g](x) \equiv \int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi$$

20.13 Convolution Theorem

A convolution of two functions can be represented as point-to-point multiplication of their Fourier transforms.

$$f * g = \sqrt{2\pi}\mathscr{F}^{-1}[\mathscr{F}(f)\cdot\mathscr{F}(g)]$$

The constant $\sqrt{2\pi}$ depends on the definition of $\mathscr{F}$

20.14 Period and & Frequency in Fourier Transform

For the angular frequency, ω, the period is $\frac{2\pi}{\omega}$

Taking the Fourier transform of this gives, a delta function with the absolute value of its position being ω

The Fourier transform absolute value of the delta result is Ω

20.15 Applications for Fourier Transform

- Target certain frequencies of a picture, audio or signal. Perform work on $\mathscr{F}$ then take $\mathscr{F}$ to recover the original
- Represents quantum state wave-function. ψ is a vector in a Hilbert space with different bases.
  - Position Basis - $\Psi(x)$. This form shows the amplitude of the particle at any point in time. Integrate $|\psi(x)|^2$ over a region of position space to find out the P(particle at x).
  - Momentum Basis - $\Psi(x)$. This form shows the amplitude of the particle at any point in time. Integrate $|\psi(p)|^2$ over a region of momentum space to find out the P(particle at p).

The movement between x-space & p-space is through the Fourier transform

$$\psi(p) = \mathscr{F}(\psi(x))\ ;\ \psi(x) = \mathscr{F}^{-1}(\psi(p))$$

20.16 The Uncertainty Principle

If a Gaussian wave packet is used to represent the position state $\psi(x)$, as $|\psi(x)|^2$ becomes narrower, $|\psi(p)|^2$ becomes more spread out.

This is the Heisenberg Uncertainty Principle

Chapter 21

Discrete & Fast Fourier Transforms

$\mathcal{DFT}$ maps functions from $\mathbb{Z}_N$ to $\mathbb{C}$. This is useful for discretely sampled functions and generated data.

Because $\mathrm{dom}(f) = \mathbb{Z}_N$, it can be represented as a vector of operation on all values within the set or a vector in $\mathbb{C}^N$.

21.1 Adapting Continuous $\mathcal{FT}$ to $\mathcal{DFT}$

- The integral becomes sums from 0 from $N-1$. There will be N discrete real-valued vector (f_k) and N complex spectrum vector (F_k)
- The factor $\frac{1}{\sqrt{2\pi}}$ replaces by $\frac{1}{\sqrt{N}}$
- The nth root of unity ω^{jk} replaces e^{isx}, where $\omega = e^{\frac{2\pi i}{N}} \therefore \omega^{jk} = e^{\frac{(2\pi i)}{N} j \cdot k}$ (bounded domain implies periodicity)

If $\psi_s(x) = e^{-isx}$ then $\phi_j(k) = \omega_N^{-jk} = \omega^{-jk}$

e^{-isx} becomes N vectors $v_j = (\omega^{-j0}, ...\omega^{-j(N-1)})^T$

j = 0, ..., N-1. k is the coordinate index & j is the parameter label

21.2 Definition of $\mathcal{DFT}$

For $f_k : \mathbb{C}^N$ with $\mathrm{dom}(\mathbb{Z}_N)$, $\mathcal{DFT}^{(N)}(f) = F_j : \mathbb{C}^N$, where

$$F_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f_k \, \omega^{-jk}, \text{for } j = 0, ..., N-1$$

21.3 Inverse $\mathcal{DFT}$

$$f_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} F_j \, \omega^{kj}, \text{for } k = 0, ..., N-1$$

Naturally, $\mathcal{DFT}^{-1} \circ \mathcal{DFT} = \mathcal{DFT} \circ \mathcal{DFT}^{-1} = \mathbb{I}$

21.4 $\mathcal{DFT}$ as Matrix

$\mathcal{DFT}$ can be represented as matrix W with $\zeta = \omega^{-1}$

$$W = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & & \dots & & 1 \\ \vdots & \zeta & \ddots & & \\ 1 & & & & \zeta^{(N-1)(N-1)} \end{pmatrix}$$

$\mathcal{DFT}(f) = W \cdot (f_k) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f_k \, \zeta^{jk} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f_k \, \omega^{-jk}$, where $\mathcal{DFT}$ is unitary

21.5 Convolution of Two Vectors

$[f * g] = \sum_{l=0}^{N-1} f_l \, f_{k-l}$

The convolution theorem also hold with $\mathcal{DFT}$

$$f * g = \sqrt{2\pi} \mathcal{DFT}^{-1}[\mathcal{DFT}(f) \odot \mathcal{DFT}(g)]$$

21.6 Translation Invariance for Vectors

$$f_{k-l} \xrightarrow{\mathcal{DFT}} \omega^{-lk} \, F_k$$

21.7 Computational Complexity of $\mathcal{DFT}$

The complexity of $\mathcal{DFT}$ is $O(N^2)$ relative to arithmetic operations for matrix computation.

21.8 Period & Frequency in $\mathcal{DFT}$

For $dom(f)$ bounded over $\mathbb{Z}_N$, if f has period T then

$$f(k + T) = f(k) \Rightarrow T \cdot = N$$

Non-zero amplitudes in the spectrum are multiples of f (frequency)

Looking at the multiples of the frequency $\Rightarrow T$

21.9 Cost Benefit of $\mathcal{FFT}$

$\mathcal{FFT}$ relies on the symmetry of the $\mathcal{DFT}$ matrix to speed up computation from $O(n^2)$ to $O(n \, log \, n)$

This is the main method for computing $\mathcal{DFT}$.

$\mathcal{FFT}$ only operates on vectors with 2^n components. If f is not in this form, simply pad 0 to the end and perform $\mathcal{FFT}$.

21.10 Recursion Equation for $\mathcal{FFT}$

Split the k index into odd and even, where

$$f_k^{even} = f_{2k} \, and \, f_k^{odd} = f_{2k+1}$$

We get,

$$\mathcal{DFT}[f]_j = \frac{1}{\sqrt{N}} \left(\sum_{k=0}^{\frac{N}{2}-1} f_k^{even} \omega^{-j(2k)} + \sum_{k=0}^{\frac{N}{2}-1} f_k^{odd} \omega^{-j(2k+1)} \right)$$

$$= \frac{1}{\sqrt{N}} \left(\sum_{k=0}^{\frac{N}{2}-1} f_k^{even} (\omega^2)^{-jk} + \omega^{-j} \sum_{k=0}^{\frac{N}{2}-1} f_k^{odd} (\omega^2)^{-jk} \right)$$

Notice that ω^2 is a N/2 root of unity, we get

$$[F^{(N)}]_j = \frac{1}{\sqrt{2}} ([F_E^{(N/2)}]_{(j \, mod \, N/2)} + \omega_N^{-j} [F_O^{(N/2)}]_{(j \, mod \, N/2)}$$

f is periodic with $\frac{N}{2}$ period, $\mathcal{DFT}[f]$ also has $\frac{N}{2}$ period.

21.11 Danielson-Lanczos Recursion Relation

With f being $\frac{N}{2}$ periodic $\Rightarrow f(\frac{N}{2} + P) = f(P)$

$$F^{(N)}[f]_j = \frac{1}{\sqrt{2}} ([F_E^{(N/2)}]_{(j \, modN/2)} + \omega_N^{-j} [F_O^{(N/2)}]_{(j \, modN/2)})$$

Once F is restricted for example F_{EOE}, the size of j is restricted to the number of elements in the group.

21.12 Recursive N logN Solution

- Bit Reversal
- Iterative Array Building
- Normalization

21.13 Bit Reversal

Recursion after log N calls gives order-one $\mathcal{DFT}$

$$\mathcal{DFT}^{(1)}(\{c\}) = \{c\}$$

For order-2 DFT, however

$$[F_{EO...OE}^{(2)}]_j = \frac{1}{\sqrt{2}} (f_p + (-1)^{-j} \, f_q)$$

If $f_p \& f_q$ is arranged in order, then $(-1)^{-j}$ can just be multiplied out & added up.

By dividing the sum into even and odd sections, we find that the new index of f_k is f_n where n is k with its bits reversed relative to the size of k.

For example, 10010 → 01001

21.14 Time Complexity

If performed independently for each input, the time complexity if n log n.

However, a static array can be produces ahead of time & is independent of the input & so therefore be constant time complexity.

21.15 Rebuilding from the Bit-Reversed Array

The input array is first bit-reversed, then built up using Danielson-Lanczos recursion relation.

$$F^{(N)}[]_j = \frac{1}{\sqrt{2}}([F_E^{(\frac{N}{2})}]_{(j\,mod\,N/2)} + \omega_N^{-j}[F_O^{(\frac{N}{2})}]_{(j\,mod\,N/2)})$$

Note that $F_E \& F_O$ repeats twice but j goes from 0 to $N-1$.

Thanks to the bit reversal step, the elements are already in order of j-index

21.16 Time Complexity

There are $logN$ outer loop until the full size N array is reached. This is because information is reduced by half as only $a+b$ is import & and not what a & b are. There is a maximum of N operation in each loop $\therefore$ the complexity is N log N.

21.17 Normalization

The array is normalized by multiplying by $\frac{1}{\sqrt{N}}$

21.18 Time Complexity

The normalization step is linear & doesn't factor into the overall complexity

Chapter 22

Quantum Fourier Transform

Discrete Fourier Transform

$$\mathcal{DFT} : \mathbb{C}^N \rightarrow \mathbb{C}^N ; \mathcal{DFT}(f_k) \rightarrow f_j$$

Fast Fourier Transform

$$\mathcal{FFT} : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n} ; \mathcal{FFT}(f_k) \rightarrow f_j$$

Quantum Fourier Transform

$$\mathcal{QFT} : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n} ; \mathcal{QFT} : \mathcal{H}_{(n)} \rightarrow \mathcal{H}_{(n)}$$

22.1 Approaches to Operator Definition

1. Actions on CBS kets
2. Actions on complex coefficients
3. Matrix for U, where $U \left|\psi\right\rangle^n \rightarrow \left|\psi\right\rangle^{n'}$

* Make sure U is linear & unitary

- $U(\left|\psi_1\right\rangle + \left|\psi_2\right\rangle) = U\left|\psi_1\right\rangle + U\left|\psi_2\right\rangle$
- $aU\left|\psi\right\rangle = U\left|a\psi\right\rangle$
- $UU^\dagger\left|\psi\right\rangle = U^\dagger U\left|\psi\right\rangle = \left|\psi\right\rangle$

22.2 Characteristics of $\mathcal{QFT}$

If $\left|\psi\right\rangle^n \leftrightarrow (c_x)_{x=0}^{N-1}$,

$$\mathcal{QFT}^{(N)}\left|\psi\right\rangle^n \leftrightarrow (\tilde{c}_y)_{y=0}^{N-1}$$

, where $N = 2^n$

$\mathcal{QFT}$ transforms a vector of amplitudes into another vector of amplitudes. Its actions is similar to that of a logic gate.

22.3 Definition 1 of $\mathcal{QFT}$

$$\mathcal{QFT}\left|\psi\right\rangle^n = \sum_{y=0}^{N-1} \tilde{c}\left|y\right\rangle^n, \text{where}$$

$$[\mathcal{QFT}]\left|\psi\right\rangle^n]_y = \tilde{c}_y = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1} c_x\omega^{xy} = {}^n\left\langle y\right| \mathcal{QFT}\left|\psi\right\rangle^n$$

Applying linearity to CBS to find $\mathcal{QFT}$ of arbitrary states

$$\mathcal{QFT}\left|x\right\rangle^n = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}\omega^{yx}\left|y\right\rangle^n$$

22.4 Definition 2 of $\mathcal{QFT}$

$$\mathcal{QFT}\left|\psi\right\rangle^n = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}(\sum_{x=0}^{N-1} c_x\omega^{xy})\left|y\right\rangle^n$$

22.5 Translation Invariance

$$\mathcal{QFT}\left|x-z\right\rangle^n = \omega^{xz}\sum_{y=0}^{N-1}\omega^{xy}\left|y\right\rangle^n = \omega^{xz}\mathcal{QFT}\left|x\right\rangle^n$$

22.6 Comparing Hadamard & $\mathcal{QFT}$

Hadamard Gate: $H^{\otimes n}\left|x\right\rangle^n = (\frac{1}{\sqrt{2}})^n\sum_{y=0}^{2^n-1}\omega_2^{x\cdot y}\left|y\right\rangle^n$, where $\omega_2 = -1$

$\mathcal{QFT}$: $\mathcal{QFT}^{(N)}\left|x\right\rangle^n = (\frac{1}{\sqrt{2}})^n\sum_{y=0}^{2^n-1}\omega^{xy}\left|y\right\rangle$, where $n = log\ N$

Notice that $\mathcal{QFT}$ equals H for $N = 2$

22.7 Quantum Fourier Basis

Basis Conversion Property

If A is an orthonormal basis & U is an unitary operator, UA = B is also an orthonormal basis.

Applying $\mathcal{QFT}^{(2^n)}$ to the preferred z-basis in $\mathcal{H}_{(n)}$ gives the Fourier basis or frequency basis

$$\mathcal{QFT}^{(2^n)}\left|z\right\rangle^n = \left|\tilde{x}\right\rangle^n$$

22.8 Applications

If $H^{\otimes n}$ is used to find $(\mathbb{Z}_2)^n$ periodicity, then $\mathcal{QFT}$ is used to find periodicity of ordinary integer.

22.9 $\mathcal{QFT}$ Circuit

For CBS in the form $x = \sum_{k=0}^{n-1} x_k 2^k$(integer form),

$\sqrt{N}\mathcal{QFT}^{(N)} |x\rangle^n = \sum_{y=0}^{N-1} \Pi_{xy} |y_{n-1}, ..., y_0\rangle$, where $\Pi_{xy} = \prod_{k=0}^{n-1} \omega^{xy_k 2^k}$

22.10 Factoring y-even Group

With $y_0 = 0, \omega^{xy_k 2^k} = \omega^{x \cdot 0 \cdot 1} = 1, \Pi_{xy} = \prod_{k=0}^{n-1} \omega^{xy_k 2^k}$

With a few minor tweaks, y-even group $\sum$ becomes

$$(\sum_{y=0}^{\frac{N}{2}-1}(\prod_{k=0}^{n-1}(\omega^2)^{(x \bmod N/2)y_k 2^k}) |y_{n-2,...,y_0}\rangle) |0\rangle$$

$$= (\sqrt{\frac{N}{2}}\mathcal{QFT}^{(N/2)} |\tilde{x}\rangle^{(n-1)}) |0\rangle \, ; \tilde{x} = x \bmod N/2$$

22.11 Factoring y-odd Group

$$\omega^x(\sum_{y=0}^{\frac{N}{2}-1}(\prod_{k=0}^{n-1}(\omega^2)^{(x \bmod N/2)y_k 2^k}) |y_{n-2,...,y_0}\rangle) |1\rangle$$

$$= \omega^x \left(\sqrt{\tfrac{N}{2}}\mathcal{QFT}^{(N/2)} |\tilde{x}\rangle^{(n-1)}\right) |1\rangle = y\text{-odd group } \sum$$

22.12 Recursion Relation

Combining the odd & even group to get

$$\mathcal{QFT}^{(2^n)} |x\rangle^n = \mathcal{QFT}^{(2^n-1)} |\tilde{x}\rangle^{n-1} (\frac{|0\rangle + \omega_2 |1\rangle}{\sqrt{2}})$$

Reapply the relation to get

$$\mathcal{QFT}^{(2^n)} |x\rangle^n = \prod_{k=1}^{n}(\frac{|0\rangle + \omega^{2^{n-k}x} |1\rangle}{\sqrt{2}}) = \sum_{y=0}^{2^n-1} \omega^{xy} |y\rangle$$

When N=2, $\mathcal{QFT}$ = H

$$H |x_0\rangle = \frac{1}{\sqrt{2}}((-1)^{0 \cdot x_0} |0\rangle + (-1)^{1 \cdot x_0} |1\rangle)$$

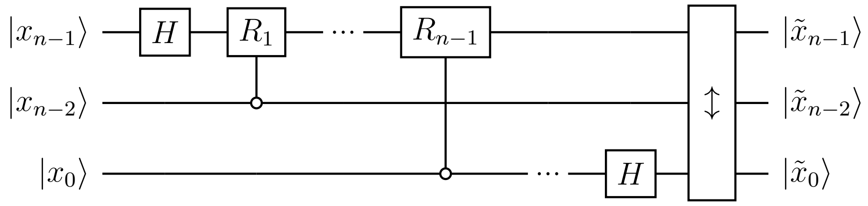
22.13 General Circuit

$$(\frac{|0\rangle + \omega^{2^{n-k}x}|1\rangle}{\sqrt{2}}) = \frac{1}{\sqrt{2}}(|0\rangle + \prod_{k=0}^{k-1} \omega^{x_k}|1\rangle)$$

Notice that if y_0 to $y_{k-2} = 0$ then the state is $H|y_{k-1}\rangle$

To perform the $\omega^{2^{n-k}x}$ multiplications, simply apply R_k gates successively,

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & \omega^{2^{n-k-1}} \end{pmatrix}$$



22.14 Computational Complexity of $\mathcal{QFT}$

The number of gates on each line is a triangle number is so $\exists$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ gates}$$

Both H & R_k is constant-time, so the complexity is $O(n^2)$

$$O(n^2) = O(log^2 N)$$

- This circuit only computes 1 CBS state. Computing N CBS takes $O(N\, log^2 N)$
- The circuit cannot be used to compute $\mathcal{DFT}$ because we need to account for the complex part of the input

Chapter 23

Shor's Algorithm

1. Shor's algorithm for period finding is a relativized exponential speed up because it doesn't include the oracle design.
2. Shor's algorithm for factoring provides a polynomial time oracle design & therefore is an absolute exponential speed up

23.1 Functions of Integers

$$f : \mathbb{Z} \rightarrow \mathcal{S}, \mathcal{S} \subset \mathbb{Z}$$

is periodic injective, if $\exists\, \alpha \in \mathbb{Z} > 0\; s.t$

$\forall x \neq y \in \mathbb{Z}, f(x) = f(y) \Leftrightarrow y = x + k\alpha, k \in \mathbb{Z}$

23.2 Functions of the Group $\mathbb{Z}_M$

$$f : \mathbb{Z}_M \rightarrow \mathcal{S}, \mathcal{S} \subset \mathbb{Z}_M$$

is periodic injective, if $\exists\, \alpha \in \mathbb{Z}_M > 0\; s.t$

$\forall x \neq y \in \mathbb{Z}_M, f(x) = f(y) \Leftrightarrow y = x + k\alpha, k \in \mathbb{Z}_M$

Injective: 1-to-1. Shor's function is injective periodic for disjoint domain.

In Simon's algorithm getting 1 pair gives α (the period), whereas in Shor's, the output is a multiple of α

23.3 Shor's Periodicity Problem

Let $f : \mathbb{Z}_M \rightarrow \mathbb{Z}$ be injective periodic. Find α

M is a bound on α, and is used to determine complexity

Again, the algorithm is relatively easy.

Assumption: conservative $\alpha < M/2$ because the algorithm still works for 1 interval of α

23.4 Domain Size

Let $2^{n-1} < M^2 \leq 2^n, [0, ..., 2^n - 1]$ be the official domain for f, or if $N = 2^n, N/2 < M^2 \leq N$.

$$\text{dom}(f) = \mathbb{Z}_{2^n}$$

23.5 Range Size

For sufficiently large $r > 0$

$$\text{ran}(f) \subseteq [0, 2^r - 1]\text{or} = \mathbb{Z}_{2^r}$$

$$0 \leq f(x) < 2^r$$

All in all,

$$f : [0, N - 1] \to [0, 2^r - 1]$$

23.6 N & M Time Complexity Equivalence

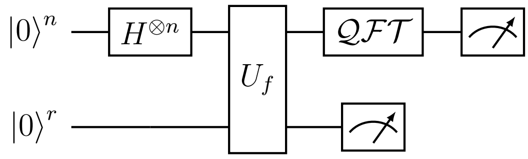
$$O(logN) = O(logM)$$

Bracketing M^2 allows the use of N to compute complexity and later replace with M.

23.7 $\mathbb{Z}_N$ - $(\mathbb{Z}_2)^n$ - CBS

- $|y \oplus f(x)\rangle^r$ is mod-2 sum with r qubits B channel
- $|x + j\alpha\rangle^n$ is ordinary addition of A channel

23.8 The Circuit



- Post-oracle A-register is processed by $\mathcal{QFT}$ instead of $H^{\otimes}n$
- The size of the B register is significantly smaller than B

23.9 Initial State Preparation

- Prepare $|0\rangle^n \otimes |0\rangle^r$ as input
- Quantum parallelism in the data channel
- Generalized Born's rule in the target channel

23.10 The Plan

- Post-oracle state of channel A goes into $\mathcal{QFT}$, which transform it from z-basis to Fourier basis. For $m = N/\alpha$, the frequency associated with period α, we will get a subset of $\{cm\}_{c=0}^{\alpha-1}$, each element with equal probability
- After a few measurements, if cm has a coprime with α, we can use that to find α. This is the easy case.
- However, the measurements will give α close to cm. This is the hard case.

23.11 Hadamard Preparation of the A Register

See Identical Section from Simon's algorithm

23.12 Quantum Oracle

See Identical Section from Simon's algorithm

23.13 GCD & Coprime

$gcd(a, b) =$ largest integer c, with $c \mid a \,\&\, c \mid b$

If $gcd(a, b) = 1 \Rightarrow a \,\&\, b$ are coprime

a ф b : $a \,\&\, b$ are coprime

$\neg a$ ф b : $a \,\&\, b$ are not coprime

23.14 Easy Case $(\alpha|N)$

This implies $\alpha m = N = 2^n \therefore 2^n/\alpha = m \;\therefore \alpha$ is in the form 2^l.

The periodicity can easily be tested by trying $\alpha = \{2^l\}_1^{n-1}$, which has $O(logN)$ complexity. This classical approach is $O(logN)$ relative to the oracle & $O(log^5 N)$ absolute.

23.15 Partitioning Domain into Cosets

Injective periodicity implies disjoint cosets size α with 1-to-1 sub-domain for f. In the easy case, there are exactly m cosets in the interval $[0, N)$.

$$[0, N-1] = [0, \alpha - 1] \cup ... \cup [(m-1)\alpha,\ m\alpha - 1]$$
$$= R \cup ... \cup R + (m-1)\alpha \text{ where}$$

$R = \{0, ..., \alpha - 1\}, \alpha$: period of f, $m = N/\alpha$

$R = j\alpha$ is the jth coset of R

$$= \bigcup_{j=0}^{m-1} \{x + j\alpha\}_{x=0}^{a-1}$$

23.16 Output of the Oracle

$\{x + j\alpha\}_{j=0}^{m-1} \to f(x) \Rightarrow \sqrt{\frac{m}{N}} \sum_{x=0}^{a-1} (\frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} |x + j\alpha\rangle^n) \, |f(x)\rangle^r$

Collapsing the B channel produce a superposition of m xs.

$$\searrow (\frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} |x_0 + j\alpha\rangle^n) \, |x_0\rangle^r$$

Fed into the $\mathcal{QFT}$ & we get non-zero amplitudes that are multiples of the frequency m.

23.17 Effect of $\mathcal{QFT}$ on A Register

$$|\psi_0\rangle \; -\boxed{\mathcal{QFT}^{(N)}}- \; \frac{1}{\sqrt{mN}} \sum_{y=0}^{N-1} \omega^{x_0 y} (\sum_{j=0}^{m-1} \omega^{j\alpha y}) \, |y\rangle^n$$

The measurement of the B channel is conceptual, so it makes the expression simpler, but won't affect the probability of A.

23.18 Easy Case Measurements Probability

1. Identify the set $C = \{y_c = cm\}_{c=0}^{a-1}$
2. Observe that each $y_c = cm$ has the same P
3. Prove that $\{c\}_{c=0}^{\alpha-1}$ ϕ α 50% of the time
4. $y = cm$ with the above c value occurs 50% of the time
5. Measure the above $y = cm$ with arbitrarily high confidence in constant time

23.19 Step 1

$$|\psi_0\rangle \; -\boxed{\mathcal{QFT}^{(N)}}- \; \frac{1}{\sqrt{mN}} \sum_{y=0}^{N-1} \omega^{x_0 y} (\sum_{j=0}^{m-1} \omega^{j\alpha y}) \, |y\rangle^n$$

Because $\omega_N = 1 = \omega^{am} = (\omega^a)^m \Rightarrow \omega^a = \omega_m$

$$\sum_{j=0}^{m-1} \omega^{j\alpha y} = \sum_{j=0}^{m-1} \omega_m^{jy} = \begin{cases} m, \text{if } y = 0 \text{ (mod m)} \\ 0, \text{if}, y \neq 0 \text{ (mod m)} \end{cases}$$

$$|\psi_0\rangle \; -\boxed{\mathcal{QFT}^{(N)}}- \; \sqrt{\frac{m}{N}} \sum_{y=0 \text{ (mod m)}} \omega^{x_0 y} \, |y\rangle^n$$

, in which

$y \equiv 0 \,(\text{mod m}) \Leftrightarrow y = \{cm\}_{c=0}^{\alpha-1}$

23.20 Step 2

The normalization factor $\sqrt{\frac{m}{N}}$ turns into $\frac{1}{\sqrt{\alpha}}$ so each cm has probability of $1/\alpha$.

After $\mathcal{QFT}$ is applied, the only possible measurements are in the frequency domain.

23.21 Step 3

$N = 2^n = m\alpha \Rightarrow \alpha = 2^l \therefore c \phi \alpha$ only if c is odd

$$P(c \phi \alpha) = \frac{\#\text{odds} < 2^l}{2^l} = \frac{2^{l-1}}{2^l} = \frac{1}{2}$$

23.22 Step 4

For $\mathcal{B} = \{y_0 | y_b = \{bm\}_{b=0}^{\alpha-1} \text{ and } b \phi \alpha\}$ & $\mathcal{C} = \{cm\}_{c=0}^{\alpha-1}$

$$P(\mathcal{B}) = P(\mathcal{B}|\mathcal{C})P(\mathcal{C}) = \frac{1}{2}$$

23.23 Step 5

Because the probability of success $P(\mathcal{B})$ is independent of N, from the CTC Theorem for Looping Algorithm, we have

$$T = \lfloor \frac{log(\mathscr{E})}{log(1-\frac{1}{2})} \rfloor + 1 : \text{with arbitrary } \mathscr{E} \text{ tolerance}$$

From this, P(getting at least 1 $c \phi \alpha$) $= 1 - (\frac{1}{2})^T$

23.24 Complexity of Euclidean Algorithm

Euclidean Algorithm takes $P > Q$ and find $gcd(P, Q)$

Use EA to find $m' = gcd(N, M)$ in $O(log^3 N)$ time

$c \phi \alpha \Rightarrow m' = gcd(N, cm) = gcd(\alpha m, cm) = m$

From that $\alpha = N/m$

However, it is unclear if $c \phi \alpha$

23.25 Full Easy Case Algorithm

- Select T as number of loops that satisfies $\mathscr{E}$ tolerance
- Repeat loop T times
  1. Apply Shor's circuit
  2. Measure output of $\mathcal{QFT}$ and get cm
  3. Comput $m' = EA(N, cm)$ and set $\alpha' = N/m'$
  4. Test a': If $f(1+\alpha') = f(1)$ then $\alpha' = \alpha$ (success), break
  5. Else continue next pass
  6. If loop ends naturally after T passes, we fail

23.26 Full Easy Case Complexity

The full complexity will be in M since $O(logN) = O(logM)$

- Hadamard: $O(logM)$
- $\mathcal{QFT}$: $O(log^2M)$
- U_f: at least $O(log^4M)$ for RSA factoring
- Outer loop: $O(T) = O(1)$
- Classical EA: $O(log^3M)$

These are done in series so they are $O(log^3M)$ relative to the oracle & $O(log^4M)$ absolute for RSA oracle.

The classical algorithm is deterministic $O(log^5M)$ for this easy case, whereas the quantum algorithm is probabilistic $O(log^4M)$

23.27 General Case

This is when $\alpha|N$ is not implied.

$N = 2^n = m\alpha +$ excess part where $m = \lfloor N/\alpha \rfloor$

23.28 Partitioning Domain into Cosets

$[0, N-1]$ is partitioned in to m cosets plus extra

$$= [0,\ \alpha - 1] \cup .... \cup [(m-1)\alpha,\ m\alpha - 1] \cup [m\alpha,\ N-1]$$

$$R \cup ... \cup R + (m-1)\alpha \cup \langle R + m\alpha \rangle \text{where}$$

$R = \{0, ..., \alpha - 1\}$ $a =$ period of f $m = \lfloor M/a \rfloor$

$R + j\alpha$ is still the jth coset of R, but now

$\langle R + m\alpha \rangle \subset [R + m\alpha]$(partial mth coset of R)

$[0, N-1] = \cup_{j=0}^{m-1} \{x + j\alpha\}_{x=0}^{\alpha-1} \cup \{x + m\alpha\}_{x=0}^{N-m\alpha-1}$

To deal with the partial coset, for some of the $f(x)$, there will be $x+1$ pre-images, therefore let $\tilde{m}$ be m or $m+1$

$$\tilde{m} = \begin{cases} m+1, \text{first few x in } [0, \alpha-1] \\ m, \text{remaining x in } [0, \alpha-1] \end{cases}$$

23.29 Output of the Oracle

With $\tilde{m}$, the output becomes

$$\approx \sqrt{\frac{\tilde{m}}{N}} \sum_{x=0}^{\alpha-1} (\frac{1}{\sqrt{\tilde{m}}} \sum_{j=0}^{\tilde{}-1} |x + j\alpha\rangle^n) \, |f(x)\rangle^r$$

The inside $\tilde{m}$ normalize each state whereas the outside is undefined between $\frac{m}{N}$&$\frac{m+1}{N}$

23.30 Hypothetical B-Measurement

The B channel values are collapsed, giving

$$|\psi_{x_0}\rangle^n = (\frac{1}{\sqrt{\tilde{m}}} \sum_{j=0}^{\tilde{m}-1} |x_0 + j\alpha\rangle^n)$$

Although f is a pure periodic function, because of $\tilde{m}$, the $\mathcal{QFT}$ measurements no longer yields exclusively $\{cm\}_{c=0}^{\alpha-1}$

However, all values with high probability appears close to $\{cm\}_{c=0}^{\alpha-1}$ in the frequency domain

23.31 Effect of the Final $\mathcal{QFT}$ on A-Register

The $\mathcal{QFT}$ state after B-collapse is:

B-measurement picked out a x_0 so $\frac{1}{\tilde{m}N}$ is precise

23.32 General Case Final Measurement Probabilities

Because $\tilde{m} \neq m$, we can't use the fact that $\omega^\alpha = \omega_n$

To show that $y_c \neq cm$ leads to a cm, from which $c \phi \alpha$ with good probability can be found, follow these steps:

1. Identify a set of $C = \{y_c\}_{c=0}^{\alpha-1}$ with high likelihood
2. Prove that $\{y_c\}_{c=0}^{\alpha-1}$ have high measurement likelihood

3. Associate $\{y_c\}_{c=0}^{\alpha-1}$ with $\{c/\alpha\}_{c=0}^{\alpha-1}$
4. Describe an $O(log^3 N)$ algorithm that produces c/α from y_c
5. Observe that y_c associated with $c \,\upphi\, \alpha$ measures in $O(1)$

23.33 Step 1: Identify set of $\{y_c\}_{c=0}^{\alpha-1}$

For $y = \{y_c\}_{c=0}^{\alpha-1}, y_c \cdot \alpha \in [cN - \frac{\alpha}{2}, cN + \frac{\alpha}{2}]$

With probability $P = 1 - \mathscr{E}$, one of the y_c can be measured in constant time T, independent of N.

The set of $\hat{y}_c$ is in $[-\frac{\alpha}{2}, \frac{\alpha}{2}]$ and is associated with y_c

$$\hat{y}_c = y_c \cdot \alpha - cN \in [-\frac{\alpha}{2}, +\frac{\alpha}{2}) = y_c \cdot \alpha \,(\text{mod N}))$$

23.34 Step 2: Prove that $\{y_c\}_{c=0}^{\alpha-1}$ have high likelihood

The A channel state is: $\frac{1}{\sqrt{\tilde{m}N}} \sum_{y=0}^{N-1} \omega^{x_0 y} (\sum_{j=0}^{\tilde{m}-1} \omega^{j\alpha y} \left|y\right\rangle^n)$

$$P(\searrow \left|y\right\rangle) = \frac{1}{\tilde{m}N} |\omega^{x_0 y}|^2 \left|\sum_{j=0}^{\tilde{m}-1} \omega^{j\alpha y}\right|^2 = \frac{1}{\tilde{m}N} \left|\sum_{j=0}^{\tilde{m}-1} \omega^{j\alpha y}\right|^2$$

Let $\mu = \omega^{\alpha y}, \sum_{j=0}^{\tilde{m}-1} \mu^j = \frac{\mu^{\tilde{m}}-1}{\mu-1} = \frac{e^{i\theta_y \tilde{m}-1}}{e^{i\theta_y}-1}$

$$P(\searrow \left|y\right\rangle) = \frac{1}{\tilde{m}N} \left|\frac{e^{i\theta_y \tilde{m}-1}}{e^{i\theta_y}-1}\right|^2, \text{where } \theta_y = \frac{2\pi\alpha y}{N}$$

23.35 Estimating Bounds for $|e^{i\phi} - 1|$

$|e^{i\phi}| \leq |\phi|$ because the chord length is always less than or equal to the arc length of a circle

$|e^{i\phi} - 1| = 2|\sin(\frac{\phi}{2})|$ using Euler's identity and double angle formula of $\phi = 2\frac{\phi}{2}$

Bounding this, we get $\frac{2|\phi|}{\pi} \leq |e^{i\phi} - 1|$

Combining both sides, $\frac{2|\phi|}{\pi} \leq |e^{i\phi} - 1| \leq |\phi \,\forall\, \phi \in [-\pi, \pi]$

$$P(\searrow \left|y_c\right\rangle) = |\frac{e^{i\theta\hat{y}_c^{\hat{m}/\alpha}} - 1}{e^{i\theta\hat{y}_c/\alpha} - 1}|^2, \hat{y}_c = y_\alpha \alpha - cN \in [-\frac{\alpha}{2}, \frac{\alpha}{2})$$

To find the lower bound of this probability, find the lower bound of the numerator & upper bound of the denominator

23.36 Estimating Probability of Step 2

The upper bound of the denominator comes from previous derivations

$$|e^{i\theta\hat{y}_c/\alpha}-1|\leq\frac{\theta\hat{y}_c}{\alpha}|$$

The lower bound of the numerator when $\tilde{m}=m$

With $2\pi m\hat{y}_c/N\in(-\pi,\pi)$, previous derivations give

$$\frac{2|\frac{m}{\alpha}\theta_{\hat{y}_c}|}{\pi}\leq|e^{i\theta_{\hat{y}_c}\tilde{m}/\alpha}|\therefore P(\searrow|\hat{y}_c\rangle)=\frac{2m}{\pi}$$

Follow this assumption for $\tilde{m}=m+1$

$$P(\searrow\text{one }y_c)>\frac{2}{\pi^2}\text{or}\frac{4}{\pi^2}-\mathscr{E}\text{as }m\rightarrow\infty$$

This satisfies CTC looping theorem. However, the assumption is invalid

So for $\tilde{m}=m+1,-\frac{3\pi}{2}<\frac{2\pi(m+1)}{N}\hat{y}_c<\frac{3\pi}{2}$

Using the same technique as before,

$$K\frac{2|\phi|}{\pi}\leq 2|\sin(\frac{\phi}{2})|,\forall\,\phi\in[-\frac{3\pi}{2},\frac{3\pi}{2}]$$

where $K=\frac{2}{3}\sin\frac{\pi}{4}=\frac{\sqrt{2}}{3}$

Substituting in $\phi=\frac{2\pi(m+1)\hat{y}_c}{N},\&\,\theta_y=\frac{2\pi\alpha y}{N}$,

The amplitude becomes $\frac{2K\tilde{m}}{\pi},\tilde{m}=m+1$

Combining for both $\tilde{m}=m\,\&\,\tilde{m}=m+1$, $\frac{2K\tilde{m}}{\pi}$ is the stronger bound

$$P(\searrow y_c)\geq\frac{1}{\tilde{m}N}\frac{4K^2\tilde{m}^2}{\pi^2}\geq\frac{m}{N}\frac{4K^2}{\pi^2}$$

$$P(\searrow y_c)\geq\frac{\alpha m}{N}\frac{4K^2}{\pi^2}>\frac{1}{2}\frac{4K^2}{\pi^2}=\frac{2k^2}{\pi^2}\approx 4.5\%$$

This is the conservative estimate

However, usually, when $\alpha<<M<N$, $\frac{m+1}{N}\approx\frac{m}{N}$

$$P(\searrow 1y_c)>\frac{4}{\pi^2}-\mathscr{E},\text{where }\mathscr{E}\rightarrow 0,m\rightarrow\infty$$

In general, $P(\searrow 1y_c)>\begin{cases}4.5\%,3\alpha>M\text{ worst case}\\40.5\%,\text{usually}\end{cases}$

Because P is independent of M, CTC theorem can be applied to get y_c within T loops ($\mathscr{E}$-tolerance)

23.37 Step 3: Associate $\{y_c\}_{c=0}^{\alpha-1}$ with $\{c/\alpha\}_{c=0}^{\alpha-1}$

Each $\frac{y_c}{N}$ is uniquely close to $\frac{c}{\alpha}$. If c is coprime to α, we can compute m & find α.

For each $[cN \pm \frac{\alpha}{2})$ around $y_c\alpha$, we have $|\frac{c}{\alpha} - \frac{y_c}{N}| \leq \frac{1}{2M^2}$

If $p, q \in \mathbb{R}\; s.t\; 0 < p, q \leq M, \forall\, \frac{l}{p}\; \&\; \frac{k}{q}, |\frac{l}{p} - \frac{k}{q}| \geq \frac{1}{M^2}$

$\therefore c/\alpha$ is closer to y_c/N than any n/d for $d \leq M$

y_c/N is uniquely close to $c/\alpha \Rightarrow y_c$ is uniquely close to $c\frac{N}{\alpha}$

where N/α is the frequency of f between $m\;\&\;m+1$

Here, $\{y_c\}$ is uniquely associated with $\{c \cdot \text{freq}\}$

23.38 Step 4: CFA that Produces c/α from y_c

Continued fraction algorithm takes $x \in \mathbb{R}$ and produce a sequence of $\{n_k/d_k\}$ that approaches x.

The $c \leftrightarrow y_c$ may not satisfies $c \,\phi\, \alpha$. $\frac{c}{\alpha} = \frac{n}{d}$ but $\frac{c}{\alpha}$ may not be in the most reduced form $\therefore$ we can't imply $c = n\;\&\;\alpha = d$

CFA can be used to produce a unique $\frac{c}{a}$ closest to $\frac{y_c}{N}$

Properties of CFA

1. CFA produces a reduced fraction $\frac{n_k}{d_k}$ at the end of kth iteration. $\frac{n_k}{d_k}$ is the kth convergent of x.

   Let $x = a_0 + \frac{1}{a_1 \ldots + \frac{1}{a_n}}$

   Each kth convergent expand up to $a_{n=k}$

2. $\forall\, x \in \mathbb{R}$, $lim_{k\to\infty}(\frac{n_k}{d_k}) = x$

3. For rational x, k is finite

4. Shor's CFA generated $\frac{n}{d}$ within $\mathscr{E}$ of x. In some case, n/d converges exactly

5. Denominators $\{d_k\}$ is strictly increasing $\forall\; x \in \mathbb{R}$ and $\leq$ denominator of x regardless of form

6. If $|\frac{n}{d} - x| < \frac{1}{d^2}$, then $\frac{n}{d} \in \left(\frac{n_k}{d_k}\right)_{k=0}^{k}$

   $\frac{n}{d}$ appears in the convergents

7. When $x = \frac{p}{q}$, CFA completes in $O(log^3 q)$

Using CFA to Produce c/a

Apply CFA to $x = y_c/N\;\&\;\mathscr{E} = \frac{1}{(2M^2)}$

$\alpha < M \therefore |\frac{c}{\alpha} - \frac{y_c}{N}| < \frac{1}{2\alpha^2}$

Since $\frac{c}{\alpha}$ is closest to $\frac{y_c}{N}\;\&\;\frac{c}{\alpha}$ is in the convergent, CFA terminates in c/α for $d_k \leq M$.

23.39 Step 5: Measuring $y_c \leftrightarrow c \,\phi\, \alpha$ in Constant Time

To show this:

- Find the ratio between the least likely & most likely y_c

To do this, bound the amplitude for the most likely y_c

Numerator upper bound: $|e^{i\theta\hat{y}_c\tilde{m}/a} - 1| \leq |\frac{\theta\hat{y}_c\tilde{m}}{a}|$

Denominator lower bound: $|e^{i\theta\hat{y}_c\tilde{m}/a} - 1| \leq L\frac{2|\theta\hat{y}_c\tilde{m}/a|}{\pi}$

Combine top & bottom, $P(\searrow y_c) \leq \frac{1}{\tilde{m}N}\frac{4L^2\tilde{m}^2}{\pi^2} \leq \frac{m+1}{N}\frac{4L^2}{\pi^2}$

With least likely probability $P(\searrow y_c) \geq \frac{m}{N}\frac{4K^2}{\pi^2}$,

$$\frac{P(\text{least likely } y_c)}{P(\text{most likely } y_c)} \geq \frac{mK^2}{(m+1)L^2} \geq \frac{2K^2}{3L^2} \approx 7.2\%$$

If $a << M < N, K, L \rightarrow 1$ & $\frac{m}{m+1} \rightarrow\ 1 \ \therefore P \geq 1 - \mathscr{E}$ usually

- Show that P($c \,\phi\, \alpha$) is constant (c, α random)

$P(c \,\phi\, \alpha) = P(\neg(2|c \wedge 2|\alpha) \wedge ... \wedge \neg(p_k|c \wedge p_k|\alpha))$,

where $p_k =$ kth prime, $= \prod_{p \in \text{ prime}}^{\text{finite}} P(\neg(p|c \wedge p|\alpha))$

With $P(p|c) = \frac{1}{p} \ \therefore\ P(\neg(p|c \wedge p|\alpha)) = 1 - \frac{1}{p^2}$

$$P(c \,\phi\, \alpha) \geq \prod_{p \,\in\, \text{prime}}^{\infty} (1 - \frac{1}{p^2}) = \zeta(2) \approx 60.7\%$$

Finally, to prove that $y_c \leftrightarrow c \,\phi\, \alpha$ in constant time, find lower bound of $P(\mathcal{B})$, where $\mathcal{B} = \{y_b |\ y_b \in \mathcal{C}\ \&\ b \,\phi\, \alpha\}$

$$P(\mathcal{B}) = P(\mathcal{B}|\mathcal{C})P(\mathcal{C}) = \sum_{b \,\in\, \mathcal{B}} P(y_b) / \sum_{c \,\in\, \mathcal{C}} P(y_c) \geq \frac{|\mathcal{B}|P(y_c \text{ min})}{|\mathcal{C}|P(y_c \text{ max})} \geq q \times .072$$

The conservative lower bound gives $P(\mathcal{B}) \approx 2\%$

Normally, $P(\mathcal{B}) \approx 26.6\%$

Because P is independent of N, we can invoke CTC theorem

In some instances, it is possible to get $d = a$ when $y_c \notin \mathcal{B}$, fix this by testing for $f(1) = f(1 + d)$ everytime

23.40 Full Algorithm

- Select T based on $\mathscr{E}$ according to number of α in [0, N-1]
- Repeat loop at most T times

1. Apply Shor's circuit
2. Measure $\mathcal{QFT}$ to get y
3. Apply CFA to y/N with $\mathscr{E} = \frac{1}{2M^2}$ to get $\frac{n}{d}$
4. Test $f(1) = f(1+d)$, if true $d = a$ (success) break loop
5. Otherwise next pass

- If loop end after T passes, failed. Else, α is found

23.41 Full General Case Complexity

- Hadamard: $O(logM)$
- $\mathcal{QFT}$: $O(log^2 M)$
- U_f: at least $O(log^4 M)$ for RSA factoring
- Outer loop: $O(T) = O(1)$
- Classical CFA: $O(log^3 M)$

These are done in series so they are $O(log^3 M)$ relative to the oracle & $O(log^4 M)$ absolute for RSA oracle.

Chapter 24

Euclidean Algorithm & Continued Fractions

24.1 Long Division

LDA does $A = qB + r$ in $O(log^2X)$, where $X = \max\{A, B\}$

24.2 Euclidean Algorithm EA(P, Q) $P > Q$

To compute $EA(P, Q)$, set $P, Q = r_0, r_1$

$$r_k = q_k \cdot r_{k+1} + r_{k+2}$$

When $r_{k+2} = 0, gcd = r_{k+1}$

EA has $logX$ loops with log^2X LDA within each loop making it log^3X

24.3 Continued Fractions Algorithm CFA

The numerator of x as a continued fraction is $\{a_k\}$

$\forall\, x = P/Q, \{a_k\} = \{q_k\}$ (From EA)

Alternatively, $x = a_0$; $a_n = (\lfloor a_n - 1 \rfloor - a_{m-1})^{-1}$

If the fraction = 0, break

24.4 Convergents

The kth convergent of x is the continued fraction up to a_k

The convergences zigzag from x. and can be written as $a_k = n_k/d_k$. To compute the kth convergence,

Set $n_0, n_1 = a_0, a_1a_0 + 1$ & $d_0, d_1 = 1, a_1$

Then, repeat for both n & k: $n = a_kn_{k-1} + n_{k+2}$

24.5 Properties of Convergences

1. $\forall\, x \in \mathbb{R}, lim_{x\to\infty}\frac{n_k}{d_k} = x$
2. For rational x, k is finite
3. Assuming $\frac{n_k}{d_k} \neq x$, $\frac{n_k}{d_k} = \begin{cases} > x, \text{odd k} \\ < x, \text{even k} \end{cases}$
4. $|x - \frac{n_k}{d_k}| \leq \frac{1}{d_k d_{k+1}}$(error bound)
5. $|x - \frac{n_k}{d_k}| \leq |x - \frac{n}{d}|\ \forall\, d \leq d_k$ (best estimate)
6. $|\frac{n_{k-1}}{d_{k-1}} - \frac{n_k}{d_k}| = \frac{1}{d_k d_{k+1}}$(bounds between convergents)
7. $d_{k+1} > d_k$ & rational $x \leq \{d_k\}$
8. If $|\frac{n}{d} - x| < \frac{1}{2d^2}, \frac{n}{d} = \frac{n_{k_0}}{d_{k_0}} \in \left\{\frac{n_k}{d_k}\right\}_{k=0}^{k}$

24.6 CFA Complexity

Shor's CFA uses property 4 with $\mathscr{E} = \frac{1}{d^2}$

Because CFA is based on EA, the time complexity is $O(log^3 X)$