Day 4: Quantum Algorithm

Recall from yesterday

Superdense Coding

Teleportation

BB84

} Communicat Protocols

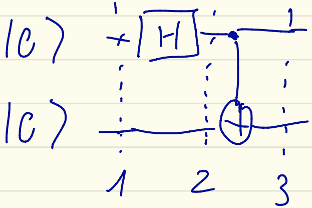
↳ Transport information using quantum phenomena

$|ab\rangle = |a\rangle \otimes |b\rangle$

· Superposition (Linear Combination)

$\underline{H|0\rangle} = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ $\qquad$ $H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

· Entanglement



At 1, $|00\rangle$

At 2, $(H \otimes I)|0\rangle|0\rangle = H|0\rangle \otimes I|0\rangle$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

At 3, $CX \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) = \frac{1}{\sqrt{2}}\left(CX|00\rangle + |10\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Quantum Algorithms

- Provide speed-up in computation using quantum phenomena

Computation: also called calculations, perform some operations on information

- Function: takes value input space -> output space

$$x \rightarrow \boxed{f} \rightarrow f(x)$$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

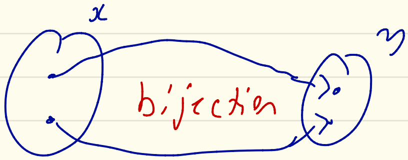
$$x = 001101 \rightarrow \text{odd}$$

$$001101 \rightarrow \boxed{f} \rightarrow 1$$

Classical Function

Quantum Function / Oracle

- unitary $\quad U^{-1} = U^{\dagger} \qquad U^{\dagger}U = UU^{\dagger} = I$
- reversible



$$y = f(a) \iff x = a$$

- input size = output size

Implement a QF is w/ an oracle

Today 2-qubit oracle

unitary

Input bit $|x\rangle$ — U_f — $|x\rangle$

Target bit $|y\rangle$ — U_f — $|y \oplus f(x)\rangle$

XOR

a	b	a ⊕ b
0	0	0
0	1	1
1	0	1
1	1	0

A classical 1-bit function (unary function)

$$f : \{0, 1\} \to \{0, 1\}$$

x	$a(x) = 0$	$b(x) = 1$	$c(x) = x$	$d(x) = \overline{x}$ (not)
0	0	1	0	1
1	0	1	1	0
	constant -[0]	constant -[1]	identity	negation (not)

$\underbrace{a(x),\ b(x)}_{\text{Constant}}$ $\qquad$ $\underbrace{c(x),\ d(x)}_{\text{balanced}}$

$$f(x) = \underline{a(x)} = 0$$

$q_0\ |x\rangle$ — U_f — $|x\rangle$

$q_1\ |0\rangle$ — U_f — $|y \oplus f(x)\rangle$

$0 \oplus 0$

$|q_0 q_1\rangle$

let $x = 0$

$|00\rangle \to |00\rangle$

let $x = 1$

$|10\rangle \to |10\rangle$

Problem: Constant vs Balanced

Statement: Given a classical unary, determine whether it's constant or balanced

$$x \;-\boxed{f}-\; f(x)$$

0	0	[C]	or	I	} 2 passes
1	1				
1	0	[C]	or	X	} 2 passes
0	0	[C]			

In quantum world, we can do this in 1 pass

Deutsch's Algorithm

- XOR Identities

$$0 \oplus a = a \,, \quad 1 \oplus a = \bar{a}$$

- X basis vectors

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

- Oracle Action

$$U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$$

Fix y to a value $\quad U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$

- $y = 0$

$$U_f|x\rangle|0\rangle = |x\rangle|0 \oplus f(x)\rangle$$
$$= |x\rangle|f(x)\rangle$$

- $y = 1$

$$U_f|x\rangle|1\rangle = |x\rangle|1 \oplus f(x)\rangle$$
$$= |x\rangle|\overline{f(x)}\rangle$$

- $y = -$

$$U_f|x\rangle|-\rangle = U_f|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$= \frac{1}{\sqrt{2}}U_f|x\rangle|0\rangle - \frac{1}{\sqrt{2}}U_f|x\rangle|1\rangle$$
$$= \frac{1}{\sqrt{2}}|x\rangle|f(x)\rangle - \frac{1}{\sqrt{2}}U_f|x\rangle|\overline{f(x)}\rangle$$

$$U_f |x\rangle|-\rangle = \frac{1}{\sqrt{2}} |x\rangle|f(x)\rangle - \frac{1}{\sqrt{2}} U_f |x\rangle|\overline{f(x)}\rangle$$

Observations:

- If $f(x) = 0$

$$U_f |x\rangle|-\rangle = \frac{1}{\sqrt{2}} |x\rangle|0\rangle - \frac{1}{\sqrt{2}} |x\rangle|1\rangle$$

- If $f(x) = 1$

$$U_f |x\rangle|-\rangle = \frac{1}{\sqrt{2}} |x\rangle|1\rangle - \frac{1}{\sqrt{2}} |x\rangle|0\rangle$$

For an arbitrary $f(x)$

$$U_f |x\rangle|-\rangle = (-1)^{f(x)} \left(\frac{1}{\sqrt{2}} |x\rangle|0\rangle - \frac{1}{\sqrt{2}} |x\rangle|1\rangle \right)$$

$$= (-1)^{f(x)} |x\rangle \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right)$$

- phase kickback

encode $f(x)$ into phase

$$= (-1)^{f(x)} |x\rangle |-\rangle$$

$$U_f |x\rangle |-\rangle = (-1)^{f(x)} |x\rangle |-\rangle$$

$$a^{bc} = (a^b)^c$$

$$\begin{aligned}
(-1)^{f(x)} U_f |x\rangle |-\rangle &= (-1)^{f(x)} \cdot (-1)^{f(x)} |x\rangle |-\rangle \\
&= (-1)^{2f(x)} |x\rangle |-\rangle \\
&= \left[(-1)^2\right]^{f(x)} |x\rangle |-\rangle \\
&= |x\rangle |-\rangle
\end{aligned}$$

$|0\rangle$

$|1\rangle$

$-\boxed{f}-$

$$U_f |x\rangle|-\rangle = (-1)^{f(x)} |x\rangle|-\rangle$$

Fix x, let $y = |-\rangle$

- $x = 0$

$$U_f |0\rangle|-\rangle = (-1)^{f(0)} |0\rangle|-\rangle$$

- $x = 1$

$$U_f |1\rangle|-\rangle = (-1)^{f(1)} |1\rangle|-\rangle$$

- $x = +$

$$U_f |+\rangle|-\rangle = \frac{1}{\sqrt{2}} U_f |0\rangle|-\rangle + \frac{1}{\sqrt{2}} U_f |1\rangle|-\rangle$$

$$= \frac{(-1)^{f(0)}}{\sqrt{2}} |0\rangle|-\rangle + \frac{(-1)^{f(1)}}{\sqrt{2}} |1\rangle|-\rangle$$

$$= \left[\frac{(-1)^{f(0)}}{\sqrt{2}} |0\rangle + \frac{(-1)^{f(1)}}{\sqrt{2}} |1\rangle \right] |-\rangle$$

$$U_f|+\rangle|-\rangle = \left[\frac{(-1)^{\cancel{f(0)}\to 0}}{\sqrt{2}}|0\rangle + \frac{(-1)^{\cancel{f(1)}\to 0}}{\sqrt{2}}|1\rangle\right]|-\rangle$$

Recall that $f(x)$ is a unary function

$$f: \{0,1\} \to \{0,1\}$$

$H|+\rangle = |0\rangle$

$H|-\rangle = |1\rangle$

- $[0]$: $f(x) = 0 \to U_f|+\rangle|-\rangle = |+\rangle|-\rangle$
- $[1]$: $f(x) = 1 \to U_f|+\rangle|-\rangle = -|+\rangle|-\rangle$

$\Big\}$ constant $|+\rangle$

- I : $f(x) = x \to U_f|+\rangle|-\rangle = |-\rangle|-\rangle$
- X : $f(x) = \bar{x} \to U_f|+\rangle|-\rangle = -|-\rangle|-\rangle$

$\Big\}$ balanced $|-\rangle$

If $f(x) = 0$, then

$$U_f|+\rangle|-\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|-\rangle$$

$$= \underline{|+\rangle|-\rangle}$$

$$U_f|+\rangle|-\rangle = \left[\frac{(-1)^{\cancel{f(0)}\to 0}}{\sqrt{2}}|0\rangle + \frac{(-1)^{\cancel{f(1)}\to 1}}{\sqrt{2}}|1\rangle\right]|-\rangle$$

If $f(x) = 1$

$$U_f|+\rangle|-\rangle = -\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|-\rangle$$

$$= -|+\rangle|-\rangle$$

↳ global phase

If $f(x) = x$

$$U_f|+\rangle|-\rangle = \left[\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right]|-\rangle$$

$$= |-\rangle|-\rangle$$

$$U_f |+\rangle|-\rangle = \left[\frac{(-1)^{f(0)}}{\sqrt{2}} |0\rangle + \frac{(-1)^{f(1)}}{\sqrt{2}} |1\rangle \right] |-\rangle$$

If $f(x) = \bar{x}$

$$U_f |+\rangle|-\rangle = \underbrace{-\left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right)}|-\rangle$$

$$= - |-\rangle|-\rangle$$

↳ global phase

Recall that $f(x)$ is a unary function

$$f : \{0,1\} \to \{0,1\}$$

$H|+\rangle = |0\rangle$

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- $[0]$: $f(x) = 0 \to U_f|+\rangle|-\rangle = |+\rangle|-\rangle$
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constant $|+\rangle$

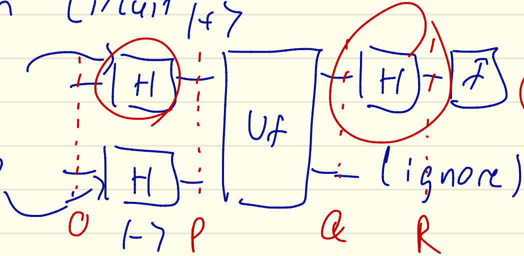
- I : $f(x) = x \to U_f|+\rangle|-\rangle = |-\rangle|-\rangle$
- X : $f(x) = \bar{x} \to U_f|+\rangle|-\rangle = -|-\rangle|-\rangle$

balanced $|-\rangle$

Deutsch's Algorithm Circuit

Input Bit $|0\rangle$

Target Bit $|1\rangle$



$|0\rangle \to$ constant

$|1\rangle \to$ balanced

- Quantum Parallelism
- Phase kickback

At O $|0\rangle|1\rangle$

At P $|+\rangle|-\rangle$

At Q, $U_f|+\rangle|-\rangle$

- $[0]$: $f(x) = 0 \to U_f |+\rangle|-\rangle = |+\rangle|-\rangle$
- $[1]$: $f(x) = 1 \to U_f |+\rangle|-\rangle = -|+\rangle|-\rangle$

constant $|+\rangle$

- I : $f(x) = x \to U_f |+\rangle|-\rangle = |-\rangle|-\rangle$
- X : $f(x) = \bar{x} \to U_f |+\rangle|-\rangle = -|-\rangle|-\rangle$

balanced $|-\rangle$

At Q, if $f(x) = [1]$

$U_f |+\rangle|-\rangle = - |+\rangle|-\rangle$

Slou bar

$$H|+\rangle = H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$
$$= \frac{1}{\sqrt{2}} H|0\rangle + \frac{1}{\sqrt{2}} H|1\rangle$$

$-(H \otimes I)|+\rangle|-\rangle = - H|+\rangle \otimes I|-\rangle$

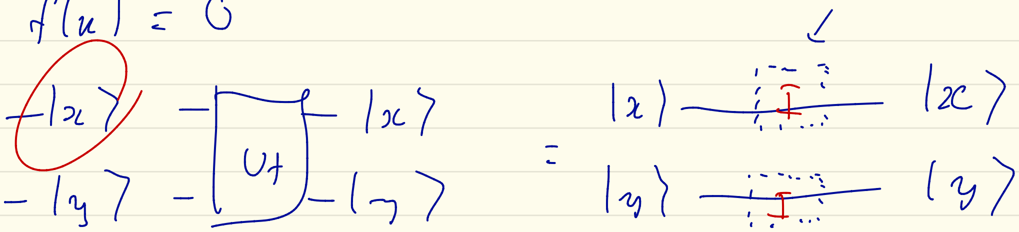
$H|-\rangle \to 1$

$= -|0\rangle|-\rangle$

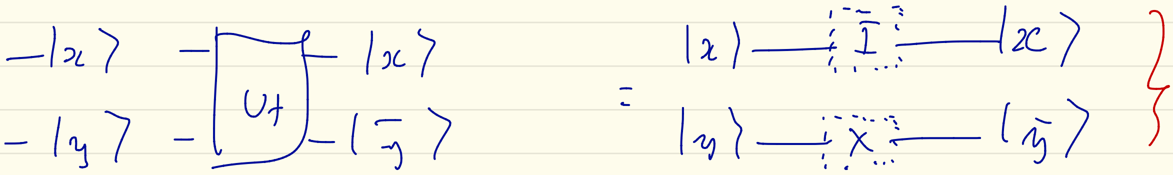
$|1\rangle|-\rangle$

1

[0] $f(x) = 0$



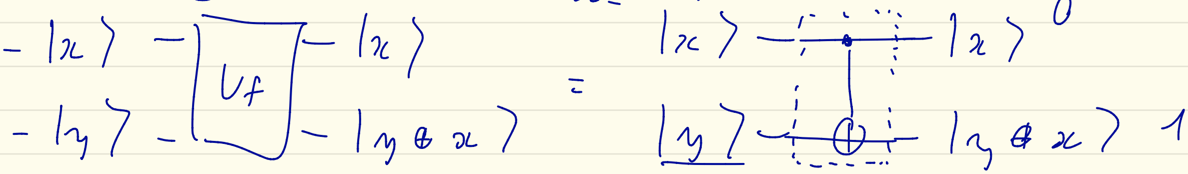
[1] $f(x) = 1$



I $f(x) = x$

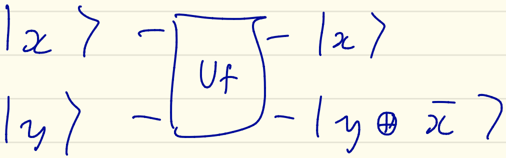
$x = 0 \quad y = 0 \qquad x = 0 \quad y = 1$

$x = 1 \quad y = 0$



CNOT $|x\rangle|y\rangle$

X $\quad f(x) = \bar{x}$



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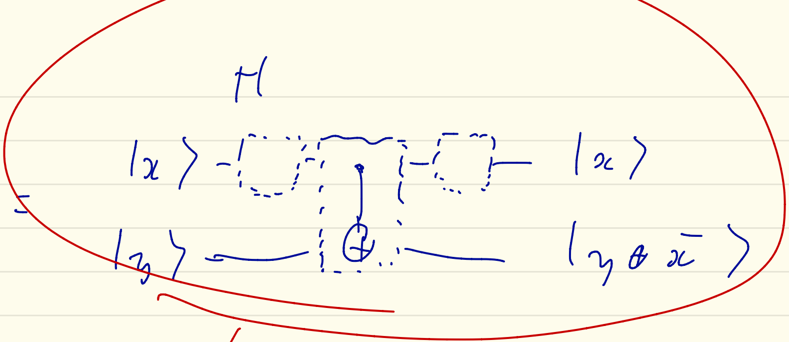

HW