Problem Set 4: Quantum Circuit and Deutsch's Algorithm

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1 Quantum Circuit

1.1 Write down the quantum states denoted point P, Q, R

- (a) $P:|1\rangle$, $Q:|-\rangle$, $R:-|1\rangle$
- $\text{(b)} \ \ P:\alpha\left|0\right\rangle-\beta\left|1\right\rangle, \quad \ Q:\alpha\left|1\right\rangle-\beta\left|0\right\rangle, \quad \ R:\frac{\alpha-\beta}{\sqrt{2}}\left|0\right\rangle+\frac{\alpha+\beta}{\sqrt{2}}\left|1\right\rangle$
- (c) $P: \alpha \gamma |00\rangle + \alpha \delta |01\rangle + \beta \gamma |10\rangle + \beta \delta |11\rangle$
 - $Q: \alpha \gamma |00\rangle + \alpha \delta |01\rangle + \beta \gamma |11\rangle + \beta \delta |10\rangle$
 - $R: \alpha \gamma |10\rangle + \alpha \delta |11\rangle + \beta \gamma |00\rangle + \beta \delta |01\rangle$
- (d) $P: \frac{1}{2}(|00\rangle |01\rangle + |10\rangle |11\rangle)$
 - $Q: \frac{1}{2}(|00\rangle |01\rangle + |11\rangle |10\rangle)$
 - $R:|11\rangle$

1.2 Simplifying Quantum Circuit

- (a) *I*
- (b) Z
- (c) iX
- (d) -Z
- (e) -iX

1.3 Quantum Oracle

Recall that the oracle action is defined as $U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$ For example, if $x = |1\rangle, y = |0\rangle, f(x) = x \wedge x$, then

$$U_f |1\rangle |0\rangle = |1\rangle |0 \oplus (1 \wedge 1)\rangle = \underline{|1\rangle |1\rangle}$$

Exercises: Calculate the output of the circuit after the oracle

(a)
$$x = |0\rangle, y = |1\rangle, f(x) = x \land x \land 1$$

(b)
$$x = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle, y = |1\rangle, f(x) = (x \land 1) \lor x$$

(c)
$$x = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle, y = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle, f(x) = \overline{(x \vee 0)}$$

2 Deutsch's Algorithm

2.1 Quantum Oracle: Calculate the output of the circuit after the oracle

(a) Note that $f(x) = x \land x \land 1 = x$

$$U_f |0\rangle |1\rangle = |0\rangle |1 \oplus 0\rangle = |0\rangle |1\rangle$$

(b) Note that $f(x) = (x \land 1) \lor x$

$$|x\rangle |y\rangle = \frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |11\rangle$$

$$\begin{aligned} U_f |x\rangle |y\rangle &= \frac{1}{\sqrt{2}} |0\rangle |1\rangle - \frac{1}{\sqrt{2}} |1\rangle |1\rangle \\ &= \frac{1}{\sqrt{2}} |0\rangle |1 \oplus 0\rangle - \frac{1}{\sqrt{2}} |1\rangle |1\rangle \\ &= \frac{1}{\sqrt{2}} |0\rangle |1\rangle - \frac{1}{\sqrt{2}} |1\rangle |0\rangle \end{aligned}$$

(c) Note that $f(x) = \overline{(x \vee 0)} = \overline{x}$

$$|x\rangle|y\rangle = |+\rangle|+\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

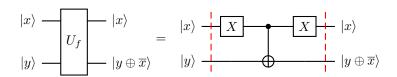
$$\begin{aligned} U_f \left| x \right\rangle \left| y \right\rangle &= \frac{1}{2} \left(U_f \left| 00 \right\rangle + U_f \left| 01 \right\rangle + U_f \left| 10 \right\rangle + U_f \left| 11 \right\rangle \\ &= \frac{1}{2} \left(\left| 0 \right\rangle \left| 0 \oplus 1 \right\rangle + \left| 0 \right\rangle \left| 1 \oplus 1 \right\rangle + \left| 1 \right\rangle \left| 0 \oplus 0 \right\rangle + \left| 1 \right\rangle \left| 1 \oplus 0 \right\rangle \right) \\ &= \frac{1}{2} \left(\left| 0 \right\rangle \left| 1 \right\rangle + \left| 0 \right\rangle \left| 0 \right\rangle + \left| 1 \right\rangle \left| 0 \right\rangle + \left| 1 \right\rangle \left| 1 \right\rangle \right) \end{aligned}$$

2.2 Deutsch's Oracle: Design an oracle for the negation function

The negation oracle has the action $f(x) = \overline{x}$. Equivalently,

$$U_f |x\rangle |y\rangle = |x\rangle |y \oplus \overline{x}\rangle$$

To perform $|y \oplus \overline{x}\rangle$, we need to add an X gate before the control bit of the CX gate. To get the $|x\rangle$ back, we apply another X gate after the control bit.



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