Problem Set 1: Math Review

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1 Exercises

1. Are the following vectors plausible state-vectors? If not, normalize them.

   (a) $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

   (b) $|\omega\rangle = \begin{pmatrix}1\\1\end{pmatrix}$

   (c) $|\phi\rangle = \begin{pmatrix}3+2i\\0\\3i\end{pmatrix}$

   (d) $|\chi\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

   (e) $|\epsilon\rangle = \begin{pmatrix}1\\1/2\\1/4\\1/8\\1/16\\ \vdots\end{pmatrix}$

2. Find the Hermitian conjugate (conjugate transpose) of the following matrices:

   (a) $A = \begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$

   (b) $B = \begin{pmatrix}3 & 7 & 9 & 7 & 0 & 8\\ 1 & 4 & 5 & 5 & 9 & 5\\ 9 & 4 & 7 & 8 & 8 & 8\\ 6 & 3 & 3 & 1 & 9 & 1\\ 2 & 1 & 5 & 6 & 4 & 4\\ 8 & 2 & 5 & 5 & 0 & 5\end{pmatrix}$

   (c) $C = \begin{pmatrix}i & 0 & 3+2i & 5i\end{pmatrix}$

3. Which of the following matrices are unitary?

   (a) $A = \begin{pmatrix} 1+i & 2-i \\ 3+i & 4-i \end{pmatrix}$

   (b) $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

   (c) $C = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i(\phi+\lambda)}\cos(\theta/2) \end{pmatrix}$

4. Find the eigenvalues, $\{\lambda\}$ and the eigenvectors $\{|\lambda\rangle\}$ of the following matrices:

   (a) $C = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$

   (b) $B = \begin{pmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$

   (c) $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

5. Calculate [1]

$$\sum_{i=0}^{1} \sum_{j=0}^{1} |i\rangle\langle j|$$

6. The product of two Pauli Matrices can be written as

$$\sigma_i \sigma_j = \delta_{ij} I + i \sum_k \epsilon_{ijk} \sigma_k$$

   where $\delta_{i,j}$ is the Kronecker-Delta[2], I is the identity matrix, and ϵ_{ijk} is a three-dimensional Levi-Civita symbol. This symbol is defined as

$$\epsilon_{ijk} = \begin{cases} +1, \text{ if } (ijk) = (1,2,3), (2,3,1) \text{ or } (3,1,2) \\ -1, \text{ if } (ijk) = (3,2,1), (1,3,2) \text{ or } (2,1,3) \\ 0 \text{ if any of the two indices are equal} \end{cases}$$

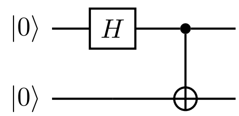
   Verify this identity for any 3 cases (e.g $\sigma_x\sigma_y$, $\sigma_y\sigma_z$, $\sigma_x\sigma_z$). Remember that $\sigma_1 = \sigma_x = X$, $\sigma_2 = \sigma_y = Y$, and $\sigma_3 = \sigma_z = Z$. Finally, the i before the sum symbol ($\sum$) denotes the imaginary number, not an index as the other i in the equation.

[1] Note: Some of you may find this notation tricky to interpret if you haven't done this kind of stuff before. Try your best to figure it out, but if you can't, just ask one of the mentors to help you out!

[2] 1 if $i = j$, 0 if $i \neq j$

2 Bonus

1. A $|\phi^+\rangle$ bell state can be prepared with the following 2-qubit circuit:



Where $H = |+\rangle\langle 0| + |-\rangle\langle 1|$ and $CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$.

Also, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

   (a) Calculate the statevector $|\phi^+\rangle$. Try to not use matrices and vectors. See how the Hadamrd gate affects qubit 0 and then how would the $CNOT$ gate act between the two qubits.

   (b) Write the resulting state as a tensor product. For example: $|01\rangle + |11\rangle = (|0\rangle + |1\rangle) \otimes |1\rangle$.[3]

[3] We're offering $100,000 to anyone who can answer this