Draft 2

Definition v in terms of p_1

We have that

$$v(t) = \exp\left(\int_0^t e^{-t+s}(\mu - p_1(s))\ ds\right) \tag{0.1}$$

Lemma 4.2 (v satisfies a certain integral equation)

For $t \geq 0$, $v(t)$ satisfies

$$v(t) = 1 - f_0(t) + f_0(0)e^{-\int_0^t \log v(s)\ ds} + \mu \int_0^t [v(s)]^{e^{-t+s}} e^{-t+s}\ ds, \tag{0.2}$$

where $f_0(t) := G(0, 1-e^{-t})$.

0.1 Lemma for Determinant Condition

Let $k \in L^1(\mathbb{R}^+; \mathbb{R})$ be a strictly decreasing and positive function such that $\hat{k}(z)$ is defined for $\Re z \geq 0$. Then $[1 + \hat{k}(z)] \neq 0$ for $\Re z \geq 0$.

Proof. We want to show that $\int_0^\infty k(t)e^{-zt}\ dt \neq -1$ for $\Re z \geq 0$. Write $z = z_1 + iz_2$. We have two cases. First, if $z_2 = 0$, then $\int_0^\infty k(t)e^{-z_1 t}\ dt > 0$. Second, if $z_2 \neq 0$, then

$$\int_0^\infty k(t)e^{-(z_1+iz_2)t}\ dt = \int_0^\infty k(t)e^{-z_1 t}\cos(z_2 t)\ dt - i\int_0^\infty k(t)e^{-z_1 t}\sin(z_2 t)\ dt.$$

Since $k(t)e^{-z_1 t}$ is strictly decreasing, the imaginary part is always non-zero. □

0.2 Lemma e^ν is the fixed point of φ

Let $\varphi : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+$ be a strictly increasing continuous function defined by

$$\varphi(x) = \mu \int_0^\infty x^{e^{-t}} e^{-t}\ dt = \begin{cases} \mu \cdot \frac{x-1}{\log x}, & x > 0, x \neq 1 \\ \mu, & x = 1 \end{cases}.$$

Let $\nu = \mu + W_0(-\mu e^{-\nu})$. We have that e^ν is the only fixed point of $\varphi(x)$.

Proof. It is easy to check that e^ν is a fixed point of $\varphi(x)$. It being the only fixed point follows from the concavity of φ. $\square$

Lemma Existence and Uniqueness of v

For $\mu > 1$, there exists a unique solution $v(t)$ to the integral equation (0.2).

Proof. Let $x(t) = v(t) - e^\nu$. We rewrite (0.2) as

$$x(t) = \underbrace{1 - f_0(t) + f_0(0)e^{-\int_0^t \log v(s)\ ds}}_{e_2(t)} + \mu \int_0^t [x(s) + e^\nu]^{e^{-t+s}} e^{-t+s}\ ds - e^\nu$$

Using Lemma 0.2, the right hand side becomes

$$\begin{aligned}
&e_2(t) + \mu \int_0^t [x(s) + e^\nu]^{e^{-t+s}} e^{-t+s}\ ds - \mu \int_0^\infty e^{\nu e^{-s}} e^{-s}\ ds \\
&= e_2(t) - \mu \int_t^\infty e^{\nu e^{-s}} e^{-s}\ dt + \mu \int_0^t \left\{ [x(s) + e^\nu]^{e^{-t+s}} e^{-t+s} - e^{\nu e^{-s}} e^{-s} \right\}\ ds \\
&= \underbrace{e_2(t) + \frac{\mu}{\nu}\left(1 - e^{\nu e^{-t}}\right)}_{f(t)} + \int_0^t \underbrace{\mu \left\{ [x(s) + e^\nu]^{e^{-t+s}} - e^{\nu e^{-t+s}} \right\} e^{-t+s}}_{-g(t-s, x(s))}\ ds
\end{aligned} \tag{0.3}$$

where $g(t,x) = -\mu \left\{ [x + e^\nu]^{e^{-t}} - e^{\nu e^{-t}} \right\} e^{-t}$. Since g is differentiable in x, we compute

$$k(t) = \frac{\partial}{\partial x} g(t,x)\bigg|_{x=0} = -\mu e^{-2t} e^{\nu(e^{-t}-1)}.$$

and let $h(t,x) = g(t,x) - k(t)x$ be the remainder. The existence and uniqueness of x (and by extension v) follows from Theorem 3.1. It remains for us to check that the integral equation (0.3) satisfies the sufficient conditions.

(i) Since $e^{\nu(e^{-t}-1)}$ is bounded, $k(t)$ is $L^1(\mathbb{R}^+, \mathbb{R})$. Next, we have that k is negative and

$$\frac{d}{dt} k(t) = -\mu e^{-3t}(2e^t + \nu) e^{\nu(e^{-t}-1)} \tag{0.4}$$

which implies that k is strictly increasing for all $t \geq 0$. By Lemma 0.1, $\det[1 + \hat{k}(z)] \neq 0$ for $\Re z \geq 0$.

(ii) Since $g(t,x)$ and $k(t)x$ is continuous, $h(t,x)$ is continuous and thus $t \mapsto h(t,x)$ is clearly Borel-measurable. Next, we check that $h(t,0) = g(t,0) - k(t) \cdot 0 = 0$.

(iii) For all $0 < \delta \leq 1$, we have that

$$
\begin{aligned}
b_\delta(t) &= \sup_{\substack{|x|,|y|\le\delta\\ x\neq y}} \frac{|h(t,x)-h(t,y)|}{|x-y|}\\
&= \sup_{\substack{|x|,|y|\le\delta\\ x\neq y}} \frac{\left|\mu\left[(x+e^\nu)^{e^{-t}}-(y+e^\nu)^{e^{-t}}\right]e^{-t}-\mu e^{-2t}e^{\nu(e^{-t}-1)}(x-y)\right|}{|x-y|}\\
&= \sup_{\substack{|x|,|y|\le\delta\\ x\neq y}} \mu e^{-t}\frac{\left|\left[(x+e^\nu)^{e^{-t}}-(y+e^\nu)^{e^{-t}}\right]-e^{-t}e^{\nu(e^{-t}-1)}(x-y)\right|}{|x-y|}\\
&= \sup_{\substack{|x|,|y|\le\delta\\ x\neq y}} \mu e^{-t}\frac{\left|\left[(x+e^\nu)^{e^{-t}}-e^{-t}e^{\nu(e^{-t}-1)}x\right]-\left[(y+e^\nu)^{e^{-t}}-e^{-t}e^{\nu(e^{-t}-1)}y)\right]\right|}{|x-y|}
\end{aligned}
$$

Recall that a function in an interval is γ-Lipschitz if its derivative is bounded by γ. Consider

$$
p(x) = (x+e^\nu)^{e^{-t}} - e^{-t}e^{\nu(e^{-t}-1)}x.
$$

Then

$$
p'(x) = e^{-t}\left[(x+e^\nu)^{e^{-t}-1} - e^{\nu(e^{-t}-1)}\right].
$$

Since $(x+e^\nu)^{e^{-t}-1}$ is decreasing in x, $p'(x)$ attains it maximum absolute value at either endpoints $x=\pm\delta$. Therefore,

$$
b_\delta(t) = \mu e^{-2t}\left[(\pm\delta+e^\nu)^{e^{-t}-1} - e^{\nu(e^{-t}-1)}\right] = Ce^{-2t}. \tag{0.5}
$$

We deduce that $b_\delta(t)$ belongs to $L^1(\mathbb{R}^+;\mathbb{R})$. We also have that

$$
\lim_{\delta\to 0}||b_\delta||_{L^1(\mathbb{R}^+)} \le \lim_{\delta\to 0}\int_0^\infty \mu e^{-2t}\left[(\pm\delta+e^\nu)^{e^{-t}-1} - e^{\nu(e^{-t}-1)}\right]\, dt = 0
$$

by the dominated convergence theorem. Finally, since

$$
0 \le 1-f_0(t) = 1-G(0,1-e^{-t}) \le \partial_z G(0,1)e^{-t} = \mu e^{-t}
$$

and

$$
\int_0^t \log v(s)\, ds \ge \int_0^t (\mu-1)(1-e^{-s})\, ds = (\mu-1)(t+e^{-t}-1) \ge (\mu-1)(t-1),
$$

we deduce that

$$
e_2(t) \le \mu e^{-t} + f_0(0)e^{-(\mu-1)(t-1)}. \tag{0.6}
$$

We conclude that f is bounded. Therefore, by Theorem 11.3.1 [1], there exists a unique solution x. $\square$

0.3 Lemma Infinite Sums of Lipschitz Continuous Functions with Bounded Lipschitz constant is Lipschitz Continuous

Let $\{a_n(t)\}_{n=0}^{\infty}$ be a sequence of Lipschitz continuous function all with Lipschitz constant less than or equal to γ. Let $\{b_n\}_{n=0}^{\infty}$ be a sequence terms that is absolutely summable. Then

$$c(t) = \sum_{n=0}^{\infty} b_n a_n(t)$$

is well-defined and Lipschitz continuous.

Proof. For $\delta > 0$ and $|t - s| < \delta$, we have that

$$|c(t) - c(s)| \leq \sum_{n=0}^{\infty} |b_n||a_n(t) - a_n(s)| \leq \sum_{n=0}^{\infty} |b_n| \cdot \gamma |t - s|.$$

Since the third series is absolutely summable, we deduce that the second one is as well and thus $c(t)$ is well-defined and Lispchitz with constant $\gamma \sum_{n=0}^{\infty} |b_n|$. $\square$

0.4 Corollary v converges to e^{ν}

We have that $x \in BC_0(\mathbb{R}^+; \mathbb{R})$.

Proof. We rewrite

$$\begin{aligned} f(t) &= e_2(t) + \frac{\mu}{\nu}\left(1 - (1 + \nu e^{-t} + O(e^{-2t}))\right) \\ &= e_2(t) + \frac{\mu}{\nu}(-\nu e^{-t} + O(e^{-2t})) \\ &= e_2(t) - \mu e^{-t} + O(e^{-2t}). \end{aligned} \tag{0.7}$$

This shows that f is bounded. From (0.6), $\lim_{t \to \infty} f(t) = 0$. It remains to show that f is continuous. First, as a solution to a well-posed system of ODEs, v is continuous. This implies that $f_0(0) \exp\left(-\int_0^t \log v(s)\ ds\right)$ is continuous. Next, recall that

$$f_0(t) = \sum_{n=0}^{\infty} p_n(0)(1 - e^{-t})^n.$$

We want to show that $a(t) = (1 - e^{-t})^n$ is Lipschitz-continuous with Lipschitz constant 1. When $n = 0, 1$, this is obvious. When $n \geq 2$ we compute the first two derivatives

$$\begin{aligned} \frac{d}{dt} a(t) &= n e^{-t}(1 - e^{-t})^{n-1} \\ \frac{d^2}{dt^2} a(t) &= -\frac{n\left(1 - e^{-t}\right)^n \left(e^t - n\right)}{\left(e^t - 1\right)^2} \end{aligned}$$

Observe that the second derivative is zero at 0 and $\ln(n)$. At $\ln(n)$, the first derivative attains the value $\left(1-\frac{1}{n}\right)^{n-1}$. This value is positive and is less than 1 for all $n \geq 2$. By Lemma 0.3, $f_0(t)$ is Lipschitz continuous. By Theorem 11.3.1(3) [1], $x(t)$ of (0.3) is in the space $BC_0(\mathbb{R}^+;\mathbb{R})$. From the definition of x, we also deduce that $\lim_{t\to\infty} v(t) = e^\nu$. □

0.5 Lemma Estimate (4.16)

We have that

$$e_2(t) \leq \mu e^{-t} + Ce^{-\nu t}.$$

Proof. From (0.6) and (0.7), we have that

$$|f(t)| \leq 2\mu e^{-t} + f_0(0)e^{-(\mu-1)(t-1)} + O(e^{-2t}).$$

From this bound, we deduce that f is $L^1(\mathbb{R}^+;\mathbb{R}^n)$. By Theorem 3.1 f being $L^1(\mathbb{R}^+;\mathbb{R})$ implies that x is also $L^1(\mathbb{R}^+;\mathbb{R}^n)$. By Lipschitzness, we get

$$\int_0^t |\log v(s) - \nu|\ ds \leq \int_0^t |v(s) - e^\nu|\ ds \leq C.$$

Therefore, we can improve that estimate of Equation 4.8 [2] to

$$\begin{aligned} e_2(t) &\leq 1 - f_0(t) + f_0(0)\exp\left(-\int_0^t \nu\ ds\right) \cdot \exp\left(\int_0^t |\log v(s) - \nu|\ ds\right) \\ &\leq \mu e^{-t} + Ce^{-\nu t}. \end{aligned}$$

□

Lemma Convergence Rate

For $t \geq 0$, we have that

$$|v(t) - e^\nu| \leq \begin{cases} Ce^{-\nu t} & \nu < 1 \\ Ce^{-ct} & \nu \geq 1 \text{ and } c < 1 \end{cases}.$$

Proof. We apply Theorem 11.3.3 [1]. Choosing e^{ct} to be our submultiplicative weight function on $\mathbb{R}^+$, we check the sufficient conditions:

(i) From (0.4), we have that $k \in L^1(\mathbb{R}^+;e^{ct};\mathbb{R})$ because

$$\int_0^\infty |k(t)e^{ct}|\ dt \leq C\int_0^\infty e^{-(2-c)t}\ dt < \infty.$$

We show that $\det[I + \hat{k}(z)] \neq 0$ for $\Re z \geq -c$. This is equivalent to $\det[I + \hat{k}(z-c)] \neq 0$ for $\Re z \geq 0$. In particular, $\hat{k}(z-c) = \widehat{e^{c\cdot}k(\cdot)}(z)$. We now have $e^{ct}k(t) = -\mu e^{-(2-c)t}e^{\nu(e^{-t}-1)}$. Since $\hat{k}$ is monotonically increasing and $\hat{k}(-1)$, by Lemma 0.1, we obtain $\det[I + \hat{k}(z)] \neq 0$ for $z > -1$.

(ii) Since $g(t,x)$ and $k(t)x$ are continuous in both variables, $x \mapsto h(t,x)$ is continuous for every $t \in \mathbb{R}^+$ and $t \mapsto h(t,x)$ is measurable for every $x \in \mathbb{R}$.

(iii) For $0 < \eta < 1$, we obtain from (0.5) that

$$b_\eta(t) := \sup_{|x|\leq\eta} \frac{|h(t,x)|}{|x|} \leq Ce^{-2t}.$$

(Theorem 3.3 only requires (iii) to be true for all sufficiently small $\delta > 0$) From this bound, it is clear that $b_\eta(t)$ belongs to $L^1(\mathbb{R}^+; e^{ct}; \mathbb{R})$. Again from (0.5), we have that

$$\lim_{\eta\to 0} ||b_\eta||_{L^1(\mathbb{R}^+;e^{ct})} \leq \lim_{\eta\to 0} \int_0^\infty \mu e^{-(2-c)t} \left[(\pm\eta + e^\nu)^{e^{-t}-1} - e^{\nu(e^{-t}-1)} \right] \, dt = 0$$

by the dominated convergence theorem.

(iv) From the proof of Lemma 0.4, $f \in BC_0(\mathbb{R}^+; \mathbb{R})$. From (0.7) and 0.5, we have that $|f(t)| \leq Ce^{-ct}$. Therefore, $e^{ct}f(t)$ is bounded.

(v) This is the result of (0.4). □

Proposition 4.9

Let $y_2(t) = \int_0^t |v(s) - e^\nu| e^{-2t+2s} \, ds$ and $c = \nu \wedge 1$. There exists a constant $C > 0$ depending on μ such that

$$y_2(t) + |v(t) - e^\nu| + |v'(t)| \leq Ce^{-ct}.$$

Proof. First, we have that

$$\begin{aligned}
y_2(t) &= \int_0^t |v(s) - e^\nu| e^{-2t+2s} \, ds \\
&\leq Ce^{-2t} \int_0^t e^{(2-c)s} \, ds \\
&= Ce^{-2t} \left(\frac{1}{c-2} - \frac{e^{2t-ct}}{c-2} \right) \\
&= \frac{Ce^{-2t}}{c-2} - \frac{Ce^{-ct}}{c-2} \\
&\leq Ce^{-ct}.
\end{aligned}$$

For the derivative estimate, we differentiate v using (0.2) and get

$$\begin{aligned}
v'(t) &= e_2'(t) + \mu v(t) - \mu \int_0^t [v(s)]^{e^{-t+s}} e^{-t+s} \, ds - \mu \int_0^t [v(s)]^{e^{-t+s}} \log(v(s)) e^{-2t+2s} \, ds \\
&= e_2'(t) + (\mu - 1)v(t) + \mu e_2(t) - \mu \int_0^t [v(s)]^{e^{-t+s}} \log(v(s)) e^{-2t+2s} \, ds,
\end{aligned}$$

in which

$$e_2'(t) = -f_0'(t) - f_0(0) \exp\left(-\int_0^t \log v(s)\ ds\right) \log v(t) = \mathcal{O}\left(e^{-ct}\right).$$

Note that $x \mapsto x^\alpha \log x$ is Lipschitz on $[1, e^\nu]$ uniformly for $\alpha \in [0,1]$, hence

$$\begin{aligned}
&\left|\mu \int_0^t [v(s)]^{e^{-t+s}} \log(v(s)) e^{-2t+2s}\ ds - \mu \int_0^t \nu e^{\nu e^{-t+s}} e^{-2t+2s}\ ds\right| \\
&\leq C\mu \int_0^t |v(s) - e^\nu| e^{-2t+2s}\ ds = C y_2(t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
|v'(t)| \leq &|e_2'(t)| + (\mu-1)|v(t) - e^\nu| + \mu e_2(t) + C y_2(t) \\
&+ (\mu-1)e^\nu - \mu\nu \int_0^t e^{\nu e^{-2s}} e^{-2s}\ ds \\
&\leq |e_2'(t)| + (\mu-1)|v(t) - e^\nu| + |\mu e_2(t)| + C y_2(t) + \mu\nu \int_t^\infty e^{\nu e^{-s}} e^{-2s}\ ds \\
&\leq C e^{-ct}.
\end{aligned}$$

This completes the proof. □

As a corollary, we deduce the following convergence rate of $p_1(t) \to \mu - \nu$ as $t \to \infty$.

Corollary 4.10

For $t \geq 0$, we have

$$|p_1(t) - \mu + \nu| \leq C e^{-ct},$$

where $c = \nu \wedge 1$.

Proof. It suffices to notice that $\mu - p_1(t)$ can be recovered from v via

$$\mu - p_1(t) = \frac{v'(t)}{v(t)} + \log v(t).$$

Since both $v'(t) \to 0$ and $v(t) \to e^\nu$ occur at this rate, the result follows. □

References

[1] Volterra Integral and Functional Equations

[2] Quantitative convergence guarantees for the mean-field dispersion process