TEST 2

UNIVERSITY OF MALAWI

SCHOOL OF NATURAL & APPLIED SCIENCES

Mathematical Sciences Department

TEST 2: Recurrence Relations / Difference Equations

(For 2nd year Science students taking MAT 212)

SOLUTION KEY

Saturday, 23rd December 2023 **Time: 2 hours (from 18:30hrs)**

Instructions

63 MARKS

(1) ***This is a closed book test*** where you are expected to do the test alone without any assistance from some other person(s) or some other form of notes or communication.

(2) ***Non-programmable calculators may be used***. However, mobile phones are not allowed. If accidentally brought in, they should be switched off and packed away.

(3) ***Show your method or reasoning***. Most marks shown in square brackets at the end of each part are allocated to the method.

(4) Start with questions that you can do comfortably first.

(5) Attempt ***ALL questions***.

Solution Key

Question 1: [30 marks]

(a) Deduce formulas for recurrence relations for each of the following problems:

(i) In how many ways can n distinct items be placed on n distinct (fixed) positions? **[6 marks]**

(ii) A $2 \times n$ rectangle is formed from joining 1×1 squares. Suppose someone has 2×1 tiles, how many different ways can that person completely (& minimally) cover the $2 \times n$ rectangle with the 2×1 tiles? **[8 marks]**

(b) Use **iteration method** to calculate the value of $f(3)$ for the recursive relation $f(k) = f(k-1) \times f(k-3) - f(k-1)$; where $f(-1) = 4, f(0) = 1, f(1) = 3$. **[4 marks]**

(c) Use **root method** to solve in terms of m the difference equation $b_m - 4b_{m-2} = 0$; $b_0 = 8$, $b_1 = 4$, $m = 2, 3, 4, \ldots$. **[8 marks]**

(d) Let b_n be the total number of sequences of n numbers that can be formed from the two digits 0 and 1. Prove that $b_1 = 2$ and hence show that $b_n = 2b_{n-1}$; $\forall\, n \geq 2$. **[4 marks]**

Question 2: [25 marks] +2 = 27+6 = 33

(a) Given that a generating function $h(z) = A_0z^0 + A_1z^1 + A_2z^2 + A_3z^3 + \ldots + A_nz^n + \ldots$ has the closed form

$$h(z) = \frac{1}{1-5z} + \frac{1}{(1+2z)^4} + 6.\exp\{z\},$$

calculate the value of A_3. **[~~8~~ 10 marks]**

(b) Use **generating functions method** to solve $a_n - 4a_{n-1} = 0$, $a_0 = 5$, $n = 1, 2, 3, \ldots$ in terms of n. **[~~6~~ marks]** 12

(c) Sequences of m digits, where the digits are chosen from the set $\{0,1,2\}$, are to be formed such that no successive 2's are to appear on any part of the sequence. Let a_m denote the number of sequences of m digits in which each digit is either 0 or 1 or 2.

a. Show that $a_1 = 3$ and $a_2 = 8$. **[4 marks]**

b. Hence, derive a recurrence relation in form of a_m for $m \geq 3$. **[7 marks]**

~~~~~ooooOOOO **End of Test 2 Questions** OOOOoooo~~~~~
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MAT 212 - TEST 2

QUESTION No. Examination No.

SOLUTION KEY

Q1 (a) 6

(i) Let a_n be the number of ways of placing n distinct items on n distinct (fixed) positions (1)

Let the n-distinct positions be

(2) $\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad} \cdots \underline{\quad}\ \underline{\quad}$

$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots \quad n-1 \quad n$

(1) If we consider the last position, we have $^nC_1$ ways of placing n letters

(1) if such a position is filled or has an item placed on it, then the number of ways of placing the remaining $n-1$ different items on $n-1$ different positions is now a_{n-1}

Hence by Multiplication Rule, we have

(2) $$\overbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad} \cdots \underline{\quad}}^{a_{n-1}} \times \underline{\ n\ }$$

$1 \quad 2 \quad 3 \quad \cdots \quad n-1 \quad n\text{th position}$

$a_n = n \times a_{n-1}$, where $a_1 = 1$

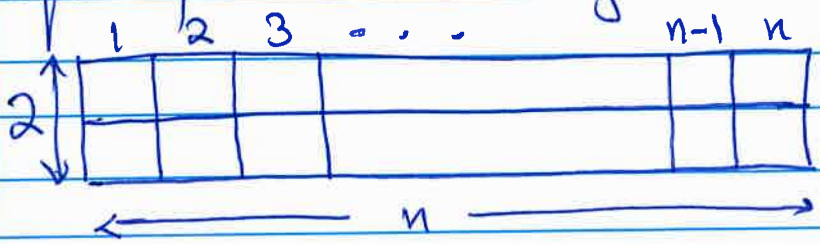
(1) $\therefore a_n = n a_{n-1}$, $a_1 = 1$, $n = 2, 3, 4, \ldots$

Q1 (a) $\boxed{8}$

(ii)

Let a_n be the number of ways of covering a $2\times n$ rectangle with 2×1 tiles such that the whole of the $2\times n$ rectangle is minimally covered. $\frac{1}{2}$

If the $2\times n$ rectangle is formed from 1×1 squares, let the design be as below



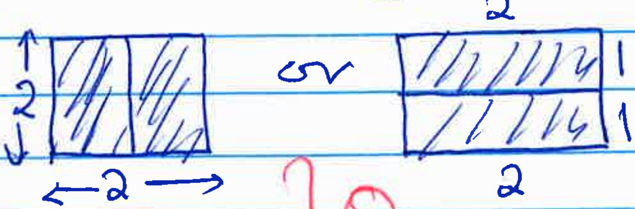
$\frac{1}{2}$

Thus, a_1 is number of ways of covering a 2×1 rectangle with 2×1 rectangle/tiles

1 

$\Rightarrow \quad a_1 = 1$

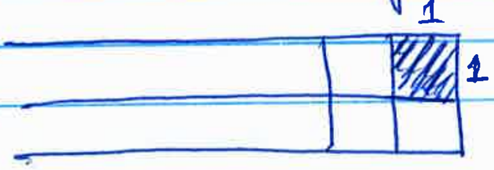
Again, a_2 is number of ways of covering 2×2 rectangle with 2×1 tiles,

Thus, 

, Two ways

Hence, $a_2 = 2$ 1

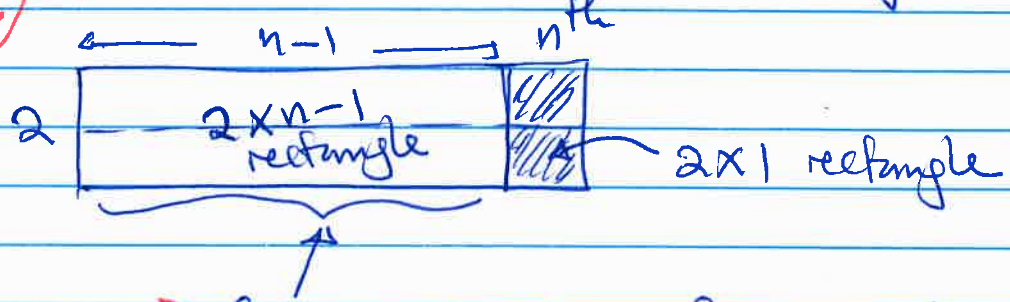
Now let's consider the last top-right 1×1 tile region as $\frac{1}{2}$



We can observe that such a square can be covered by a 2x1 rectangle, placed either vertically or horizontally.

Case 1: Suppose a 2x1 rectangle covers a 1x1 top-right tile vertically, i.e.

$\frac{1}{2}$

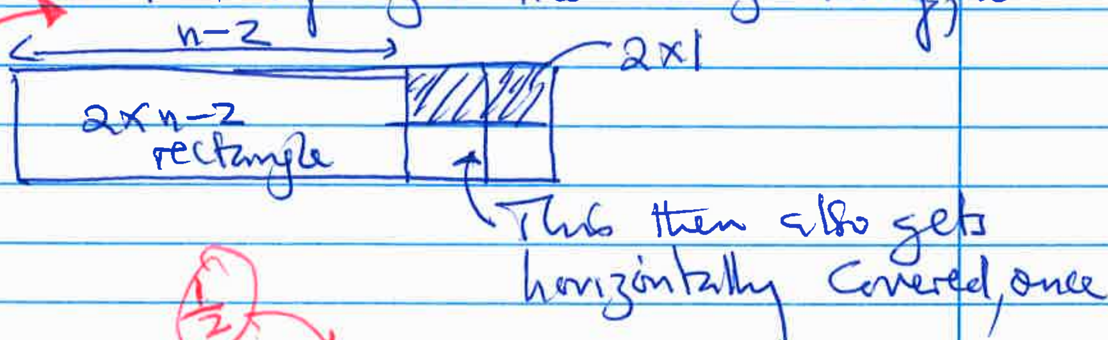


1

a_{n-1} will thus be number of ways of covering the rest of the 1x1 tiles

Case 2: Suppose a 2x1 rectangle covers a 1x1 top-right tile horizontally, i.e.

1



$\frac{1}{2}$

Thus, the remaining 2xn-2 rectangle can be covered in a_{n-2} ways.

∴ Applying the SUM RULE, for the two cases,

$\frac{1}{2}$

we have

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 1, \; a_2 = 2$$

number of ways of covering a 2xn rectangle with the 2x1 tiles.

1

QUESTION No. Q. 1 (b) Examination No.

(b) Given the recursive relation

4

$$f(k) = f(k-1) \times f(k-3) - f(k-1)$$

where $f(-1) = 4$, $f(0) = 1$, and $f(1) = 3$

$$f(3) = f(2) \times f(0) - f(2)$$ ½

but $f(2) = f(1) \times f(-1) - f(1)$

½

Thus

$$f(3) = \{f(1) \times f(-1) - f(1)\} \times f(0) - \{f(1) \times f(-1) - f(1)\}$$

1

½

$$= f(1)f(-1)f(0) - f(1)f(0) - f(1)f(-1) + f(1)$$

½

$$= (3)(4)(1) - (3)(1) - (3)(4) +$$

$$f(3) = 12 - 3 - 12 + 3 = 0$$

1

Q.1

(c) Using the root method,

8

given $b_m - 4b_{m-2} = 0$; $b_0 = 8$, $b_1 = 4$

Let $b_m = \lambda^m$, and that $b_{m-2} = \lambda^{m-2}$

½ ½

Then $b_m - 4b_{m-2} = \lambda^m - 4\lambda^{m-2} = 0$

½

$$\Rightarrow \lambda^{m-2}(\lambda^2 - 4) = 0$$

1

when either $\lambda^{m-2} = 0$ or $\lambda^2 - 4 = 0$

but $\lambda^{m-2} \neq 0$, hence $\lambda^2 - 4 = 0$ $\frac{1}{2}$

Solving the quadratic equation (1)

$$(\lambda^2 - 4) \equiv (\lambda - 2)(\lambda + 2) = 0$$

So that either $\lambda_1 = 2$ or $\lambda_2 = -2$, hence the roots are distinct such that the general solution is

$$b_m = K_1 \lambda_1^m + K_2 \lambda_2^m$$ (1)

Now, using $b_0 = 8$ and $b_1 = 4$

$$b_0 = K_1 \lambda_1^0 + K_2 \lambda_2^0 = 8$$

$$\Rightarrow \quad K_1 + K_2 = 8 \quad \ldots\ldots (i)$$

$$b_1 = K_1 \lambda_1^1 + K_2 \lambda_2^1 = 4$$

$$\Rightarrow \quad 2K_1 - 2K_2 = 4 \quad \ldots\ldots (ii)$$

1

Thus, simultaneously solving (i) and (ii) for K_1 and K_2, we have $K_1 = 8 - K_2$, substitute into (ii), we have $K_2 = 3$. Hence, $K_1 = 5$. (1)

$\therefore$ our general solution is

$$b_m = (5)(2)^m + (3)(-2)^m$$ (1)

for $m = 2, 3, 4, \ldots$

QUESTION No. Q. 1 Examination No.

(d)

4

Let b_n be the total number of sequences of n numbers that can be formed from the two digits 0 and 1

b_1 means # of sequences of length 1 that can be formed from two digits 0 and 1

$\frac{1}{2}$

$\frac{1}{2}$

i.e, either <u>0</u> or <u>1</u>, hence by sum Rule, we have $b_1 = 2$

Let design a sequence of length n as follows

$\frac{1}{2}$

___ ___ ___ . . . ___ ___

1 2 3 n−1 n − positions

Lets consider the last position − nth position, this can be filled in two-ways by either a 0 or a 1

$\frac{1}{2}$

$$\text{Thus}\quad \overbrace{\underline{\quad}_1\ \underline{\quad}_2\ \underline{\quad}_3\ \cdots\ \underline{\quad}_{n-1}}^{b_{n-1}}\ \underset{n\,-\,\text{position}}{\underline{\ 2\ }}$$

$\frac{1}{2}$

The remaining ~~of~~ n−1 positions can be filled by two digits 0 and 1 in b_{n-1} number of ways

$\frac{1}{2}$

Hence, by Multiplicative Rule, we have
$b_n = 2 \times b_{n-1} = 2b_{n-1}$ ways of form sequences of n numbers using 0's and 1's.

1

QUESTION No. Q-2 Examination No.

(a) 10

Given that the generating function $h(z) = A_0 z^0 + A_1 z^1 + A_2 z^2 + A_3 z^3 + \dots + A_n z^n + \dots$ has a closed form

$$h(z) = \frac{1}{1-5z} + \frac{1}{(1+2z)^4} + 6.\exp(z)$$

then from the knowledge that 1

$$z^0 + z^1 + z^2 + z^3 + \dots + z^n + \dots = \frac{1}{1-z}$$

then

$$\frac{1}{1-5z} = x^0 + x^1 + x^2 + x^3 + \dots \equiv \frac{1}{1-x}$$

$\frac{1}{2}$ 1

where $x = 5z$, so that

$\frac{1}{1-5z}$ is a closed form of the generating function $(5z)^0 + (5z)^1 + (5z)^2 + (5z)^3 + \dots$

1

$$\equiv z^0 + 5z^1 + 5^2 z^2 + 5^3 z^3 + \dots$$

Again, since we know that

1

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

Then

$$\frac{1}{(1+2z)^4} = \sum_{n=0}^{\infty} \binom{n+4-1}{4-1} (2z)^n = \sum_{n=0}^{\infty} \binom{n+3}{3} 2^n z^n$$

$1\frac{1}{2}$

Then

$$\frac{1}{(1+2z)^4} = \sum_{n=0}^{\infty} \binom{n+3}{3} (2z)^n$$

1

$$= \binom{3}{3}(2z)^0 + \binom{4}{3}(2z)^1 + \binom{5}{3}(2z)^2 + \binom{6}{3}(2z)^3 + \cdots$$

$$\therefore h(z) = \frac{1}{1-5z} + \frac{1}{(1+2z)^4} + 6.\exp(z)$$

And Since $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

1

then $6.\exp(z) = 6\left\{\frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right\}$

from $h(z) = A_0 z^0 + A_1 z^1 + A_2 z^2 + A_3 z^3 + \cdots$,

we are looking for A_3 which is coefficient of z^3

$\therefore$ grouping all terms of the form $A_3 z^3$

$\frac{1}{2}$

from $h(z) = \frac{1}{1-5z} + \frac{1}{(1+2z)^4} + 6.\exp(z)$

we have $5^3 z^3 + \binom{6}{3}(2z)^3 + 6 \times \frac{z^3}{3!}$

$$\therefore A_3 z^3 \equiv \left\{5^3 + \binom{6}{3} 2^3 + \frac{6}{3!}\right\} z^3 = 286\, z^3$$

$\therefore A_3 = 286$ $\frac{1}{2}$ 1

QUESTION No. 2 Examination No.

(b) 12

Given $a_n - 4a_{n-1} = 0$, $a_0 = 5$, $n = 1, 2, 3, \ldots$

Using generating function,

let $f(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$ $\frac{1}{2}$

be the generating function of the sequence $\{a_n\}_{n=0}^{\infty}$ $\frac{1}{2}$

But a_n is a coefficient of x^n $\frac{1}{2}$

Hence, let $a_n = C[x^n]f(x)$ $\frac{1}{2}$

Then multiplying $f(x)$ by x, we get

$$xf(x) = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots + a_{n-1}x^n + a_nx^{n+1} + \cdots$$ $\frac{1}{2}$

So that a_{n-1} is new coefficient of x^n

$$\Rightarrow a_{n-1} = C[x^n]xf(x)$$ $\frac{1}{2}$

Similarly, multiplying $f(x)$ by x^2, we get

$$x^2f(x) = a_0x^2 + a_1x^3 + a_2x^3 + \cdots + a_{n-2}x^n + a_{n-1}x^{n+1} + \cdots$$ $\frac{1}{2}$

$$\Rightarrow a_{n-2} = C[x^n]x^2f(x)$$ $\frac{1}{2}$

$\therefore$ from $a_n - 4a_{n-1} \equiv C[x^n]f(x) - 4C[x^n]xf(x) = 0$ $\frac{1}{2}$

So that

$$C[x^n]f(x) - 4C[x^n]xf(x) = 0$$ $\frac{1}{2}$

$$\Rightarrow f(x) - 4xf(x) = 0$$ $\frac{1}{2}$

But $f(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \ldots$ 1

$-4xf(x) = -4a_0x^1 - 4a_1x^2 - 4a_2x^3 - 4a_3x^4 \ldots$

$$f(x) - 4xf(x) = a_0 + (a_1 - 4a_0)x^1 + (a_2 - 4a_1)x^2 + (a_3 - 4a_2)x^3 + \ldots$$

$\frac{1}{2}$

Since $a_0 = 5$, then from $a_n - 4a_{n-1} = 0$

$\frac{1}{2}$ $\{ a_1 = 20,\ a_2 = 80,\ a_3 = 320, \ldots$

Hence
$$\left.\begin{aligned} a_1 - 4a_0 &\equiv 20 - 20 = 0 \\ a_2 - 4a_1 &\equiv 80 - 80 = 0 \\ a_3 - 4a_2 &\equiv 320 - 320 = 0 \end{aligned}\right\} \frac{1}{2}$$

$\frac{1}{2}$

$$\therefore f(x) - 4xf(x) = a_0 + 0 + 0 + \ldots$$

But $f(x)\{1 - 4x\} = a_0$, where $a_0 = 5$

$\frac{1}{2}$

we have

$$f(x) = \frac{5}{1 - 4x}$$

1

But, since $\frac{1}{(1-z)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} z^n$

Then $f(x) = 5\left\{\frac{1}{1-4x}\right\} = 5 \times \sum_{n=0}^{\infty} \binom{n}{0}(4x)^n$

1

$$= 5 \times \sum_{n=0}^{\infty} (4x)^n$$

$$\therefore a_n = C[x^n]f(x) = 5 \times 4^n$$ 1

QUESTION No. Q-2 Examination No.

(c) Let a_m be the number of sequences of m-digits in which each digit is either 0 or 1 or 2.

(i) 4

$\Rightarrow$ a_1 means sequences of length 1, i.e. $\frac{1}{2}$

0 or 1 or 2

By SUM RULE, $a_1 = 3$ possible sequences 1

Again a_2 means sequences of length 2, i.e.

00 02 22 × Not allowed

01 20 21 $1\frac{1}{2}$

8 possible number of sequences { 10 11 12

∴ By the SUM RULE, $a_2 = 8$ 1

(ii) 7

Now, let a_m^0 be the number of such sequences that end with a 0 $\frac{1}{2}$

Let a_m^1 be the number of such sequences that end with a 1 $\frac{1}{2}$

Let a_m^2 be the number of such sequences that end with a 2 $\frac{1}{2}$

Then by SUM RULE, $a_m = a_m^0 + a_m^1 + a_m^2$ $\frac{1}{2}$

CASE 1: Let's consider the sequence positions

1	2	3	. . .	m-2	m-1	m

— positions

For a case of a_m^0, if the m^{th} digit is a 0, then the ~~nth~~ $m-1^{th}$ position can be a 0, or a 1 or a 2.

...	$m-2$	$m-1$	m
			0

1

$$\therefore a^0_m = a^0_{m-1} + a^1_{m-1} + a^2_{m-1} \equiv a_{m-1}$$

Case 2: Assume the m^{th} position has a 1

Then on the $m-1^{th}$ position, we can also have a 0 or a 1 or a 2.

1

$$\therefore a^1_m = a^0_{m-1} + a^1_{m-1} + a^2_{m-1} \equiv a_{m-1}$$

	$m-2$	$m-1$	m
			1

Case 3: Suppose the m^{th} position has a 2

then on the $m-1^{th}$ position, we can only have a 0 or a 1 to avoid having successive 2's

$\frac{1}{2}$

But on $m-2^{th}$ position, we can have a 0 or a 1 or a 2. $\frac{1}{2}$

$$\therefore a^2_m = a^2_{m-2} + a^1_{m-2} + a^0_{m-2} = 2a_{m-2}$$

	$m-2$	$m-1$	m
			2

1

$m-2$: Any

$m-1$: 0 or 1 (must)

$\therefore$ Using sum rule, since $a_m = a^0_m + a^1_m + a^2_m$

$$\Rightarrow a_m = a_{m-1} + a_{m-1} + 2a_{m-2}$$

$$a_m = 2a_{m-1} + 2a_{m-2}$$

1