Unit 1: Counting Techniques

Introduction

In other previous courses such as College Algebra, we have probably come across results on the size of finite sets. We shall present in this unit further results along this line, mainly when we are faced with a real life problem that would need counting of elements or obtaining some finite set. For instance, if we know that set B has got size m. We might wish to know the number of distinct subsets of the set B, which we have called it before, the power set of B, $P(B)$. Furthermore, we might wish to know the number of subsets that are of size p, among all subsets of B. For example, if in Malawi a cabinet is composed of 20 members, then we call B the set of 20 cabinet ministers, such that the number of subsets is $P(B) = 2^{20}$, which is equivalent to the number of different parliamentary committees the ministers can form.

There are many situations in which it would be too difficult and/or too tedious to list all of the possible outcomes in a sample space. In this unit, we will learn various ways of counting the number of elements in a sample space without actually having to identify the specific outcomes. In this unit we shall explore the techniques of counting and other related problems in the context of multiplication rule, permutations and combinations.

Unit Objectives

On successful completion of the Unit, students should be able to:

- Apply the multiplication principle to solving related real life problems
- Use the permutation formula to count the number of ordered arrangements
- Use the combination formula to count the number of unordered subsets of *r* objects taken from *n* objects
- Use factorial formula to count the number of distinguishable permutations of *n* objects
- Solve counting application problems

Key Terms

As you go through this unit, ensure that you understand the key terms or phrases used in this unit as listed below:

- Product
- Sum
- Permutation/arrangement
- Combination
- Order
- Distinguishable/ non-distinguishable

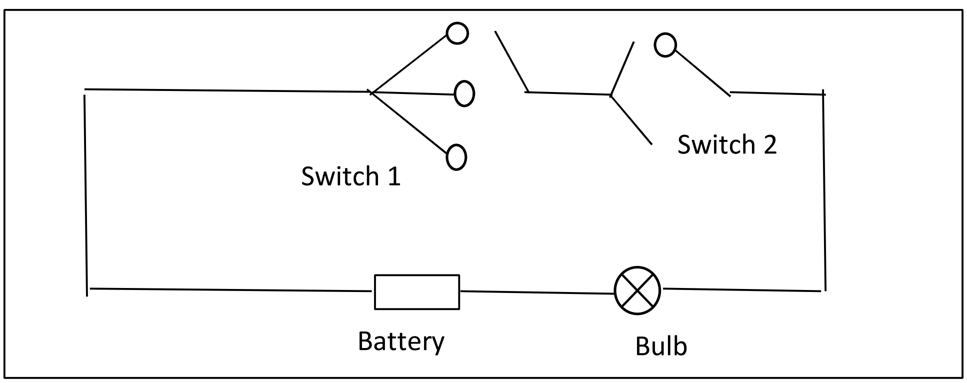
Product Rule of counting

By an **event** we shall mean something that takes place, for instance, placing a ball in a box, selecting a representative among a group of ministers, assigning offices to teachers at a newly built school, and connecting a switch to have light on the bulb within some circuit, are examples of events. However, these events occur differently. For example, in a case of placing a single ball in each of the several boxes, the number of ways of placing the ball is equal to the number of boxes available. Throughout our discussion in this lesson, when we are to consider the occurrence of several events, we shall be guided by certain underlying principles stated here in as product rule and sum rule of counting.

Product Rule of Counting

Example

How many different ways can one switch on the light in the circuit shown below?



<u>Solution</u>

- If on switch 1 , "a" is connected, then one has 2 possible ways of connecting switch 2
- Else, if on switch 1, "b" is connected, one has 2 possible ways of connecting switch 2
- Similarly for connecting "c" on switch 1

Hence, we have 3×2 possibilities of switching the bulb on. Thus, 6 possibilities.

Example

Suppose $A = \{a, b, c\}$ and $B = \{1,2\}$. How many ways can we choose 2 members from the 2 sets such that one member is from set A and another from set B.

Solution

Since making a pair requires choosing any member from set A and another from set B, then we have 3 choices from set A followed by 2 choices from set B, making it a total of 3×2 pairs.

THEORY

Let A and B be two finite sets. The number of ways of choosing pairs of members such that that one member comes from A and the other from B is

$n(A) \times n(B)$ Or $|A| \times |B|$

Where " $n(A) \times n(B)$" means number of members in set A multiply by number of members in set B or "$|A| \times |B|$" means modulus of A multiply by B.

This is called the PRODUCT RULE/MUTLIPLICATION RULE OF COUNTING

Example

A road traffic authority issues vehicle license number plates of two letters followed by any 4 digits. What is the maximum numbers of vehicles they can issue number plates on?

Solution

Let's suppose we consider six places/positions where the first two positions are to be filled by any two letters and the last four places are to be filled by any digits.

$$1st\ ,2nd, 3rd, 4th, 5th, 6th$$

Since we have 26 letters from English Alphabet, position 1 can have a chance of any of the 26 possible letters and so is with position 2 since it's possible to have a license number plate with repeated letter e.g. KK 6606. Then for the remaining four positions, any digit 0 up to 9 can fill in the places, with possibilities of repetitions as well, such as KK 7777. Hence, each of the remaining four positions can be filled by 10 digits$\{0,1,2,3,4,5,6,7,8,9\}$.

Hence, applying the product rule, we have

$$26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^2 \times 10^4 = 6{,}760{,}000$$

In the next example, suppose one wishes to count number of 4-digit whole numbers. Before we proceed looking at the solution, let's first remind ourselves of what it means to have either a one-digit, two-digit, three-digit or n-digit number which is formed from the 10 digits 0 up to 9.

Any n-digit number, where n is the length of such number, e.g. 345, a number of length/positions 3, is a number formulated from digits 0 up to 9 with repetitions allowed but with one major restriction, i.e. a number should not begin with a zero digit. This is the case following the rule of naming number in accordance with "Thousands", "Hundreds", "Tens" and "Ones/Units" i.e. $Th\, H\, T\, O$

By such principle, then one can strongly argue that while "19" is a two-digit number, "09" is NOT a two-digit number but rather a one-digit number. Let's proceed by now looking at the following example.

Example

How many 4-digit whole numbers are there?

Solution

Here, the same principles that have been discussed before shall apply where a number such as 0123 does not qualify to be called a 4-digit number but rather a 3-digit number. Hence, formulating an n-digit number there requires having n-positions where the "first" position does not have a chance of "a digit zero" appearing on it, while the rest of the $n-1$ remaining positions, any digit can stand a chance of being placed. Thus for

$$Th\, H\, T\, O$$

Only digits 1 up to 9 have a chance of appearing on "Th" while any of the 10 digits $\{0,1,2,3,4,5,6,7,8,9\}$ can appear on the remaining positions "H", "T" and "O". Hence, applying the product rule, we have

$$9 \times 10 \times 10 \times 10 = 9 \times 10^3 = 9{,}000$$

Thus, we have got 9000 four-digit whole numbers in total.

Such an example demonstrates the possible number of 4-digit whole numbers we can get, if repetition of digits at any of the positions is allowed. How about if repetition is not allowed? Let's see the following example.

Example

How many 4-digit integers (positive integers) are there such that no digit is repeated?

Solution

Without loss of generality, we shall again assign "Th", followed by "H", followed by "T", and finally "U" and consider each way we can fill these positions while keeping in mind of the restrictions.

$$Th\,H\,T\,O$$

In the “Th” position, the default restriction of not involving a digit “zero” is to be followed first. Hence we have 9 ways of filling the “Th” position using digits 1 up to 9. However, for the “H” position, we first need to remove one digit which should have been placed on “Th” already since such a digit does not have to repeat itself on other positions if it was already placed on “Th” position first. Thus, taking out a digit from the set$\{1,2,3,4,5,6,7,8,9\}$, we then have 8 remaining digits to be placed on “H”. But, we also need to include a digit “zero” which had no chance on first position, hence we have 9 ways of placing a digit on “H” position. Next, would be the “T” position which should now have (9-1) remaining possible ways. Lastly, the Unit “O” position would have to be filled in (8-1) remaining possible ways, making it 9 9 8 7 .

Therefore, by applying the multiplication rule of counting, we have $9 \times 9 \times 8 \times 7 = 4536$ possible 4-digit whole numbers.

How about if we had already started by eliminating a digit zero from the 10-digits and only having 9 digits $\{1,2,3,4,5,6,7,8,9\}$ such that we are interested in forming a 9-digit whole number where repetition is not allowed as well? Let’s see the following example which may assist us into getting the answer to such a question.

Example

How many 9-digit positive whole numbers are there that use digits $1,2,3,\cdots,9$ such that no digit is repeated?

<u>Solution</u>

Applying same principles as in the previous examples, here we consider 9 positions

$$1st\,,2nd,3rd,4th,5th,6th,7th,8th,9th$$

Such that with no repetitions allowed, we have 9 ways on 1^{st} position, (9-1) ways on second position, and (9-1-1) ways on 3^{rd} position, and so on and so forth until we have got only one digit remaining for the last (9^{th}) position. Hence, applying the product rule of counting, we have

$$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362{,}880$$

But in first year college algebra, we may have come across a notation such as 5! to mean “5 factorial” defined by $5 \times 4 \times 3 \times 2 \times 1$. Likewise, one can say that the total number of 9-digit positive whole numbers that one can form from digits 1 up to 9 such that no digit is repeated is basically $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 9!$

Example

How many different ways can you arrange n different items on a straight line?

Solution

Suppose we draw a line where these items are to be arranged, having boxes labelled 1, 2, up to n as below;

$$1 - 2 - 3 - - - - - - (n-2) - (n-1) - (nth)$$

Let's think of n items to be contained in a box. We put them on the 1st line by choosing one at a time and placing them on positions $1,2,3,\cdots,n$ respectively. The possibilities are

$$n \; n-1 \; n-2 \; \cdots 2 \; 1$$

Where by the product rule of counting, we have $n(n-1)(n-2) \times \cdots \times 3 \times 2 \times 1$.

NOTATION

Remember that $n! = n(n-1)(n-2) \times \cdots \times 3 \times 2 \times 1$. Hence the problem of counting number of times one can arranged n different items on n different positions is basically $n!$

Let us consider another example of the similar manner below

Example

Suppose we have the digits $1,2,3 \cdots,9$. How many 4-digit positive integers are there with no digit being repeated?

Solution

Just as in previous examples, our plan would be to consider four positions, labelled "Th", "H", "T" and "O" for Thousand, Hundreds, Tens and Units, respectively, where to assign the digits so as to come up with a 4-digit number.

$$Th \; H \; T \; O$$

Since, no repetition is allowed, then we have 9 possible digits for "Th" position, (9-1) possible digits for "H" position, (9-1-1) possible digits for "T" position and lastly (9-1-1-1) possible digits for "O" position, making it $9 \times 8 \times 7 \times 6$ options of integers, by multiplication rule of counting

Activity 1 a

a) Calculate the number of 3-digit numbers that one can form using the odd digits.

b) Calculate the number of ways of choosing 2 boys from 6 boys and 3 girls from 7 girls.

c) Calculate the number of 6-letter words (not necessarily meaningful words) that one can form using the English alphabet such that the word has two vowels.

d) A four-digit positive integer is formed using the digits 1, 2, 3, 5, 6, 7 and 9. Calculate the total number of four-digit positive even integers that can be formed.

e) Suppose we have seven different courses offered in the morning and five different courses offered in the afternoon for those willing to do a Diploma in Statistics course at Chancellor College. How many choices would one student have if he/she wishes to enroll in one course in the morning and one in the afternoon?

Permutation

With reference to the last example in lesson 1.1, here, we wish to deduce a formula that can also be used as a counting technique principal which we may have been introduced in the first year, under college algebra. The principal of permutation.

Example

How many sequences of r letters are there where the r letters are chosen from n different letters and no letter is repeated $(r \leq n)$

Solution

This problem is equivalent to that of formulating a 4-digit sequence from 1 to 9 digits where no digit should be repeated. Say, for instance, if we consider the r positions as

$$1 - 2 - 3 - - - - - - (r-2) - (r-1) - (rth)$$

Such that we have

- n possibilities of a letter appearing on position 1
- $n-1$ possibilities of a letter appearing on position 2
- $n-2$ possibilities of a letter appearing on position 3
- $n-3$ possibilities of a letter appearing on position 4

$\vdots$ $\vdots$

- $n-(r-1)$ possibilities of a letter appearing on position r [by reading the pattern]

Thus, by multiplication rule of counting, we have

$n(n-1)(n-2)(n-3) \times \cdots \times (n-r+1)$ possible sequences.

But we may recall from the concept of permutation in College Algebra course that $n(n-1)(n-2)(n-3) \times \cdots \times (n-r+1)$ is essentially P_r^n which defines "number of ways of choosing r different items from n different items and arranging (permutting) them on r positions.

Example

In P_2^3, here $n = 3$ and $r = 2$ such that from $P_r^n = n(n-1)(n-2)(n-3) \times \cdots \times (n-r+1)$, $(n-r+1) = (3-2+1) = 2$

Such that $P_2^3 = 3 \times 2 = 6$

Thus, if are to consider having three different letters A, B, C, then P_2^3 suggests that we choose 2 letters from A, B, C at a time and place them as a sequence. Hence if we consider $r = 2$ positions where to permute any chosen 2 letters at a time, then for the possible pairs $(AB), (AC)$ and (BC), we have 2! possible arrangements for each pair, making it a total of $3 \times 2! = 6$ options.

ALTERNATIVE FORMULA FOR P_r^n

Since $P_r^n = n(n-1)(n-2)(n-3) \times \cdots \times (n-r+1)$, then taking $n! = n(n-1)(n-2) \times \cdots \times (n-r+1) \times (n-r) \times (n-r-1) \times (n-r-2) \times \cdots \times 3 \times 2 \times 1$ and dividing by $(n-r) \times (n-r-1) \times (n-r-2) \times \cdots \times 3 \times 2 \times 1$, we have $n(n-1)(n-2) \times \cdots \times (n-r+1)$, then we have

$$n(n-1)(n-2) \times \cdots \times (n-r+1)$$
$$= \frac{n(n-1)(n-2) \times \cdots \times (n-r+1) \times (n-r) \times (n-r-1) \times (n-r-2) \times \cdots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times (n-r-2) \times \cdots \times 3 \times 2 \times 1}$$

But, we know that $n(n-1)(n-2) \times \cdots \times (n-r+1) = P_r^n$, $n(n-1)(n-2) \times \cdots \times (n-r+1) \times (n-r) \times (n-r-1) \times (n-r-2) \times \cdots \times 3 \times 2 \times 1 = n!$ and $(n-r) \times (n-r-1) \times (n-r-2) \times \cdots \times 3 \times 2 \times 1 = (n-r)!$, then

$$P_r^n = \frac{n!}{(n-r)!}$$

Hence, conventionally, we can say that

- $P_r^n = \frac{n!}{(n-r)!}$ is permutation without repetition,
- $P_n^n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$, where $0! = 1$ (as permutation with repetition)

Example

In how many ways can 3 persons be born in an ordinary year so that no two persons should share the same birthday?

Solution

This problem is equivalent to that of formulating sequences of r items from n distinct items such that there should be no repetition, which should essentially be $P_r^n = \frac{n!}{(n-r)!}$.

Thus for $n = 365$ days and $r = 3$ persons, then to have different birthdays, we have to choose 3 different days from 365 days and permute them on the three persons.

Therefore, $P_3^{365} = \frac{365!}{(365-3)!} = \frac{365!}{362!} = 365 \times 364 \times 363 = 48{,}228{,}180$

Activity 1 b

a) In how many ways can we place three balls coloured red, blue, and white in ten boxes numbered if each box can hold only one ball at a time?

b) In how many ways can three examinations be scheduled within a five-day period so that no two examinations are scheduled in the same day?

c) Determine the number of four-digit decimal numbers that contain no repeated digits.

d) Suppose that we have seven rooms and want to assign four of them to four programmers as offices and use the remaining three rooms for computer terminals. In how many ways can you carry out such an assignment procedure?

e) In how many ways can one make up strings of four distinct letters followed by three distinct digits?

Sum Rule of Counting

Here, our goal is to introduce another rule of counting for handling of joint events that occur independently, the sum rule of counting.

THEOREM

Given that A and B are two disjoint sets such that $A \cap B = \emptyset = \{\}$, then

$$|A \cup B| = |A| + |B|$$

This is called the SUM RULE OF COUNTING

Let's look at this example below to see how both the multiplication rule of counting and sum rule of counting can apply.

Example

How many 5-digit integers (positive integers) are there where the last digit is an odd number or a 4.

Solution

If we are to consider five positions where to place any of these integers as $1,2,3,4,5$ then for the principal of formulating digit numbers of certain length, the restriction of not including a "zero" digit on first position such that we have $\{1,2,3,4,5,6,7,8,9\}$ potential digits to be placed on first position. On the last position, we are asked to have an odd number or a 4, i.e. $\{1,3,5,7,9\}$ or a $\{4\}$, having a total of 6 potential digits to be placed on last position. However, for the 2nd, 3rd, and 4th positions, we have a potential of all the 10 digits appearing on them.

Hence in total, we have the following options, $9\ 10\ 10\ 10\ 6$, such that by multiplication rule of counting, we have $9 \times 10 \times 10 \times 10 \times 6 = 54{,}000$ possible digits.

ALTERNATIVELY

Let A_1 be the set of all 5-digit integers that have an odd number as a last digit

Let A_2 be the set of all 5-digit positive integers that end with a 4.

We want to count a number of members in A_1 or A_2. Thus, we want $|A_1 \cup A_2|$. Clearly $A_1 \cap A_2 = \emptyset$ because there cannot be a 5-digit number that can end with both an odd number and a 4.

Therefore, by sum rule of counting $|A_1 \cup A_2| = |A_1| + |A_2|$.

Hence, for all those that end with an odd number, we have by multiplication rule of counting $|A_1| = 9 \times 10 \times 10 \times 10 \times 5 = 9 \times 10^3 \times 5 = 45{,}000$. Again, for those that end with a 4, we have $|A_2| = 9 \times 10 \times 10 \times 10 \times 1 = 9 \times 10^3 \times 1 = 9{,}000$.

Thus, by the sum rule, $|A_1 \cup A_2| = |A_1| + |A_2| = 45{,}000 + 9{,}000 = 54{,}000$

Let's divert a little bit and look at the aspect of merely choosing r items from n different items and without necessarily permuting them.

Example

How many ways can one choose 2 different items from 3 different items?

Solution

As of now, we only know that we can choose r items from n different items and permute them, which should have been P_2^3. But this is NOT what we want here.

We may wish to recall that the problem of choosing 2 items from 3 different items and permuting them was dealt with before but in a slightly different manner as in that of permuting 2 letters from three letters namely A, B, C, where we had exhaustively three choices to make

- Choice of A, B and permuting/re-arranging them to have $2! = 2$ options
- Choice of A, C and permuting them to have again $2! = 2$ options
- Choice of B, C and permuting them to have again $2! = 2$ options

Such that we had $P_2^3 = 6$, i.e. $(2! + 2! + 2!)$

However, the question required us only to count the number of r set of choices we make, i.e. AB, AC, and BC. Hence, in total we only have 3 choices of $r = 2$ items from $n = 3$ different items.

Therefore, the answer is 3.

What about if we want to choose 3 items from 4 items?

We know P_3^4, using the sum rule, we have

- 1st choice of 3 items, resulting into 3! permutations
- 2nd choice of 3 items, resulting into 3! permutations
- 3rd choice of 3 items, resulting into 3! Permutations

  $\vdots$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\vdots$

- x^{th} choice of 3 items, resulting into 3! Permutations (where this is the last choice to make)

If our goal is to count the number of choices of 3 items without permuting, then

$P_3^4 = 3! + 3! + \cdots + 3!$ (where the right hand side terms occur x times). Therefore

$$P_3^4 = x3!$$

Such that $x = \frac{P_3^4}{3!}$

But $P_3^4 = \frac{4!}{(4-3)!}$. Hence, $x = \frac{4!}{(4-3)!3!}$

Hence, for the problem of choosing 3 items from 4 items, we have $\frac{4!}{(4-3)!3!}$ number of ways.

In general, what would be if we wanted to count number of ways we can choose r different items from n distinct items?

Solution

Suppose we adopt the notation C_r^n to stand for the number of ways of choosing r items from n. Then we want to find C_r^n, but only with knowledge of P_r^n.

Just as in the previous example, we shall approach the problem of counting number of times we choose through summing different x possible r permutations, i.e.

- 1st choice of r items gives rise to $r!$ permutations
- 2nd choice of r items gives rise to $r!$ permutations
- 3rd choice of r items gives rise to $r!$ permutations

  $\vdots \qquad\qquad\qquad \vdots$

- x^{th} choice of r items gives rise to $r!$ permutations

Such that $P_r^n = r! + r! + r! + \cdots + r!$, (where the right hand side terms occur x times).

Hence, $P_r^n = xr!$

Then $x = \frac{P_r^n}{r!}$. Hence, $x = \frac{n!}{(n-r)!r!}$

But we should remember that x counts the number of choices of r items which is essentially, C_r^n.

Therefore, we can as well say $C_r^n = \frac{n!}{(n-r)!r!}$, as out desired formula for counting number of choices of r items from n items.

Activity 1 c

a) Calculate the number of ways of formulating four-digit positive integers using the digits 1,2,3,5,6,7 and 9 such that the integers are even or odd.

b) Determine possible number of six letter sequences formed from a set $\{A, B, C, \cdots, G\}$ such that either BD or BG appears together.

c) A class has 10 girls and 8 boys. Members of this class enter a competition draw where a single prize of an air ticket for two to South Africa is to be won. Given that the two winners have to be of the same sex, calculate the different number of ways the prize can be won.

d) Suppose we have seven different courses offered in the morning and five different courses offered in the afternoon for those willing to do a Diploma in Statistics course at Chancellor College. How many choices would one student have if he/she wishes to enroll in only one course?

e) In how many ways can you select a representative for the junior class and a representative for the senior class in some committee if either of the members are from a junior class of 52 students or a senior class of 49 students?

Combinations

We now discuss the principles of counting, namely combinations and permutations in detail. We first present the notations and symbols to be familiar with throughout this section as follows;

NOTATION

$$C_r^n = \binom{n}{r}$$

NAMES

$P_r^n \rightarrow$ Permutations

$C_r^n \rightarrow$ Combinations

Example

How many ways can one form a sequence of 4 $a's$ and 3 $b's$?

Solution

A typical sequence would be as : $abaaabb$.

Let's have the positions of the letters as shown

$$1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7th$$

A sequence; $abaaabb$ can be gotten by choosing positions 1,3,4,5, to allocate $a's$ with the understanding that the unchosen positions will automatically be $b's$.

Thus in general, the problem of choosing any 4 positions from 7 positions to allocate $a's$ would be C_4^7. If $b's$ were in consideration first, we would similarly get C_3^7.

Thus

$$C_4^7 = \frac{7!}{(7-4)!\,4!}$$

$$= \frac{7 \times 6 \times 5 \times 4!}{3! \times 4!}$$

$$= \frac{7 \times 6 \times 5}{3!}$$

$$= \quad 35$$

$$\text{Or } C_3^7 = \frac{7!}{(7-3)!3!}$$

$$= \frac{7 \times 6 \times 5 \times 4!}{4! \times 3!}$$

$$= \frac{7 \times 6 \times 5}{3!}$$

$$= \quad 35$$

THEOREM

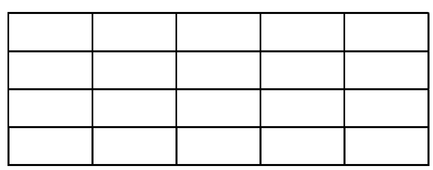
$$\binom{n}{r} = \binom{n}{n-r}$$

Choosing r items from n is equivalent to choosing $n - r$ unwanted items from n to discard.

Hence $C_r^n \equiv C_{n-r}^n$

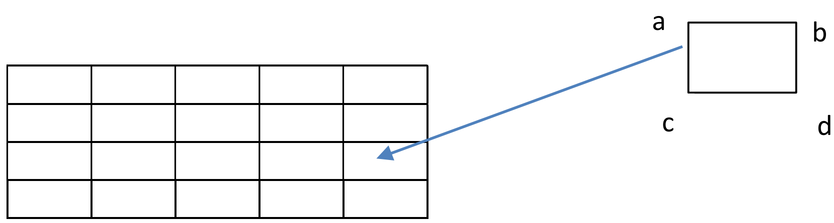
Example

Use counting techniques to find the total number of rectangles in the figure below:



<u>Solution</u>

Let's label the horizontal lines as 1,2,3,4,5 going downwards and vertical lines as P, Q, R, S, T, U going to the right as in the figure below and also spot any rectangle a, b, c, d extracted from the figure.



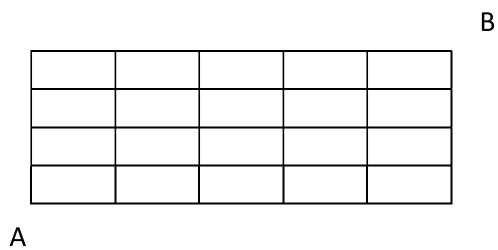
A rectangle a, b, c, d can be gotten by choosing S and T among the vertical lines and choosing 3 and 4 lines among the horizontal lines, i.e. the point of intersection forms a rectangle a, b, c, d.

Thus, every rectangle is UNIQUELY determined this way; by choosing 2 horizontal and 2 vertical lines. In our diagram, the number of ways of choosing 2 horizontal lines is C_2^5 and that of choosing 2 vertical lines is C_2^6

Hence, by product rule of counting, we have $C_2^5 \times C_2^6$ ways of choosing 2 horizontal lines and 2 vertical lines, which is equivalent to getting the total number of rectangles in the diagram.

Example

Let's consider the same diagram as before, and now translate the lines as straight roads intersecting at right angles. If someone wants to travel along the roads from junction A to junction B in the shortest route, in how many ways can this be done?



<u>Solution</u>

Take a typical route, say that of moving straight from point A to the right, passing through "5 Easting slots" up to the far right-bottom corner and then move upwards, passing through "4 Northing slots" to reach point B. Then such a route is $EEEEENNNN$, where E stands for "Easting slot" and N, for "Northing slot", as per campus direction.

Alternatively, if you decided to move from point A, straight upwards and then turn right, move straight to point B, then you may have a route defined by $NNNNEEEEE$.

But if you decided otherwise, by starting from point A, move up one Northing slot, then turn right, move one Easting slot, then turn up for another Northing slot, in a zig-zag manner up until you hit point B, then your route probably becomes $NENENENEE$.

Thus in either way you may think of moving, your route is probably described as a sequence of "$E's$" and "$N's$" e.g. $EEEEENNNN$ or $NENENENEE$. Another thing to note is that in either case, each sequence is made up of 5 E's and 4N's.

Therefore, to count the number of shortest routes that one would possibly take, we need only to count the total number of sequences of 5 E's and $4N$'s, i.e.

$$C_4^{4+5} = C_4^9 \equiv C_5^9 = 126$$

Now let's turn to something else, where the principal of counting can also be used to count the possible number of increasing or strictly increasing sequences. But before we do that, let's first define or remind us ourselves of these two definitions

Definition:

1) a sequence is said to be STRICTLY increasing if say for the terms in the sequence $a_1, a_2, a_3, a_4, \cdots$, one has $a_n < a_{n+1}, \ \forall\, n = 1,2,3, \cdots$
2) a sequence is said to be INCREASING if and only if $a_n \leq a_{n+1}, \ \forall\, n = 1,2,3, \cdots$

Thus a sequence $7,7,8,9,10,11,13$ is NOT strictly increasing but rather increasing. On the other hand, the sequence 1,3,4,5,7 is a STRICTLY increasing sequence.

Now let's turn back to our original problems, which of counting possible number of strictly increasing or just increasing sequence.

Example

How many strictly increasing sequence of r terms can one form from the numbers $1,2,3, \cdots, n$?

<u>Solution</u>

Since by definition, in a strictly increasing sequence, $a_n \neq a_{n+1}$ in a $a_n < a_{n+1}$ manner, then, any choice of r different numbers from $\{1,2,3, \cdots, n\}$ gives rise to One-and-only-ONE strictly increasing sequence, when arranged in ascending order.

So the problem is equivalent to that of choosing r different items from n different items, i.e. C_r^n strictly increasing sequences.

How about that of counting the possible number of merely increasing sequences? Let's respond to such a question by looking at the principle of "non-negative solutions" to a linear equation of countable number of variables. Consider the example below for easier of understanding what we want to learn from it.

Example

How many non-negative integer solutions has the equation $a + b + c + d = 10$?

Solution

Since from the equation, any letter can take up any non-negative integer value, as long as they SUM up to 10, then potentially, we can have the following possible combinations

$$\begin{array}{cccc} a & b & c & d \\ 0 & 0 & 0 & 10 \\ 1 & 1 & 1 & 7 \\ -1 & -2 & 0 & 13 \end{array}$$

Is not part of this since we want non-negative values ONLY

Now think of a, b, c, d as 4 boxes and 10 as pencils such that we could be dropping pencils in such 4 boxes.

$$a \;\; b \;\; c \;\; d$$

Alternatively, if we arrange the 10 pencils on a straight line and introduce 3 splitters, e.g. for $a = 2$, $b = 3$, $c = 4$ and $d = 1$, it can be represented as

$$\uparrow\uparrow\ \Delta\uparrow\uparrow\uparrow\ \Delta\uparrow\uparrow\uparrow\uparrow\ \Delta\uparrow$$

For $a = 1$, $b = 0$, $c = 5$ and $d = 4$, it can be represented as

$$\uparrow\ \Delta\ \ \Delta\uparrow\uparrow\uparrow\uparrow\uparrow\ \Delta\uparrow\uparrow\uparrow\uparrow$$

If we then regard the splitters as $\uparrow$ as well, then we have $10 + 3 = 13$ pencils altogether. Hence, the solution of seeking for possible numbers of pencils in each of the 4 boxes to sum, as long as they sum up to 10, is equivalent to that of choosing 3 pencils among the 13 to change into splitters and come up with possible numbers.

Therefore, C_3^{13} are the total possible ways. Hence our equation, $a + b + c + d = 10$ has got

$$\binom{13}{3} \equiv \binom{n+k-1}{k-1}$$

total possible solutions where for $n = 10$, $k = 4$.

THEOREM

The number of non-negative integer solutions to the equation $x_1 + x_2 + x_3 + \cdots + x_k = n$ is

$$C_{k-1}^{n+k-1} \equiv \binom{n+k-1}{k-1}$$

Example

In how many ways can n indistinguishable items be put in k containers?

Solution

Let x_j be the number of items in container j, such that for $j = 1,2,3,\cdots,k$

$$x_1 + x_2 + \cdots + x_k = n$$

Therefore, we want the number of non-negative integer solutions to the equation $x_1 + x_2 + \cdots + x_k = n$

i.e. $\binom{n+k-1}{k-1}$.

Now we can possibly turn back to our problem of interest, and answer back the question of fining the number of increasing sequences of r integers (terms) formed from the numbers $1,2,3,\cdots,n$.

Solution

Let's consider some typical sequences of r terms such as

$1111111\cdots$ or $11122\cdots33$

Let $a_1, a_2, a_3, \cdots, a_r$ be an increasing sequence that uses numbers $1,2,3,\cdots,n$ and let x_i be the number of times number i is repeated on the sequence $a_1, a_2, a_3, \cdots, a_r$ for all $i = 1,2,\cdots,n$. For instance, in a typical sequence $1111222\cdots33555\cdots n$, then $x_1 = 4$, $x_2 = 3$, $x_3 = 2$, $x_4 = 0$, $x_5 = 3$, and so on and so forth.

Therefore,

$$x_1 + x_2 + \cdots + x_n = r$$

Thus in a problem of counting number of increasing sequences, we basically want the number of non-negative integer solutions to the equation $x_1 + x_2 + \cdots + x_n = r$, i.e.

$$C_{n-1}^{r+n-1} \equiv \binom{r+n-1}{n-1}$$

Activity 1 d

(a) List all the possible number of members in a subset formed from the set, $\{1, 2, 3, \ldots, m\}$.

(b) A netball league consists of 20 teams. A tournament is to be held so that each team plays every other team once on home ground and once on away ground. Calculate the maximum possible number of games that will be played in the tournament (a) using first principles (b) applying some counting formulae.

(c) Deduce, via combinatorial argument, that the total possible number of subsets formed from the set $\{a_1, a_2, a_3, \ldots, a_k\}$ is ${}^kC_0 + {}^kC_1 + {}^kC_2 + \ldots. + {}^kC_{k-1} + {}^kC_k = 2^k$.

(d) How many non-negative integer solutions has the equation $x + y + z = 30$ got?

(e) How many ways can one share twenty K1 coins to John, Yohane and Johane?

Binomial Theorem of Expansion

BINOMIAL THEOREM

Here, we wish to remind ourselves of the concept of binomial theorem that we may have probably already encountered in first year college algebra course. The famous binomial theorem of expanding the terms of the form $(a+b)^n$ is stated as

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$$

$$= \binom{n}{0} x^0 a^{n-0} + \binom{n}{1} x^1 a^{n-1} + \binom{n}{2} x^2 a^{n-2} + \cdots + \binom{n}{n-1} x^{n-1} a^1 + \binom{n}{n} x^n a^0$$

Alternatively, is the Pascal's triangle, which can also be used to expand such problems of the form $(a+b)^n$ where the most important thing is to determine the coefficients $\binom{n}{r}$ of the terms $a^r b^{n-r}$ in advance through the triangle of different levels of n powers. Let's see how we arrive at the Pascal's triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Where

Power	Expression	Coefficients
$n=1$	$(a+b)^1$	1 1
$n=2$	$(a+b)^2$	1 2 1
$n=3$	$(a+b)^3$	1 3 3 1
$n=4$	$(a+b)^4$	1 4 6 4 1
$n=5$	$(a+b)^5$	1 5 10 10 5 1
$n=6$	$(a+b)^6$	1 6 15 20 15 6 1

Puzzle

Let's think of this interesting puzzle. Suppose you come in a certain class room where there are n students in total and you decide to choose any r group of students. Suppose further that in such a class, there is a boy named "John". In how many ways can you choose this group of r students from n?

<u>Solution</u>

We know that the problem of just making random choice of r from n is merely that of $\binom{n}{r}$. however, in this group of r chosen students, two possibilities arise,

- John is not among the chosen r or
- John is among the chosen r

Hence, by the SUM RULE of counting, choosing a group of r students from n has the possibitlity of John being there plus possibility of John not being there. Thus we have two cases

- Case 1: John is not among the chosen r
- John is among the chosen r

Such that

$$\binom{n}{r} \equiv CASE\ 1 \quad + CASE\ 2$$

CASE 1: Choosing r items without John. Exclude John and choose r individuals from the remaining $n-1$ persons, i.e. the required number is $\binom{n-1}{r}$.

CASE 2: Choosing r items with John INCLUSIVE. Choose John first and then look for the remaining $r-1$ individuals from the remaining $n-1$ persons, i.e. the required number is $\binom{n-1}{r-1}$.

Therefore,

$$\boxed{\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}}$$

PASCAL'S TRIANGLE

Both, the binomial theorem of expansion and Pascal's triangle can be used to determine specific desirable coefficients of the terms $a^r b^{n-r}$ in an expansion $(a+b)^n$.

Example

Find the coefficient of x^3y^4 in the expansion of $(x+y)^7$

Solution

Common sense tells us that $(x+y)^7$is essentially the process of multiplying $(x+y)$ terms, 7 times such that to have $x^3y^4 \equiv xxyyxyy$ simply means x appears three times, i.e. 3 brackets and so does y appear 4 times, i.e. 4 brackets.

Therefore, to have x^3y^4, then we would need to count total number of ways of choosing 3 x's from 3 brackets out of 7 brackets and have the rest of the remaining $(7-3)$ brackets contributing y's Or count total number of ways of choosing 4 y's from 4 brackets out of 7 brackets and have the rest of the remaining $(7-4)$ brackets contributing x's.

Thus,

$$\binom{7}{3} \quad or \quad \binom{7}{4}$$

Hence, the coefficient of x^3y^4 is 35.

Example

Find the coefficient of x^3y in the expansion of $\left(x^{\frac{1}{2}}+2y\right)^7$

Solution

Looking at our term of interest, and the variables in the expansion $\left(x^{\frac{1}{2}}+2y\right)^7$, we should be convinced that

- y comes from a $2y$ hence it come from 1 bracket
- x^3 must have originated from $x^{\frac{1}{2}}$. Hence, $x^{\frac{1}{2}}$ must have been chosen from the remaining 6 brackets

Since $x^3 = x^{\frac{1}{2}} \times x^{\frac{1}{2}} \times x^{\frac{1}{2}} \times x^{\frac{1}{2}} \times x^{\frac{1}{2}} \times x^{\frac{1}{2}}$

Therefore, the corresponding term was

$$\binom{7}{6}\left(x^{\frac{1}{2}}\right)^6 (2y)^1$$

$$= \binom{7}{6} \times 2 \times x^3y$$

Hence, the coefficient of x^3y is $\binom{7}{6} \times 2 = 7 \times 2 = 14$.

THEOREM

For any non-negative integer $n \geq 0$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Proof

Using the binomial theorem of expansion, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Then, let $2^n \equiv (a+b)^n$. Then without loss of generality, $2^n \equiv (1+1)^n$such that

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i}$$

where $a = 1$ and $b = 1$. Hence

$$\sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i}$$

$$= \binom{n}{0} 1^0 1^{n-0} + \binom{n}{1} 1^1 1^{n-1} + \binom{n}{2} 1^2 1^{n-2} + \cdots + \binom{n}{n-1} 1^{n-1} 1^1 + \binom{n}{n} 1^n 1^0$$

But then, a number 1 raised to any power is basically 1. Hence any term of the form $1^r 1^{n-r}$ is equal to 1. Thus we have

$$\sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Alternatively, suppose we have a set of numbers $\{1,2,3,\cdots,n\}$ and we are interested to know how many subsets one can make out of such a set.

Applying the SUM RULE of counting, formulation of subsets has the following possible make

- Some subsets have 0 members e.g. an empty set
- Some subsets have 1 member e.g. $\{n-2\}$
- Some subsets have 2 members e.g. $\{2, n-2\}$
- Some subsets have 3 members e.g. $\{3, n-1, n-3\}$

$\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$

- Some subsets have n members.

Thus for each subset, we make r choice of members out of n elements for $0 \leq r \leq n$. Hence, in total (by SUM RULE), we have

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

Applying MULTIPLICATION Rule,

For each of the n elements, we have got two choices that we make each time we need to formulate a subset; choosing an element or leaving it. Thus, if we consider aligning the n elements on a straight line,

$$1\,, 2, 3, \cdots, n-1, n$$

Then for n positions, we have 2 choices on each to obtain

$$2\,, 2, 2, \cdots, 2, 2$$

Thus by multiplication rule of counting, we have $2 \times 2 \times 2 \times \cdots \times 2 = 2^n$.

Previously, we were able to work out on the problem of counting the number of sequences one would have from say $4a$'s and $3b$'s to be $C_4^7 \equiv C_3^7$. However, what if the group of letters go beyond 2, say formulating sequences from $4a$'s, $3b$'s and $2c$'s?

Example

How many sequences can be formed from $4a$'s, $3b$'s and $2c$'s?

Solution

A sequence of such form would be $aaaabbbcc$ or $ababababcc$. Thus, in which ever way we think a sequence would be, we have $4 + 3 + 2 = 9$ letters to be allocated on 9 positions such that 4 positions are reserved from allocating a's; 3 positions are reserved from allocating b's; and remaining 2 positions are reserved from allocating c's.

But allocation of a's is independent of allocation of b's and c's. Thus, if a's are allocated first, then we have

$$\binom{9}{4}$$

for allocating b's, we have

$$\binom{9-4}{3} \equiv \binom{5}{3}$$

And lastly, for allocating c's , we have

$$\binom{9-4-3}{2} \equiv \binom{2}{2}$$

Hence, by the multiplication rule of counting, we have

$$\binom{9}{4} \times \binom{5}{3} \times \binom{2}{2} = 126 \times 10 \times 1 = 1260$$

Therefore, we have 1260 possible sequences formed from $4a$'s, $3b$'s and $2c$'s.

Activity 1 e

a) Find the coefficient of x^2y^2 in the expansion of $\left(2x + y^{\frac{1}{3}}\right)^8$.

b) Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \ldots + \binom{n}{r}\binom{m}{0}, \; 0 \le r \le n, 0 \le r \le m.$$

[**HINT:** Use **(a)** combinatorial argument and **(b)** binomial expansions of $(1+x)^{n+m} = (1+x)^n(1+x)^m$

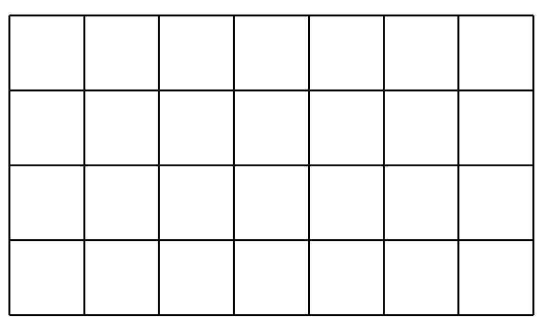
c) How many sequences would one form from 5 $T's$ and 3 $W's$?

d) Use (a) combination formulae and (b) a combinatorial argument, to prove that

$$\binom{m}{n}\binom{n}{r} = \binom{m}{r}\binom{m-r}{n-r}.$$

[**HINT:** For combinatorial approach, relate the number of ways of choosing *n* different persons from *m* persons to put on a set of *n* indistinguishable uniforms (every person putting one uniform) and then choosing *r* of the *n* worn uniforms to stick indistinguishable label on them to the l.h.s. of the equation. Study the r.h.s. and explain an equivalent way of achieving the same thing by starting with choosing *r* persons that are to wear uniforms that will be stuck with the label]

e) Suppose the figure below was showing a floor with square tiles. Given that you have to colour 7 tiles red; 9 tiles black and 12 tiles white, calculate the total number of different ways the floor can be coloured.



Multinomial Theorem of Expansion

You may recall that the last example under lesson 5 was a problem where we actually knew the type of letters in use, and the numbers in them, i.e. $4a$'s, $3b$'s and $2c$'s. How about if we had no clue about which letters and how many they are in each category, but just in general, for a_j letters, each occurring in n_j number of times for $1 \leq j \leq k$, for some non-negative integer j? Let us look at this example.

Example

Consider the problem of counting possible number of sequences formed from a_j letters such that each letter occurs n_j number of times for $1 \leq j \leq k$, for some non-negative integer j, i.e.

Letter	Number of repetitions
a_1	n_1
a_2	n_2
a_3	n_3
$\vdots$	$\vdots$
a_k	n_k

How many sequences can we have?

<u>Solution</u>

Here, we have each letter a_j occurring in n_j times, e.g. for the sequence that was formed from $4a$'s, $3b$'s and $2c$'s, we would regard $a_1 \equiv a$ and $n_1 = 4$; $a_2 \equiv b$ and $n_2 = 3$, and $a_3 \equiv c$ and $n_3 = 2$, such that the total number of allocations for such a sequence would be $4 + 3 + 2 \equiv n_1 + n_2 + n_3$.

Thus, using the reasoning in the previous example, here in our generalized problem, we need to use $n_1 + n_2 + n_3 + \cdots + n_{k-1} + n_k$ allocations for which

STEP 1: Allocating a_1's gives

$$\binom{n_1 + n_2 + n_3 + \cdots + n_{k-1} + n_k}{n_1}$$

STEP 2: Allocating a_2's gives

$$\binom{n_2 + n_3 + \cdots + n_{k-1} + n_k}{n_2}$$

STEP 3: Allocating a_3's gives

$$\left(\frac{n_3 + \cdots + n_{k-1} + n_k}{n_3}\right)$$

STEP $k - 1$: Allocating a_{k-1}'s gives

$$\left(\frac{n_{k-1} + n_k}{n_{k-1}}\right)$$

STEP $k - 1$: Allocating a_k's gives

$$\left(\frac{n_k}{n_k}\right)$$

Hence, by multiplication rule of counting, we have

$$\left(\frac{n_1 + n_2 + n_3 + \cdots + n_{k-1} + n_k}{n_1}\right) \times \left(\frac{n_2 + n_3 + \cdots + n_{k-1} + n_k}{n_2}\right) \times \left(\frac{n_3 + \cdots + n_{k-1} + n_k}{n_3}\right) \times \cdots \times \left(\frac{n_{k-1} + n_k}{n_{k-1}}\right) \times \left(\frac{n_k}{n_k}\right)$$

$$= \frac{(n_1 + n_2 + n_3 + \cdots + n_{k-1} + n_k)!}{(n_2 + n_3 + \cdots + n_{k-1} + n_k)!\, n_1!} \times \frac{(n_2 + n_3 + \cdots + n_{k-1} + n_{k)!}}{(n_3 + \cdots + n_{k-1} + n_k)!\, n_2!} \times \frac{(n_3 + \cdots + n_{k-1} + n_{k!})!}{(n_4 + \cdots + n_{k-1} + n_{k)!} n_3!} \times \cdots \times \frac{(n_{k-1} + n_k)!}{n_{k-1}!\, n_k!} \times \frac{n_k!}{n_k!\, 0!}$$

Now, by closely looking at all the NEIGHBOURING terms in the numerators and denominators, we see certain common terms e.g. $(n_2 + n_3 + \cdots + n_{k-1} + n_{k)!}$ appearing on numerator of second term, and at the same time, denominator of first term, and just as other terms. We can therefore perform some cancellation as

$$= \frac{(n_1 + n_2 + n_3 + \cdots + n_{k-1} + n_k)!}{\cancel{(n_2 + n_3 + \cdots + n_{k-1} + n_k)!}\, n_1!} \times \frac{\cancel{(n_2 + n_3 + \cdots + n_{k-1} + n_{k)!}}}{\cancel{(n_3 + \cdots + n_{k-1} + n_k)!}\, n_2!} \times \frac{\cancel{(n_3 + \cdots + n_{k-1} + n_{k!})!}}{\cancel{(n_4 + \cdots + n_{k-1} + n_{k)!}} n_3!} \times \cdots \times \frac{\cancel{(n_{k-1} + n_k)!}}{n_{k-1}!\, \cancel{n_k!}} \times \frac{\cancel{n_k!}}{n_k!\, 0!}$$

giving us

$$= \frac{(n_1 + n_2 + n_3 + \cdots + n_{k-1} + n_k)!}{n_1!\, n_2!\, n_3! \cdots n_{k-1}!\, n_k!}$$

Since $0! = 1$.

Thus, the number of number of sequences formed from a_j letters such that each letter occurs n_j number of times for $1 \leq j \leq k$, for some non-negative integer j is

$$\boxed{\frac{(n_1 + n_2 + n_3 + \cdots + n_{k-1} + n_k)!}{n_1!\, n_2!\, n_3! \cdots n_{k-1}!\, n_k!}}$$

Example (revisited)

How many sequences can be formed from $4a$'s, $3b$'s and $2c$'s?

Solution

Let $n_1 = 4$, $n_2 = 3$ and $n_3 = 2$ such that total number of sequences of $4a$'s, $3b$'s and $2c$'s would be

$$\frac{(n_1 + n_2 + n_3)!}{n_1!\, n_2!\, n_3!} = \frac{(4+3+2)!}{4!\, 3!\, 2!} = \frac{9!}{4!\, 3!\, 2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 6 \times 2} = 9 \times 4 \times 7 \times 5$$

Example

How many sequences can be formed from $3a$'s, $2b$'s and $4c$'s?

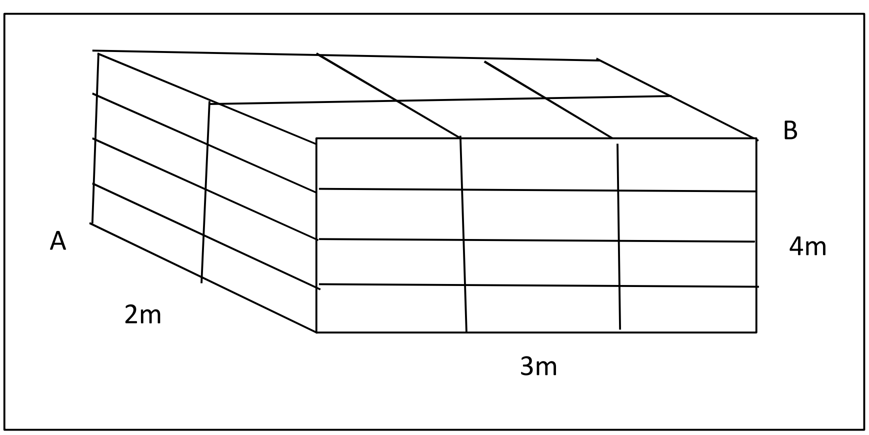
Solution

Similarly for $n_1 = 3$, $n_2 = 2$ and $n_3 = 4$, we have the number of sequences being

$$= \frac{(3+2+4)!}{3!\, 2!\, 4!} = \frac{9!}{3!\, 2!\, 4!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 6 \times 2} = 1260$$

Example

A 3 metre by 2 metre by 4 metre cuboid is formed by stacking 1m by 1m by 1m cubes as shown:



Let A and B be points on two furthest corners as shown. How many ways can one trace edges of cubes and travel from point A to B in the shortest distance possible?

Solution

Let h be a horizontal edge of a 1 by 1 by 1 cube

Let v be a vertical edge of a 1 by 1 by 1 cube

Let f be a forward edge of a 1 by 1 by 1 cube

Hence, to move from A to B, one needs 2 steps of h, 3 steps of f and 4 steps of v, e.g. $hhfffvvvv$ is a unique path. Thus, to find all unique shortest paths, we just need to count the number of sequences of $2h$'s, $3f$'s and $4v$'s, i.e.

$$\frac{(2+3+4)!}{2!\,3!\,4!}$$

Example

Find the coefficient of $x^2y^3z^4$ in the expansion of $(x+y+z)^9$

Solution

From the term we are interested in, we see that

- x^2 means x comes from 2 brackets in the expansion

$$(x+y+z)^9 = (x+y+z) \times (x+y+z) \times \cdots \times (x+y+z)$$

- y^3 means y comes from 3 of the remaining (9-2) brackets, and
- z^4 means z comes from 4 of the remaining (9-2-3) brackets

where getting $x^2y^3z^4$, is now equivalent to reading off the possible sequences generated from 3 different types of letters x, y, and z such that x appears 2 times, y appears 3 times, and z appears 4 times, which is equivalent to reading off the term;

$$\frac{(n_1+n_2+\cdots+n_k)!}{n_1!\,n_2!\,n_3!\cdots n_k!}$$

where $n_1 = 2$, $n_2 = 3$, and $n_3 = 4$. Thus, we have

$$\frac{(2+3+4)!}{2!\,3!\,3!}$$

Possible ways of getting such a sequence. Therefore the coefficient of $x^2y^3z^4$ is

$$\frac{(2+3+4)!}{2!\,3!\,3!} = 1260$$

THEOREM (MULTINOMIAL EXPANSION)

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_k \geq 0} x_1^{n_1} \times x_2^{n_2} \times \cdots \times x_k^{n_k} \frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

For $n_1 + n_2 + n_3 \cdots + n_k = n$

Activity 1 f

a) Find the coefficients of $v^3w^2xy^5$, w^2x^4y and $v^5w^2x^4y$ in $(5+ v - 5w + 3x + 2y)^{11}$.

b) Calculate the coefficient of z^2y^{-6} in $\left(\frac{1}{y} + 2 - z\right)^{10}$.

c) Calculate the coefficient of z^3y^{-6} in $\left(\frac{1}{y} + 2 - z\right)^{10}$

d) A family of 14 children has 2 tall children light in complexion, 4 short children light in complexion, 3 tall dark children and 5 short dark children. How many different birth orders can give rise to such a family?

e) Calculate is the number of ways of painting 12 offices so that 3 of them are painted green, 2 of them are painted pink, 2 of them are painted yellow, and the remaining ones are painted white?

Inclusion-Exclusion Principle

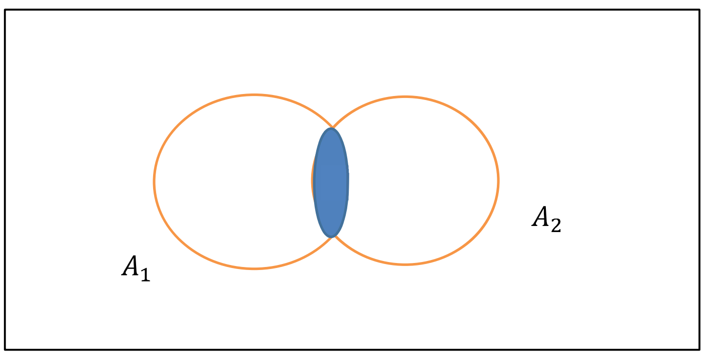
Lastly, the principal to look at in this unit, is that of counting elements with the process of including and excluding, formerly termed as "Inclusion-Exclusion" principle.

INCLUSION-EXCLUSION PRINCIPLE OF COUNTING

Previously, we discussed the SUM RULE, where for disjoint sets A_i, such that

$$|A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{i=1}^{k} |A_i|$$

If not two sets have common member, i.e. $(A_i \cap A_j) = \emptyset,\ i \neq j,\ \forall i, j$. However, what if $(A_i \cap A_j) \neq \emptyset$, e.g.



Then,

$$|A_1| + |A_2| - |A_1 \cap A_2| = |A_1 \cup A_2|$$

which implies that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Example

How many integers in the set $\{1,2,3, \cdots ,1000\}$ are divisible by 2 or 3?

Solution

Let A_1 be the subset of all numbers that are divisible by 2

Let A_2 be the subset of all numbers that are divisible by 3

Since we have some numbers, e.g. 6, 12, 18 from the set $\{1,2,3,\cdots,1000\}$, then $A_1 \cap A_2$ that describes the subset of all numbers that are divisible by both 2 and 3, then $A_1 \cap A_2 \neq \emptyset$. Hence we want to find

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

To find $|A_1|$,

We shall apply knowledge of the integer part functions, possibly introduced in the first year college algebra under topic of functions, which define

- $f(x) = \lfloor x \rfloor \equiv \textit{the largest integer less than or equal to } x$, and
- $f(x) = \lceil x \rceil \equiv \textit{the smallest integer greater than or equal to } x$

NOTATION

- $\lfloor x \rfloor$ is pronounced as the "floor of x", e.g. $\lfloor 4.1 \rfloor = 4$, $\lfloor 5 \rfloor = 5$
- $\lceil x \rceil$ is pronounced as the "ceiling of x"

Back to our problem of finding $|A_1|$, since we have $(1000 - 1 + 1) = 1000$ numbers from the consecutive list of integers beginning at 1 and ending at 1000, then

$$|A_1| = \lfloor \frac{1000}{2} \rfloor = 500$$

$$\text{Similarly, } |A_2| = \lfloor \tfrac{1000}{3} \rfloor = 333$$

Since $A_1 \cap A_2$ is the set of all numbers that are divisible by both 2 and 3, then we need the LCM (lowest common multiple of 2 and 3), such that

$$|A_1 \cap A_2| = \lfloor \frac{1000}{LCM(2,3)} \rfloor = \lfloor \frac{1000}{6} \rfloor = \lfloor 166.67 \rfloor = 166$$

Therefore,

$$|A_1 \cup A_2| = 500 + 333 - 166 = 667$$

Example

How many integers in the set $\{3,4,5,\cdots,998,999,1000,1001\}$ are divisible by 4 or 6?

Solution

We define A_1 to be the subset of all numbers that are divisible by 4 and A_2 to be the subset of all numbers that are divisible by 6, such that $A_1 \cap A_2$ defines the subset of all numbers that are divisible by both 4 and 6.

$$\text{Then we want } |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Where

$$|A_1| = \lfloor \frac{999}{4} \rfloor = 249$$

Since we have $(1001 - 3 + 1) = 999$ numbers from the given consecutive list of integers.

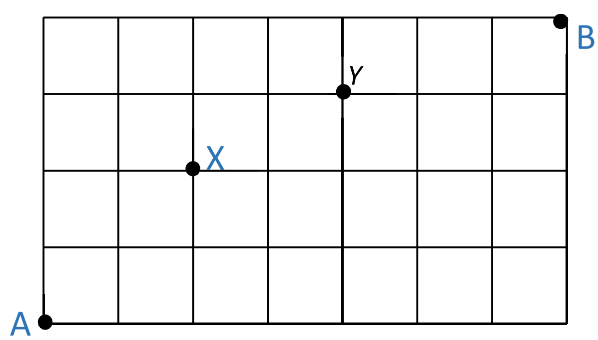
$$\text{Then } |A_2| = \lfloor \frac{999}{6} \rfloor = 166$$

$$\text{And } |A_1 \cap A_2| = \lfloor \frac{999}{LCM(4,6)} \rfloor = \lfloor \frac{999}{12} \rfloor = \lfloor 83.25 \rfloor = 83$$

$$\text{Therefore, } |A_1 \cup A_2| = 249 + 166 - 83 = 332$$

Example

Consider the diagram below which is assumed to describe a network of roads intersecting at junctions.



How many shortest routes are there from A to B that pass through X or Y?

Solution

If we consider A_1 to be set of all routes that pass through X, and A_2 be the set of all routes that pass through Y.

We want,

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$|A_1| = \binom{4}{2} \times \binom{7}{2}$$

$$|A_2| = \binom{7}{3} \times \binom{4}{1}$$

$A_1 \cap A_2$ describes the subset of routes that pass through X and through Y

Therefore,

$$|A_1 \cap A_2| = \binom{4}{2} \times \binom{3}{2} \times \binom{4}{1}$$

Such that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$|A_1 \cup A_2| = \left\{\binom{4}{2} \times \binom{7}{2}\right\} + \left\{\binom{7}{3} \times \binom{4}{1}\right\} - \left\{\binom{4}{2} \times \binom{3}{2}\binom{4}{1}\right\}$$

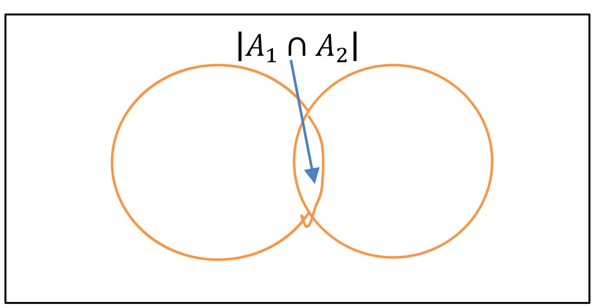
$$= 126 + 140 - 72 = 194$$

We have looked at an extension of the SUM rule for two sets. How about when we have three sets or more? We therefore wish to describe or derive the general formula for computing

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{k-1} \cup A_k| = ?$$

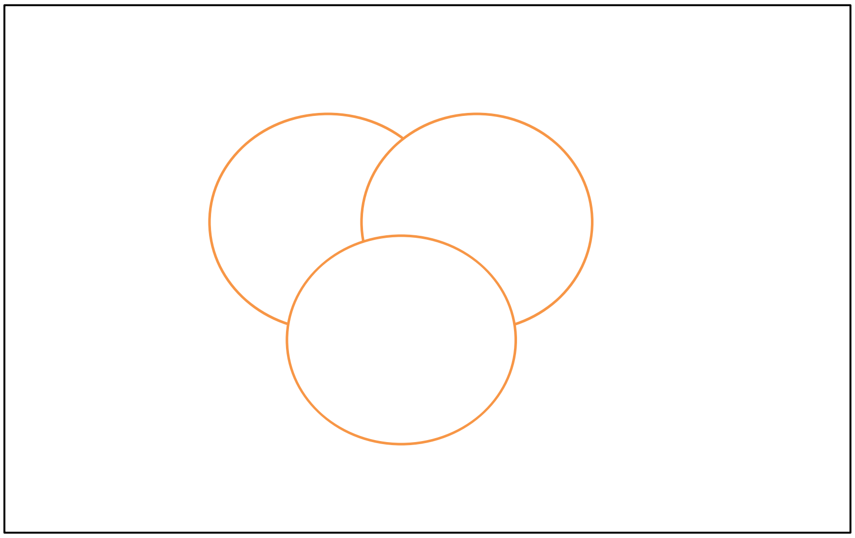
For $k = 2$, (two sets)$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$,

Was found by considering that when counting total number of elements in the union of set A_1 and set A_2, if the two sets are not disjoint, by the diagram below, then we need to add total number of elements in the two sets A_1 and A_2, then eliminate or take away the number of elements that are commonly found in set A_1 and A_2, as these may have been counted twice.



Hence, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

For $k = 3$, (three sets)



Likewise, by the diagram, we see that counting number of elements in the union of three sets A_1, A_2, and A_3, actually means counting twice elements belonging to $A_1 \cap A_2$, $A_2 \cap A_3$ and $A_1 \cap A_3$, therefore we should remove such counts to have

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{k-1} \cup A_k| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$$

But then, by doing that, we should have now some elements not in the count of the union any more; those that were existing in $A_1 \cap A_2 \cap A_3$. These have been taken out together with the taking out of each $|A_i \cap A_j|$. Hence, we need such count of elements back to obtain

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Hence for $k = 4$ (four sets)

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = &\sum_{i=1}^{4} |A_i| \\ &- \sum_{i_1 < i_2} |A_{i_1} \cap A_{i_2}| \\ &+ \sum_{i_1 < i_2 < i_3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\ &- |A_1 \cap A_2 \cap A_3 \cap A_4|\end{aligned}$$

For $\underline{k = 5}$ (five sets)

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = \sum_{i=1}^{5} |A_i| \\
- \sum_{i_1 < i_2} |A_{i_1} \cap A_{i_2}| \\
+ \sum_{i_1 < i_2 < i_3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
- \sum_{i_1 < i_2 < i_3 < i_4} |A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}| \\
+ \ |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|
\end{aligned}
$$

Thus, by reading off the pattern, we have a theorem called Generalized Inclusion-Exclusion principle.

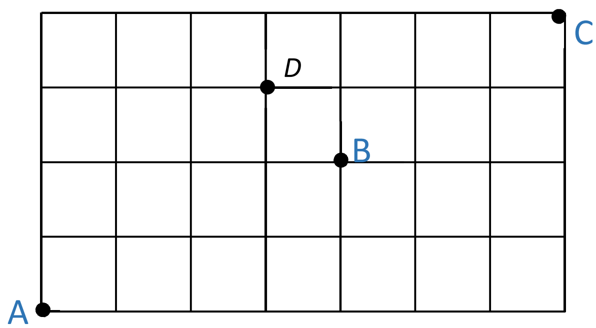
THEOREM (GENERALISED INCLUSION-EXCLUSION PRINCIPLE)

For any k sets,

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4 \cup \cdots \cup A_k| = \sum_{i=1}^{k} |A_i| \\
- \sum_{i_1 < i_2} |A_{i_1} \cap A_{i_2}| \\
+ \sum_{i_1 < i_2 < i_3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
- \sum_{i_1 < i_2 < i_3 < i_4} |A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}| \\
\vdots \\
+ (-1)^{k-1} \ |A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k|
\end{aligned}
$$

Activity 1 g

a) The diagram below shows a network of streets intersecting at right angles.



Calculate the total number of shortest routes (via the streets) from A to C via either junction B or D.

b) How many numbers are multiples of 2, 3, or 5 in the list 1, 2, 3, …, 6000?

c) How many sequences of two $A's$, two $C's$, two $T's$ and two $G's$ are there such that either the $A's$ appear together or the $T's$ appear together?

d) Determine the possible number of six letter sequences that can be formed from a set $\{A, B, C, \cdots, G\}$, such that either AD or AG appears together.

e) Calculate the number of integers in the list 301, 302, 303, …, 6998, 6999, 7000 that are either divisible by 6 or 8?

Summary/Let Us Sum Up

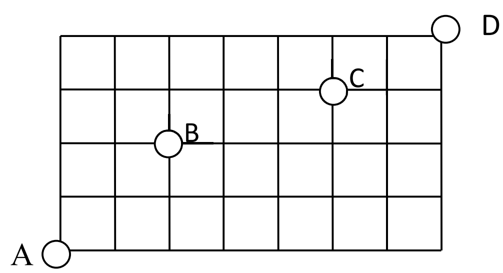
To sum up, this unit was meant to introduce the student to the notion of counting, which is a key principle and very fundamental to most real life application problems. With principles of counting techniques, the SUM RULE is noted to be useful for counting events that are made of collection of independent events. On the other hand, the COMBINATORIAL RULE gives us a chance to look at problems of making choices in life, which if we want to obtain such related unique choices, we engage in the PERMUTATION RULE. Below, the reader finds further reading reference materials not only for this Unit, but also in other subsequent units.

Suggestions for Further Reading/ Reading Assignment

- Susanna S. Epp. (2011). *Discrete Mathematics with Applications* (4th Edition)**,** Boston: Brooks Cole. [Chapter 9: pp. 516-592]
- Susanna S. Epp. (2011). *Discrete Mathematics with Applications: Student Solutions Manual* (4th Edition)**,** Belmont: Cengage Learning.
- Kenneth H. Rosen. (2012). *Discrete Mathematics and its Applications* (7th Edition)**,** New York: McGraw-Hill Education. [Chapter 6: pp. 385-439, Chapter 8: pp. 552-558]
- Kenneth H Rosen, & Jerrold W. Grossman Professor. (2011). *Student's Solutions Guide to Accompany Discrete Mathematics and Its Applications*, (7th Edition). New York: McGraw-Hill Education.
- Balakrishnan, V.K. (2010). *Introductory Discrete Mathematics*. New York: Dover Publications, Inc. [Chapter 1: pp. 35-69]
- Oscar Levin. (2017). *Discrete Mathematics: An Open Introduction*, (2nd Edition). Colorado: CreateSpace Independent Publishing Platform. [Chapter 1]
- Seymour Lipschutz, & Marc Lipson. (2009). *Schaum's Outline of Discrete Mathematics*, (3rd Edition). New York: McGraw-Hill Education. [Chapter 5, Chapter 6]
- Richard Kohar. (2016). *Basic Discrete Mathematics: Logic, Set Theory, and Probability.* Singapore: World Scientific Publishing Company. [Chapter 5, Chapter 7: pp. 277]

Unit 1 Test

1. Calculate the number of different messages that can be represented by sequences of three dashes and two dots.
2. In how many ways can three examinations be scheduled within a five-day period?
3. In how many ways can 4 persons be born in an ordinary year so that no two share the same birthday?
4. How many 5-digit positive integers are there such that the last digit is either an even number or a 5?
5. How many subsets can one form from the set {1, 2, 3, …, m} such that one has exactly r members (0 δ r δ m)? Write your answer using factorials only.
6. Determine the number of integers between 1 and 1000 that are coprime to 5, 7 and 13.
7. Calculate the number of integers in the list 701, 702, 703, …, 5998, 5999, 6000 that are either divisible by 4 or 6?
8. Determine the coefficient of w^6z^6 in $(5 - 2w^3 + 3z^2)^8$.
9. How many permutations of the letters A, B, C, D, E are there where A appears next to B or E appears next to B? **[HINT:** blocks like AB and BA allowed}.
10. Let the notation $T(x, m)$ stand for the number of ways of selecting a team of x students from a class of m students $(1 \le x < m)$. Use combinatorial logic only to deduce that $T(x, m) = T(x, m-1) + T(x-1, m-1)$.
11. Figure below shows networks of streets intersecting at right angles.



    Calculate the number of shortest routes from junction A to junction D along the streets that pass through junctions B and C.**[HINT:** Apply multiplication rule]
12. How many ways can one share 50 one kwacha coins to Maria, Mary and Mariam such that Maria gets at least K10 or Mary gets at least K11 or Mariam gets at least K19?
13. In the diagram of Q11, how many rectangular boxes are there?
14. How many non-negative integer solutions has the equation $x - 6.9 + y + z - 8.1 = 0$ got?

Selected answers to Unit Activities

ACTIVITY	**(a)**	**(b)**	**(c)**	**(d)**	**(e)**
Activity 1 a	125		$26 \times 25 \times 24^4$		35
Activity 1 b		60		840	
Activity 1 c	2401		73		2548
Activity 1 d		190		496	
Activity 1 e	112		56		3.480×10^{11}
Activity 1 f		5040		2,522,520	
Activity 1 g	100		1080		1674