SOLUTION KEY

UNIVERSITY OF MALAWI

SCHOOL OF NATURAL & APPLIED SCIENCES

Mathematical Sciences Department

TEST 1: Counting Techniques & Pigeonhole Principle

(For 2nd year Science students taking MAT 212)

Sunday, 19th November 2023 **Time: 2 hours (from 14:00hrs)**

Instructions

(1) ***This is a closed book test*** where you are expected to do the test alone without any assistance from some other person(s) or some other form of notes or communication.

(2) ***Non-programmable calculators may be used***. However, mobile phones are not allowed. If accidentally brought in, they should be switched off and packed away.

(3) ***Show your method or reasoning***. Most marks shown in square brackets at the end of each part are allocated to the method.

(4) Start with questions that you can do comfortably first.

(5) Attempt ***ALL questions***.

+3 = 33 marks

Question 1: [30 marks]

(a) A telephone company uses 8-digit telephone numbers where numbers for district D are designated to start with the three digits 015. How many 8-digit telephone numbers are possible for this district D? **[4.5]**

(b) A tutorial class has 6 girls and 5 boys. How many ways can we choose two students of the same sex from this class? **[4]**

(c) Find the coefficient of $a^2b^3c^5$ in $(2+ a^{1/2} - b^3 + c^{2.5})^8$. **[5.5]**

(d) In how many ways can three examinations be scheduled within a five-day period so that no two examinations are scheduled in the same day? **[3]**

(e) Find the number of integers between 1 and 2,000 that are either divisible by 6 or 9. **[5]**

(f) Ten indistinguishable prizes can each be won by 5 competitors. Find the total number of different ways that the 10 prizes can be won. **[4]**

(g) How many 5-digit integers (positive integers) are there where the last digit is an odd number or a 4. **[4]**

Question 2: [25 marks]

(a) Prove that for any choice of 367 babies born in 2008, there were at least two babies who were born on the same day. **[3]**

(b) Deduce that when 27 gifts are issued to five beneficiaries, then there will be a beneficiary receiving not less than 6 gifts. **[3]**

(c) Show that if one chooses 101 numbers from the list $\{1, 2, 3, \ldots, 200\}$; then there exists at least two numbers among the chosen 101 numbers that divide each other. **[5.5]**

(d) A school hired three buses to ferry 103 students. Justify why one of the buses must have taken no more than 34 students in it. **[3]**

(e) Seven points are marked inside a circle of radius 50cm. By splitting the angle at the centre of the circle into six congruent angles, prove that at least two of the seven points will lie no longer than 50cm apart. **[5.5]**

(f) Let the notation $T(x,m)$ stand for the number of ways of selecting a team of x students from a class of m students $(1 \leq x < m)$. Use combinatorial logic only to deduce that $T(x,m) = T(x,m - 1) + T(x - 1,m - 1)$. **[5]**

~~~~~ooooOOOO **End of Test 1 Questions** OOOOoooo~~~~~
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MAT212 – ANSWER GUIDE

Q 1

QUESTION No. .... Q 1 .... Examination No. ....

(a) Let the 8-digit sequence of a telephone number be designed as

1

$$\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}\;\underline{\quad}$$
$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$$

Using the digits $\{0, 1, 2, \ldots, 9\}$

1

Since the first 3 positions are fixed at once for $\boxed{0}\boxed{1}\boxed{5}$, Then from the remaining 5-positions, we have

$$\underbrace{\boxed{0}\boxed{1}\boxed{5}}_{\text{once}} \quad \underset{4}{\underline{{}^{10}C_1}} \quad \underset{5}{\underline{{}^{10}C_1}} \quad \underset{6}{\underline{{}^{10}C_1}} \quad \underset{7}{\underline{{}^{10}C_1}} \quad \underset{8}{\underline{{}^{10}C_1}}$$

1

$\frac{1}{2}$

Hence by Multiplicative Rule of Counting, we have

1

$$1 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^5$$ possible 8-digit telephone numbers for district D.

$\frac{1}{2}$

(b) Let A be a set of 6 girls and B be a set of 5 boys.

Let the two students of same sex to be chosen be presented as two positions

$\frac{1}{2}$

$$\underline{\quad\;}\;\;\underline{\quad\;}$$
$$1 \qquad 2$$

1

We can choose two of same sex (Boys only) or two of same sex (females only)

Q.1 – Contd...

Thus

either ${}^6C_2$ or ${}^5C_2$ $\frac{1}{2}$

Hence by the SUM RULE, we have $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$${}^6C_2 + {}^5C_2 = \underline{\underline{25}} \text{ ways in total}$$

possible ways of choosing two students from 6 girls and 5 boys of the same sex.

OR one can choose to use $|A|+|B|-|A\cap B|$, where $|A\cap B|=0$, $(A\cap B)=\phi$

(e)

from the expansion $(2+a^{\frac{1}{2}}-b^3+c^{2.5})^8$,

the coefficient $a^2b^3c^5$ can be obtained when

$a^{\frac{1}{2}}$ Comes from 4 brackets $\frac{1}{2}$

b^3 Comes from 1 bracket $\frac{1}{2}$

$c^{2.5}$ Comes from 2 brackets $\frac{1}{2}$

and 2 Comes from the remaining $8-(4+1+2)$ brackets

i.e. 2 Comes from 1 bracket $\frac{1}{2}$

Hence, if we let $n_1=4$, $n_2=1$, $n_3=2$ and $n_4=1$, by multinomial Rule of Counting, we have $\frac{1}{2}$

$$\binom{4+1+2+1}{4,1,2,1}(a^{\frac{1}{2}})^4(-b^3)^1(c^{2.5})^2(2)^1 \quad 1.5$$

$$= -\frac{8!}{4!.1!.2!.1!}\times 2^1 a^2 b^3 c^5 \quad 1$$

$\therefore$ the Coefficient of $a^2b^3c^5$ is $\dfrac{-8!}{4!} = \underline{-1680}$ $\frac{1}{2}$

Q-1

QUESTION No.. Examination No..

(d) Let the 3 examinations be regarded as distinct positions.

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$

1 2 3 exams

Let the 5-day period (5 days) be regarded as different items.

Then the problem is equivalent to counting number of ways of arranging 5 different items on 3 different positions

i.e. $^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \underline{60 \text{ ways}}$

(e) Let $|A|$ be number of integers between 1 and 2000 divisible by 6

Let $|B|$ be number of integers between 1 and 2000 divisible by 9

Thus, we want $|A \cup B| = |A| + |B| - |A \cap B|$

Where $|A \cap B|$ is number of integers divisible by both 6 and 9

Since between 1 and 2000, there are 2000 integers, we want

$$|A \cup B| = \left\lfloor \frac{2000}{6} \right\rfloor + \left\lfloor \frac{2000}{9} \right\rfloor - \left\lfloor \frac{2000}{LCM(6,9)} \right\rfloor$$

$|A \cup B| = \lfloor 333.33 \rfloor + \lfloor 222.2 \rfloor - \lfloor 111.11 \rfloor = \underline{444 \text{ ways}}$

Q. 1

(f) Let the 5 ~~Competitors~~ notations be represented by x_1, x_2, x_3, x_4, and x_5, respectively for each number of prizes won by Competitor 1, 2, 3, 4, and 5. $\frac{1}{2}$

Since we have 10 prizes to be won, then

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10 \quad \frac{1}{2}$$

Since the prizes are identical, then this is equivalent to computing total possible non-negative integer solutions where

1 $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 1$, $x_4 \geq 1$, and $x_5 \geq 1$

Let $x_1 = a+1$, $x_2 = b+1$, $x_3 = c+1$, $x_4 = d+1$ and $x_5 = e+1$ 1

Then $x_1 + x_2 + x_3 + x_4 + x_5 \equiv a+1+b+1+c+1+d+1+e+1 = 10$ $\frac{1}{2}$

$$\Rightarrow a+b+c+d+e+5 = 10$$

$$\Rightarrow a+b+c+d+e = 5 \quad \frac{1}{2}$$

Thus, for $n=5$, $K=5$, we need 1

$$\binom{n+K-1}{K-1}$$ possible ~~colours~~ ways of winning the prizes

$$= \binom{5+5-1}{5-1} \; 1 \; = \binom{5+4}{4} = \binom{9}{4} \; \frac{1}{2}$$

$\frac{1}{2}$ $= 126$ different ways

QUESTION No. Q.1 Examination No.

(g) For a 5-digit Integer, let the 5-positions be

$\sqcup \quad \sqcup \quad \sqcup \quad \sqcup \quad \sqcup$
1 2 3 4 5

(1/2)

(1/2)

Since we have 10 digits {0, 1, 2, 3, ..., 9},
for position 1, 0 is not allowed (1/2) (1/2)
for Position 5, we have {1, 3, 5, 7, 9, 4} possible
digits to take from any odd number or 4

Hence, we have

$\underset{1}{\sqcup^{9_{C_1}}} \quad \underset{2}{\sqcup^{10_{C_1}}} \quad \underset{3}{\sqcup^{10_{C_1}}} \quad \underset{4}{\sqcup^{10_{C_1}}} \quad \sqcup^{6_{C_1}}$ (1)

Thus by multiplicative rule of Counting we have $9_{C_1} \times 10_{C_1} \times 10_{C_1} \times 10_{C_1} \times 6_{C_1}$ (1)

(1/2)

$= 9 \times 10^3 \times 6 = 54{,}000$ possible integers

OR

One can apply the SUM RULE,

(1/2) $|A|$ - number of 5 digit-integers where last digit ends with an odd number

(1/2) $|B|$ - number of 5 digit-Integers where last digit ends with a 4

$|A \cap B|$ - number of 5 digit-Integers where last digit ends with an odd number and a 4.

(1/2)

(1)

$\Rightarrow$ We need $|A \cup B| = |A| + |B| - |A \cap B|$

where $|A| = 9 \times 10 \times 10 \times 10 \times 5 = 45{,}000$ (1/2)

$|B| = 9 \times 10 \times 10 \times 10 \times 1 = 9000$ (1/2)

But $|A \cap B| = 0$, Since $(A \cap B) = \phi$

(1/2)

$\therefore |A \cup B| = 45000 + 9000 - 0 = 54{,}000$

QUESTION No. Q.2 Examination No.

(a) The year 2008 was a leap year with 366 days $\frac{1}{2}$

Hence let the 366 days be pigeonholes $\frac{1}{2}$ and let the 367 babies be pigeons $\frac{1}{2}$

Thus, the number of pigeons is one more than pigeonholes, i.e., $\frac{1}{2}$ $\frac{1}{2}$

$n+1 = 367$, $n = 366$

∴ By simplest form of pigeonhole principle, ∃ a pigeonhole with more than one pigeon $\frac{1}{2}$ ≡ at least 2 pigeons.
Hence, indeed there were at least two babies who were born in the same day.

(b) $\frac{1}{2}$

Let the 27 gifts be pigeons and let the ~~5~~ 5 beneficiaries be pigeonholes $\frac{1}{2}$

Since $27 \equiv 5 \times 5 + 2$ $\frac{1}{2}$

Then $27 = nr + 2$, where $n = 5$ and $r = 5$.

$\frac{1}{2}$

Thus by regular form of pigeonhole principle ∃ a pigeonhole with more than 5 pigeons ≡ at least 6 pigeons $\frac{1}{2}$

$\frac{1}{2}$

⇒ Indeed no pigeonhole has < 6 pigeons ≡ no beneficiary recieves less than 6 gifts.

OR Others may attempt it using extended regular form / generalised form / principles as in $\frac{1}{2}$ $\frac{1}{2}$

Let $m = 27$ gifts / letters
$K = 5$ beneficiaries / boxes $\frac{1}{2}$

then $\exists$ a box / beneficiary with more than $\left\lfloor \frac{m-1}{K} \right\rfloor$ items $\frac{1}{2}$

i.e. $> \left\lfloor \frac{27-1}{5} \right\rfloor = \left\lfloor \frac{26}{5} \right\rfloor = \lfloor 5.2 \rfloor = 5 \equiv \geq 6$ $\frac{1}{2}$ $\frac{1}{2}$

Indeed no box / beneficiary has ~~more~~ less than 6 gifts

(c) Let the 101 chosen numbers be
$x_1, x_2, x_3, \ldots, x_{101}$ $\frac{1}{2}$

Let each number be expressed as
$$x_i = 2^{n_i} \times d_i$$ $\frac{1}{2}$

where each d_i is an odd integer eg
$8 = 2^3 \times 1$, $34 = 2^1 \times 17$ etc
and $n_i \in \{0\} \cup \mathbb{N}$

then we have possibly

$$\left.\begin{aligned} x_1 &= 2^{n_1} d_1 \\ x_2 &= 2^{n_2} d_2 \\ x_3 &= 2^{n_3} d_3 \\ &\vdots \\ x_{101} &= 2^{n_{101}} d_{101} \end{aligned}\right\} 1$$

QUESTION No. Q-2 Examination No.

(c) continued...

But from the list $\{1, 2, 3, \ldots, 200\}$, we have 100 even and 100 odd numbers

$\therefore$ possibly $d_1, d_2, d_3, \ldots, d_{101}$ correspondingly comes from $\{1, 3, 5, 7, \ldots, 199\}$

Thus, we have 100 of these d_i's against 101 chosen numbers $x_i = 2^{n_i} d_i$'s

$\Rightarrow$ By simplest form of pigeonhole principle, $\exists$ two d_i's that are equal say d_i and d_j for ~~two~~ numbers

$$x_i = 2^{n_i} d_i \quad \text{and} \quad x_j = 2^{n_j} d_j$$

Since $d_i = d_j$, then $x_i / x_j = 2^{n_i - n_j}$

or $x_j / x_i = 2^{n_j - n_i}$

$\therefore$ Indeed, $\exists$ at least two numbers among the chosen 101 numbers from a set $\{1, 2, 3, \ldots, 200\}$ ~~with~~ where one divides the other.

Q2

(d) Let the 103 students be pigeons $\frac{1}{2}$
and 3 buses be pigeonholes $\frac{1}{2}$

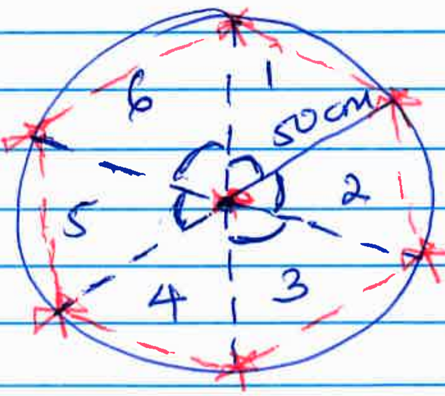
Since $103 = 34 \times 3 + 1 \equiv nr + 1$
where $n = 3$, $r = 34$ 1

Then by regular form of pigeonhole principle, $\exists$ a 2 pigeonhole with more than 34 students in it.

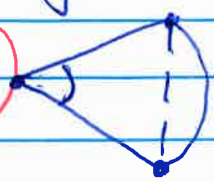
$\frac{1}{2}$
But at minimum ~~each~~ Some bus therefore holds ≤ 34 students, if one bus takes more than 34 students in it.

(e) Let's consider a circle with radius 50cm split into 6 congruent angles

2



Then complete connecting the 6 congruent triangle, e.g. as follows; $\frac{1}{2}$



If we place 7 points on each corner of a triangle, then certainly 1 one point will fall on the centre
$\Rightarrow$ Any maximum distance between this centre point 1 and any around creates at least 2 points lying no more/longer than 50 cm distance apart.

QUESTION No. Q-2 Examination No.

(e)

Thus, by simplest form of the pigeonhole principle, Indeed, placing 7 points on 6 ~~triangles~~, $\exists$ at least two points lying no longer than 50 cm apart.

(f)

If $T(x,m)$ stands for number of ways of selecting a team of x students from a class of m students $(1 \leq x \leq m)$

Suppose we have 1 student r in this class who can either be chosen among the x students or not.

$\boxed{\text{Case (a)}}$: Suppose r belongs to x students then from the $m-1$ students we choose the remaining $x-1$ students of the team to which r belongs already
$\Rightarrow$ By multiplicative rule ~~case~~

$$1 \times \binom{m-1}{x-1} = T(x-1, m-1)$$

$\boxed{\text{Case (b)}}$: Suppose r does not belong to this team of x students

Then, removing this r^{th} student from $m-1$ students, we have to choose all x students from the remaining $m-1$ students
$\Rightarrow$ By multiplicative rule, we have

$$1 \times \binom{m-1}{x} = T(x, m-1)$$

By sum Rule, we have $T(x,m) = T(x,m-1) + T(x-1, m-1)$