



## Selected answers to Unit Activities

ACTIVITY	(a)	(b)	(c)	(d)	(e)
Activity 1 a	125		$26 \times 25 \times 24^4$		35
Activity 1 b		60		840	
Activity 1 c	2401		73		2548
Activity 1 d		190		496	
Activity 1 e	112		56		$3.480 \times 10^{11}$
Activity 1 f		5040		2,522,520	
Activity 1 g	100		1080		1674

# Unit 2: Pigeon-hole Principle



## Introduction

A pigeon-hole principle is one well-known proving technique in mathematics which is based on the idea of having “many” pigeons and “a few” pigeonholes where informally there must be some pigeonhole occupied by two or more pigeons. Formally the principle is described by two finite sets  $A$  and  $B$ , such that if  $|A| > |B|$ , then for any function  $f$  from  $A$  to  $B$ , there exists  $a_1, a_2 \in A$  such that  $f(a_1) = f(a_2)$ . Pigeonhole principle has got more other general forms and their applications have been seen in a far wide across various science and non-science disciplines, e.g. one can challenge that among any 13 people, there are at least 2 of them who were born in the same month where the 13 people are the pigeons and the 12 months are the pigeonholes. In this unit, we shall look at various forms of pigeonhole principle and also their fascinating applications. The pigeonhole principle is also well known as the *shoe box argument* or *Dirichlet drawer principle*, based on the issue that if one decides to select 11 shoes from 10 pairs of shoes in a drawer, there must be a pair of matched shoes among the selection where the 11 shoes are the pigeons and the 10 pairs are the pigeonholes.

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## Unit Objectives

On successful completion of the Unit, students should be able to:

- State and describe various forms of the pigeonhole principle
- Prove theories of pigeonhole principle
- Relate different forms of pigeonhole principles where applicable
- Apply pigeonhole principles to solve real life problems



## Key Terms

As you go through this unit, ensure that you understand the key terms or phrases used in this unit as listed below.

- Pigeons
  - Pigeonholes
  - Simplest form
  - Regular form
  - Extended regular form
  - Generalized form
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## Simplest Form of Pigeonhole Principle

### THEOREM (Simplest form of Pigeonhole Principle)

If  $n + 1$  letters are placed in  $n$  pigeonholes, there will be a pigeonhole with MORE THAN one letter in it.

#### *Proof*

Suppose the theorem was false, then by placing same  $n + 1$  letters in  $n$  pigeonholes, each/every pigeonhole would have AT MOST one letter in it, i.e.

$$[\leq 1], [\leq 1], [\leq 1], \dots, [\leq 1]$$

This implies that the total number of letters cannot exceed  $1 + 1 + 1 + \dots + 1 = n$  as contained in the pigeonholes. However, this contradicts with the fact that we had  $n + 1$  letters in total.

Hence, our assumption that the theorem is false cannot hold. Therefore, by contradiction, the simplest form of the pigeonhole principle must be ALWAYS true.

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### Example

Prove that in Malawi, among those who have auto-teller cards, there exists at least two (2) persons who have the same pin code.

#### Solution

In Malawi, we have several banks, e.g. NBM, NBS, STD, FDH, FMB, NED, etc. that provide auto-teller cards to its customers each bearing a 4-digit code.

Since a pin code of a card uses 4-digit number, by the process of counting technique, we have

$$10, 10, 10, 10$$

Such that by product rule, we have  $10 \times 10 \times 10 \times 10 = 10^4$  different possible pin-codes.

Thus, we took these 10,000 different possible pin-codes as pigeonholes, we only need a minimum of 10,001, so that there should be a pin-code shared by two individuals. But common sense tells us that there are over 10,000 persons in Malawi who have bank cards with any of the banks, then by simplest form of the pigeonhole principle, we should indeed find at least two persons sharing same auto-teller cards pin codes.

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### Example

Let  $N = 21$ , and  $M = 617$ . Prove that there exists a number of the form  $212121 \cdots 2100 \cdots 0$  which is divisible by 617.

### Solution

Let's formulate a sequence as follows

- Term 1 = 21  $\leftarrow$  one 21
- Term 2 = 2121  $\leftarrow$  two 21's
- Term 3 = 212121  $\leftarrow$  three 21's
- $\vdots$
- Term 618 = 212121  $\cdots$  2121  $\leftarrow$  618 21's

Let  $r_i$  be the remainder when the term  $T_i$  is divided by 617, such that we have  $r_1, r_2, r_3, r_4 \cdots, r_{617}$

If any of the  $r_i = 0$ , then there exists a number 212121  $\cdots$  21 which is divisible by 617. Then the result that a number of the form 212121  $\cdots$  210000 is divisible by 617 is trivial.

But if none of the remainders is 0, then the remainders range from 1 to 616 such that

$$1 \leq r_1, r_2, \cdots, r_{617} \leq 616$$

Therefore, by simplest form of the pigeonholes, we have number of pigeonhole (616), one-less than the pigeons (617), and thus there exists 2 remainders that are the same, say

$$r_i = r_j, \forall i < j$$

Therefore,

$$T_i = 617 \times W + r_i$$

and

$$T_j = 617 \times Z + r_j, (W, Z \in \{0\} \cup \mathbb{N})$$

This implies that  $T_j$  is a term with more 21's than  $T_i$  such that

$$\begin{aligned} T_j - T_i &= 2121 \cdots 2100 \cdots 0 = 617 \times Z - 617 \times W \\ &= 2121 \cdots 2100 \cdots 0 = 617(Z - W) \end{aligned}$$

Hence, 2121  $\cdots$  2100  $\cdots$  0 is therefore a multiple of 617 and therefore divisible by 617.

### Example

For any grouping of  $n$  persons, there exists at least two persons who greet the same number of persons within the group ( $n \geq 2$ ).

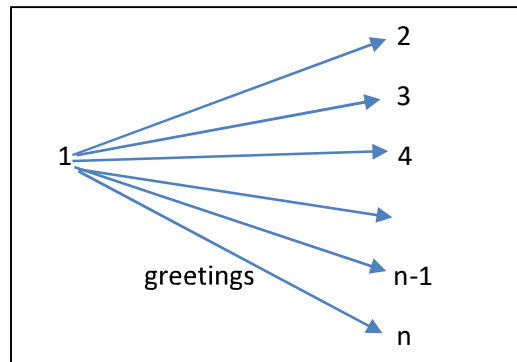
### Solution

In a group of  $n$  persons who decide to greet each other, there are two possible cases

- CASE 1: Someone greets everyone
- CASE 2: None has greeted everyone

Suppose we have CASE 1

Then we mean that there exists someone who has made  $n - 1$  greetings, i.e. the one who greeted everyone



Therefore, No one has made no greeting. Hence, the possible number of greetings for the  $n$  persons are

$$1, 2, 3, \dots, n - 1$$

Let the  $n$  persons be regarded as letters and let the number of possible greetings be regarded as pigeonholes.

$$[1], [2], [3], \dots, [n - 1]$$

If we let  $n$  persons (letters) go into the  $n - 1$  (pigeonholes) that correspond to the number of greetings each person has made, then by simplest form of the pigeonhole principal, some pigeonhole will have more than 1 letter (i.e. at least 2 letters in it).

Hence, these persons have made the same number of greetings.

CASE 2: There doesn't exist anyone who greets everyone

For  $n \geq 2$  persons, then the maximum possible number of greetings is  $n - 2$  and the minimum possible number of greetings is 0, such that we have  $n - 1$  total possible greetings.

If we correspond these greetings to boxes or pigeonholes, then we have

$$[0], [1], [2], \dots, [n - 2]$$

Let the  $n$  persons go into the  $n - 1$  pigeonholes that corresponds to the number of greetings each person makes, then there is none who has greeted everyone.

Therefore, in the group, there are at least 2 persons who have greeted the same number of people within the group.



### Activity 2 a

- Prove that for any choice of 367 babies born in 2008, there were at least two babies who were born on the same day.
- Calculate the minimum possible number of persons one can randomly choose to ensure that at least some pair of persons will have the same first initials for their first names using the English alphabet.
- Show that if one chooses 101 numbers from the list  $\{1, 2, 3, \dots, 200\}$ , then there exists at least two numbers among the chosen 101 numbers that have the highest common factor (HCF) as 1. [**HINT:** Deduce the HCF of any consecutive integers]
- Prove that if  $P$  is any positive prime number, some integer of the form  $777\dots7000\dots0$  is divisible by this prime number  $P$ .
- Prove that if one chooses any five points inside a  $2\sqrt{2}$  cm by  $2\sqrt{2}$  cm square, some pair of points will lie no further than 2cm apart. [**HINT:** Split the square into identical pigeonholes]



## Regular form of pigeonhole principle

### THEOREM (Regular form of Pigeonhole Principle)

If one places  $nr + 1$  letters in  $n$  pigeonholes, then there exists a pigeonhole with MORE THAN  $r$  letters.

#### Example

Take for example any 49 persons. Deduce that there exist greater than or more than 4 persons among the 49 who were born in the same month.

#### Solution

Regard the 49 persons =  $12 \times 4 + 1$  persons as letters and the 12 months in an ordinary year as pigeonholes.

Let the persons (letters) move into the pigeonholes corresponding to the month they were born.

$$[Jan], [Feb], [Mar], \dots, [Nov], [Dec]$$

By regular form of the pigeonhole principle, there exists a month with MORE THAN  $r = 4$  persons, i.e. for  $n = 12, r = 4$ ,

$$nr + 1 = 12 \times 4 + 1$$

Before we move into the third theorem, let's review the integer part function that was probably introduced in first year college algebra and also reviewed in the previous topic.

Suppose we have a function  $f$  that maps elements from the domain  $R$  to range  $Z$  with element  $x$  in  $R$  and  $y$  in  $Z$ . Thus  $y$  is the largest integer  $\leq x$ .

i.e. for  $x = 4.1$ ,  $y$  is the largest integer  $\leq 4.1$ . Therefore,  $y = 4$ .

Thus if  $f: R \rightarrow Z$ , then  $f(4.1) = 4$ ,  $f(4) = 4$ , and  $f(99.76) = 99$

We shall denote the integer function as  $f(x) = \lfloor x \rfloor$ , e.g.  $\lfloor 5.5 \rfloor = 5$ ,  $\lfloor \frac{8}{3} \rfloor = 2$  and  $\lfloor \frac{8}{2} \rfloor = 4$ .

Now let's investigate something interesting,

$$\lfloor \frac{8}{2} \rfloor \times 2 = 8 \quad \lfloor \frac{8}{3} \rfloor \times 3 = 6 \quad \lfloor \frac{6}{3} \rfloor \times 3 = 6$$



Hence, we can deduce that

### THEOREM

For any  $x, r \in \mathbb{Z}$ ,

$$\left\lfloor \frac{x}{r} \right\rfloor \times r \leq x$$

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### Activity 2 b

- a) Prove that if one studied for 4 hours in 3 days, then there must be a day that the person studied no more than 1 hour 20 minutes.
- b) Prove that if one studied for 4 hours in 3 days, then there must be a day that the person studied more than 1 hour.
- c) Hence, prove that if one studied for 4 hours in 51 consecutive days, there must be some 17 consecutive days that the person studied more than 1 hour.
- d) Prove that if one chooses 11 numbers among the numbers 1, 2, 3, ..., 20, then some pair among the chosen numbers are such that one is divisible by the other.
- e) Prove that in a group of 731 students, at least 2 persons share the same birthday day in an ordinary year.



## Extended Regular form of Pigeonhole Principle

### **THEOREM (Extended regular form of pigeonhole principle)**

If  $n$  letters are placed in  $k$  pigeonholes, then there exist a pigeonhole with MORE THAN  $\lfloor \frac{n-1}{k} \rfloor$  letters.

### **Proof**

Suppose the theorem was false, then every pigeonhole has AT MOST (less than or equal to)  $\lfloor \frac{n-1}{k} \rfloor$  letters, i.e. for  $k$  pigeonholes,

$$\leq \lfloor \frac{n-1}{k} \rfloor, \leq \lfloor \frac{n-1}{k} \rfloor, \leq \lfloor \frac{n-1}{k} \rfloor, \dots, \leq \lfloor \frac{n-1}{k} \rfloor, \leq \lfloor \frac{n-1}{k} \rfloor$$

Then the maximum number of letters contained in all the  $k$  pigeonholes does not exceed

$$\lfloor \frac{n-1}{k} \rfloor \times k \leq n-1$$

But this result contradicts with our given proposition of having  $n$  letters in total. Therefore the theorem cannot be false.

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### **Example**

A secondary school has four forms (classes) and the total students' population at the school is 401. Deduce that one form or class has at least 101 students.

### **Solution**

For  $n = 401$  students, and we have  $k = 4$  pigeonholes (classes 1 to 4). If these  $n = 401$  students gather together according to their  $k = 4$  classes, then by extended regular form of the pigeonhole principle, there exists a class with more than

$$\lfloor \frac{n-1}{k} \rfloor = \lfloor \frac{401-1}{4} \rfloor = 100$$

i.e.  $> 100$  students, which is equivalent to at least 101 students.

### Example

Suppose that 100 mangoes were placed in 3 baskets. Deduce that there was a basket with at least 34 mangoes.

### Solution

Let  $n = 100$  mangoes and  $k = 3$  baskets.

Therefore, from the extended regular form of the pigeonhole principle, there exists a basket with

$$> \lfloor \frac{n-1}{k} \rfloor = \lfloor \frac{100-1}{3} \rfloor = 33$$

Since the number of mangoes is countable, then  $> 33 \equiv \geq 34$  mangoes.

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### Remark:

The extended regular form of the pigeonhole principle generalizes the first two principles, namely: the simplest form and the regular form of the pigeonhole principles.

### Verification

Lets consider the extended regular form of the pigeonhole principle which states that If  $n$  letters are placed in  $k$  pigeonholes, then there exist a pigeonhole with MORE THAN  $\lfloor \frac{n-1}{k} \rfloor$  letters.

Now suppose that

1. CASE 1: The number of letters are equal to those described in the simplest form, and so are the pigeonholes, i.e.

$$n \equiv n + 1 \text{ letters}$$

$$k \equiv n \text{ pigeonholes}$$

Then, by the same extended regular form of the pigeonhole principle, if  $n + 1$  letters are placed in  $n$  pigeonholes, then there exist a pigeonhole with MORE THAN  $\lfloor \frac{(n+1)-1}{n} \rfloor = \lfloor \frac{n}{n} \rfloor = 1$  letters (which is essentially the result of the simplest form)

2. CASE 2: The number of letters are equal to those described in the regular form, and so are the pigeonholes, i.e.

$$n \equiv nr + 1 \text{ letters}$$

$$k \equiv n \text{ pigeonholes}$$

Then, by the same extended regular form of the pigeonhole principle, if  $nr + 1$  letters are placed in  $n$  pigeonholes, then there exist a pigeonhole with MORE THAN  $\lfloor \frac{(nr+1)-1}{n} \rfloor = \lfloor \frac{nr}{n} \rfloor = r$  letters (which is essentially the result of the regular form)

Hence, indeed the extended regular form generalizes the previous two theorems.



### Activity 2 c

- a) A student studied for 97 hours over a period of 12 days. Prove, without applying pigeonhole principle, that on some pair of successive days, the student studied more than 16 hours. Could we apply the pigeonhole principle on **uncountable** measurement (97 hours) just as we did with **countable** numbers (97 letters)?
- b) In Problem (a) above, prove, via some pigeonhole principle, that on some three consecutive days, the student studied more than 24 hours.
- c) Show that if a certain college in Blantyre has 6000 students where at least one if from each of the 50 townships located in Blantyre, then there must be a group of 120 students coming from the same township.
- d) Show that the extended regular form generalises the first two forms of the pigeonhole principle.
- e) Use the extended regular form of the pigeonhole principle to show that no matter which year we pass through, at least two persons in any group of 367 persons will have same birth days.

Before we look at the last theorem of the pigeonhole principle, let us have a look at a separate theorem, which talks about the average of numbers in relation to any of the numbers whose average was calculated upon.

Suppose we define a sequence of numbers  $x_1, x_2, x_3, \dots, x_{n-1}, x_n$ . then we define the average  $\underline{x}$  as

$$\underline{x} = \frac{x_1 + x_2 + x_3 + \dots + x_{n-1} + x_n}{n}$$

### ***THEOREM***

Let  $x_1, x_2, x_3, \dots, x_{n-1}, x_n$  be any numbers, then

- a) There exists a number among these that is not larger than their Average, i.e.  $\exists i, x_i \leq \bar{x}$
- b) There exists a number among these that is not smaller than their Average, i.e.  $\exists j, x_j \geq \bar{x}$
- c) In short the Average  $\bar{x}$  must lie somewhere in between minimum and maximum numbers, i.e.

$$x_i \leq \bar{x} \leq x_j$$

### ***Proof***

We shall demonstrate the proof of the first part (part a) and the other shall be left for the student to verify or prove as an exercise.

#### Part A

Let the average of such number be defined as

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \dots + x_{n-1} + x_n}{n}$$

Suppose the statement was false. Then it means that every number is greater than the average, i.e.  $\forall i = 1, 2, 3, \dots, n, x_i > \bar{x}$ . Hence, using inequalities,

$$x_1 > \bar{x}$$

$$x_2 > \bar{x}$$

$$x_3 > \bar{x}$$

$$\vdots$$

$$x_n > \bar{x}$$

Such that, by adding all the  $n$  inequalities, we have

$$x_1 + x_2 + \dots + x_n > \bar{x} + \bar{x} + \dots + \bar{x}$$

Which implies that  $x_1 + x_2 + \dots + x_n > n\bar{x}$

$$\Rightarrow \frac{x_1 + x_2 + \dots + x_n}{n} > \bar{x}$$

Since by definition,

$$\frac{x_1 + x_2 + x_3 + \dots + x_{n-1} + x_n}{n} = \bar{x}$$

Then  $\bar{x} > \bar{x}$ , which is a contradiction. Hence the part (a) of the statement is true always.



## Generalized form of Pigeonhole Principle

### THEOREM (Generalised form of Pigeonhole Principle)

If one places  $q_1 + q_2 + \dots + q_r - r + 1$  letters in  $r$  pigeonholes, then

Pigeonhole 1 has at least  $q_1$  letters or

Pigeonhole 2 has at least  $q_2$  letters or

Pigeonhole 3 has at least  $q_3$  letters or

$\vdots$

Pigeonhole  $r$  has at least  $q_r$  letters, where  $q_i \in \mathbb{N}$ .

In short as an example the theorem says if there is  $3 + 17 + 15 + 7 - 4 + 1 = 39$ .

#### *Proof*

Suppose the theorem is false, then it means none of the  $r$  statements could be true. Then each of the  $r$  pigeonholes could have less than  $q_r$  letters or at most  $q_r - 1$  letters, i.e.

$$\leq q_1 - 1, \leq q_2 - 1, \leq q_3 - 1, \dots, \leq q_{r-1} - 1, \leq q_r - 1$$

Then

Pigeonhole 1 has at most  $q_1 - 1$  letters and

Pigeonhole 2 has at most  $q_2 - 1$  letters and

Pigeonhole 3 has at most  $q_3 - 1$  letters and

$\vdots$

and Pigeonhole  $r$  has at most  $q_r - 1$  letters.

This implies that the maximum numbers of letters in the  $r$  pigeonholes is

$$\begin{aligned} &\leq (q_1 - 1) + (q_2 - 1) + (q_3 - 1) + \dots + (q_r - 1) \\ &= q_1 + q_2 + q_3 + \dots + q_r - r \quad \text{where } r \text{ is the number of ones (1's)} \end{aligned}$$

However, this contradicts with the fact we had,

$$q_1 + q_2 + \dots + q_r - r + 1 \text{ letters in total.}$$

Therefore the theorem can never be false.

**Example**

A school has 4 classes with 321 students. Deduce that class 1 has at least 100 students or class 2 has at least 101 students or class 3 has at least 100 students or class 4 has at least 23 students.

**Solution**

Suppose we have  $q_1 = 100$ ,  $q_2 = 101$ ,  $q_3 = 100$ , and  $q_4 = 23$ , then for  $r = 4$ , by the generalized pigeonhole principle,

$$q_1 + q_2 + \cdots + q_r - r + 1 = 100 + 101 + 100 + 23 - 4 + 1 = 321$$

**Example**

Calculate the minimum number of persons that one can be assured of finding at least 3 persons being born in Jan 2003, or at least 3 persons being born in Feb 2003, or ..., or at least 3 persons being born in December 2003.

**Solution**

Let

$$N = q_1 + q_2 + \cdots + q_r - r + 1$$

persons be placed in  $r = 12$  (Jan, Feb, ..., Dec) boxes such that  $q_1 = 3$ ,  $q_2 = 3$ ,  $q_3 = 3$ , ...,  $q_{12} = 3$ , then

$$N = q_1 + q_2 + \cdots + q_r - r + 1 = 3 + 3 + \cdots + 3 - 12 + 1 = 25.$$



### Activity 2 d

- a) Let  $a_1, a_2, a_3, \dots, a_n$  be a list of  $n$  numbers and  $v$  be the arithmetic mean (average) of these numbers.
- Express  $v$  in terms of  $a_1, a_2, a_3, \dots, a_n$ .
  - Deduce that if the maximum of the numbers  $a_1, a_2, a_3, \dots, a_n$  is below the mean  $v$  then all values  $a_1, a_2, a_3, \dots, a_n$  lie below.
  - Prove by contradiction that the maximum of the numbers  $a_1, a_2, a_3, \dots, a_n$  is *never* below the arithmetic mean  $v$ .
  - Likewise prove that the minimum of the numbers maximum of the numbers  $a_1, a_2, a_3, \dots, a_n$  is never above the arithmetic mean  $v$ .
- b) Determine the minimum number of one Kwacha coins that an employer needs to have at the end of each month if she promises that, at the end of every month, she will either give a bonus of at least 500 one Kwacha coins to her longest serving worker or at least 400 one Kwacha coins to the second longest serving worker or at least 300 one Kwacha coins to the third longest serving worker.
- c) Show that you need to have a minimum number of 33 individuals to guarantee a cock-tail party attended by at least 10 Malawians, or at least 16 Indians or at least 9 Germans.





## Summary/Let Us Sum Up

In this unit we have discussed key principles behind mathematics of item distributions, that of pigeon-hole principle. We have specifically discussed various forms of pigeon-hole principle beginning with the simplest form, then regular, extended regular, and finally generalized form. The simplest form states that If  $n + 1$  letters are placed in  $n$  pigeonholes, there will be a pigeonhole with MORE THAN one letter in it. On the other hand, the regular form states that If one places  $nr + 1$  letters in  $n$  pigeonholes, then there exists a pigeonhole with MORE THAN  $r$  letters which is then **extended** by the fact that If  $n$  letters are placed in  $k$  pigeonholes, then there exist a pigeonhole with MORE THAN  $\lfloor \frac{n-1}{k} \rfloor$  letters. Lastly, the generalized form states that If one places  $q_1 + q_2 + \dots + q_r - r + 1$  letters in  $r$  pigeonholes, then Pigeonhole 1 has at least  $q_1$  letters or Pigeonhole 2 has at least  $q_2$  letters or Pigeonhole 3 has at least  $q_3$  letters or Pigeonhole  $r$  has at least  $q_r$  letters, where  $q_i \in \mathbb{N}$ . We have demonstrated their relevance and various applications to solving real life related problems concerning minimum and maximum item distribution. Below, the reader finds further reading reference materials not only for this Unit, but also in other subsequent units.



## Suggestions for Further Reading/ Reading Assignment

- Susanna S. Epp. (2011). *Discrete Mathematics with Applications* (4<sup>th</sup> Edition), Boston: Brooks Cole. [Chapter 9: pp. 554]
- Susanna S. Epp. (2011). *Discrete Mathematics with Applications: Student Solutions Manual* (4<sup>th</sup> Edition), Belmont: Cengage Learning.
- Kenneth H. Rosen. (2012). *Discrete Mathematics and its Applications* (7<sup>th</sup> Edition), New York: McGraw-Hill Education. [Chapter 6: pp. 399]
- Kenneth H Rosen, & Jerrold W. Grossman Professor. (2011). *Student's Solutions Guide to Accompany Discrete Mathematics and Its Applications*, (7<sup>th</sup> Edition). New York: McGraw-Hill Education.
- Balakrishnan, V.K. (2010). *Introductory Discrete Mathematics*. New York: Dover Publications, Inc. [Chapter 1: pp. 53]
- Seymour Lipschutz, & Marc Lipson. (2009). *Schaum's Outline of Discrete Mathematics*, (3<sup>rd</sup> Edition). New York: McGraw-Hill Education. [Chapter 5: pp. 94, Chapter 6: pp. 110]
- Richard Kohar. (2016). *Basic Discrete Mathematics: Logic, Set Theory, and Probability*. Singapore: World Scientific Publishing Company. [Chapter 5: pp. 137]



## Unit 2 Test

1. State and describe the last two theorems about the pigeonhole principle.
2. Use method of contradiction to show that the first form of the pigeonhole principle is true.
3. Show that in any finite set of numbers, there is always a number whose value is at most the average.
4. A party of  $m$  ( $m > 2$ ) persons took place at University graduation ceremony. Some people shook hands with one another (without repeating or shaking their own hands). Show that there were at least two persons who made the same number of handshakes during the party.
5. Show that if one chooses 7 points inside a circle of diameter 10 cm, then there must be at least two points that are never more than 5 cm apart.
6. Show that in a cafeteria where 465 students are dining on 95 tables, one knows that some table has more than four students dining on it.
7. Show that if one chooses 101 numbers from the list 1, 2, 3, ..., 198, 199, 200, then at least two of the chosen numbers will be consecutive.
8. Assuming that if person  $x$  is a friend of  $y$  then  $y$  is also a friend of  $x$ , show that in any group of  $n$  people, there are at least two persons having the same number of friends.
9. Find the minimum number of persons one can choose from a crowd to be sure that more than three persons among the group have surnames starting with the same letter of English alphabet.
10. Demonstrate that if 51 points are placed on and inside a square of side 1, then there must be three points that fits inside a circle of radius  $\frac{1}{7}$



## Selected answers to Unit Activities

ACTIVITY	(a)	(b)	(c)	(d)	(e)
Activity 3 a	$n + 1 = 367;$ $n = 366$		$n + 1 = 101;$ $2n = 200,$ $n = 100$		diagonal = 2 = $\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2}$
Activity 3 b		$4 = nr + 1;$ $n = 3,$ $r = 1$		$n = 365, r = 2,$ $nr + 1 = 731$	
Activity 3 c	$n - 1 = 96, k = 12,$ $\lfloor \frac{96}{12} \rfloor = 8$		$n - 1 = 5999; k = 50,$ $\lfloor \frac{5999}{50} \rfloor = 120$		$n = 367, k = 365$
Activity 3 d		1198			