Section 5.2
Confidence Intervals for a Proportion

Confidence intervals

- A plausible range of values for the population parameter is called a *confidence interval*.
- Using only a sample statistic to estimate a parameter is like fishing in a murky lake with a spear, and using a confidence interval is like fishing with a net.

We can throw a spear where we saw a fish but we will probably miss. If we toss a net in that area, we have a good chance of catching the fish.

- If we report a point estimate, we probably won't hit the exact population parameter. If we report a range of plausible values we have a good shot at capturing the parameter.

Photos by Mark Fischer (http://www.flickr.com/photos/fischerfotos/7439791462) and Chris Penny (http://www.flickr.com/photos/clearlydived/7029109617) on Flickr.

Facebook's categorization of user interests

Q: *Most commercial websites (e.g. social media platforms, news outlets, online retailers) collect data about their users' behaviors and use these data to deliver targeted content, recommendations, and ads. To understand whether Americans think their lives line up with how the algorithm-driven classification systems categorizes them, Pew Research asked a representative sample of 850 American Facebook users how accurately they feel the list of categories Facebook has listed for them on the page of their supposed interests actually represents them and their interests. 67% of the respondents said that the listed categories were accurate. Estimate the true proportion of American Facebook users who think the Facebook categorizes their interests accurately.*

https://www.pewinternet.org/2019/01/16/facebook-algorithms-and-personal-data/

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Q: *Which of the following is the correct interpretation of this confidence interval?*
We are 95% confident that

(a) *64% to 70% of American Facebook users in this sample think Facebook categorizes their interests accurately.*
(b) *64% to 70% of all American Facebook users think Facebook categorizes their interests accurately*
(c) *there is a 64% to 70% chance that a randomly chosen American Facebook user's interests are categorized accurately.*
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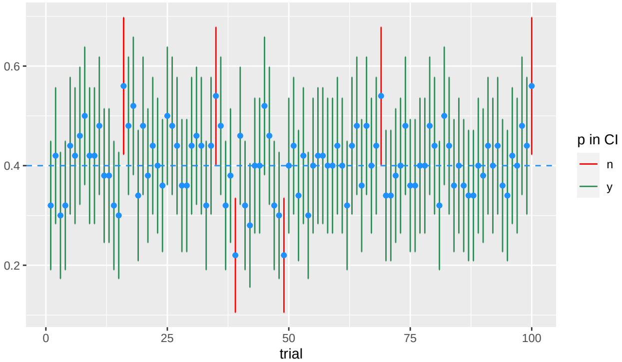
What does 95% confident mean?

- Suppose we took many samples and built a confidence interval from each sample using the equation *point estimate* $\pm 1.96 \times SE$.
- Then about 95% of those intervals would contain the true population proportion (p).

What does 95% confident mean?

Simulation (100 repetitions): $p = 0.40$, determine a 95% confidence interval for p based on a sample of size $n = 50$.



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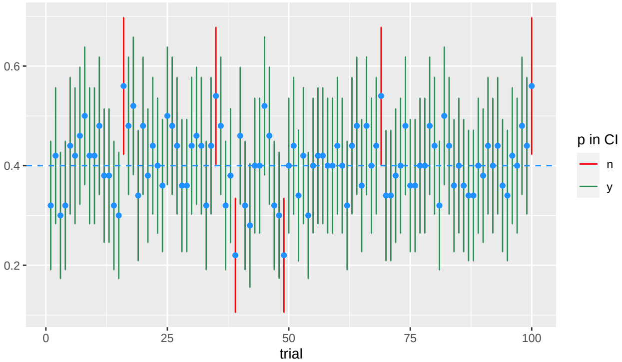

We see that 94 of the 100 confidence intervals in this simulation actually contain the population proportion $p = .4$.

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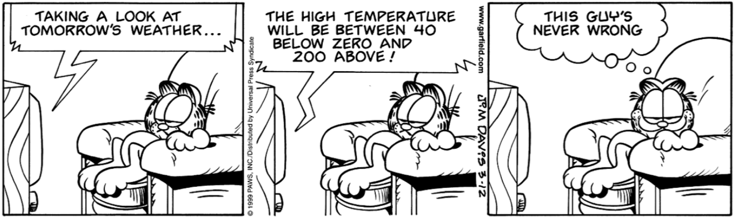

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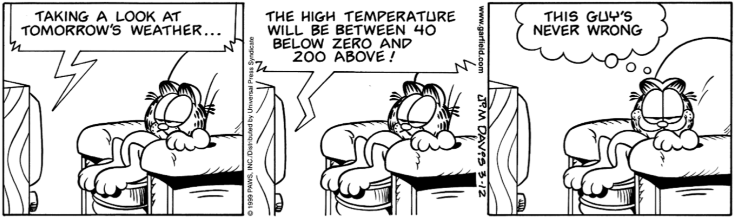

If the interval is too wide it may not be very informative.

Image source: http://web.as.uky.edu/statistics/users/earo227/misc/garfield_weather.gif

Changing the confidence level

$$point\ \ estimate \pm z^{\star} \times SE$$

- In a confidence interval, $z^{\star} \times SE$ is called the *margin of error*, and for a given sample, the margin of error changes as the confidence level changes.
- In order to change the confidence level we need to adjust $z^{\star}$ in the above formula.
- Commonly used confidence levels in practice are 90%, 95%, 98%, and 99%.
- For a 95% confidence interval, $z^{\star} = 1.96$.
- However, using the standard normal (z) distribution, it is possible to find the appropriate $z^{\star}$ for any confidence level.

Q: *Which of the below Z scores is the appropriate $z^{\star}$ when calculating a 98% confidence interval?*

(a) $Z = 2.05$

(b) $Z = 1.96$

(c) $Z = 2.33$

(d) $Z = -2.33$

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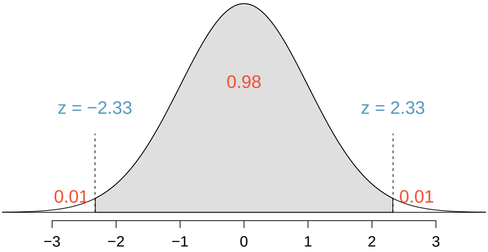
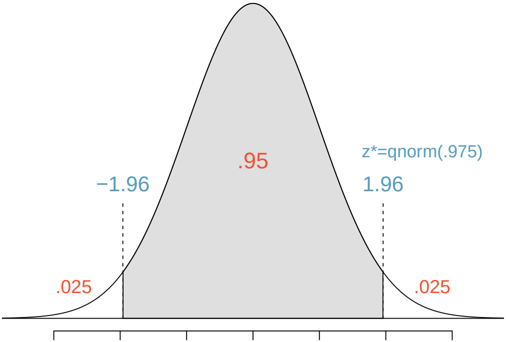


Finding z^* for a given confidence level

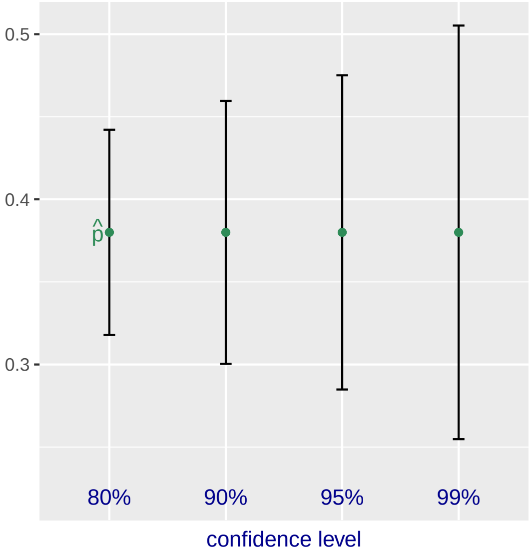
A level L confidence interval for a population proportion p

$$\hat{p} \pm z^* SE$$

$\hat{p}$ - sample proportion from an independent sample of size n
SE $= \sqrt{\hat{p}(1-\hat{p})/n}$, and $z^* = \texttt{qnorm}(L + (1-L)/2)$.



Confidence levels impact the size of the confidence interval



Interpreting confidence intervals

Confidence intervals are ...

- always about the population
- not probability statements
- only about population parameters, not individual observations
- only reliable if the sample statistic they're based on is an unbiased estimator of the population parameter

Confidence interval for a single proportion

Once you've determined a one-proportion confidence interval would be helpful for an application, there are four steps to constructing the interval:

Prepare. Identify $\hat{p}$ and n, and determine what confidence level you wish to use.

Check. Verify the conditions to ensure $\hat{p}$ is nearly normal. For one-proportion confidence intervals, use $\hat{p}$ in place of p to check the success-failure condition.

Calculate. If the conditions hold, compute SE using $\hat{p}$, find $z^{\star}$, and construct the interval.

Conclude. Interpret the confidence interval in the context of the problem.

Example: Solar Energy

In a Pew Research poll in 2018 about solar energy, 84.8% of respondents supported expanding the use of wind turbines (see text p. 186). Follow the four steps for creating a 99% confidence interval for the level of American support for expanding the use of wind turbines for power generation.

Example: Solar Energy

In a Pew Research poll in 2018 about solar energy, 84.8% of respondents supported expanding the use of wind turbines (see text p. 186). Follow the four steps for creating a 99% confidence interval for the level of American support for expanding the use of wind turbines for power generation.

Prepare. Here, we are being asked to use 99% confidence level, we have $n = 1000$ and $\hat{p} = .848$.

Example: Solar Energy

Check. The Pew survey was a random sample, and we can check that both counts (in favor and opposed) are at least 10:

$$n\hat{p} = 848 \text{ and } n(1-\hat{p}) = 152.$$

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Calculate. $z^* = \texttt{qnorm}(.995) = 2.576$, SE $= \sqrt{\frac{.848(1-.848)}{1000}} = 0.0114$. So the margin of error MOE $= 2.576 \cdot 0.0114 = 0.0294$ and the 99% confidence interval is:

$$0.848 \pm 0.0294 \quad \text{or} \quad (0.8186, 0.8774).$$

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Conclude. We are 99% confident the proportion of American (in 2018) who support expanding the use of wind turbines is between 81.9% and 87.7%.

Choosing a sample size to obtain desired margin of error

The margin of error for a confidence interval is:

$$\text{MOE} = z^* \cdot \sqrt{\frac{p(1-p)}{n}}.$$

If we specify a confidence level, say 95%, then this determines z^*, ($z^* = 1.96$ in this case).
The other factor we control is our choice of sample size, n. As n increases, MOE decreases.

Example

Find the sample size n that will give a MOE of .03 in a 95% confidence interval, assuming $p = .5$.

Solution

Solve this equation for n:

$$.03 = 1.96 \cdot \sqrt{\frac{(.5)(.5)}{n}}$$
$$\frac{.03}{1.96} = \sqrt{\frac{.25}{n}}$$
$$\left(\frac{.03}{1.96}\right)^2 = \frac{.25}{n}$$
$$n = \frac{(.25)(1.96)^2}{(.03)^2} \approx 1067.11$$

Since n must be a whole number, we round up to 1068.

General Formula for specifying sample size n to produce desired MOE

We can solve the MOE formula for n we obtain the following general formula:

Finding sample size for desired MOE

$$n = p(1-p)\left(\frac{z^*}{\text{MOE}}\right)^2$$

If you don't know p, letting $p = 1/2$ to produce an n that will work regardless of the actual value of p, in which case the formula for n becomes

$$n = \frac{1}{4}\left(\frac{z^*}{\text{MOE}}\right)^2$$