Chapter 3: Probability

Math 140
Based on content in OpenIntro Stats, 4th Ed

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Section 3.1: Defining Probability

Random Process

- Roll a 6 sided die.
- Measure a patient's systolic blood pressure.
- Record how long it takes you to run one mile.
- Record how many texts you send each day.

These are examples of *random processes*, an event whose outcome is unknown ahead of time, but has a predictable set of possible outcomes.

Random Variable

- A *random variable* is a variable (commonly X) used to indicate an outcome of a random process if the outcomes are numerical.
- Perhaps X represents a patient's systolic blood pressure.
- As a health care provider, we want to check whether $X > 140$ since values above 140 indicate hypertension.
- Or perhaps X represents my time running the mile.

Discrete vs Continuous Random Variables

- A *discrete* random variable that can only take numerical values with jumps.
- A *continuous* random variable is one that can take all values over an interval of numbers.

Example (Discrete or Continuous?)

- Roll a 6 sided die
- Measure this patient's systolic blood pressure
- Record how long it takes you to run one mile
- Record how many texts you send each day

ANSWER: The first and 4th are discrete. Blood pressure and mile time can be measured to as many decimals as the measuring instruments allow, so they are continuous.

Probability

Definition

The *probability* of an outcome in a random process is the proportion of times the outcome would occur if we observed the random process an infinite number of times.

Example (*Rolling a fair 6-sided die*)

- It is reasonable to suppose each of the 6 numbers has the same chance of coming up.
- In the long run, when I roll many, many times, I would expect each number to come up about $1/6$th of the time.
- That is, it is reasonable to suppose the probability of rolling any value is $1/6$.

Discrete vs Continuous Probabilities

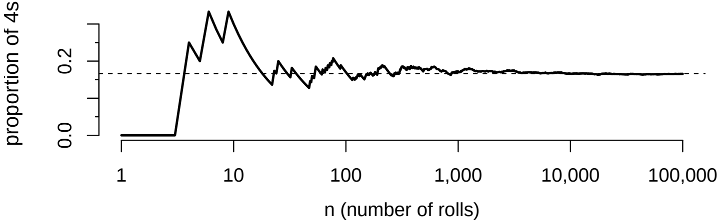
- Probabilities associated to a discrete distribution can often be presented via a table, as we shall see
- Probabilities associated to a continuous distribution are often represented via a density curve, as we discuss in Section 4.1
- For the remainder of this chapter, we'll focus on discrete distributions

Simulation: Rolling a fair 6-sided die

Q: If I roll a fair, 6-sided die, what is the probability that a 4 comes up?

We can approximate this probability with a *simulation*: Roll the die many, many times, recording each time whether we roll a 4. Also, as we go, we can record the proportion of rolls up to that point that have resulted in a 4:

- Results of first 10 rolls ('0' means not a 4, '1' means 4): 0 0 0 1 0 1 0 0 1 0
- First 10 sample proportions: 0 0 0 0.25 0.2 0.33 0.29 0.25 0.33 0.3
- A plot of the sample proportion up through 100,000 rolls



The Law of Large Numbers

Law of Large Numbers

As more observations are collected, the proportion $\hat{p}$ of occurrences with a particular outcome converges to the probability p of that outcome.

- p - (theoretical) probability
- $\hat{p}$ - the proportion of times a result occurs in a number of trials
- Law of Large Numbers says: As the number of trials gets larger and larger, $\hat{p} \to p$.

Probability Model: Rolling a fair 6-sided die

- **Random Variable** X. The variable we use to denote the values we can roll.
- **Sample Space** $\{1, 2, 3, 4, 5, 6\}$. The set of possible outcomes.
- **Probability Model**. A description of the probabilities associated to the values in the sample space.

X	1	2	3	4	5	6
$P(X)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

Rules for Probability Distributions

Probability Distribution

A probability distribution is a list of the possible outcomes with corresponding probabilities that satisfies three rules:

1. The outcomes listed in the sample space must be disjoint.
2. Each probability must be between 0 and 1.
3. The probabilities must total 1.

Probability Notation and Terms

Suppose X is a random variable with sample space S.

- $P(X = a)$ and $P(a)$ denote the probability that item a in the sample space occurs.
- If A is a subset of the sample space, we call A an **event**,
- $P(A)$ denotes the probability that an outcome in the subset A occurs.
- A^c is called the **complement** of A. It consists of all values in the sample space that are *not* in A.
- Two events A and B are **disjoint**, or **mutually exclusive**, if they cannot both happen - they have no outcomes in common.

Example: Rolling a fair 6-sided Die

- Sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
- $P(a) = 1/6$ for each value in the sample space.
- Let A denote the event that I roll an even number. What is $P(A)$?
- **Answer**: $P(A) = P(2 \text{ or } 4 \text{ or } 6) = 1/6 + 1/6 + 1/6 = 1/2$.
- Let B denote the event that I roll a 1 or a 2. Then $P(B) = 1/3$.
- A and B are not disjoint events since they have an outcome in common (the outcome of rolling a 2).
- B^c is the event that I roll a 3, 4, 5, or 6, and $P(B^c) = 4/6 = 2/3$.

Probability Summation Rules

Two Handy Properties of Probability

- $P(A^c) = 1 - P(A)$.
- If A and B are disjoint events then $P(A \text{ or } B) = P(A) + P(B)$.

Example

- If the probability that it rains today is 0.3, then the probability that it *doesn't* rain is 0.7.
- If X denotes the result of rolling a fair 6-sided die, then

$$P(X = 2 \text{ or } 4) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

Funny Dice

Example (A strange die)

Here is most of the probability model for a strange die.

1. What must the probability be of rolling a 3?
2. If I roll this strange die 10,000 times, which is more likely, rolling a 4, or rolling a number less than 4?

X	1	2	3	4	5	6
$P(X)$	0.1	0.1		0.5	0.1	0.2

ANSWERS:

1. $P(3) = 0$ since the other probabilities already sum to 1.
2. In 10,000 rolls I would expect about 5,000 4s. On the other hand, I should expect about 2,000 rolls to give a value less than 4 (about 1,000 1s, 1,000 2s, and 0 3s). Rolling a 4 seems much more likely than rolling a number less than 4.

Conditional Probability

Scene

You have two events A and B associated with a random process. Then we might be interested in the following probabilities

- $P(A)$, the probability that event A occurs.
- $P(B)$, the probability that event B occurs.
- $P(A \cap B)$, the probability that both A and B occur.
- $P(A \cup B)$, the probability that either A or B occurs (or possibly both).
- $P(A \mid B)$, the probability that A occurs given that event B has occurred.

The last probability, $P(A \mid B)$ is called the **conditional probability** that A occurs given that B has occurred.

Conditional Probability Example: Roll a 6-sided Die

Recall the earlier example of rolling a fair 6-sided die. We considered two events:

- $A = \{2, 4, 6\}$ (we roll an even number), and $P(A) = 3/6 = 1/2$
- $B = \{1, 2\}$ (we roll a 1 or 2), and $P(B) = 2/6 = 1/3$

Questions: Find $P(A \cup B)$, $P(A \cap B)$, and $P(A \mid B)$.

- $A \cup B = \{1, 2, 4, 6\}$, so $P(A \cup B) = 4/6 = 2/3$.
- $A \cap B = \{2\}$, so $P(A \cap B) = 1/6$.

What about $P(A \mid B)$, the probability that we roll an even number, knowing that event B has occurred?
$P(A \mid B) = 1/2$ because, assuming B has occurred, this means we rolled a 1 or 2, so there is a 50 percent chance we rolled an even number.

Two general probability rules

Two general probability rules

Suppose A and B are two events associated to a random process.

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and
2. $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, assuming $P(B) > 0$.

Independence

Definition

Two *processes* are **independent** if knowing the outcome of one provides no useful information about the outcome of the other.
Two *events* A and B are independent if either

$$P(A \mid B) = P(A) \qquad \text{or} \qquad P(B \mid A) = P(B)$$

Activity: Random Phones

Scene

After class I find 4 phones in the classroom. The next day I randomly return the 4 phones to the 4 students who misplaced them. What is the probability that all 4 students get their own phone back?

Three Strange Dice

Scene

You have 3 dice on a table. You and a friend will each roll one of them. Whoever rolls the higher number wins.

- blue die: 1, 1, 4, 4, 4, 4.
- red die: 2, 2, 2, 2, 5, 5.
- purple die: 3, 3, 3, 3, 3, 6.

Which die should you choose to roll?