

# Chapter 3: Probability

Math 140 · Fall '21

Based on content in OpenIntro Stats, 4th Ed

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## Section 3.1: Defining Probability

# Random Process

- Roll a 6 sided die.
- Measure a patient's systolic blood pressure.
- Record how long it takes you to run one mile.
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These are examples of *random processes*, an event whose outcome is unknown ahead of time, but has a predictable set of possible outcomes.

# Random Variable

- A *random variable* is a variable (commonly  $X$ ) used to indicate an outcome of a random process if the outcomes are numerical.
- Perhaps  $X$  represents a patient's systolic blood pressure.
- As a health care provider, we want to check whether  $X > 140$  since values above 140 indicate hypertension.

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- Perhaps  $X$  represents a patient's systolic blood pressure.
- As a health care provider, we want to check whether  $X > 140$  since values above 140 indicate hypertension.
- Or perhaps  $X$  represents my time running the mile.

# Discrete vs Continuous Random Variables

- A *discrete* random variable that can only take numerical values with jumps.
- A *continuous* random variable is one that can take all values over an interval of numbers.

## Example (Discrete or Continuous?)

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- Record how long it takes you to run one mile
- Record how many texts you send each day

ANSWER: The first and 4th are discrete. Blood pressure and mile time can be measured to as many decimals as the measuring instruments allow, so they are continuous.



# Probability

## Definition

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## Example (*Rolling a fair 6-sided die*)

- It is reasonable to suppose each of the 6 numbers has the same chance of coming up.
- In the long run, when I roll many, many times, I would expect each number to come up about  $1/6$ th of the time.
- That is, it is reasonable to suppose the probability of rolling any value is  $1/6$ .

# Discrete vs Continuous Probabilities

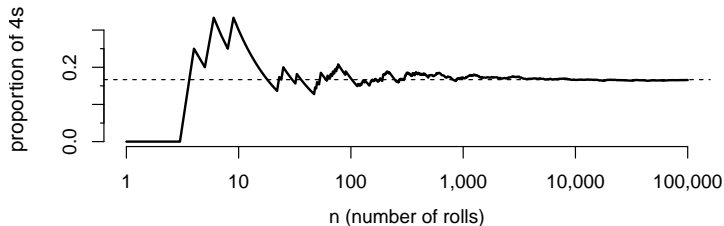
- Probabilities associated to a discrete distribution can often be presented via a table, as we shall see
- Probabilities associated to a continuous distribution are often represented via a density curve, as we discuss in Section 4.1
- For the remainder of this chapter, we'll focus on discrete distributions

## Simulation: Rolling a fair 6-sided die

Q: If I roll a fair, 6-sided die, what is the probability that a 4 comes up?

We can approximate this probability with a *simulation*: Roll the die many, many times, recording each time whether we roll a 4. Also, as we go, we can record the proportion of rolls up to that point that have resulted in a 4:

- Results of first 10 rolls ('0' means not a 4, '1' means 4): 0 0 0 1 0 1 0 0 1 0
- First 10 sample proportions: 0 0 0 0.25 0.2 0.33 0.29 0.25 0.33 0.3
- A plot of the sample proportion up through 100,000 rolls



# The Law of Large Numbers

## Law of Large Numbers

As more observations are collected, the proportion  $\hat{p}$  of occurrences with a particular outcome converges to the probability  $p$  of that outcome.

- $p$  - (theoretical) probability
- $\hat{p}$  - the proportion of times a result occurs in a number of trials
- Law of Large Numbers says: As the number of trials gets larger and larger,  $\hat{p} \rightarrow p$ .

# Probability Model: Rolling a fair 6-sided die

- **Random Variable  $X$ .** The variable we use to denote the values we can roll.
- **Sample Space  $\{1, 2, 3, 4, 5, 6\}$ .** The set of possible outcomes.
- **Probability Model.** A description of the probabilities associated to the values in the sample space.

$X$	1	2	3	4	5	6
$P(X)$	1/6	1/6	1/6	1/6	1/6	1/6

# Rules for Probability Distributions

## Probability Distribution

A probability distribution is a list of the possible outcomes with corresponding probabilities that satisfies three rules:

- 1 The outcomes listed in the sample space must be disjoint.
- 2 Each probability must be between 0 and 1.
- 3 The probabilities must total 1.

# Probability Notation and Terms

Suppose  $X$  is a random variable with sample space  $S$ .

- $P(X = a)$  and  $P(a)$  denote the probability that item  $a$  in the sample space occurs.
- If  $A$  is a subset of the sample space, we call  $A$  an **event**,
- $P(A)$  denotes the probability that an outcome in the subset  $A$  occurs.
- $A^c$  is called the **complement** of  $A$ . It consists of all values in the sample space that are *not* in  $A$ .
- Two events  $A$  and  $B$  are **disjoint**, or **mutually exclusive**, if they cannot both happen - they have no outcomes in common.



## Example: Rolling a fair 6-sided Die

- Sample space is  $S = \{1, 2, 3, 4, 5, 6\}$ .
- $P(a) = 1/6$  for each value in the sample space.
- Let  $A$  denote the event that I roll an even number. Then  $P(A) = P(2 \text{ or } 4 \text{ or } 6) = 1/6 + 1/6 + 1/6 = 1/2$ .
- Let  $B$  denote the event that I roll a 1 or a 2. Then  $P(B) = 1/3$ .
- $A$  and  $B$  are not disjoint events since they have an outcome in common (the outcome of rolling a 2).
- $B^c$  is the event that I roll a 3, 4, 5, or 6, and  $P(B^c) = 4/6 = 2/3$ .

# Probability Summation Rules

## Two Handy Properties of Probability

- $P(A^c) = 1 - P(A)$ .
- If  $A$  and  $B$  are disjoint events then  $P(A \text{ or } B) = P(A) + P(B)$ .

## Example

- If the probability that it rains today is 0.3, then the probability that it *doesn't* rain is 0.7.
- If  $X$  denotes the result of rolling a fair 6-sided die, then

$$P(X = 2 \text{ or } 4) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

# Funny Dice

## Example (A strange die)

Here is most of the probability model for a strange die.

- 1 What must the probability be of rolling a 3?
- 2 If I roll this strange die 10,000 times, which is more likely, rolling a 4, or rolling a number less than 4?

$X$	1	2	3	4	5	6
$P(X)$	0.1	0.1		0.5	0.1	0.2

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ANSWERS:

- 1  $P(3) = 0$  since the other probabilities already sum to 1.
- 2 In 10,000 rolls I would expect about 5,000 4s. On the other hand, I should expect about 2,000 rolls to give a value less than 4 (about 1,000 1s, 1,000 2s, and 0 3s). Rolling a 4 seems much more likely than rolling a number less than 4.

# Independence

## Definition

Two processes are **independent** if knowing the outcome of one provides no useful information about the outcome of the other.

# Activity: Random Phones

## Scene

After class I find 4 phones in the classroom. The next day I randomly return the 4 phones to the 4 students who misplaced them. What is the probability that all 4 students get their own phone back?