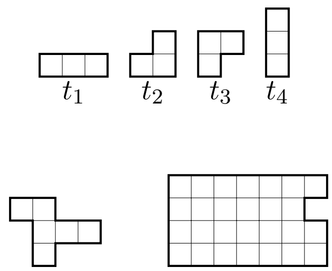
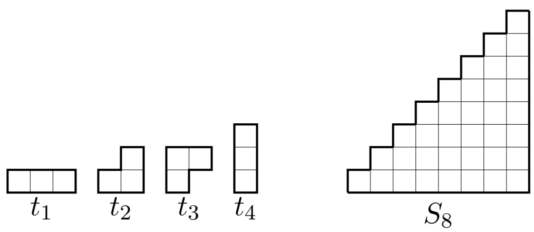
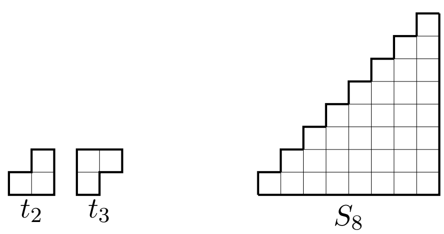
1. Can the set of ribbon tile trominoes tile either of the regions below?



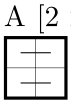
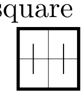
2. Construct a tiling of the staircase region S_8 by the set of ribbon tile trominoes.



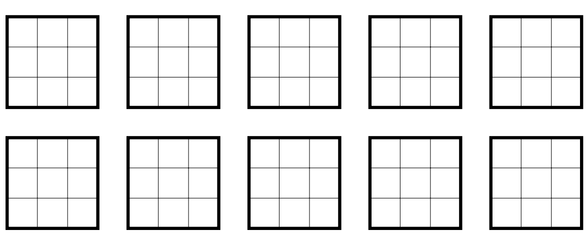
3. Repeat the previous problem, but use just the two L-shaped trominoes.



4. A $[2 \times 2]$ square can be tiled by dominoes in two different ways, as pictured.

 and 

How many different ways can a $[3 \times 3]$ square be tiled by the four ribbon tile trominoes in Q1 (No rotating or flipping the tiles.)

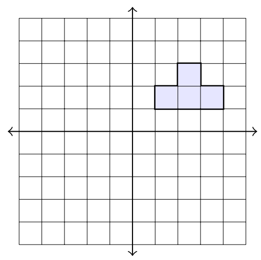


Tiling Notes iFOCUS - Fall '21

Professor Hitchman

Basics

Our tiles and regions live in the integer lattice, and are viewed as unions of 1×1 squares, which we call cells.



If T is a tile set, and R is a region, we say that T tiles R if we may place copies of the tiles in T in the region R so that each cell in R is covered exactly once, and no cell outside of R is covered.

Rules of the Game

In our effort to tile a region R by a tile set T:

1. We have an unlimited supply of each type of tile in T
2. We may **not** rotate or flip the tiles listed in the set T.

Area Argument

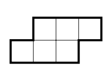
The area of a region equals the number of cells it contains. A first test in a tiling question is to check whether a region's area is compatible with the tile areas.

Area Invariant

Suppose each tile in the tile set T has area k. If the area of a region R is not divisible by k then T does not tile R.

For instance, dominoes will never tile a region that has odd area.

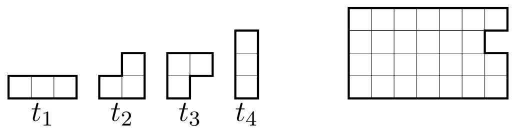
Note that the converse of this fact is false: Even if a region R has area divisible by k, T might not tile R. For instance, dominoes do not tile the following region even though its area is an even number:



Coloring Arguments

If we can "color" each cell with an integer in such a way that the color sum of each tile in T is the same wherever the tile is placed in the lattice (or has the same remainder when divided by a certain integer), then this gives us a condition on what a region R must look like if it can be tiled by T.

Example 1. *Can the set of ribbon tile trominoes tile the region below?*



We can assign integers 0, 1, and 2, to the cells of the integer lattice in the pattern below. Note that each tile, wherever it is placed has a color sum of 3 (in fact, will cover one 0, one 1, and one 2). So, if a region R can be tiled by this tile set, the color sum of R must also be divisible by 3.

0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0

> **Theorem**. If the set of ribbon tile trominoes tiles the region R then the color sum of R (for the coloring above) must be divisible by 3.

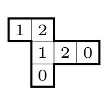
So, if the color sum of a region R is not divisible by 3 then ribbon tile trominoes do not tile R. As we see below, the region given in this problem has a color sum not divisible by 3, so ribbon tile trominoes do not tile this region.

0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0

color sum

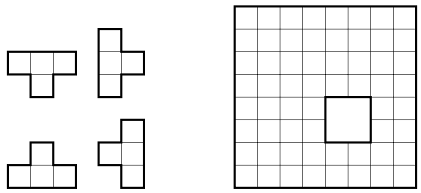
$$10 \cdot 0 + 9 \cdot 1 + 8 \cdot 2 = 25$$

Note, even if a region has color sum divisible by 3, ribbon tile trominoes might not tile it:



So, as with the area invariant, the converse of this coloring theorem is false.

Example 2. *Can T-tetrominoes tile the region obtained by removing a* $[2\times 2]$ *square from the chessboard* $[8\times 8]$*?*



Begin by coloring the cells of the lattice in a checkerboard pattern with 1's and 5's as pictured below.

5	1	5	1	5	1	5	1
1	5	1	5	1	5	1	5
5	1	5	1	5	1	5	1
1	5	1	5	1	5	1	5
5	1	5	1	5	1	5	1
1	5	1	5	1	5	1	5
5	1	5	1	5	1	5	1
1	5	1	5	1	5	1	5

Notice that the color sum of each T-tetromino, wherever it is placed, is divisible by 8, giving us a powerful necessary condition on a region to be tileable by the set T. In particular, we have the following

Theorem. For this coloring, if the color sum a region R is not divisible by 8, then the set T of T-tetrominoes does not tile R.

Proof. Suppose a region R is tileable by the set T. Then the region's color sum can be counted by summing the individual cell values, of course, but it can also be counted by summing the color sums of each tile used in a tiling of R. Since each tile in T has a color sum that is divisible by 8, so, too, will any sum of tile color sums be divisible by 8. That is, the color sum of the region R must be divisible by 8 if it is tileable by T. Logically, then, it follows that if R has a color sum that is not divisible by 8 (for this coloring of 1s and 5s), then it is not tileable by T. $\square$

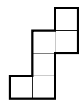
So, for this coloring of 1s and 5s, let's determine the color sum of the chessboard with a $[2\times 2]$ square removed from it. The chessboard, wherever it is placed in the lattice will cover 32 1s and 32 5s. Moreover, any $[2\times 2]$ square, placed anywhere in the chessboard, covers two 1s and two 5s. So, the color sum of the $[8\times 8]$ with a $[2\times 2]$ removed is $30\cdot 1+30\cdot 5=180$. More importantly, this color sum is not divisible by 8. By the theorem, then, the region is not tileable by T-tetrominoes.

Note: The clever part of this solution is finding an effective coloring. Choosing 1s and 5s probably seems like it came out of a magician's hat. It did. Different questions call for different colorings, generally speaking, and trial and error is a fine way to proceed.

Ribbon Tile Invariants

Definition: A ribbon tile of area n consists of n cells, laid out in a path such that from an initial cell, each step either goes up or to the right. A ribbon tile of area n can be indexed by a binary signature of length $n-1$, where each 0 represents a step to the right, and each 1 represents a step up.

For instance, the binary signature 01101 defines this ribbon tile with 6 cells:



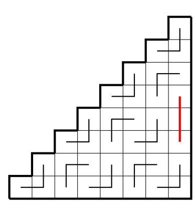
01101

- We let T_n denote the set of area n ribbon tiles.
- A height-1 ribbon tile is one that has an odd number of 1s in its binary signature.
- A height-0 ribbon tile has an even number of 1s in its binary signature.
- We let T_n^1 denote the subset of T_n consisting of the height-1 ribbon tiles of area n.
- The tile set in questions **1** and **2** of the preamble is T_3, and the tile set in question **3** is T_3^1.

Tileability Test for T_n^1: Suppose there exists a tiling of a simply connected region R by T_n in which an odd number of height-0 tiles are used. Then the set T_n^1 does not tile R.

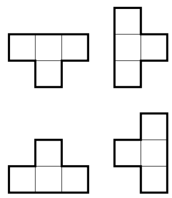
Example 3. *No tiling exists for question* **3** *of the preamble.*

Proof: Here's a tiling of S_8 by the set T_3 that uses an odd number of height-0 tiles:

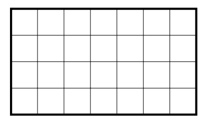


Breakout Questions

1. Given the tile set T consisting of the 4 T-tetrominoes pictured below, we let $[a \times b]$ denote the rectangle with height a and width b. For which values of a and b is $[a \times b]$ tileable by T?

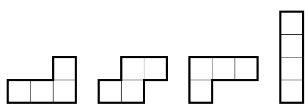


T

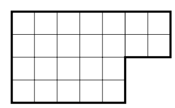


$[4 \times 7]$

2. Given the tile set S consisting of the 4 ribbon tiles pictured below, we let $\Gamma_2(a, b)$ denote the rectangle $[a \times b]$ that has had a 2×2 square removed from its lower-right corner. For which values of a and b is $\Gamma_2(a, b)$ tileable by S?



S

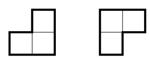


$\Gamma_2(4, 7)$

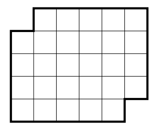
3. The set T_2 of dominoes tiles the $[2 \times 2]$ square in 2 ways. The set T_3 of ribbon tile trominoes tiles the $[3 \times 3]$ square in 6 ways. How many ways does the set T_4 of ribbon tile tetrominoes tile the $[4 \times 4]$ square? How many ways does the set T_n tile the $[n \times n]$ square?

Second Breakout Question

Given the tile set L consisting of the two L-shaped ribbon trominoes pictured below, we let $M(a,b)$ denote the modified rectangle obtained by removing the upper-left and lower-right cells from an $[a \times b]$ rectangle. For which values of a and b is $M(a,b)$ tileable by L?



L



$M(5,6)$