ECE 174 Mini Project 1

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0.1 Part 1: The Normal Equation

Given $A \in \mathbb{C}^{\gtrdot \cap \ltimes}$ and $y \in \mathbb{C}^{\gtrdot}$ show that $\hat{x}$ solves $\min_x \|y - Ax\|_2$ if it is a solution to the following set of linear equations:

$$A^H A y = A^H x$$

0.1.1 (a) The Normal Equation Always has a Solution

First, let $B = A^H A$ and $C = A^H y$. Then $C = Bx$.

$C = Bx$ will have a solution when $C \in Range(B) = Range(A^H A)$.

But $C = A^H y \implies C \in Range(A^H)$.

If $Range(A^H A) = Range(A^H)$ then $C \in Range(A^H A)$ and $C \in Range(A^H)$ will always be true

Let $v \in Range(A^H A)$ and u be a vector $\in \mathbb{C}^n$. Then:

$v = A^H u = A^H A u = A^H z$ where $z = Au \implies v \in Range(A^H) \implies Range(A^H A) \subseteq Range(A^H)$

From Homework 2, we know $rank(A^H A) = rank(A^H)$

Given $Range(A^H A) \subseteq range(A^H)$ and $rank(A^H A) = rank(A^H)$, we can conclude that $C \in Range(A^H A)$ and $C \in Range(A^H)$ will always be true

Thus, $A^H y \in Range(A^H)$ and $A^H y \in Range(A^H A) \implies A^H y = A^H A x$ will always have a solution

0.1.2 (b) The Normal Equation can have Infinite Solutions

From part (a), we know that the normal equations ($A^H A y = A^H x$) will always have at least one solution x.

If A is rank deficient $\implies \exists v \neq 0, v \in \mathbb{R}^{\ltimes}$ s.t. $Av = 0$

For all $\alpha \in \mathbb{R}$, $x + \alpha v$ is also a solution to the normal equation.

Therefore, there can be infinite solutions

0.1.3 (c) Singular Value Decomposition

Suppose $rank(A) < n$. Then finding a solution to the normal equation is much harder because A will not be invertible.

One way to work around this is to use the pseudo-inverse of A, denoted $A^\dagger$.

$A^\dagger = (A^H A)^{-1} A^H$ when A is full rank, but because A is not full rank, we can instead use Singular Value Decompostion (SVD) to find $A^\dagger$

Let $A = U \Sigma V^H$ be the singular value decompostion of A

Let $rank(A) = r$, then $A^\dagger = V \Sigma^{-1} U^H$, where $U \in \mathbb{C}^{m \times r}$, $\Sigma \in \mathbb{C}^{r \times r}$, $V \in \mathbb{C}^{n \times r}$

Using this method, the pseudo-inverse of $A^T A$ can be found and used to calculate the solution to the normal equation.

1 Part 2: Least Squares Classifier

1.1 Import Packages

[16]:
```
import numpy as np
import scipy as sp
from scipy import io
from scipy import linalg
import numpy.ma as ma
import pandas as pd
from matplotlib import pyplot as plt
```

1.2 Binary Class

[17]:
```
# Binary Classifier using Linear Least Squares
class Binary:

# Init Binary object
# Takes training set and label set as arguments
# alpha, beta, data, and labels are attributes
# beta -  vector containing all the parameters used in the predicition
# alpha - the first eleement in the beta vector
# data - training dataset, used for child classes
# labels - labels (self-explanitory)
# Uses ls_constants method to derive alpha and beta values

    def __init__(self, X: np.array, Y: np.array):
        (self.alpha, self.beta) = self.ls_constants(X, Y)
        self.data = X
        self.labels = Y

# returns array of signed linear regression results as the integer type'
    def predict(self, test:np.array):
        return np.sign(self.linear_regression(test)).astype(int)

# returns array of values from linear regression using the parameters and
↪alphas from ls_constants'
    def linear_regression(self, test:np.array):
```

```
        return test @ self.beta + self.alpha

# First, method will create an array of ones and add it as the first column in␣
 ↪the data array (A)
# method will then find b_hat solving the normal equation using the␣
 ↪pseudo_inverse of A
# The result will be a vector (b_hat), which contains a 'bias' element and the␣
 ↪rest of the
# parameters for linear regression. The bias element will be the first element␣
 ↪in the vector, (by construction).
# Method will return the bias element as alpha and the rest of the parameters␣
 ↪as beta

    def ls_constants(self, X:np.array, Y:np.array):
        ones = np.ones([len(X), 1])
        A = np.hstack([ones, X])
        b_hat = np.linalg.pinv(A.T@A) @ A.T @ Y
        alpha = b_hat[0]
        beta = b_hat[1:]
        return alpha, beta

# Here is where I would use SVD to find the pseudo-inverse of (A.T @ A).
# Unfortunately, the np.linalg.svd would not run on my machine due to lack of␣
 ↪memory.
# As a result, the np.linalg.pinv is used instead. Included below the Binary␣
 ↪class is how would have
# implemented SVD to find the solution
```

1.3 SVD Implementation to Solve Normal Equation

Below is the function I would use to calculate the pseudo-inverse to find **beta** and **alpha** in the **ls_constants** function inside the Binary class. It is commented out to avoid errors. This method would replace the use of **np.linalg.pinv**.

[18]:
```
'''
def pseudo(self, X:np.array):
    #We want the pseudo-inverse of X.T @ X
    ATA = X.T @ X

    #Get U, Sigma, and V.T
    U, Sig, VT = np.linalg.svd(ATA)

    #Sig is a vector of the values, so we need to put the values along the␣
 ↪diagonal of a matrix
    sigMatrix = np.diag(s)

    #Calculate Pseudo-Inverse
```

```
    pseu = V.T @ np.inv(Sig) @ U.T

    return pseu
'''
print(" ")
```

1.4 One Versus All (OVA) Class

[19]:
```
# Child class that inherits from Binary
class OVA(Binary):

# For a single given input, the predict method will run the linear regression␣
 ↪method from the Binary class,
# but instead of returning only ones and negatives ones, it will instead return␣
 ↪an array of 10 elements,
# all elements being one or negative one. The argmax will return the index of␣
 ↪the location of the single
# positive 1 as our prediction. This method will return an array of our␣
 ↪predictions.

    def predict(self, test: np.array):
        return np.argmax(self.linear_regression(test), axis = 1)

# Init OneVsAll object
# OneVsAll will inherit methods from Binary, but for the label inputs,
#it use the resulting output from the k_classes method within OneVsAll

    def __init__(self, X: np.array, Y: np.array):
        super().__init__(X, self.k_classes(Y))

# The k_classes method will first extract the unique labels (digits 0 through␣
 ↪9) from the input array of size mx1
# Then, it creates a mx(number of unique labels) matrix, filled with -1.
# It then will iterate over the Y array, and it will set k equal to the value␣
 ↪of the element in Y.
# Our for loop goes row by row in the s matrix, and changes the value of the␣
 ↪kth column to 1.
# It will return an nxn (10x10 in our case) array of the individual labels for␣
 ↪our classifier
# It esstenially expands the test vector into a matrix of -1s and 1s, with the␣
 ↪location of the
# column index of the positive 1 corresponding the the actual value

    def k_classes(self, Y: np.array):
        labels = np.unique(Y)
```

```
        output_matrix = np.where(np.zeros((Y.shape[0], len(labels))) == 0, -1,␣
↪0)
        for i, k in enumerate(Y):
            output_matrix[i, k] = 1
        return output_matrix
```

2 One Versus One (OVO) Class

[20]:
```
class OVO(Binary):

#Like we did with the predict method in the OneVsAll class, we rewrite the␣
↪least squares
# method of finding the parameters for linear regression. In the Binary class,␣
↪the ls_constants method
# created alpha and beta for the entire training set at once. Here, we are␣
↪performing Boolean classification
# between pairs of classes instead of comparing all the classes at once. We␣
↪also redefine the predict method
# to compare between two classes.

# This ls_constants will create 2 different lists of 45 pairs of numbers, where␣
↪val corresponds to the
# index i of  the class (i.e. (0,0)) val2 corresponds to val + 1 (val1 = 0,␣
↪val2 = 1). We create a boolean mask
# of size 60,000x1. The elements of the mask will be "True" if an the element␣
↪in Y is equal to val1 or val2.
# temp_array returns an array of size mx1, where m is the number of "True"␣
↪values  in the mask, and will set
# an element to 1 if the value of Y at that index is equivalent to val1; else␣
↪it will be -1.
# mask and temp are then passed through the ls_constants method from the Binary␣
↪class to output the required
# parameters for te pairwise classificaiton. It will repeat this process for␣
↪every pair of classes
# (45 total classifiers). The resulting alpha and beta vector are then stored␣
↪in the empty alpha and beta lists
# within the method. The method returns alpha and beta as NumPy arrays, alpha␣
↪being 45x1, and beta 45x784

    def ls_constants(self, X:np.array, Y:np.array):
        classes = np.unique(Y)
        alpha = []
        beta = []
        for i, val1 in enumerate(classes):
            for j, val2 in enumerate(classes[i + 1:], i + 1):
```

```
                mask = np.logical_or(Y == val1, Y == val2)
                temp = np.where(Y[mask] == val1, 1, -1)

            # Create a 1x60000 vector with ascending values from 0 to 60000 and␣
↪transpose it
                v = np.array([range(0,60000)])
                v = v.T

            #Indices from x_train needed
                indices = v[mask]

                (a, b) = super().ls_constants(X[indices], temp)
                alpha.append(a)
                beta.append(b)

        return (np.array(alpha), np.array(beta))

    def predict(self, test:np.array):
        classifier = 0
        labels = np.unique(self.labels)
        votes = np.zeros((test.shape[0], labels.shape[0]))

    # iterates over all 45 classifiers
        for i, val1 in enumerate(labels):
            for j, val2 in enumerate(labels[i+1:], i+1):

                alpha = self.alpha[classifier]
                beta = self.beta[classifier]

            # Instead of calling the linear regression method from the Binary␣
↪class, we hard code the equation
            # because we only want to use the specific alpha and beta␣
↪parameters corresponding to the classifier
            # we are using.

                lr_classifier_binary = np.sign(test @ beta + alpha).flatten()

            # Take the signed regression output and create two masks. mask1␣
↪take....,
            # mask1 will take the -1 values as a 0 vote and 1 as a +1 vote
            # mask 2 will take the -1 values as +1 vote because the -1 values␣
↪correspond
            # to classiying as the other value.

                mask1 = np.where(lr_classifier_binary == 1, 1, 0)
                mask2 = np.where(lr_classifier_binary == 1, 0, 1)
```

```
            # In the ith column (value i), add the values of mask1
            # In the jth column( value j), add the values of mask 2

                votes[:, i] += mask1
                votes[:, j] += mask2
                classifier += 1

            # Count the total number of 'votes' in each row to the voting␣
↪matrix
            # and return the label of the column with the most votes'''

        return np.argmax(votes, axis = 1)
```

2.1 Error Rate and Confusion Matrix Functions

[21]:
```
# To compute the error rate, the error_rate method compares the elements in the␣
↪'predicted' and 'actual' arrays
# and counts the instances where they do not match/ It then divdes by the total␣
↪number of elements in
# the 'actual' array

def error_rate(predict: np.array, actual: np.array):
    predict = predict.flatten()
    actual = actual.flatten()
    return np.count_nonzero(predict != actual)/ len(actual)

# To create the confusion matrix, the confusion method extracts the labels from␣
↪the 'actual' array
# and then creates a nxn square matrix of zeroes called 'cmatrix', where n is␣
↪the number of labels + 1. The
# 'predict' and 'actual' arrays are then flattened into a 1-dimensional array,␣
↪called 'predictions'
# and 'act_vals' respectively. 'predictions' and 'act_vals' are then tupled␣
↪together in a zip object,
# which is used to iterate over the indices of 'cmatrix'. Finally, the values␣
↪in the matrix are converted to integers
# and the matrix is placed in a pandas dataframe and returned.

def confusion(predict: np.array, actual: np.array):
    labels = np.unique(actual)
    cmatrix = np.zeros((labels.shape[0] + 1, labels.shape[0] + 1))
    predictions = predict.flatten()
    act_vals = actual.flatten()

    combined = zip(predictions, act_vals)
    #list(combined)
```

```
    for i, j in combined:
        cmatrix[j, i] += 1   # Add to the correct cell
        cmatrix[-1, -1] += 1 # Add to the total
        cmatrix[j, -1] += 1  # Add to the bottom row
        cmatrix[-1, i] += 1  # Add to the right column

    cmatrix = cmatrix.astype(int)

    return pd.DataFrame(cmatrix)
```

3 Implementation of Standard Linear Least Squares (LLS) |Classifier

3.0.1 Import/Extract Data Sets

[22]:
```
data = sp.io.loadmat('mnist.mat')
```

[23]:
```
x_train = data.get('trainX').astype(float)/255
y_train = data.get('trainY').T
x_test = data.get('testX').astype(float)/255
y_test = data.get('testY').T

x_train.shape, y_train.shape, x_test.shape, y_test.shape
```

[23]:
```
((60000, 784), (60000, 1), (10000, 784), (10000, 1))
```

3.0.2 Train Least Squares Classifiers and Predict

[24]:
```
OneVsAll = OVA(x_train, y_train)
OVAprediction = OneVsAll.predict(x_test)
```

[25]:
```
OneVsOne = OVO(x_train, y_train)
OVOprediction = OneVsOne.predict(x_test)
```

3.0.3 Evaluation Metrics of One Versus All Classifiers

[26]:
```
error_rate(OVAprediction, y_test)
```

[26]:
```
0.1397
```

[27]:
```
confusion(OVAprediction, y_test)
```

[27]:

	0	1	2	3	4	5	6	7	8	9	10
0	944	0	1	2	2	7	14	2	7	1	980
1	0	1107	2	2	3	1	5	1	14	0	1135
2	18	54	813	26	15	0	42	22	37	5	1032

3	4	17	23	880	5	17	9	21	22	12	1010
4	0	22	6	1	881	5	10	2	11	44	982
5	23	18	3	72	24	659	23	14	39	17	892
6	18	10	9	0	22	17	875	0	7	0	958
7	5	40	16	6	26	0	1	884	0	50	1028
8	14	46	11	30	27	40	15	12	759	20	974
9	15	11	2	17	80	1	1	77	4	801	1009
10	1041	1325	886	1036	1085	747	995	1035	900	950	10000

3.0.4 Evaluation Metrics of One Versus One Classifiers

[13]:
```
error_rate(OVOprediction, y_test)
```

[13]: 0.0703

[14]:
```
confusion(OVOprediction, y_test)
```

[14]:

	0	1	2	3	4	5	6	7	8	9	10
0	961	0	1	1	0	6	8	3	0	0	980
1	0	1120	3	3	1	1	4	1	2	0	1135
2	9	18	936	12	10	5	10	10	22	0	1032
3	9	1	18	926	2	20	1	7	21	5	1010
4	2	4	6	1	931	1	7	4	3	23	982
5	7	5	3	30	8	800	17	2	15	5	892
6	6	5	12	0	5	19	908	1	2	0	958
7	1	16	17	3	11	1	0	955	1	23	1028
8	7	17	8	23	10	36	10	10	840	13	974
9	6	5	1	11	30	12	0	21	3	920	1009
10	1008	1191	1005	1010	1008	901	965	1014	909	989	10000

3.0.5 Evaluation of Both Types of Classifiers

The "One Vs One" classifier had a lower error rate than the "One vs All" classifier. There was about a 7% difference in the error rate. This is not to say the "One vs One" classifier is better overall, but rather it performed better with this given dataset. Both classifiers generalized well, which can be quantified by the low error rates on both. The easiest digits for the classifiers to recognize were 0 and 1, and the haredest to identify were 8 and 9.

4 Part 3: Randomized Features Based Least Squares Classifiers

For this part, the classes defined in the earlier are modified to find the best linear regession model in *feature space* instead of input space. Given N training data points, the goal of here is to solve the following least squares problem:

$$\min_x \sum_{i=1}^{N} (y_i - \beta^T h(x_i) - \alpha)^2$$

We rewrite the **ls__constants** and **linear__regression** methods from the **Binary** class, now rewritten as **R__Binary** (R denoting ”Randomized Features). We also modify **OneVsAll** and **OneVsOne** classes as **R__OneVsAll** and **R__OneVsOne** to work with the **R__Binary**

4.1 Non-Linear Functions

[15]:
```
def g_map_identity(W: np.array):
    return W

def g_map_sigmoid(W: np.array):
    return 1/(1+ np.exp(W))

def g_map_sinu(W: np.array):
    return np.sin(W)

def g_map_relu(W:np.array):
    return np.maximum(W, 0)
```

4.2 R.Features Binary

[28]:
```
class R_Binary:

# L = Number of features
# G = integer denoting type of non-linear function we want to use
    # 1: Identity
    # 2: Sigmoid
    # 3: Sinusodial
    # 4: ReLU

    def __init__(self, X: np.array, Y: np.array, L, G):
        (self.alpha, self.beta, self.W, self.b) = self.ls_constants(X, Y, L, G)
        self.data = X
        self.labels = Y
        self.features = L
        self.function = G

    def predict(self, test:np.array):
        return np.sign(self.linear_regression(test, self.features)).astype(int)

    def linear_regression(self, test:np.array, L):
        A = self.W @ test.T + self.b

        if self.function == 1:
            H = g_map_identity(A.T)
        if self.function == 2:
            H = g_map_sigmoid(A.T)
        if self.function == 3:
```

```
            H = g_map_sinu(A.T)
        if self.function == 4:
            H = g_map_relu(A.T)

        return H @ self.beta + self.alpha

    # In our new ls_constants, the method creates the W matrix of size␣
↪Lxd(number of columns in X) and b vector os size Lx1.
    # The method append the b vector to W matrix to create w_hat, a matrix of␣
↪size Lx(d+1).
    # Then the method create x_bar of size mx(d+1), the original X matrix with␣
↪a columns of ones.
    # The method finds the dot product of w_hat and x_bar tranpose to get a␣
↪matrix A of size Lxm
    # Depending on the value of G, we put A tranposed through a non-linear␣
↪function to get H, a matrix of size Lxm
    # We then again add a column of ones to H for out alpha value.
    # We then find our beta and alpha by solving for the normal equation.␣
↪Again, this is where SVD would have been used.
    # The method then return beta and alpha, but also the original W and b, so␣
↪that they may be used in linear regression
    # with the test data.

    def ls_constants(self, X:np.array, Y:np.array, L, G):
        W = np.random.normal(0, 1, size = (L, X.shape[1]))
        b = np.random.normal(0, 1, size = (L, 1))
        w_hat = np.append(W, b, 1)
        ones = np.ones([len(X), 1])
        x_bar = np.hstack([ones, X])

        A = w_hat @ x_bar.T

        if G == 1:
            H = g_map_identity(A.T)
        if G == 2:
            H = g_map_sigmoid(A.T)
        if G == 3:
            H = g_map_sinu(A.T)
        if G == 4:
            H = g_map_relu(A.T)

        ones2 = np.ones([len(H), 1])
        h_bar = np.hstack([ones, H])

        b_hat = np.linalg.pinv(h_bar.T @ h_bar) @ h_bar.T @ Y
        alpha = b_hat[0]
```

```
        beta = b_hat[1:]

        return alpha, beta, W, b
```

4.3 R.One Vs All

[29]:
```
class R_OVA(R_Binary):

    def predict(self, test: np.array):
        return np.argmax(self.linear_regression(test, self.features), axis = 1)

    #Added L and G arguments to be compatible with the R_Binary class

    def __init__(self, X: np.array, Y: np.array, L, G):
        super().__init__(X, self.k_classes(Y), L, G)

    def k_classes(self, Y: np.array):
        labels = np.unique(Y)
        output_matrix = np.where(np.zeros((Y.shape[0], len(labels))) == 0, -1,␣
 ↪0)
        for i, k in enumerate(Y):
            output_matrix[i, k] = 1
        return output_matrix
```

4.4 R.One Vs One

[30]:
```
class R_OVO(R_Binary):

# Ths ls_constants function does the exact smae thing as the ls_constants in␣
 ↪the original OVO class, but
# the main difference here is that for each classfier, we are storing the␣
 ↪associated randomized W matrix
# and b vector inside a 3-dimensional array, so that when the predict method in␣
 ↪the OVO class is called,
# it will iterate over the 3-dimensinal array and use the radomized matrix and␣
 ↪vector used to train the model.

    def ls_constants(self, X:np.array, Y:np.array, L, G):
        classes = np.unique(Y)
        alpha = []
        beta = []
        b_vex = []
        W_mats = [[[]]] #3D Lsit
        num = 0

        for i, val1 in enumerate(classes):
            for j, val2 in enumerate(classes[i + 1:], i + 1):
```

```
                mask = np.logical_or(Y == val1, Y == val2)
                temp = np.where(Y[mask] == val1, 1, -1)

                v = np.array([range(0, X.shape[0])])
                v = v.T

                indices = v[mask]

                (a, b, W_matrix, b_vec) = super().ls_constants(X[indices],␣
↪temp, L, G)
                alpha.append(a)
                beta.append(b)
                W_mats.insert(num, W_matrix)
                b_vex.append(b_vec)

                num+=1

        return (np.array(alpha), np.array(beta), np.array(W_mats,␣
↪dtype=object), np.array(b_vex))

    def predict(self, test:np.array):
        classifier = 0
        labels = np.unique(self.labels)
        votes = np.zeros((test.shape[0], labels.shape[0]))

    # iterates over all 45 classifiers
        for i, val1 in enumerate(labels):
            for j, val2 in enumerate(labels[i+1:], i+1):

                A = self.W[classifier] @ test.T + self.b[classifier]

                if self.function == 1:
                    H = g_map_identity(A.T)
                if self.function == 2:
                    H = g_map_sigmoid(A.T)
                if self.function == 3:
                    H = g_map_sinu(A.T)
                if self.function == 4:
                    H = g_map_relu(A.T)

                alpha = self.alpha[classifier]
                beta = self.beta[classifier]

                lr_classifier_binary = np.sign(H @ beta + alpha).flatten()

                mask1 = np.where(lr_classifier_binary == 1, 1, 0)
                mask2 = np.where(lr_classifier_binary == 1, 0, 1)
```

```
                votes[:, i] += mask1
                votes[:, j] += mask2
                classifier += 1

        return np.argmax(votes, axis = 1)
```

5 Implementation of Random Features LLS Classifier

5.0.1 One Vs All: Identity

[21]:
```
#Identity OneVsAll
ROneVsAll1 = R_OVA(x_train, y_train, 1000, 1)
ROVAprediction1 = ROneVsAll1.predict(x_test)
error_rate(ROVAprediction1, y_test)
```

[21]: 0.6612

[22]:
```
confusion(ROVAprediction1, y_test)
```

[22]:

	0	1	2	3	4	5	6	7	8	9	10
0	961	0	0	0	0	0	3	16	0	0	980
1	993	0	0	0	0	0	127	15	0	0	1135
2	610	1	111	0	0	0	162	138	0	10	1032
3	728	0	0	7	0	0	64	190	0	21	1010
4	556	0	0	0	0	0	56	260	0	110	982
5	759	0	0	1	0	0	48	73	0	11	892
6	137	0	0	0	0	0	818	3	0	0	958
7	38	0	0	1	0	0	11	973	0	5	1028
8	801	0	0	0	0	0	92	65	7	9	974
9	120	0	0	0	0	0	7	371	0	511	1009
10	5703	1	111	9	0	0	1388	2104	7	677	10000

5.0.2 One Vs All: Sigmoid

[23]:
```
#Sigmoid OneVsAll
ROneVsAll2 = R_OVA(x_train, y_train, 1000, 2)
ROVAprediction2 = ROneVsAll2.predict(x_test)
error_rate(ROVAprediction2, y_test)
```

[23]: 0.1255

[24]:
```
confusion(ROVAprediction2, y_test)
```

[24]:

	0	1	2	3	4	5	6	7	8	9	10
0	879	0	33	19	3	14	10	8	4	10	980
1	1	973	8	3	0	20	47	1	77	5	1135

2	6	2	932	27	9	1	16	8	29	2	1032
3	0	2	19	872	4	59	4	15	20	15	1010
4	0	24	5	0	846	3	21	1	18	64	982
5	12	1	7	37	6	754	20	14	27	14	892
6	10	0	14	1	9	26	886	2	5	5	958
7	0	44	13	3	7	6	6	915	10	24	1028
8	8	4	11	40	7	37	19	3	827	18	974
9	2	7	9	11	29	11	5	40	34	861	1009
10	918	1057	1051	1013	920	931	1034	1007	1051	1018	10000

5.0.3 One Vs All: Sinusodial

[25]:
```
#Sinusodial OneVsAll
ROneVsAll3 = R_OVA(x_train, y_train, 1000, 3)
ROVAprediction3 = ROneVsAll3.predict(x_test)
error_rate(ROVAprediction3, y_test)
```

[25]: 0.8929

[26]:
```
confusion(ROVAprediction3, y_test)
```

[26]:

	0	1	2	3	4	5	6	7	8	9	10
0	72	281	83	92	60	57	65	117	69	84	980
1	79	385	104	95	83	45	65	132	69	78	1135
2	77	350	64	103	63	51	78	107	65	74	1032
3	77	280	77	106	75	52	89	91	75	88	1010
4	59	297	103	104	67	39	71	92	73	77	982
5	70	266	77	65	69	41	75	92	57	80	892
6	72	280	79	94	77	51	70	90	74	71	958
7	76	317	83	93	75	49	84	119	59	73	1028
8	82	305	67	81	80	53	73	102	73	58	974
9	78	319	72	113	62	44	62	109	76	74	1009
10	742	3080	809	946	711	482	732	1051	690	757	10000

5.0.4 One Vs All: ReLU

[27]:
```
#ReLU OneVsAll
ROneVsAll4 = R_OVA(x_train, y_train, 1000, 4)
ROVAprediction4 = ROneVsAll4.predict(x_test)
error_rate(ROVAprediction4, y_test)
```

[27]: 0.1106

[28]:
```
confusion(ROVAprediction4, y_test)
```

[28]:

	0	1	2	3	4	5	6	7	8	9	10
0	892	0	19	12	6	4	17	25	2	3	980
1	1	997	4	4	11	6	38	4	70	0	1135

2	3	2	910	38	13	1	22	18	24	1	1032
3	0	2	6	901	1	54	14	21	7	4	1010
4	1	23	1	1	889	1	14	12	13	27	982
5	10	1	3	26	7	761	21	37	20	6	892
6	3	0	5	0	6	8	927	5	3	1	958
7	0	29	10	5	5	0	0	966	7	6	1028
8	9	2	5	41	8	19	19	15	850	6	974
9	4	20	5	14	35	3	2	98	27	801	1009
10	923	1076	968	1042	981	857	1074	1201	1023	855	10000

5.0.5 One Vs One: Identity

[29]:
```
#Identity OneVsOne
ROneVsOne1 = R_OVO(x_train, y_train, 1000, 1)
ROVOprediction1 = ROneVsOne1.predict(x_test)
error_rate(ROVOprediction1, y_test)
```

[29]: 0.852

[30]:
```
confusion(ROVOprediction1, y_test)
```

[30]:

	0	1	2	3	4	5	6	7	8	9	10
0	934	0	0	0	0	1	45	0	0	0	980
1	408	0	0	0	0	0	727	0	0	0	1135
2	389	4	2	1	8	13	601	1	2	11	1032
3	660	1	11	1	28	13	268	15	8	5	1010
4	561	4	0	0	0	0	400	0	7	10	982
5	777	2	1	3	3	4	98	0	3	1	892
6	460	2	0	1	0	2	493	0	0	0	958
7	230	53	124	78	0	6	400	1	71	65	1028
8	679	0	0	0	0	0	289	0	6	0	974
9	290	12	39	19	0	3	586	0	21	39	1009
10	5388	78	177	103	39	42	3907	17	118	131	10000

5.0.6 One Vs One: Sigmoid

[31]:
```
#Sigmoid OneVsOne
ROneVsOne2 = R_OVO(x_train, y_train, 1000, 2)
ROVOprediction2 = ROneVsOne2.predict(x_test)
error_rate(ROVOprediction2, y_test)
```

[31]: 0.0863

[32]:
```
confusion(ROVOprediction2, y_test)
```

[32]:

	0	1	2	3	4	5	6	7	8	9	10
0	928	0	21	2	1	16	4	4	1	3	980

1	2	1053	5	1	0	32	11	0	31	0	1135
2	6	1	973	8	5	8	4	4	23	0	1032
3	1	2	28	870	2	87	0	9	11	0	1010
4	0	1	8	1	904	4	8	6	12	38	982
5	5	0	8	15	4	852	3	0	2	3	892
6	10	0	21	1	5	22	897	0	2	0	958
7	0	17	37	5	7	1	0	937	15	9	1028
8	8	0	9	41	4	42	6	6	858	0	974
9	2	5	6	14	41	8	1	37	30	865	1009
10	962	1079	1116	958	973	1072	934	1003	985	918	10000

5.0.7 One Vs One: Sinusodial

[33]:
```
#Sinusodial OneVsOne
ROneVsOne3 = R_OVO(x_train, y_train, 1000, 3)
ROVOprediction3 = ROneVsOne3.predict(x_test)
error_rate(ROVOprediction3, y_test)
```

[33]: 0.8941

[34]:
```
confusion(ROVOprediction3, y_test)
```

[34]:

	0	1	2	3	4	5	6	7	8	9	10
0	145	40	126	150	90	59	90	129	69	82	980
1	200	84	176	155	104	48	91	126	83	68	1135
2	168	45	157	174	95	58	76	111	75	73	1032
3	171	36	155	167	107	46	86	104	70	68	1010
4	147	48	139	160	113	38	85	124	62	66	982
5	157	38	142	124	93	51	71	98	56	62	892
6	142	47	157	141	93	47	100	113	52	66	958
7	170	44	154	167	106	40	91	106	77	73	1028
8	159	55	150	149	93	49	86	92	64	77	974
9	161	43	137	176	97	51	88	110	74	72	1009
10	1620	480	1493	1563	991	487	864	1113	682	707	10000

5.0.8 One Vs One: ReLU

[35]:
```
#ReLU OneVsOne
ROneVsOne4 = R_OVO(x_train, y_train, 1000, 4)
ROVOprediction4 = ROneVsOne4.predict(x_test)
error_rate(ROVOprediction4, y_test)
```

[35]: 0.0686

[36]:
```
confusion(ROVOprediction4, y_test)
```

[36]:

	0	1	2	3	4	5	6	7	8	9	10
0	949	0	24	1	0	2	1	2	1	0	980

1	1	1076	1	0	0	21	11	0	25	0	1135
2	4	4	990	4	2	0	7	6	15	0	1032
3	0	5	17	919	0	54	0	7	4	4	1010
4	1	3	7	0	932	1	3	1	5	29	982
5	5	1	4	30	7	826	4	2	7	6	892
6	11	0	11	0	5	10	919	0	2	0	958
7	0	41	31	2	9	1	0	897	8	39	1028
8	5	0	13	31	8	23	7	0	884	3	974
9	2	5	7	6	33	1	1	12	20	922	1009
10	978	1135	1105	993	996	939	953	927	971	1003	10000

5.1 Analysis of Classifiers

Out of all the choices of non-linear functions for the feature space classifier, both the Sigmoid and ReLU functions, performed significantly better than the Identity and Sinusodial. The Identity and Sinusodial functions both had relatively high error in comparision to the Linear Least Sqaures Classfier (LLS). The LLS had roughly 14% error for "One vs All" and 7% error for "One vs One". The Randomized Feature LLS with the Sigmoid Function had roughly 14% error for "One Vs All" and about 9% error for "One vs One. The Randomized Feature LLS with the ReLU Function had roughly 10% error for"One Vs All" and about 7% error for "One vs One. In general, both Sigmoid and ReLU generalized well. The Random Feature LLS with the ReLU function seemed to perform better on the training data than the Standard LLS, while the Sigmoid seemed to perform just as well as Standard LLS. But, since we only worked with one training set and one test set, one cannot say for sure given how close in error rate wthey were.

5.2 Effect of the Number of Features on Error Rate

[137]:
```
i_error = []

for i in range(0, 2200, 200):
    ROneVsOne1 = R_OVO(x_train, y_train, i, 1)
    ROVOprediction1 = ROneVsOne1.predict(x_test)
    e = error_rate(ROVOprediction1, y_test)
    i_error.append(e)

i_error = np.array(i_error)
```

[138]:
```
sig_error = []

for i in range(0, 2200, 200):
    ROneVsOne2 = R_OVO(x_train, y_train, i, 2)
    ROVOprediction2 = ROneVsOne2.predict(x_test)
    e = error_rate(ROVOprediction2, y_test)
    sig_error.append(e)

sig_error = np.array(sig_error)
```

[139]:
```
sin_error = []

for i in range(0, 2200, 200):
    ROneVsOne3 = R_OVO(x_train, y_train, i, 3)
    ROVOprediction3 = ROneVsOne3.predict(x_test)
    e = error_rate(ROVOprediction3, y_test)
    sin_error.append(e)

sin_error = np.array(sin_error)
```

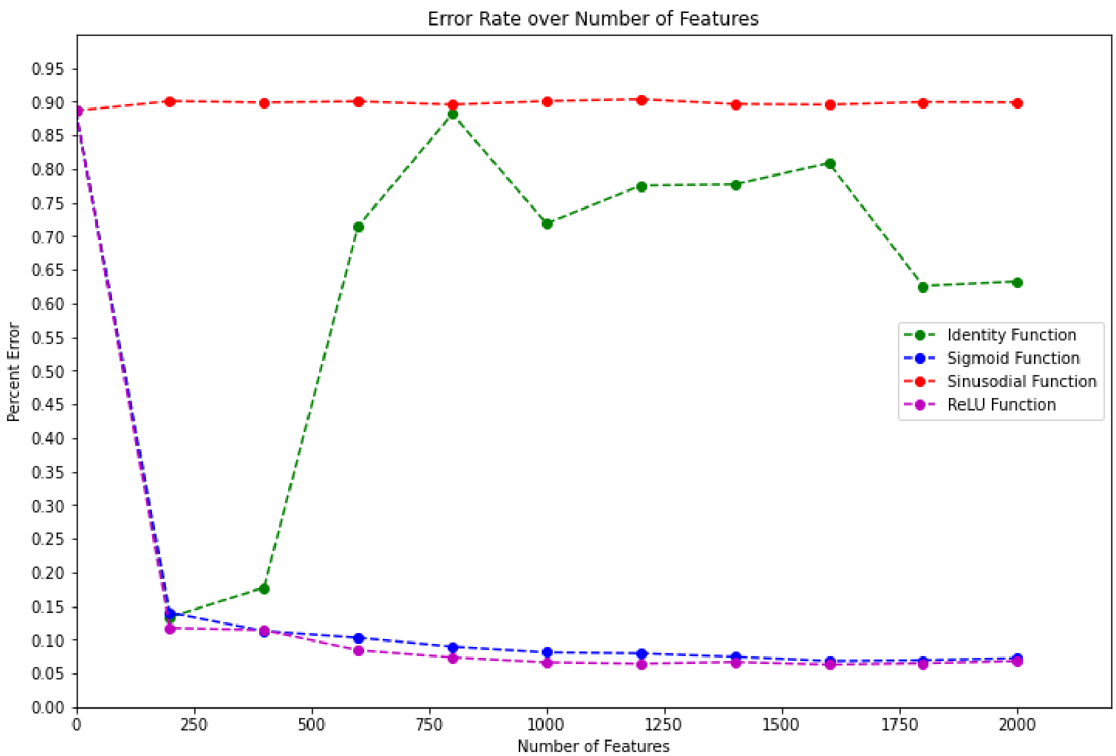
[140]:
```
r_error = []

for i in range(0, 2200, 200):
    ROneVsOne4 = R_OVO(x_train, y_train, i, 4)
    ROVOprediction4 = ROneVsOne4.predict(x_test)
    e = error_rate(ROVOprediction4, y_test)
    r_error.append(e)

r_error = np.array(r_error)
```

[141]:
```
i_error, sig_error, sin_error, r_error
```

[141]:
```
(array([0.8865, 0.1334, 0.1776, 0.7149, 0.8819, 0.7188, 0.7754, 0.7771,
        0.8088, 0.6261, 0.6327]),
 array([0.8865, 0.1404, 0.1127, 0.1029, 0.0896, 0.0812, 0.0797, 0.0746,
        0.0684, 0.0695, 0.0724]),
 array([0.8865, 0.901 , 0.899 , 0.9006, 0.8958, 0.901 , 0.9038, 0.8966,
        0.8957, 0.8997, 0.8991]),
 array([0.8865, 0.1173, 0.1142, 0.0844, 0.0736, 0.0663, 0.0644, 0.0668,
        0.063 , 0.065 , 0.0682]))
```

[146]:
```
feature_vals = np.arange(0,2200,200)

y = np.arange(0, 1, 0.05)

plt.figure(figsize=(12, 8))

plt.plot(feature_vals, i_error, '--go')
plt.plot(feature_vals, sig_error, '--bo')
plt.plot(feature_vals, sin_error, '--ro')
plt.plot(feature_vals, r_error, '--mo')

plt.xlabel("Number of Features")
plt.ylabel("Percent Error")
plt.title("Error Rate over Number of Features")

plt.yticks(y)
```

```
plt.axis([0, 2200, 0, 1])
plt.ylim(0,1)
plt.legend(["Identity Function", "Sigmoid Function", "Sinusodial Function",␣
↪"ReLU Function"])

plt.show()
```



The effect of adding more features to our classifier model will vary depending on the choice on non-linear function used in the computation of the feature mapping. Initially, with 0 features, the choice offunction does not matter. With zero features, the classifier has an error rate of about 89%. In the case of the Identity function, the error rate drops significantly with 200 features, with an error rate of about 13%, but the error quickly grows as L (the number of features) grows. It maxes out at 1000 features and begins to trend downard as L increases. The Sinusoidal function had a very consistent error rate of about 90% no matter the size of L. The best performing functions were the Sigmoid and ReLU fnctions. As L increases, their error tends to stay low and stagnate around 7%.