Characteristic Scales in Shock/Turbulence Interaction

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A semi-empirical estimate of the time-averaged thickness of a planar shock embedded in a turbulent mean flow is presented in an effort to quantify the characteristic time and length scales for turbulence modeling. Simplified Favre-averaged Navier-Stokes equations are reformulated and combined with the Rankine-Hugoniot relations to obtain an equation that relates the shock structure and its characteristic thickness to the upstream turbulent and mean flow quantities. The only mean flow and turbulence quantities needed to compute the shock thickness or velocity profile through the shock are: upstream Mach number (M_u), turbulent Mach number (M_t), and the Taylor-based Reynolds number (Re_λ). The accuracy of method for the shock structure is validated against direct numerical simulations of shock/turbulence interaction and shows a very good agreement over a wide range of Mach, Reynolds and turbulent Mach numbers. Finally, the obtained ratio of characteristic shock time scale to the upstream turbulence time scale is shown to be proportional to the turbulent kinetic energy amplification through the shock and supported through direct numerical simulations.

Nomenclature

M	=	Mach number (–)
M_t	=	turbulent Mach number (–)
$\bar{u}_i$, $\tilde{u}_i$	=	Reynolds and Favre-average velocity in the i^{th} direction (m/s)
ρ	=	local fluid density (kg/m^3)

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p	=	pressure (N/m^2)
R	=	universal gas constant ($J/kg \cdot K$)
γ	=	heat capacity ratio (–)
e, E	=	internal energy and total internal energy (m^2/s^2)
h, H	=	enthalpy and total enthalpy (m^2/s^2)
$\mathbf{m}$, $\mathbf{V}$, $\mathbf{I}$	=	Conserved quantities from integration
k	=	Turbulent kinetic energy (m^2/s^2)
R_{ij}	=	Reynolds stress of components i, j
$\overline{\rho u'' u''}$	=	Reynolds stress based on a Favre decomposition ($kg/m \cdot s^2$)
Re_λ	=	Reynolds number based on λ (–)
Subscripts		
u,d	=	upstream, downstream quantity
λ	=	Taylor length scale

I. Introduction

Shocks are present in a number of engineering applications (scramjet engines, rocket propulsion, inertial confinement fusion), astrophysical phenomenon (supernovae explosions, coronal mass ejection) and can even be used for medical purposes (lithotripsy). Generally, shocks are embedded in a turbulent mean flow resulting in a strong coupling between the turbulent fluctuations and the shock wave. Locally, the shock wave represents a near discontinuity in the velocity and thermodynamic properties as it occurs on the order of the mean free path of the molecules. The exact laminar structure of the viscous shock can be accurately described by the Boltzmann equations or, under a second-order Chapman-Enskog approximation, the Burnett equations [1]. The numerical complexity of these equations limit their usefulness for relevant engineering problems, although some efforts have been made to alleviate the numerical stability issues of these equations [2]. Despite the known limitations, the Navier-Stokes (NS) equations provide a surprisingly accurate description of the inner shock structure [3–5]. Resting on a linear Chapman-Enskog approximation, the NS equations have been the basis for a number of theoretical studies of the laminar shock structure [6–9]. Using order of magnitude arguments and some simplification of the governing equations, Howarth [9] derived a simple analytical relation to obtain the velocity profile within a shock and defined the characteristic laminar shock thickness, δ_l.

The analytical, laminar shock structure has provided a better understanding of the inner shock physics and has helped to validate compressible solvers [10, 11]. But, relative to the massive interest it sparked in the 1950 and 1960s research community, the insight gleaned from this finding has been of very little use for engineering relevant applications with shock waves. The main reason is that nearly all shocks are found within a turbulent flow. The local velocity and thermodynamic fluctuations cause the shock surface to oscillate, “wrinkle” and/or “break”. In a time-averaged sense, the turbulence causes the shock to have a much wider structure than the instantaneous laminar shock thickness would suggest—two, three, or even four orders of magnitude difference are to be expected. Therefore, we can think of the time-averaged shock thickness in a turbulent flow as a probability density function of the discrete shock location. Simple inferences would suggest that an increase of the turbulent Mach number should result in a wider shock structure through a more vigorous interaction between shock and turbulence; similarly a higher upstream Mach number has the opposite effect for the same level of upstream fluctuations. A few papers have proposed a quantitative model of the time-averaged shock thickness in a turbulent flow [12–15]. This is a particularly important quantity since the turbulent shock thickness, $\langle \delta_t \rangle$, provides a much needed length scale (and concomitantly a time scale, knowing the upstream velocity) for the modeling of turbulence passing through a shock [16]. Furthermore, grid independent solutions in Reynolds Averaged Navier-Stokes (RANS) simulations with shocks are not achievable for most common two-equation turbulence models [17–19] as the turbulent kinetic energy production term is dependent on the mean flow gradient which is not bound for the typical grid resolutions used. Therefore, an estimate of the time-averaged shock thickness in a turbulent flow can provide an upper bound on the turbulent production and thus allow for a physically-consistent grid independence to be achieved (here we leave out the discussion on the validity of the two-equations, equilibrium turbulence models within the shock). A number of scaling approaches have been suggested in the literature. Ribner [20], in the 1950s, first considered the interaction of an upstream perturbation with a shock wave; this analytical analysis was conducted under a thin shock assumption. More recently, Zank et al. [21] proposed that the shock thickness in a turbulent flow is proportional to the characteristic root-mean-square of the upstream fluctuating quantities. Donzis [22] used a similarity scaling to define a ratio of the laminar shock thickness, δ_l, and the Kolmogorov scale, η (which is found to be proportional to the upstream Mach number, M, turbulent Mach number, M_t and the speed of sound Reynolds number, Re_c) and related this parameter to the amplification factor through the shock. Donzis [13] further extended the analysis to show an incomplete similarity of the turbulent shock thickness. Despite these interesting findings, the characteristic spatial scale of the interaction is the shock thickness, $\langle \delta_t \rangle$; this quantity encapsulates all the important upstream quantities. Ryu and Livescu [12] showed, through a careful study of the parameter space, that amplification results converge to the linear interaction analysis (LIA) predictions when δ_l/η tends towards zero—a regime in which the linear interactions dominate, this work supported many of the claims by [13, 22]. The analysis was further extended by [14] who proposed a transition to linear interaction analysis regime and quantified the threshold of δ_l/η at which this occurs. Very recent work by [15], using the DNS data from [14], proposed a semi-empirical model for turbulent shock thickness that is based on a physical understanding of the dominant mechanisms in the shock-turbulence interaction. Other than this recent contribution and the work by [12], a quantitative estimate of the time-averaged shock thickness in a turbulent flow remains rather unexplored in the literature.

In light of the recent large-scale direct numerical simulations of the interaction between vortical isotropic turbulence and a planar shock wave [12, 14, 23–26], we derive a quantitatively accurate model to estimate of the structure and thickness of a shock embedded in a turbulent mean flow. This work is done with the objective of characterizing temporal and spatial scales of shock-turbulence interaction for an accurate modeling of the turbulence amplification for applied RANS simulations and, potentially, new insight into sub-grid scale shock-models for LES. In this work, we simplify the Favre-averaged Navier-Stokes equations to derive a semi-analytical expression of the turbulent shock structure which is used to estimate the characteristics thickness based uniquely on the upstream description of the flow. The objective is an accurate, simple and rapid estimate of the characteristic scales. Finally, we show that the ratio of the characteristic shock time scale to the eddy-turnover time of the upstream turbulence can be used to define the turbulent kinetic energy amplification through a normal shock. Thus, the computation of the characteristic scales in a turbulent shock can be used to characterize the turbulence amplification through a normal shock.

II. Model Development

A mathematical model is developed starting from the steady-state, Favre-averaged conservation equations for mass, momentum, and energy. The objective of the model is to relate the maximum streamwise velocity derivative in the shock to the upstream mean and turbulence quantities for a shock/turbulence interaction problem. From a mathematical perspective, the inviscid shock represents a discontinuity in the Euler equations; the width of a laminar viscous shock is determined by the molecular viscosity and heat conductivity of the fluid [27]. The time-averaged thickness of the shock interacting with turbulence is governed, primarily, by the wrinkling and unsteadiness of the shock. For typical engineering turbulence levels, the time-averaged shock thickness is orders of magnitude larger than the laminar shock thickness (quantified further in the text), therefore we neglect the laminar viscosity from the governing equations from this point onward in an effort to simplify the mathematical treatment.

The interaction of vortical, isotropic turbulence with a planar shock can be abstracted to a one-dimensional problem in the shock normal direction. Therefore we assume $\tilde{u}_2 = \tilde{u}_3 = 0$ and $\partial/\partial x_2 = \partial/\partial x_3 = 0$). The one-dimensional Favre-averaged equations read:

$$\frac{\partial \bar{\rho}\tilde{u}}{\partial x} = 0 \tag{1}$$

$$\frac{\partial \bar{\rho}\tilde{u}\tilde{u}}{\partial x} = -\frac{\partial}{\partial x}\left[\bar{p} + \overline{\rho u'' u''}\right] \tag{2}$$

$$\frac{\partial \bar{\rho}\tilde{u}H}{\partial x} = -\frac{\partial}{\partial x}\left[\overline{q} + \tilde{u}\overline{\rho u'' u''} + \overline{c_p \rho u'' T''} + \overline{\rho u'' k}\right] \tag{3}$$

The first approximation is to assume that the triple correlation term, for example $\overline{\rho u'' k}$, is small and can be neglected. This approximation allows us to simplify $\overline{\rho u'' u''} \approx \bar{\rho}\overline{u'' u''}$. This is justified on the observation that $(M_t/M)^2 << 1$; the turbulent kinetic energy is small compared to the mean flow kinetic energy. Consequently, a higher order correlation of the turbulent kinetic energy is expected to be even smaller. By integrating the conservative form of the governing equations across the shock, we obtain:

$$\bar{\rho}\tilde{u} = \bar{\rho}_u\tilde{u}_u = \bar{\rho}_d\tilde{u}_d = \mathbf{m} \tag{4a}$$

$$\mathbf{m}\tilde{u} + p + \overline{\rho}R_{11} = \mathbf{m}\mathbf{V} \tag{4b}$$

$$\mathbf{m}H + \mathbf{m}R_{11} + \overline{c_p\rho u''T''} - \kappa\frac{\partial\overline{T}}{\partial x} = \mathbf{m}\mathbf{I} \tag{4c}$$

where $\mathbf{m}$, $\mathbf{I}$, $\mathbf{V}$ are conserved quantities arising from the integration constants and indices u and d denote the upstream and downstream quantities with respect to the shock. The total enthalpy is written in the form $H = E + \frac{p}{\rho} = \tilde{e} + \frac{\overline{p}}{\overline{\rho}} + \frac{\tilde{u}\tilde{u}}{2} + k$. Note, the triple correlation term, $\overline{\rho u''k}$, represents a third-order fluctuating term, which can be neglected based on the previous reasoning. The governing equations are supplemented with a perfect gas equation of state which can be related to the conservative quantities in the shock as:

$$p = \rho\left(c_p - c_v\right)T = \rho\left(\tilde{h} - \tilde{e}\right) = \frac{\mathbf{m}}{\tilde{u}}\left(\tilde{h} - \tilde{e}\right) = \frac{\mathbf{m}}{\tilde{u}}\tilde{e}\left(\gamma - 1\right) \tag{5}$$

Inspired by the approach used in a laminar shock by Becker [6] and Howarth [9], the momentum (4b) and energy (4c) equations, in integral form, can be combined and the pressure dependence removed with the modified state equation (5) to obtain a *modified energy equation*:

$$\mathbf{m}\tilde{e} - \mathbf{m}\left(\frac{\tilde{u}\tilde{u}}{2} - k\right) + \tilde{u}\mathbf{m}\mathbf{V} + c_p\overline{\rho}\overline{u''T''} - \kappa\frac{\partial\overline{T}}{\partial x} = \mathbf{m}\mathbf{I} \tag{6}$$

Inspired by turbulence modeling, we express $\overline{u''T''}$ proportional to the gradient of temperature as:

$$\overline{u''T''} = -\epsilon_H\frac{\partial\widetilde{T}}{\partial x} \tag{7}$$

where $\epsilon_H = \nu_t/Pr_t$ which imposes a modeling of the eddy viscosity (discussed later). We acknowledge that the above relation will not hold across nor in the immediate downstream vicinity of the shock wave. But since we are interested in characterizing the shock thickness, we take this crude approximation–which is consistent with standard RANS modeling framework– in order to estimate the turbulent heat flux at the shock. More importantly, this modeling assumption allows us to relate the thermodynamic fluctuations to the mean and turbulence quantities in the flow. By replacing the above term in Equation (6) and re-writing the temperature in terms of internal energy, we obtain:

$$\frac{\partial\tilde{e}}{\partial x} = \frac{1}{\mathcal{A}}\left[\tilde{e} - \mathbf{I} - \left(\frac{\tilde{u}\tilde{u}}{2} - k\right) + \tilde{u}\mathbf{V}\right] \tag{8}$$

where $\mathcal{A} = \left(\frac{\gamma}{\mathbf{m}}\overline{\rho}\epsilon_H + \frac{\kappa}{c_v \mathbf{m}}\right)$.

By multiplying $\widetilde{u}$ to the momentum equation (4b), replacing the pressure term, and differentiating both sides of the equation, the *modified momentum equation* takes the following form:

$$2\widetilde{u}\frac{\partial \widetilde{u}}{\partial x} + \underbrace{(\gamma - 1)}_{\mathcal{B}}\frac{\partial \widetilde{e}}{\partial x} + \frac{\partial R_{11}}{\partial x} = \mathbf{V}\frac{\partial \widetilde{u}}{\partial x} \tag{9}$$

By isolating the energy derivative in Eqs. (8) and (9), we obtain:

$$\frac{1}{\mathcal{B}}\left[\mathbf{V}\frac{\partial \widetilde{u}}{\partial x} - 2\widetilde{u}\frac{\partial \widetilde{u}}{\partial x} - \frac{\partial R_{11}}{\partial x}\right] = \frac{1}{\mathcal{A}}\left[\tilde{e} - \mathbf{I} - \left(\frac{\tilde{u}\tilde{u}}{2} - k\right) + \tilde{u}\mathbf{V}\right] \tag{10}$$

Noting that $(\gamma - 1)\widetilde{e} = \widetilde{u}\mathbf{V} - \widetilde{u}^2 - R_{11}$, we can remove all thermodynamic terms from the above equation which yields:

$$\mathcal{A}\left[\mathbf{V}\frac{\partial \widetilde{u}}{\partial x} - 2\widetilde{u}\frac{\partial \widetilde{u}}{\partial x} - \frac{\partial R_{11}}{\partial x}\right] = \left(\widetilde{u}\mathbf{V} - \widetilde{u}^2 - R_{11}\right) - (\gamma - 1)\left[\frac{\widetilde{u}^2}{2} - k - \widetilde{u}\mathbf{V} + \mathbf{I}\right] \tag{11}$$

The resulting equation is free of any primitive thermodynamic quantities.

The modeling of R_{11} in shock-turbulence interaction rests on the Reynolds stress model proposed by Vemula and Sinha [28]. The physical mechanism contributing to the amplification of R_{11} is modeled as $\partial R_{11}/\partial x = \Gamma R_{11} \partial \ln \widetilde{u}/\partial x$, where $\Gamma = -2 + 2b_1' + 4/3c_2$ in terms of the shock-unsteadiness parameter b_1' and the coefficient of rapid pressure strain correlation $C_2 = 0.55$. These models are described in Vemula and Sinha [28] and rest on empirically determined constant but the equations are derived from the consideration of shock unsteadiness. By isolating $\mathbf{V}$ from the momentum equation and recasting pressure with the Mach number, yields:$\mathbf{V} = \widetilde{u} + \dfrac{\widetilde{u}}{\gamma M^2} + \dfrac{R_{11}}{\widetilde{u}}$ which allows us to re-write the above equation as:

$$\mathcal{A}\left[\frac{\widetilde{u}}{\gamma M^2} - \widetilde{u} + \frac{(1-\Gamma)R_{11}}{\tilde{u}}\right]\frac{\partial \widetilde{u}}{\partial x} = -\left[(\gamma + 1)\frac{\widetilde{u}^2}{2} - \gamma\widetilde{u}\mathbf{V} - (\gamma - 1)\mathbf{I}\right] + (\gamma - 1)k - R_{11} \tag{12}$$

Next we replace the terms in the second brackets following the simplification proposed Howarth [9], which recognizes the cubic form of the velocity with constants. Assuming isotropic flow, the resulting term in the numerator $(\gamma - 1)k - R_{11}$ is perfectly null for a monoatomic gases and very small for more general cases (about $2/5R_{11}$). We neglect this term as it remains small. This leads to:

$$\frac{\partial \widetilde{u}}{\partial x} = \frac{-\frac{\gamma+1}{2}(\widetilde{u} - \widetilde{u_u})(\widetilde{u} - \widetilde{u_d})}{\mathcal{A}\left[\dfrac{\widetilde{u}}{\gamma M^2} - \widetilde{u} + \dfrac{(1-\Gamma)R_{11}}{\tilde{u}}\right]} \tag{13}$$

Furthermore, by noting that:

$$\mathcal{A} = \frac{\gamma}{\tilde{u}} \left(\epsilon_H + \frac{\kappa}{\rho c_p} \right) = \frac{\gamma}{\tilde{u}} \underbrace{\left(\frac{\nu_t}{Pr_t} + \frac{\nu}{Pr} \right)}_{C} \tag{14}$$

The modeling of ν_t in the above equation is discussed below. The final form of the velocity gradient is:

$$\frac{\partial \widetilde{u}}{\partial x} = \frac{-\frac{\gamma+1}{2}(\widetilde{u} - \widetilde{u_u})(\widetilde{u} - \widetilde{u_d})}{C \left[\frac{1}{M^2} - \gamma + \frac{\gamma(1-\Gamma)R_{11}}{\tilde{u}^2} \right]} \tag{15}$$

By numerically integrating $\tilde{u}$ in the above equation, we find the mean velocity profile of a shock-wave interacting with vortical turbulence. Other than the velocity variable in the numerator, the other terms (e.g. M, R_{11}, including the velocity in the denominator) are assumed constant and selected as their upstream value. The detailed derivation of the model is presented in the Supplementary Material along with a validation of the important modeling assumptions.

A. Shock thickness estimate

Although equation (15) is relevant to compute the velocity profile, we seek to define a simple model to estimate the shock thickness without the need to numerically integrate the above equation. We define the shock thickness, similar to [15], as:

$$\langle \delta_t \rangle = \frac{(\widetilde{u_d} - \widetilde{u_u})}{\partial \widetilde{u}/\partial x|_{\max}} \tag{16}$$

where the maximum velocity gradient is assumed to be located at the center of the shock. To compute the maximum velocity gradient from equation (15), we take the mean velocity and density quantities at the mid-shock location ($\tilde{u} = 0.5(\tilde{u}_u + \tilde{u}_d)$); as previously stated, the other quantities are taken as the upstream, constant values. Given the known upstream values, the downstream mean flow quantities can be found either by using the Rankine-Hugoniot (RH) relations or by relying on previously developed semi-empirical models.

The modeling of the terms in C in Eq.(14) is now considered. More specifically, one term need to be approximated: ν_t. The eddy viscosity is modeled following the classical RANS approach where $\rho\nu_t = c_\theta k^2/\epsilon$, where $c_\theta = 0.15$ and k/ϵ are calculated from the upstream conditions. A slightly higher value of c_θ, compared to the classical value of 0.09, is selected to account for the stronger turbulence effects in the shock. This corresponds to a reasonable coefficient based on observation, but has not been rigorously derived. Although we admit that the formal validity of this RANS equation is questionable within the shock, it can nonetheless provides a useful characteristic scale for the shock-turbulence interaction. The remaining terms in C, namely: mu, Pr and Pr_t are considered as constants based on their upstream quantities. In the present work, we set standard values of $Pr = 0.72$ and $Pr_t = 0.89$. By replacing velocity gradient through the shock and re-organizing the equations, we obtain:

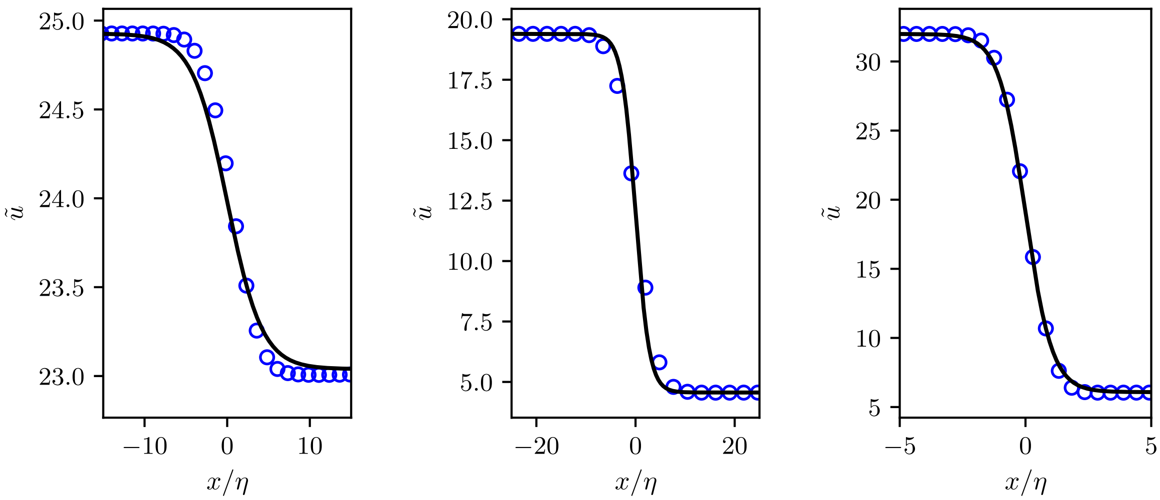

Fig. 1 Comparison of velocity profile obtained from the simplified equation (15) **and time-averaged DNS [24] at various conditions: (left)** $M = 1.05$**,** $M_t = 0.05$**,** $Re_\lambda = 39$**; (middle)**$M = 3.5$**,** $M_t = 0.16$**,** $Re_\lambda = 74$**; (right)** $M = 6.0$**,** $M_t = 0.23$**,** $Re_\lambda = 42$**.**

$$\langle \delta_t \rangle = \frac{\widetilde{u_d} - \widetilde{u_u}}{\partial \widetilde{u}/\partial x|_{max}} = \frac{8C\left[\dfrac{1}{M_u^2} - \gamma + \dfrac{\gamma(1-\Gamma)R_{11,u}}{\tilde{u}_u^2}\right]}{(\gamma+1)(\widetilde{u}_d - \widetilde{u_u})} \tag{17}$$

III. Results and Analysis

The accuracy of the time-averaged shock thickness is compared against the direct numerical simulation (DNS) of shock-turbulence interaction from Larsson et al. [24]. Despite highly-resolved nature of the DNS, the local, instantaneous shock thickness remains influenced by the shock capturing numerical method used in the simulation (high-order WENO scheme). We recall that the laminar shock thickness is $\delta_l/\eta = O(0.01 - 0.1)$ for the cases under consideration. On the other hand, the time-averaged shock thickness resulting from the turbulence interaction is well captured (at least 25 points in the turbulent shock structure). For comparative purposes, the shock thickness is normalized by the Kolmogorov length scale, which is defined immediately upstream of the shock, and is related to the kinetic viscosity and dissipation through: $\eta = \left(\nu^3/\epsilon\right)^{1/4}$ [29]. The velocity profiles obtained from the integration of equation (15) show very good agreement with the DNS over a wide range of upstream M, M_t and Re_λ numbers, see figure 1. For full transparency, a Jupyter notebook solving these equations and generating all figures in this paper is available at: `https://git.uwaterloo.ca/jphickey/shockthickness_aiaa.git`.

Now, we assess some of the modeling assumptions made in the previous subsection. It should be noted that the ratio of turbulent to laminar viscosity is about one order of magnitude larger for very low-Reynolds number flows; for engineering relevant applications, there are usually many orders of magnitude separation between both terms. For the

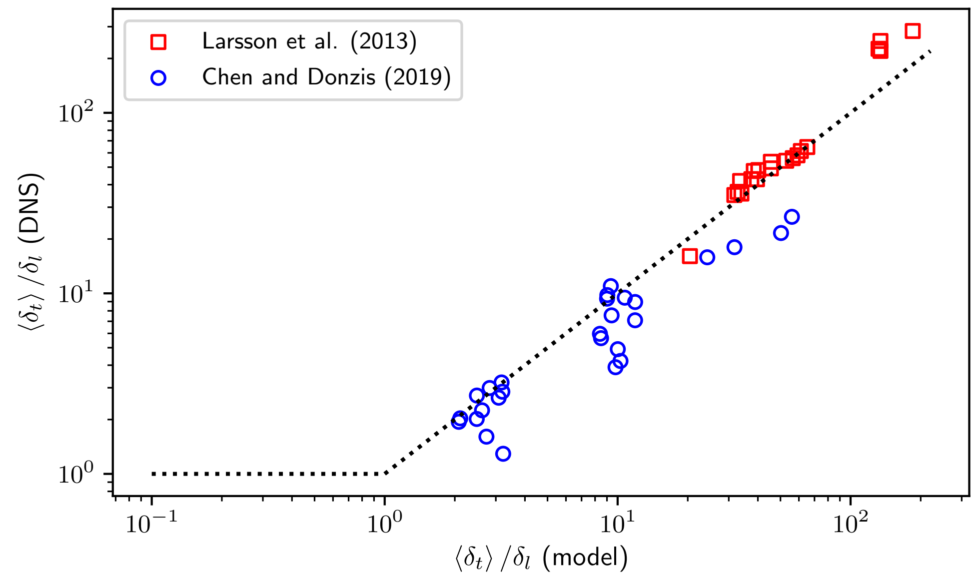

Fig. 2 Comparison of modeled shock thickness, computed from Eq. (17)**, and the exact direct numerical simulations data from [24] and [14] (isotropic cases only).**

shock-turbulence simulation data used herein, the Re_L defined in [24] corresponds directly to the ratio μ_T/μ and, for all cases presented, varies from 180 to 670. Therefore neglecting the laminar viscosity is fully justified to a first-order approximation (although we note that this assumption is not as well satisfied with the data from [14] which is at a lower Reynolds number). Second, the ratio of the turbulent to the mean flow kinetic energy is defined by the ratio $(M_t/M)^2$. If the ratio tends towards zero, the contribution of the turbulent kinetic energy to the total kinetic energy is small. For most engineering relevant applications, the ratio will remain far below unity; for all cases investigated, the ratio remains below 0.0625 (which occurs for the case at $M_t = 0.38$ at $M = 1.52$ in [24]). As a result, we can easily justify neglecting the higher-order terms.

A further assessment of the model is done by comparing the model shock thickness estimate with the numerically computed DNS results from both [24] and [14] (isotropic datasets), as shown in Figure 3. These simulation results a very large range of Taylor-based Reynolds numbers ($Re_\lambda = [10 - 75]$) and turbulent Mach numbers ($M_t = [0.05 - 0.54]$). For both datasets, the only values used are: M, M_t, and Re_λ. Similar to the approach by Chen and Donzis. [14], we normalized the shock thickness by the laminar shock thickness defined as: $\delta_l = 2\langle k_u\rangle\langle\mu_u\rangle/(\langle\rho_u\rangle\langle c_u\rangle\Delta M)$. The model and DNS results for all simulations in [24] are presented in Table 1. Although we note a good trend, some thickness estimates are underpredicted, especially at higher Reynolds numbers.

The simplicity of the Equation (17) allows for a parametric study of the quantities influencing the shock thickness. Figure 3 shows the Mach number effects at three different turbulent Mach numbers of: 0.16, 0.22, and 0.31. For the parametric study, we set the upstream Reynolds number to $Re_L = 200$ (which corresponds to the average Re_L of the

Table 1 Properties of the DNS cases from Larsson et al. [24]. The computed shock thickness is compared with the measured DNS result.

Re_λ	M	M_t	$<\delta_t>/\delta_l$ **(DNS)**	$<\delta_t>/\delta_l$ **(model)**	$\tau_{\text{shock}}/\tau_{\text{turbulence}}$	Amplification
39	1.05	0.05	16.074	20.431	3.85	1.03
38	1.28	0.15	35.043	31.695	1.31	1.17
39	1.28	0.22	35.686	33.998	1.86	1.14
38	1.28	0.26	36.497	32.747	2.34	1.12
38	1.28	0.31	41.919	33.550	3.09	1.10
38	1.5	0.15	42.918	37.498	0.65	1.30
39	1.5	0.22	42.850	39.655	0.90	1.26
39	1.51	0.31	48.187	40.163	1.43	1.22
39	1.51	0.37	47.619	38.408	1.83	1.20
39	1.87	0.22	49.229	45.484	0.48	1.40
39	1.87	0.31	53.559	45.476	0.78	1.35
40	2.5	0.22	54.251	52.936	0.27	1.50
40	3.5	0.16	56.089	56.472	0.12	1.58
41	3.5	0.23	58.145	59.171	0.16	1.55
42	4.7	0.23	61.430	61.169	0.12	1.59
42	6	0.23	64.621	65.2037	0.10	1.62
73	1.5	0.14	226.216	131.89	0.92	1.28
73	1.5	0.22	220.234	134.393	1.28	1.26
72	1.52	0.28	250.078	134.335	2.47	1.16
74	3.5	0.15	283.948	184.874	0.18	1.57

cases under consideration) and turbulent Mach number. We then vary the upstream Mach number M. By doing so, the relative velocity fluctuations are decreasing with Mach number given that M_t/M. The modeled shock thickness is compared against the DNS data for the cases at at M_t of about 0.16, 0.22, and 0.31 (as shown in Table 1). The developed model is compared to the functional form of the models proposed by [12] and [15]. The parametric study reveals that, as expected, the shock thickness becomes very large (relative to the η) as M tends toward unity. In the limit case, as the upstream Mach number approaches unity, the Rankine-Hugoniot relations suggest that the velocity difference upstream and downstream of a normal shock should also tend towards zero. Additionally, we note that the model proposed by [15] reverts back to the laminar shock thickness as M_t tends toward zero. Our proposed model does not have this feature as we neglected the viscous effects in the derivation of the model (the viscous term only arises through the heat conductivity term). Yet, the developed model does show a consistent relative thickness of the turbulent shock as the Mach tends to large numbers.

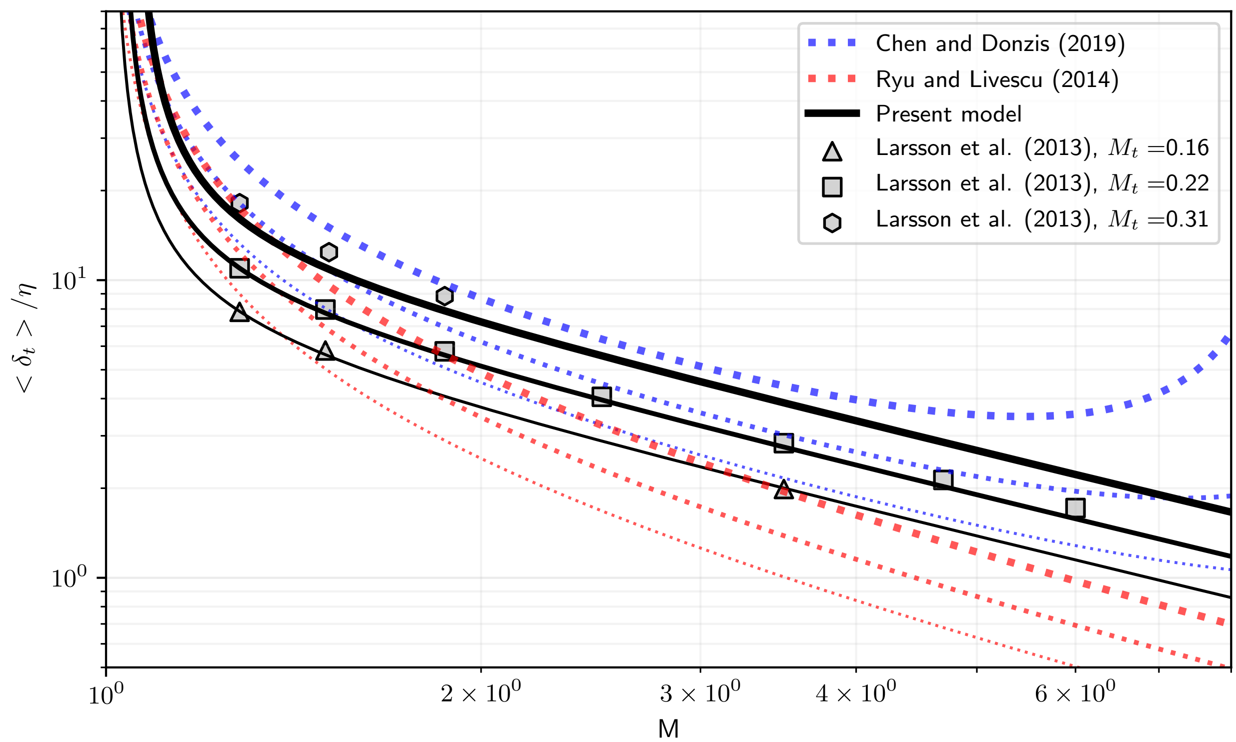

Fig. 3 Functional comparison of current model Eq. (17) **with existing shock thickness models by [12] and [15]. In the [15] model, we replaced the viscosity/velocity correlation term with M_t/M. To compare the functional forms, we assumed isotropic turbulence upstream of the shock and set $Re_L = 200$ and span the Mach number at three different turbulent Mach numbers: 0.16, 0.22, 0.31 (shown with increasing line thickness). Note that we multiplied the Ryu [12] and Chen [15] by empirically determined constant to better fit the data, respectively of 4 and 100.**

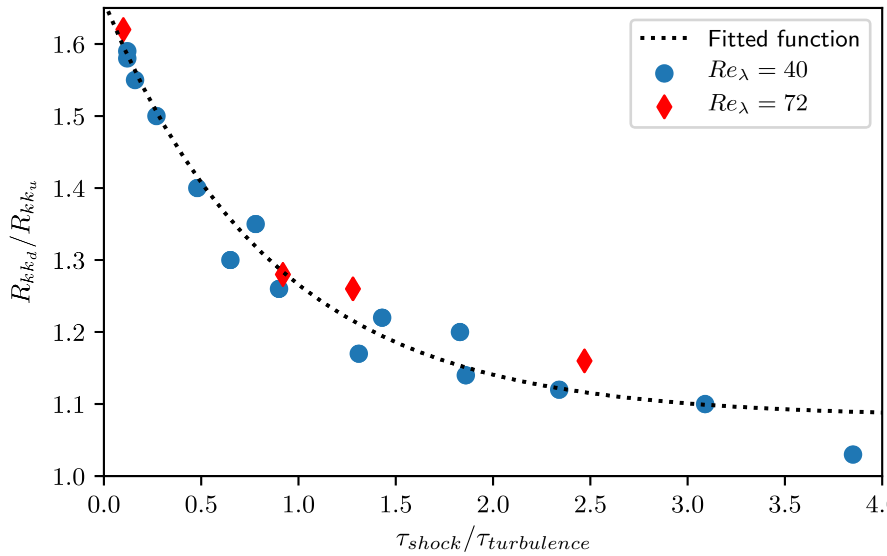

Fig. 4 Turbulence amplification through the shock as a function of the ratio of the characteristic time scale of the shock and turbulence. Only the DNS data is used covering $M = [1.05, 6.0]$, $M_t = [0.05, 0.4]$ and $Re_\lambda = [40, 72]$. See [24] for the exact description of the individual cases. Here the line represents a curve fit of the available DNS data.

A. Turbulence amplification and characteristic scales

From a RANS or LES perspective, the knowledge of a turbulent shock thickness provides a new characteristic time and length scale that can be used for modeling the shock/turbulence interaction. Previous works have proposed corrections to the turbulent kinetic energy production terms in classical two-equation models (see *e.g.* Sinha et al. [17]). These models must resort to empirical fits to accurately account for the turbulence amplification through the shock. Other works have estimated the turbulence amplification on the basis of rapid distortion theory (see e.g. Jacquin et al. [30], Mahesh et al. [31]). Donzis [22], suggests that the shock normal amplification factor is directly related to the laminar shock thickness over the Kolmogorov scale.

Here we propose that the amplification of the turbulent kinetic energy through a shock oscillating from the interaction of vortical turbulence is related to the ratio of the characteristic time scale of the shock ($\tau_{shock} = 1/\,\partial\tilde{u}/\partial x|_{max}$) to the integral time scale of the incoming turbulence ($\tau_{turb} = L_{turb}/\tilde{u}_u$). This concept is analogous to the "rapidity" of the distortion used to characterize the turbulence amplification with rapid distortion theory [30]. In simple terms, if the time scale of the shock is small with respect to the characteristic eddy turnover time, the turbulence amplification will be large. The time scale ratio and amplification factors of all direct numerical simulations [24] are presented in figure 4. The amplification factor is defined herein as k_d/k_u where the upstream value is taken immediately before the shock and the downstream quantity corresponds to the peak post-shock turbulent kinetic energy. All the results in figure 4 are directly post-processed from the temporal and spatial averaged three-dimensional DNS simulations, the model is not used herein. As a result, the accurate and simple estimation of the turbulent shock thickness (using equation (17)) can be used as a direct method to estimate the amplification factor in a shock. This approach extends from the LIA framework used by Sinha et al. [17] as it directly accounts for influence of turbulence on the shock and provides a robust estimate over a large range of Re, M and M_t.

The physical time-scale ratio argument made in the previous paragraph can be shown to be directly related to the amplification factor in the shock. We recall that the shock thickness is defined as: $\delta_t = (u_d - u_u)/\,\partial\widetilde{u}/\partial x|_{max}$. The mean density ratio across a shock wave can be re-written, by recalling the shock and turbulent time scales defined in the previous paragraph, as

$$r = \frac{\rho_d}{\rho_u} = \frac{u_u}{u_d} = \frac{u_u}{u_u + \delta_t\ \partial\widetilde{u}/\partial x|_{max}} = \frac{1}{1 + \frac{\tau_{turb}}{\tau_{shock}}\frac{\delta_t}{L_{turb}}} \tag{18}$$

From previous works, it can be shown that amplification is directly tied to r through: $k_d/k_u = r^{2/3(1-b_1')}$ [17]. Hence the amplification across shock waves can be shown to be directly related to a ratio of characteristic time scales of shock and incoming turbulence.

IV. Conclusion

Using a limited number of simplifying assumptions of the one-dimensional, Favre-averaged Navier-Stokes equations, a relation is obtained which describes the time-averaged shock structure in a turbulent flow. The model allows the estimation of the time-averaged shock profile, within a turbulent flow, based uniquely on the knowledge of three upstream mean and turbulence quantities, namely: Mach number, turbulent Mach number and the Reynolds number. This modeled equation can be used to either obtain a velocity profile inside the shock or to compute the time-averaged turbulent shock thickness. The estimated mean velocity profile and characteristic thickness are compared against the three-dimensional direct numerical simulations and shows very good agreement with all the cases investigated (up to a Mach number of 6). Furthermore, the average turbulent thickness provides a characteristic spatial, and concomitantly a characteristic time scale, for shock-turbulence interaction problems. This is particularly important in the RANS or LES context. This key information can play an important role in the understanding the intrinsic physics of the interaction and can potentially provide insight into the elaboration of a shock-turbulence model. We show that the turbulence amplification rate through the shock is a direct function of the time scale ratio τ_{shock}/τ_{turb}. This ratio can be easily approximated and is valid over a wide range of Mach and turbulent Mach numbers, and can provide an effective method to model the post-shock turbulent kinetic energy jump. Future extensions to this work will consider flows in which the shocks have an even stronger thermodynamic coupling [32], for example using non-ideal state equations.

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