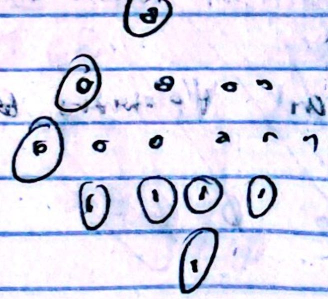
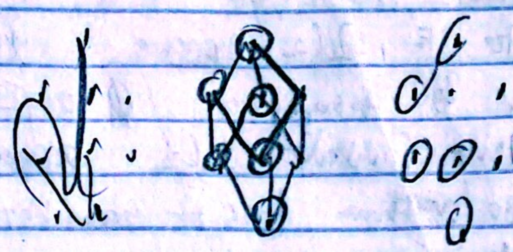
~~X~~ idea

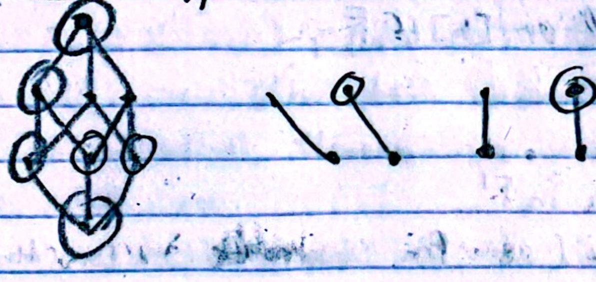
1. need to take $\emptyset, [n]$, as will immediately be non disjoint, 2
2. then take $\binom{n}{\lfloor \frac{n}{2} \rfloor}$
3. choose specific element in middle layer, considering picking specific element in different layers but middle layer will have least number of connections, then to guarantee disjointness, not sure #
4. then what ever is added will not have connection to middle layer element so $\exists$ $2C_2$

~~5.~~ does not have $2C_2$ as





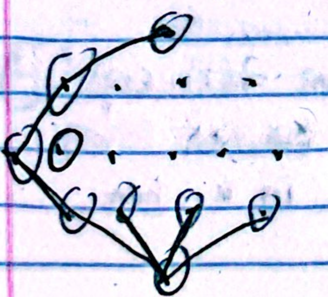
$2C_2$ strategy



either way $2C_2$

top

$\{1,2,3\}$



bot

any of these $\exists$ some singleton $\notin$ of new element $\{a,b\}$

Cases

1. $1 \in, 2 \notin \{a,b\}$, $(\{2\},\{1,2,3\}), (\{b\},\{a,b\})$
   $a=1,\ b \neq 1,2$
2. by symmetry to 1.
3. $1,2 \notin \{a,b\}$, $(\{1\},\{1,2,3\}), (\{a\},\{a,b\})$
   as $\{a\} \notin \{1,2,3\}$, $\{1\} \notin \{a,b\}$

top $\exists \{i\} \notin$ new element, but must be in already chosen top, so $\{i\}$, (chosen top), $\{j\} \notin$ (chosen top), (new element)

loop applies for each layer.

F induced $2C_2$ saturated $\Rightarrow \{\emptyset, [n]\} \subseteq F$

assume F induced $2C_2$ saturated F is not induced $2C_2$ saturated

by $\Rightarrow \not\in$ $\{\emptyset, [n]\} \notin F$

so it's given that F induced $2C_2$ saturated $\Rightarrow$

1. $2C_2$ not present in F 2. any additional element added to F, $2C_2$ appears

$\Rightarrow$ by assumption $\emptyset$ or $[n] \notin F$ and $\forall$ elements $\notin F$ when added $2C_2$ appears

so without loss of generality consider $\emptyset \notin F$,

by assumption

Proof of F induced $2C_2$ saturated $\Rightarrow \{\emptyset, [n]\} \subseteq F$

by contradiction, $\Rightarrow \not\in$, $\emptyset$ or $[n] \notin F$

by assumption

1. $2C_2$ not in F

2. $\forall F'$, $F \subset F'$, $2C_2$ in F'

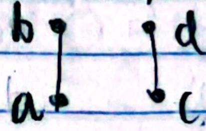
without loss of generality (as B_n symmetric), consider case $\emptyset \notin F$

by 2 the Family $F \cup \{\emptyset\}$ must contain $2C_2$, and by 1 $2C_2$ did not exist in F before, so F' to contain $2C_2$ the $\emptyset$ must exist in a new disjoint chain

2 cases

1. there does not exist another C_2 in F' other than the new chain created by adding $\emptyset$

2. $\exists$ some other C_2, so let two arbitrary chain denoted (a, b) with $a \preceq b$

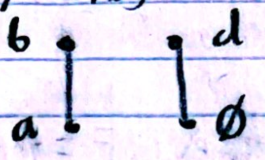


$b \neq \emptyset$ as the only set S, such that, $S \subseteq \emptyset$ is the $\emptyset$, which create $(\emptyset, \emptyset)$ which is not a C_2

$a \neq \emptyset$ as the 2 chains must be disjoint by def of $2C_2$,

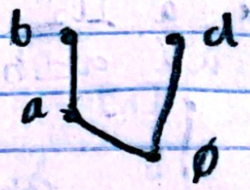
by the same logic as $b \neq \emptyset \Rightarrow d \neq \emptyset \Rightarrow c = \emptyset$

so $\emptyset \notin (a, b)$



$b \neq d$ as disjoint

but since $a \neq \emptyset$ and $b \neq \emptyset$ and $\forall S,\ S \in B_n / \{\emptyset\}$ $\emptyset \subseteq S$, so $\emptyset \subseteq a, b$ which definition of $2C_2 \Rightarrow\!\Leftarrow$

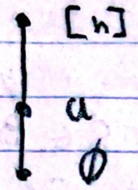


1. want to prove that F must contain chain (a, b)
   $\emptyset, [n] \notin (a, b)$, for $n = 3$ maximal
2. for $n=3$, once established maximal chain, prove F needs additional element as any F' would create comparable chains
3. then counter examples to show that need additional element to get new chain $\forall$ F's
4. prove in same row? not necessarily, not necessary proof, already 6

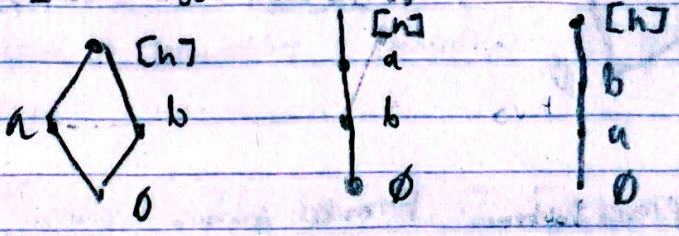
So proven so far $\{\emptyset, [n]\} \subseteq F$. so $|F| \geq 2$

Since $2C_2$ contains 4 distinct vertices, by definition,
if $|F'| = 3$, then F' is not induced $2C_2$ saturated,
so $|F| \geq 3$.

So for $n \geq 3$, F must contain $\emptyset$, $[n]$, and at least 1 other element
let this new element be denoted a, since $\forall S \in B_n$
$\emptyset \subseteq S \wedge S \subseteq [n]$, $\emptyset \subseteq a \subseteq [n]$ or



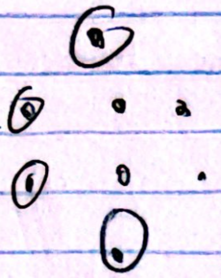
let this family be denoted $F = \{\emptyset, [n], a\}$
this structure does not contain a $2C_2$ as
all elements are comparable, but adding
in a new element b will not create a $2C_2$
as $\emptyset \subseteq b \subseteq [n]$ so whether or not $a \preceq b$ or $b \preceq a$
or neither



either way no $2C_2$

So F must contain at least 1 more element. b
$F = \{\emptyset, [n], a, b\}$

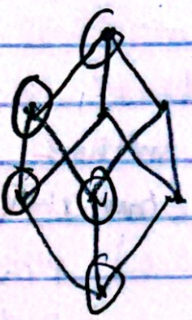
F induced $2C_2$ saturated $\Rightarrow$ F contains maximal chain?
F induced $2C_2$ saturated $\Rightarrow$ $\exists C_2$ $\emptyset, [n] \notin C_2$
$\forall C_2 \in F$, $\emptyset \in C_2 \vee [n] \in C_2 \Rightarrow F$ not induced $2C_2$ saturated
for a $2C_2$ structure by def need 2 disjoint C_2's,
consider that for each F' must create an additional C_2
to form a $2C_2$, this new element at most would
create 1 C_2 to be used to construct a $2C_2$, any
additional chains created by the new element would
be by definition not disjoint, so the other disjoint
chain must exist in the original F family, let this
chain be denoted (a, b), by assumption $\emptyset$ or $[n] \in (a,b)$
so for any chain in F it would not be disjoint to
the new chain created by F'



So $n=3$ must contain maximal chain

$sat^*(3, 2C_2) \ge 4$

Consider that for every F', such that $|F'| = |F| + 1$. So a single new element is added, for a $2C_2$ to be created, an addition C_2 must be created, however, this chain will contain an original element in F, since $\forall S_1, S_2 \in F$, $S_1 \| S_2$ by definition of maximal chain (or the generalized family defined so that $F = \{\emptyset, [n], a, b\}$ where $a \| b$), thus new chain will not be disjoint, so $sat^*(3, 2C_2) \ge 5$



So we have a maximal chain with one additional element

This new element will fall into 2 cases

1. it is comparable the element in the other layer
2. or not

1. Consider the case in which the 2 are $\|$. for any F', if the new element is chosen to be on the same layer as 2, this will only create a C_2 with $\emptyset$ or $[n]$, not a $2C_2$, and is guaranteed to not be $\|$ with the elements in the same and other layer, so the other layer.
for a $2C_2$, $\emptyset, [n] \notin 2C_2$ as not disjoint, so there are only 4 points left to make a $2C_2$, however we are given that one of the points $\|$ to the 2 others on the opposing layer. so not disjoint
2. may work.

Since case 1 does not work not induced $2C_2$ saturated

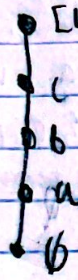
$sat^*(3, 2C_2) \ge 6$, proven that $sat^*(3, 2C_2) \le 6$

So $sat^*(3, 2C_2) = 6$

prove that F must contain maximal chain to be induced $2C_2$ saturated

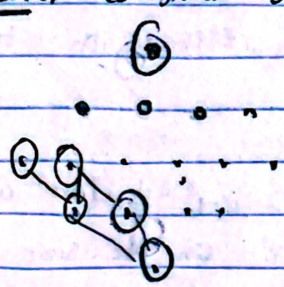
So for $F = \{\emptyset, [n], a, b\}$ where $a \| b$
consider the F' where c, new element, $a \| c$ and $b \| c$, in this case, no disjoint C_2's exist as all elements

constructing proof? do not think it valid



are related, so to avoid ~~this~~ avoid this case c must be in F, this reasoning can be applied inductively until all elements that relate to every $s \in F$, we added creating a maximal chain

F induced $2C_2$ saturated $\Rightarrow$ F contains maximal chain?
Already proven contains at least one chain without $\emptyset, [n]$, consider situation where F' extends that chain



must not contain $2C_2$
somehow prove that chain extended
not disjoint
2 cases

1. only chain exists
2. multiple chains exists but not disjoint, either contain some $\emptyset, [n]$ or

Since no chains maximal can always find F' extends existing chain without $\emptyset, [n]$
Since this new element creates 1 chain that will be used in $2C_2$, 2 cases

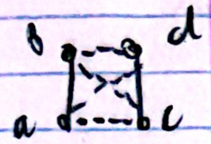
1. the extended chain is disjoint to other C_2, and but C_2 extended on would have been disjoint as share vertex, use extended
2. new element creates new chain that is not contain the original chain and disjoint to other existing chain but a new chain.. use diagonal

X

X

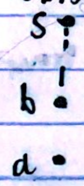
$(a,b) \Rightarrow a \| b,\ C_2$

* $\forall (a,b), (c,d) \in F,\ (a,b) \neq (c,d),\ a \| c \vee b \| d \vee a \| d \vee b \| c$, C_2's are never fully disjoint as if $\exists$ pair then $2C_2$



in someway not disjoint

if no maximal chain, can find F' which extends existing chain, so (b,s), where $a \| b$, $a \| s$



and since $(a,b) \in F \Rightarrow (b,s) \in F$ by transitivity, passes 1st

since not disjoint to all C_2's in F, no $2C_2$

not induced $2C_2$ saturated fails 2nd condition

distinction between adding to existing C_2 without $\emptyset$, $[n]$ vs with?

* families $\supseteq$ induced $2C_2$ saturated F's, prove that satisfies condition on relations, by definition of not containing $2C_2$ as would fail 1st condition of saturation

—

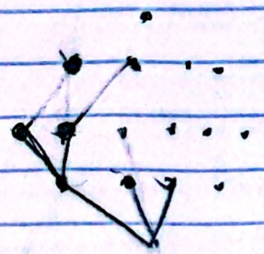
F contains horizontal chain, now full layer

only C_2's in F either $\emptyset$ or $[n]$, or connect with maximal chain

—

So know that F contains $[n]$, $\emptyset$, at least one C_2 without $\emptyset$, $[n]$, bound to number of C_2's without $\emptyset$, $[n]$, categorize C_2's to fully/partially related, one end relates vs both

So $\exists$ at least one C_2 without $\emptyset$, $[n]$, must fully related?

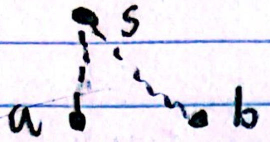


extending existing C_2 without $\emptyset$, $[n]$ never disjoint? too many vertices assume it created $2C_2$, so

$\forall (a,b), (c,d) \in F,\ \emptyset, [n] \notin (a,b), (c,d).\ (a,b) \| (c,d)$

if not, $2C_2$

So adding to existing C_2 without $\emptyset, [n]$ to $2C_2$, not p subtractal
So creates new chain's, any chain from then $\emptyset, [n]$.



points either directly connected to $\emptyset, [n]$, or not connected then must be <u>in same chain group</u> as a. and therefore no disjoint one created
need to prove that any chain created in same group never disjoint

no maximal chain, can always find vertex that || to all of a group, prove always related ?

looking for B_3's in larger B_n's ?