Forbidden Posets: Small Posets on Small Lattices

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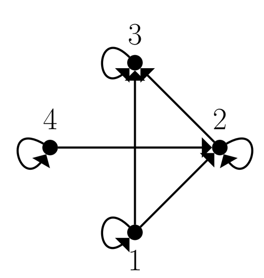
Definitions

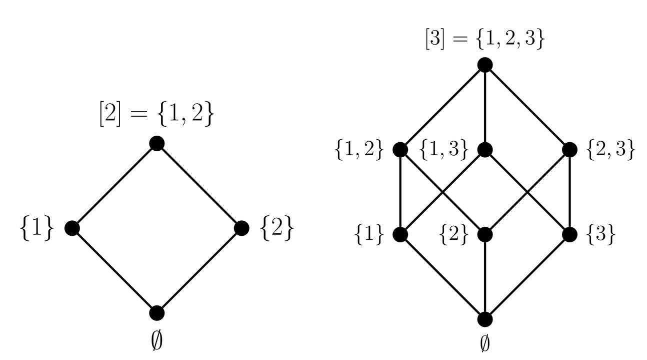
A relation R defined on a set P is called a **partial ordering** or **partial order** if it is *reflexive*, *antisymmetric*, and *transitive*.

A **partially ordered set** or **poset** $\mathcal{P} = (P, R)$ is a set P together with its partial ordering R.

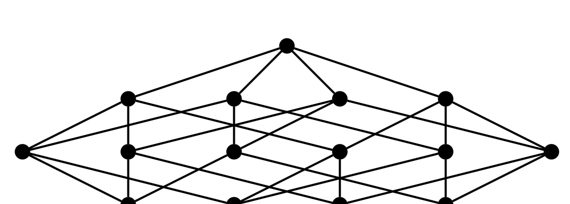
Example: The poset $\mathcal{P} = (P, R)$ where
$P = \{1, 2, 3, 4\}$ and
$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3), (4, 2)\}$.



The n-**dimensional Boolean lattice** $\mathcal{B}_n$ is the poset $(2^{[n]}, \subseteq)$ where $2^{[n]}$ denotes the set of all subsets of $[n] := \{1, 2, ..., n\}$ and $\subseteq$ is the subset inclusion relation.



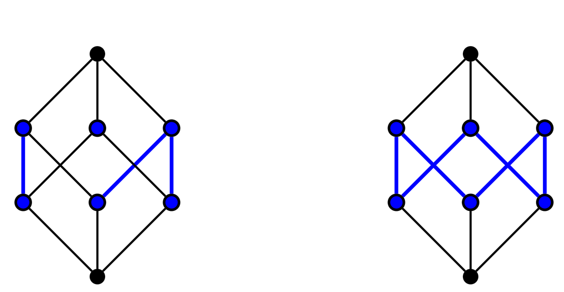
$\mathcal{B}_2$ $\mathcal{B}_3$



$\mathcal{B}_4$

A **subposet** $\mathcal{P}$ is a subset of a poset that inherits the order relation from the poset.

An **induced subposet** $\mathcal{P}$ is a subposet where the partial order is exactly the same as in the original poset, with no alterations.



Subposet Induced Subposet

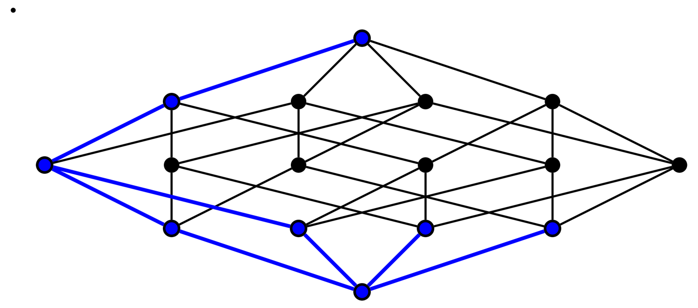
Main Research Goal

Investigate the minimum size of an induced-poset-saturated family in the n-dimensional Boolean lattice $\mathcal{B}_n$.

Background

A family $\mathcal{F} \in \mathcal{B}_n$ is said to be $\mathcal{P}$-**saturated** if it does not contain a copy of $\mathcal{P}$, but every proper superset contains a copy of $\mathcal{P}$.

An induced family $\mathcal{F} \in \mathcal{B}_n$ is **induced-$\mathcal{P}$-saturated** if it does not contain an *induced* copy of $\mathcal{P}$, but every proper induced superset of $\mathcal{F}$ contains an induced copy of $\mathcal{P}$.

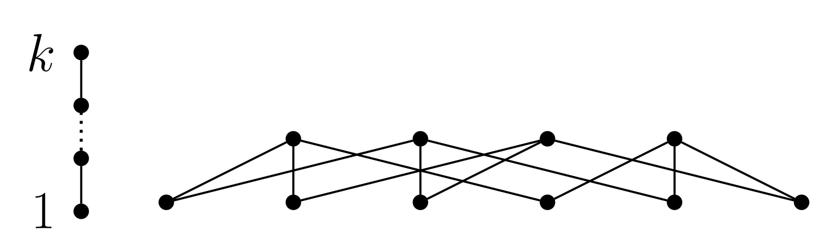


Induced$-2\mathcal{C}_2-$saturated family

La$(n, \mathcal{P})$ denotes the maximum size of a $\mathcal{P}$-saturated family.

Theorem (Sperner). La$(n, \mathcal{P}_2) = \binom{n}{\lfloor n/2 \rfloor}$ where $\mathcal{P}_k$ is a chain on k vertices.

Theorem (Erdős). La$(n, \mathcal{P}_k) \approx (k-1)\binom{n}{\lfloor n/2 \rfloor}$.



$\mathcal{P}_k$ La$(4, \mathcal{P}_3) = 10$

An example of a maximum size $\mathcal{P}_3$-saturated family in $\mathcal{B}_4$.

sat$(n, \mathcal{P})$ denotes the minimum size of a $\mathcal{P}$-saturated family.

sat*$(n, \mathcal{P})$ denotes the minimum size induced-$\mathcal{P}$-saturated family.

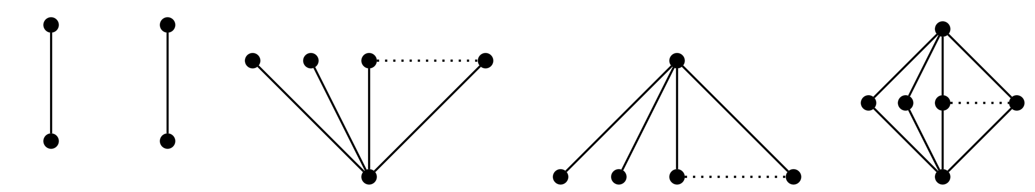
Theorem (Gerbner et al.). For n sufficiently large, $2^{k/2-1} \leq$ sat$(n, \mathcal{P}_{k+1}) \leq 2^{k-1}$.

Theorem (Ferrara et al.). If $n \geq 2$, then sat*$(n, \mathcal{V}_2) = n + 1$.

Theorem (Ferrara et al.). If $n \geq 2$, then $\lceil log_2 n \rceil \leq$ sat*$(n, \mathcal{D}_2) \leq n + 1$.

Theorem (Keszegh et al.). For any integer $n \leq 3$, $n + 2 \leq$ sat*$(n, 2\mathcal{C}_2) \leq 2n$.

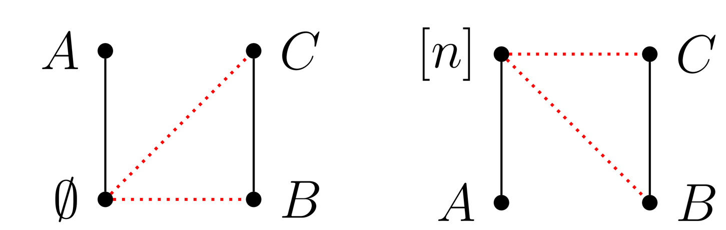
Subposets of Interest



$2\mathcal{C}_2$ $\mathcal{V}_k$ Λ_k $\mathcal{D}_k$

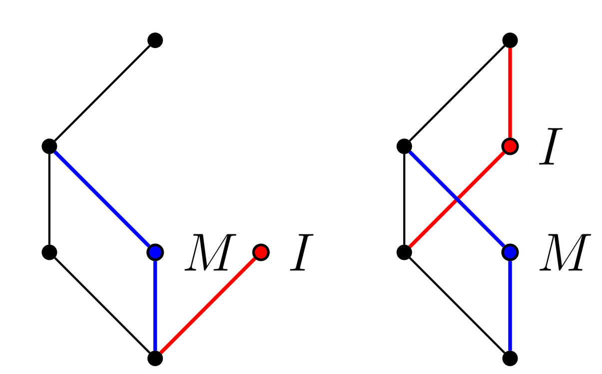
Results

Lemma: The poset $2\mathcal{C}_2$ does not contain $\emptyset$ or $[n]$.

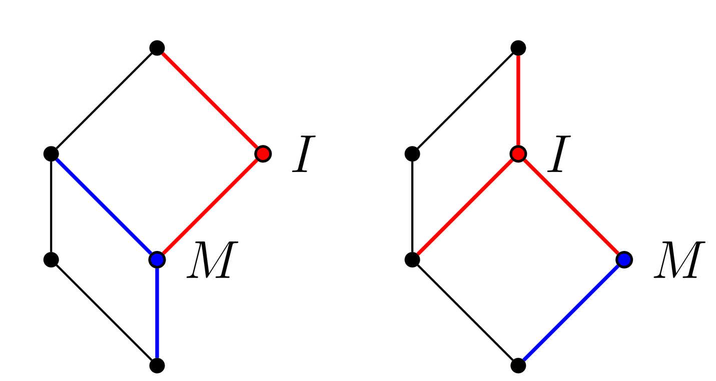


Note: $\emptyset$ and $[n]$ are comparable with every element in the Boolean lattice.

Theorem: For $n = 3$, sat*$(3, 2\mathcal{C}_2) \neq 5$. Therefore, sat*$(3, 2\mathcal{C}_2) = 6$.

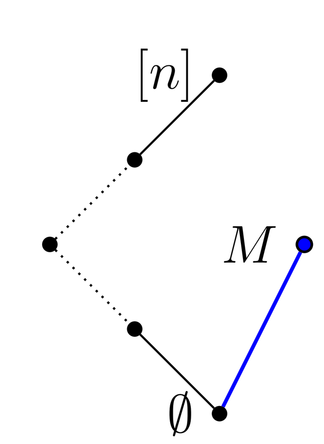


Case 1



Case 2 Case 3

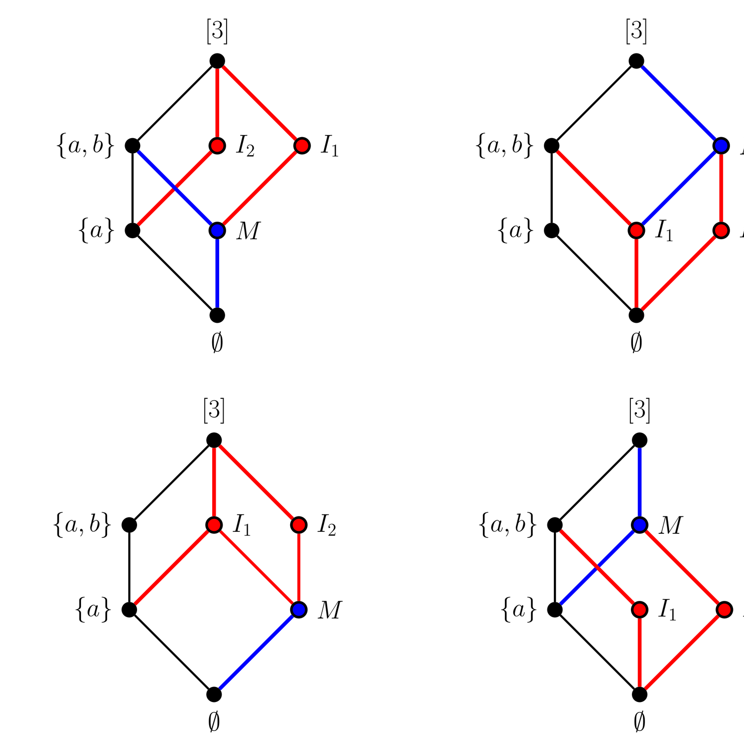
Theorem: Let $n > k$ and let $\mathcal{P} \in \{\mathcal{V}_k, \Lambda_k, \mathcal{D}_k\}$. If $\mathcal{F}$ is an induced-$\mathcal{P}$-saturated family in $\mathcal{B}_n$ and $\mathcal{F}$ contains a maximal chain, then $|\mathcal{F}| > n + 1$.



The family $\mathcal{F}$ is comprised of a maximal chain of size $n + 1$ and any element M.

More Results

Theorem: For $n = 3$, if $\mathcal{F}$ is an induced-$\mathcal{V}_3$-saturated family in $\mathcal{B}_3$ and $\mathcal{F}$ contains a maximal chain, then $|\mathcal{F}| = 6$.



These cases also apply to its dual Λ_3.

Theorem: For $n = 3$, if $\mathcal{F}$ is an induced-Λ_3-saturated family in $\mathcal{B}_3$ and $\mathcal{F}$ contains a maximal chain, then $|\mathcal{F}| = 6$.

Acknowledgements

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