Intro Slides

poset = set P with binary relation $\preceq$

reflexive, $a \preceq a$

antisymmetric, $a \preceq b \wedge b \preceq a \Rightarrow a = b$

transitive, $a \preceq b \wedge b \preceq c \Rightarrow a \preceq c$

n-dimensional boolean lattice (B_n) = poset, $(2^{[n]}, \subseteq)$

$2^{[n]} = \mathcal{P}([n])$

subposet = P', if injection that preserves partial ordering,

subgraph vs subposet?

family = essentially set but with ordering, set of sets with indexes

F is family lies in B_n does not contain subposet $\Rightarrow$ F is P free

lies in B_n each set in F is $\subseteq B_n$

$La(n,P)$ = largest size of P free family of $\subseteq$'s of $[n]$

$B(n,k)$ $\subseteq$'s of $[n]$, with $|B| = k$, $\binom{[n]}{k}$, "kth layer"

$\Sigma(n,k) = |B(n,k)|$

N poset, contains W, X, Y, Z distinct sets,

$W \subset X$, $Y \subset X$, $Y \subset Z$

binary word = vector length n 0,1's

binary code of length n, C, $\subseteq$ binary words length n

$c \in C$, c = code word

$|C| = m$, order of m

weight of c = number of 1's

hamming distance between c_1, c_2 = number of positions where entries differ

hamming distance of code = smallest hamming distance over all pairs of $c_1, c_2 \in C$

$A(n, 2\delta, k)$ = max number of C's of length n
all have weight k
any hamming distance between 2 codewords at least 2δ

Questions
1. Subposet vs subgraph?
2. Family vs set?
3. F lives in B_n = $F \subseteq B_n$?
4. Connection to code theory?
5. proof?

Scanned with CamScanner

Saturation

Sub graph = given graph G, subgraph is $\subseteq V(G), E(G)$, with only valid e's $\in E(G)$
induced $\leq$ subgraph with all valid edges preserved

Subposet = $\subseteq$ of poset, hamiliarily partial order of universal set
$(Q, \preceq) \subseteq (P, \preceq)$, if $x \preceq y \in P \Rightarrow x \preceq y \in Q$, not necessarily inherits relations
induced = same order relations present in P, must inherit relations

given poset P, family $F \subseteq B_n$, P saturated if

1. F contains no copy of P as subposet
2. every strict superset of F contains copy of P

$La(n, p)$ = max(| p saturated family |), max size
$\mathrm{sat}(n, p)$ = min | p saturated family |, min size

does not contain P
not necessarily induced
add 1 element to P saturated, contains P

Example

$P = P_2$!
$La(n, P_2)$ = middle layer as none related and element above added connected, middle layer max size
$\mathrm{sat}(n, P_2)$ = any one subset, $[n]$ vacuously true as no proper $\supsetneq$ possible

comparable = $a \preceq b$ related

induced subposet = $\exists f$ injective: $Q \to P$ $Q \leq P$
$w \preceq v \Leftrightarrow f(w) \preceq f(w)$

Sat can conveniently choose relations conveniently
! Sat* ... all relations, watch distinction
2.

family of induced P saturated
1. F contains no induced copy of P as subposet
2. every strict ⊃ contains induced copy of P

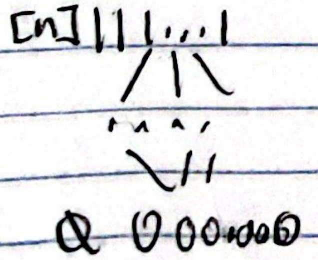
$La^{\#}(n,P)$ = max |induced P saturated F|
$sat^{*}(n,P)$ = min |induced P saturated F|

<u>Introduction</u> (Revisited)
n dimensional Boolean lattice = visual representation of poset. $(2^{[n]}, \subseteq)$,
set of all ⊆'s of [n], and their relations
subposet = ⊆ of poset, can choose relations
induced subposet: all relations preserved

Family = collection of sets, here ⊆'s in B_n.
F is p-free if ⊆'s in F do not have subposet P, structure of P
does not appear in any subposet of F

binary word = vector length n of 0's and 1's
binary code = set of binary words length n, element in C
called code word, |C| = m, order of m

weight of code word = # of 1's, layer-1 with ∅ being layer 1
0's and 1's can be mapped to lattice.
{1, 2, ..., n}
_ _ ... _ , each corresponding space represents presence of
element, if 1 present, 0 absent in subset



hamming distance = # spots where codes differ, or symmetric difference of $\subseteq$'s
hamming distance of Code, C = min hamming distance for all pairs

$A(n, 2\delta, k)$ = max number codewords of length n, weight k, hamming distance of any pair $\geq 2\delta$.

Paper:
if P not induced subposet, $Sat^*(n, P) = 2^n = |B_n|$, vacuously
$n \geq |P| \Rightarrow Sat^*(n, P) < 2^n$, when structure of P has less vertices or $\subseteq$'s than total, never need all points in B_n to construct P, as such

if P is induced subposet of $Q \not\Rightarrow Sat^*(n, P) \leq Sat^*(n, Q)$, just because smaller does not mean less

$x \| y \Rightarrow x \not\leq y$ and $y \not\leq x$, $x, y \in P$, poset
Cover: given poset P, $x, y \in P$, $x \leq y$, and $\subseteq$'s directly adjacent

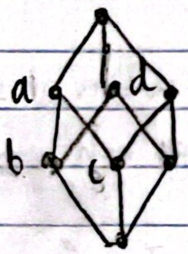
Unique Cover twin Property = $\forall S \in P$ with one cover, T, $\exists S', S' \neq S$, such that T covers S

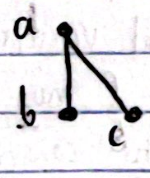
Presentation

1. induced subposet: P' is induced subposet if $\exists$ some $f: P' \to P$
$u \preceq v$ in P' $\Leftrightarrow$ $f(u) \preceq f(v)$ in P
essentially like a normal subposet, it is a subgraph or substructure in P, but the distinction is that existing relations in P must be preserved in P', Just keep all edges

Example
Consider B_3 . let P_1 be

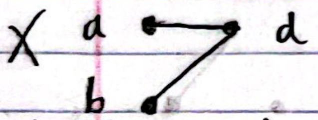
use simpler example? B_2



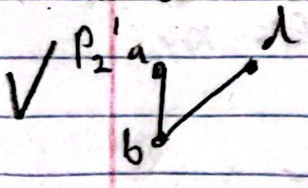


is an induced subposet as all the valid edges in the structure, appear as in B_3, and P_1 exists in B_3

let P_2 be



this is not an induced subposet or subposet for 2 reasons, 1. $a \npreceq d$ and $d \npreceq a$ in the original poset P so it should definitely not show up here and 2. $b \preceq a$ in P so to be an induced subposet it should show up here



Our research moving forward will be dealing with induced subposets, induced vs non induced is essentially creating more rigid rules, non induced offers more leniency as the subposets need not to be exact.

2. Saturation
We say a family, F, is P-saturated (or induced-P-saturated) if
1. the subposet (or induced subposet) P does not appear in F
2. for any F', $F \subset F' \subseteq Q$ (the host poset, B_n)
   P does appear

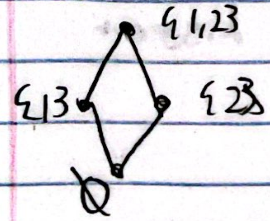
in other words, for a family, a collection of ⊆'s, or host poset or vertices in the lattice, P does not appear anywhere in lattice/host poset but if you add in any one ⊆ or additional vertex, P appears

this idea has an almost identical presence in general graph theory, and has a definition for the induced and non induced cases, we are dealing with mostly the induced version

3. $sat^*(n,P)$

$sat^*(n,P)$ = defined on B_n and for some induced poset P, the minimum size family such that F is induced-p-saturated
$*$ = induced

Example
let $P = C_2$ and $n=2$



so minimum number of elements such that F does not contain P, but add any one

in this case $sat^*(2,C_2) = 1$, as let $F = \{\emptyset\}$
or $F = \{[2]\}$

this is the minimum, but another induced p saturated F is $\{\{1\},\{2\}\}$, this is actually $La^*(2,C_2)$
as you can see there are many induced P saturated families
we are trying to prove upper and lower bounds on the minimum size of, $sat^*(n,P)$

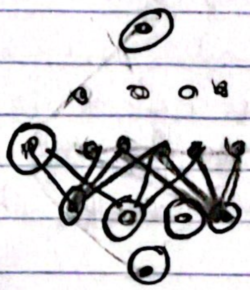
$Sat^*(n, 2C_2)$: $2C_2 =$ | | disjoint C_2's

$n+2 \le Sat^*(n, 2C_2) \le 2n$ - proof?

have minimum size

$\{i\}$'s ⊙ ... • $n+1$

⊙ Q



$F = \{\emptyset, [n], \{i\}'s, \{i_1, i_2\}\}$

$n+3$

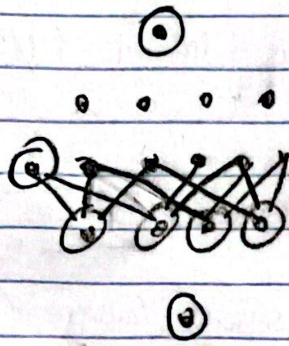
if middle layer added ✓

$\binom{n}{\lfloor n/2 \rfloor}$

$\{i\}$'s ⊙ ... ⊙ ~~$F = \{Q, \{i_1\}, \{i_2\}, [n]\}$~~

in F have to take $\emptyset, [n]$ as will remove disjoint cannot find induced-p-saturated for $p = 2C_2$

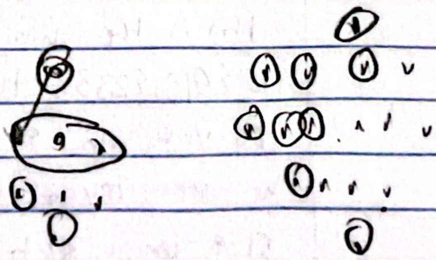
layer will have symmetric due $\mathbb{Z}2$



$\{1,3\}$ $\{1,2\}$

$\{3\}$ $\{2\}$

middle layer ✓

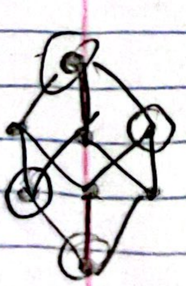


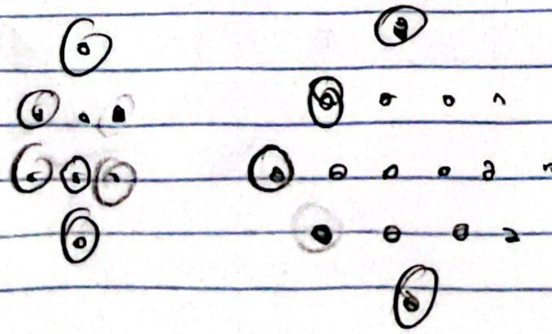
upper X

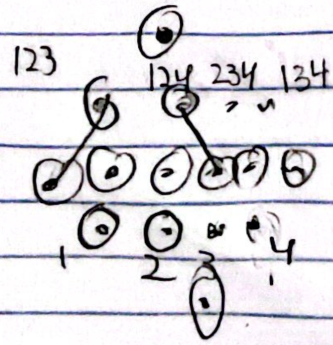
$\{1,2,3\}$ $\{1,2\}$

take $\emptyset, [n]$

$\binom{n}{\lfloor n/2 \rfloor}$,







idea

1. need to take $\emptyset, [n]$, as will immediately be non disjoint, 2
2. then take $\binom{n}{\lfloor \frac{n}{2} \rfloor}$
3. choose specific element in middle layer, considering picking specific element in different layer, but middle layer will have least number of connections, then to guarantee disjointness, not sure #.
4. then what ever is added will not have connection to middle layer element so $\exists\ 2C_2$

~~5.~~ does not have $2C_2$ as