Results on $\mathcal{N}$ poset

Shanise Walker

Clark Atlanta University
Department of Mathematical Sciences

May 29, 2024

Definitions

Notation: $[n] := \{1, 2, \ldots, n\}$

Definition

A partially ordered set (poset) is a set P and a binary relation "$\preceq$" such that for all $a, b, c \in P$

- *$a \preceq a$ (reflexivity).*
- *if $a \preceq b$ and $b \preceq a$, then $a = b$ (antisymmetry).*
- *if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity).*

Examples

1. $P = \{1, 2, \ldots, \}$ with binary relation $a \leq b$.
2. $P = \{1, 2, \ldots, \}$ with binary relation $a \preceq b$ if a divides b.

Definitions

Notation: $[n] := \{1, 2, \ldots, n\}$

- The *n*-dimensional Boolean lattice, $\mathcal{B}_n$, denotes the partially ordered set (poset) $(2^{[n]}, \subseteq)$. $2^{[n]}$ denotes the set of subsets of $[n]$.

3-dimensional Boolean Lattice

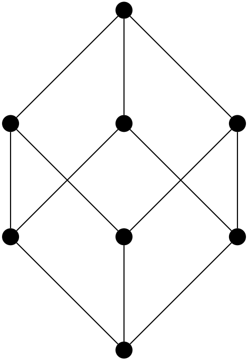


Figure: Hasse diagram of $\mathcal{B}_3$

Example of a subposet

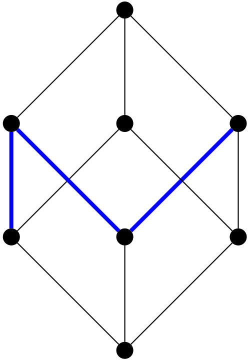


Figure: Hasse diagram of $\mathcal{B}_3$

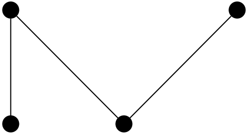


Figure: subposet of $\mathcal{B}_3$

Definitions

Notation: $[n] := \{1, 2, \ldots, n\}$

- The *n*-dimensional Boolean lattice, $\mathcal{B}_n$, denotes the partially ordered set (poset) $(2^{[n]}, \subseteq)$. $2^{[n]}$ denotes the set of subsets of $[n]$.

Definition

For posets, $\mathcal{P} = (P, \preceq_{\mathcal{P}})$ and $\mathcal{P}' = (\mathcal{P}', \preceq_{\mathcal{P}'})$, we say $\mathcal{P}'$ is a subposet of $\mathcal{P}$ if there exists an injection $f : \mathcal{P}' \to \mathcal{P}$ that preserves the partial ordering, meaning that whenever $u \leq' v$ in $\mathcal{P}'$, we have $f(u) \leq f(v)$ in $\mathcal{P}$.

Problems of Interest

Goal

Estimate the maximum size of a family of subsets of $\mathcal{B}_n$ which does not contain the subposet $\mathcal{P}$.

Definitions

- If $\mathcal{F}$ is a family that lies in $\mathcal{B}_n$ such that $\mathcal{F}$ contains no subposet $\mathcal{P}$, we say $\mathcal{F}$ is $\mathcal{P}$-free.

Definitions

- If $\mathcal{F}$ is a family that lies in $\mathcal{B}_n$ such that $\mathcal{F}$ contains no subposet $\mathcal{P}$, we say $\mathcal{F}$ is $\mathcal{P}$-free.
- $\mathrm{La}(n, \mathcal{P})$ denotes the largest size of a $\mathcal{P}$-free family of subsets of $[n]$.

Definitions

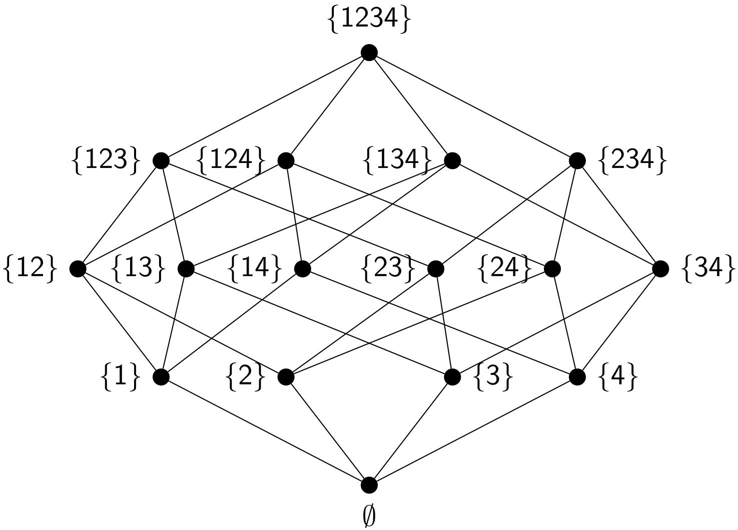
- If $\mathcal{F}$ is a family that lies in $\mathcal{B}_n$ such that $\mathcal{F}$ contains no subposet $\mathcal{P}$, we say $\mathcal{F}$ is $\mathcal{P}$-free.
- $\mathrm{La}(n, \mathcal{P})$ denotes the largest size of a $\mathcal{P}$-free family of subsets of $[n]$.
- $\mathcal{B}(n, k)$ denotes the collection of subsets of $[n]$ of the k middle layers of $\mathcal{B}_n$.
  ▶ k^{th} layer of $\mathcal{B}_n$ is the collection of all subsets of $[n]$ of size k, denoted by $\binom{[n]}{k}$.

Definitions

- If $\mathcal{F}$ is a family that lies in $\mathcal{B}_n$ such that $\mathcal{F}$ contains no subposet $\mathcal{P}$, we say $\mathcal{F}$ is $\mathcal{P}$-free.
- $\mathrm{La}(n, \mathcal{P})$ denotes the largest size of a $\mathcal{P}$-free family of subsets of $[n]$.
- $\mathcal{B}(n, k)$ denotes the collection of subsets of $[n]$ of the k middle layers of $\mathcal{B}_n$.
  ▶k^{th} layer of $\mathcal{B}_n$ is the collection of all subsets of $[n]$ of size k, denoted by $\binom{[n]}{k}$.
- If n is a fixed integer, $\sum(n, k)$ denotes the sum of the k largest binomial coefficients of the form $\binom{n}{\ell}$.

$$\sum(n, k) := |\mathcal{B}(n, k)|$$

The 4-dimensional Boolean Lattice $\mathcal{B}_4$



The Boolean Lattice

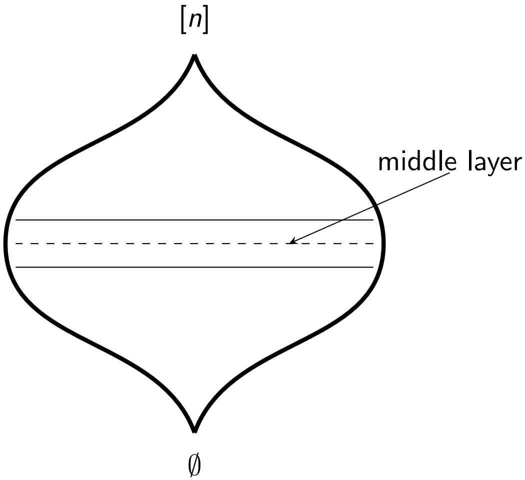
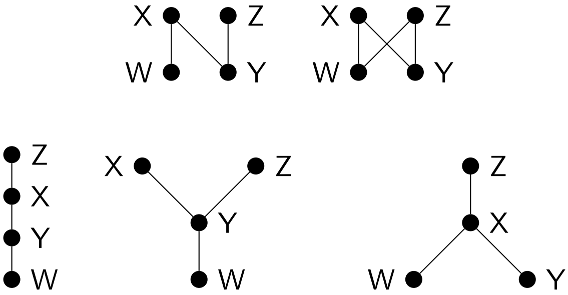


Figure: The n-dimensional Boolean lattice $\mathcal{B}_n$.

$\mathcal{N}$ poset

Definition

The $\mathcal{N}$ poset consists of four distinct sets W, X, Y, Z such that $W \subset X$, $Y \subset X$, and $Y \subset Z$ where W is not necessarily a subset of Z.



$\mathcal{N}$ poset

Question

Consider the $\mathcal{N}$ poset in the n-dimensional Boolean lattice $\mathcal{B}_n$. What can we say about the size of a largest $\mathcal{N}$-free family?

Definitions-Coding Theory Background

- A binary word is a $\{0,1\}$-vector of length n.
- A binary code of length n, say C, is a subset of all binary words of length n. An element of C is called a codeword.
- If $|C| = m$, then C is of order m.
- The weight of a codeword is the number of ones in the codeword.

$$a\text{=}[0,1,0,0,1] \qquad b\text{=}[1,1,1,1,0]$$

Figure: a and b are binary words of length 5. The weight of $a = 2$ and the weight of $b = 4$.

Definitions-Coding Theory Background

1. The Hamming distance between two codewords of equal length is the number of positions at which the corresponding entries differ.
2. The Hamming distance of a code is the smallest Hamming distance over all pairs of codewords in that code.

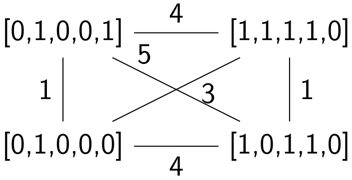


Figure: $C = \{[0,1,0,0,1],[1,1,1,1,0],[0,1,0,0,0],[1,0,1,1,0]\}$
The Hamming distance of C is 1.

$A(n, 2\delta, k)$

Let $A(n, 2\delta, k)$ denote the maximum number of codewords in any binary code of length n, such that:

- all codewords have constant weight k,
- and the Hamming distance between any two codewords is at least 2δ.

Theorem (Graham and Sloane (1980))

$A(n, 4, k) \geq \frac{1}{n}\binom{n}{k}$.

Note: $A(n, 4, k)$ computes the size of a single-error-correcting (SEC) code with constant weight k.

Bounds for $\mathcal{N}$-free families

Theorem (Griggs and Katona (2008))

$$\binom{n}{\lfloor n/2 \rfloor}\left(1+\frac{1}{n}+\Omega\left(\frac{1}{n^2}\right)\right) \leq \mathrm{La}(n,\mathcal{N}) \leq \binom{n}{\lfloor n/2 \rfloor}\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right).$$

Theorem (Katona and Tarján (1980))

$$\mathrm{La}(n,\mathcal{N}) \geq \binom{n}{\lfloor n/2 \rfloor} + A(n,4,\lfloor n/2 \rfloor + 1).$$

Bounds for $\mathcal{N}$-free families

Theorem (Martin and W., 2017)

$$\mathrm{La}(n,\mathcal{N}) \geq \binom{n}{\lfloor n/2 \rfloor} + A(n,4,\lfloor n/2 \rfloor).$$

Theorem (Katona and Tarján, 1980)

$$\mathrm{La}(n,\mathcal{N}) \geq \binom{n}{\lfloor n/2 \rfloor} + A(n,4,\lfloor n/2 \rfloor + 1).$$

Note: When n is odd, the results are the same by symmetry.

Lower bound for $\mathcal{N}$-free families

Theorem (Martin and W., 2017)

$$\mathrm{La}(n, \mathcal{N}) \geq \binom{n}{\lfloor n/2 \rfloor} + A(n, 4, \lfloor n/2 \rfloor).$$

Proof: Given $k = n/2$, let C be a constant weight SEC code of size $A(n, 4, k)$. Define $C_{\mathrm{up}} := \{c \cup \{i\} : c \in C,\ i \notin c\}$ and $C_{\mathrm{down}} := \{c - \{i\} : c \in C,\ i \in c\}$.

Lower bound for $\mathcal{N}$-free families

Theorem (Martin and W., 2017)

$$\mathrm{La}(n, \mathcal{N}) \geq \binom{n}{\lfloor n/2 \rfloor} + A(n, 4, \lfloor n/2 \rfloor).$$

Proof: Given $k = n/2$, let C be a constant weight SEC code of size $A(n, 4, k)$. Define $C_{\mathrm{up}} := \{c \cup \{i\} : c \in C,\ i \notin c\}$ and $C_{\mathrm{down}} := \{c - \{i\} : c \in C,\ i \in c\}$.

Observe the following:

- Both C_{up} and C_{down} are SEC codes with constant weight $k+1$ and $k-1$, respectively.
- If $c'' \in C_{\mathrm{up}}$ and $c' \in C_{\mathrm{down}}$, $c' \not\subseteq c''$.

Proof continued

Claim: The family $\mathcal{F} := \binom{[n]}{k} \cup C_{\mathrm{up}} \cup C_{\mathrm{down}}$ is $\mathcal{N}$-free.
Proof: Suppose there is a subposet $\mathcal{N}$ with elements W, X, Y, Z where $W \subset X$, $Y \subset X$ and $Y \subset Z$.

Proof continued

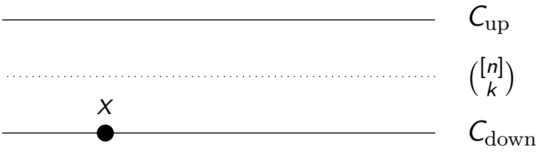
Claim: The family $\mathcal{F} := \binom{[n]}{k} \cup C_{\mathrm{up}} \cup C_{\mathrm{down}}$ is $\mathcal{N}$-free.
Proof: Suppose there is a subposet $\mathcal{N}$ with elements W, X, Y, Z where $W \subset X$, $Y \subset X$ and $Y \subset Z$.

- Where is the element X?

Proof continued

Claim: The family $\mathcal{F} := \binom{[n]}{k} \cup C_{\mathrm{up}} \cup C_{\mathrm{down}}$ is $\mathcal{N}$-free.
Proof: Suppose there is a subposet $\mathcal{N}$ with elements W, X, Y, Z where $W \subset X$, $Y \subset X$ and $Y \subset Z$.

- Where is the element X?

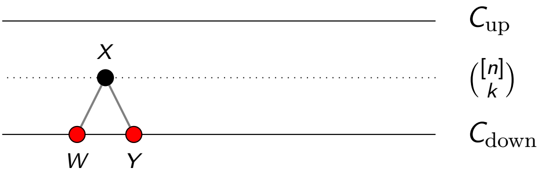


$X \notin C_{\mathrm{down}}$ because it has to have elements below it and the elements of C_{down} are all minimal in $\mathcal{F}$.

Proof of Claim continued

Claim: The family $\mathcal{F} := \binom{[n]}{k} \cup C_{\mathrm{up}} \cup C_{\mathrm{down}}$ is $\mathcal{N}$-free.
Proof: Suppose there is a subposet $\mathcal{N}$ with elements W, X, Y, Z where $W \subset X$, $Y \subset X$ and $Y \subset Z$.

- Where is the element X?

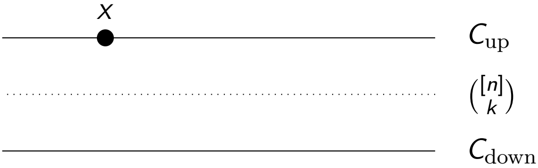


$X \notin \binom{[n]}{k}$ because that would force $W, Y \in C_{\mathrm{down}}$ and have symmetric difference of size 2.

Proof of Claim continued

Claim: The family $\mathcal{F} := \binom{[n]}{k} \cup C_{\mathrm{up}} \cup C_{\mathrm{down}}$ is $\mathcal{N}$-free.
Proof: Suppose there is a subposet $\mathcal{N}$ with elements W, X, Y, Z where $W \subset X$, $Y \subset X$ and $Y \subset Z$.

- Where is the element X?

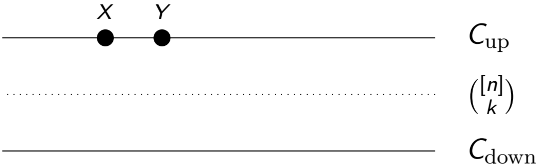


$$X \in C_{\mathrm{up}}.$$

Proof of Claim continued

Claim: The family $\mathcal{F} := \binom{[n]}{k} \cup C_{\mathrm{up}} \cup C_{\mathrm{down}}$ is $\mathcal{N}$-free.
Proof: Suppose there is a subposet $\mathcal{N}$ with elements W, X, Y, Z where $W \subset X$, $Y \subset X$ and $Y \subset Z$.

- Where is the element Y?

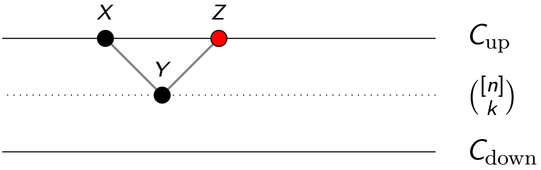


$$Y \notin C_{\mathrm{up}} \text{ because } Y \subset X.$$

Proof of Claim continued

Claim: The family $\mathcal{F} := \binom{[n]}{k} \cup C_{\mathrm{up}} \cup C_{\mathrm{down}}$ is $\mathcal{N}$-free.
Proof: Suppose there is a subposet $\mathcal{N}$ with elements W, X, Y, Z where $W \subset X$, $Y \subset X$ and $Y \subset Z$.

- Where is the element Y?

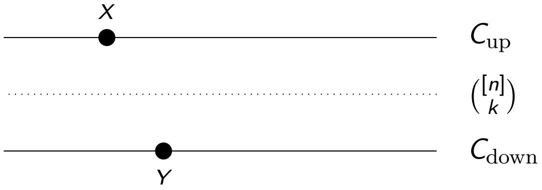


$Y \notin \binom{[n]}{k}$ because that would force $X, Z \in C_{\mathrm{up}}$ and have symmetric difference of size 2.

Proof of Claim continued

Claim: The family $\mathcal{F} := \binom{[n]}{k} \cup C_{\mathrm{up}} \cup C_{\mathrm{down}}$ is $\mathcal{N}$-free.
Proof: Suppose there is a subposet $\mathcal{N}$ with elements W, X, Y, Z where $W \subset X$, $Y \subset X$ and $Y \subset Z$.

- Where is the element Y?

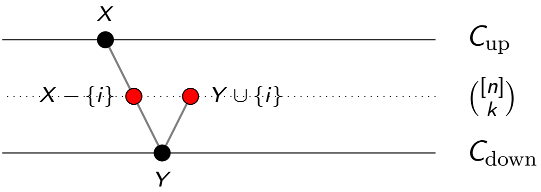


$$Y \in C_{\mathrm{down}}.$$

Proof of Claim continued

Claim: The family $\mathcal{F} := \binom{[n]}{k} \cup C_{\text{up}} \cup C_{\text{down}}$ is $\mathcal{N}$-free.
Proof: Suppose there is a subposet $\mathcal{N}$ with elements W, X, Y, Z where $W \subset X$, $Y \subset X$ and $Y \subset Z$.

- For $\mathcal{N}$ to exist, $Y \subset X$ implies that $Y \subset X - \{i\}$.



Note that $Y \cup \{i\}$ and $X - \{i\}$ have symmetric difference of size 2, but both are elements of C.

$\square$