

# A note on the size of $\mathcal{N}$ -free families

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## Abstract

The  $\mathcal{N}$  poset consists of four distinct sets  $W, X, Y, Z$  such that  $W \subset X$ ,  $Y \subset X$ , and  $Y \subset Z$  where  $W$  is not necessarily a subset of  $Z$ . A family  $\mathcal{F}$ , considered as a subposet of the  $n$ -dimensional Boolean lattice  $\mathcal{B}_n$ , is  $\mathcal{N}$ -free if it does not contain  $\mathcal{N}$  as a subposet. Let  $\text{La}(n, \mathcal{N})$  be the size of a largest  $\mathcal{N}$ -free family in  $\mathcal{B}_n$ . Katona and Tarján proved that  $\text{La}(n, \mathcal{N}) \geq \binom{n}{k} + A(n, 4, k+1)$ , where  $k = \lfloor n/2 \rfloor$  and  $A(n, 4, k+1)$  is the size of a single-error-correcting code with constant weight  $k+1$ . In this note, we prove for  $n$  even and  $k = n/2$ ,  $\text{La}(n, \mathcal{N}) \geq \binom{n}{k} + A(n, 4, k)$ , which improves the bound on  $\text{La}(n, \mathcal{N})$  in the second order term for some values of  $n$  and should be an improvement for an infinite family of values of  $n$ , depending on the behavior of the function  $A(n, 4, \cdot)$ .

**Keywords:** forbidden subposets, error-correcting codes

## 1 Introduction

The  $n$ -dimensional Boolean lattice,  $\mathcal{B}_n$ , denotes the partially ordered set (poset)  $(2^{[n]}, \subseteq)$ , where  $[n] = \{1, \dots, n\}$  and, for every finite set  $S$ ,  $2^S$  denotes the set of subsets of  $S$ . For posets,  $P = (P, \preceq)$  and  $P' = (P', \preceq)$ , we say  $P'$  is a (weak) subposet of  $P$  if there exists an injection  $f : P' \rightarrow P$  that preserves the partial ordering. That is, whenever  $u \preceq' v$  in  $P'$ , we have  $f(u) \preceq f(v)$  in  $P$ . If  $\mathcal{F}$  is a subposet of  $\mathcal{B}_n$  such that  $\mathcal{F}$  contains no subposet  $P$ , we say  $\mathcal{F}$  is  $P$ -free.

$P$ -free posets (or  $P$ -free families) have been extensively studied, beginning with Sperner's theorem in 1928. Sperner [7] proved that the size of the largest antichain in  $\mathcal{B}_n$  is  $\binom{n}{\lfloor n/2 \rfloor}$ .

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Erdős [2] generalized this result to chains. Katona and Tarján [6] addressed the problem of  $\mathcal{V}$ -free families and got an asymptotic result. Griggs and Katona [5] addressed  $\mathcal{N}$ -free families, obtaining Theorem 1 below. See Griggs and Li [4] for a survey of the progress on  $P$ -free families. Let  $\text{La}(n, P)$  denote the size of the largest  $P$ -free family in  $\mathcal{B}_n$ .

The main result of this note is Theorem 4, in which, for some values of  $n$ , we improve the bounds on  $\text{La}(n, \mathcal{N})$  in the second-order term. The poset  $\mathcal{N}$  consists of four distinct sets  $W, X, Y, Z$  such that  $W \subset X$ ,  $Y \subset X$ , and  $Y \subset Z$ . However,  $W$  is not necessarily a subset of  $Z$ . See Figure 1. The earliest extremal result on  $\mathcal{N}$ -free families is Theorem 1.

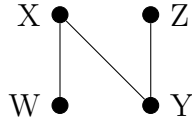


Figure 1: The  $\mathcal{N}$  poset.

**Theorem 1** (Griggs and Katona [5]).

$$\binom{n}{\lfloor n/2 \rfloor} \left( 1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right) \right) \leq \text{La}(n, \mathcal{N}) \leq \binom{n}{\lfloor n/2 \rfloor} \left( 1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right) \right).$$

The construction for the lower bound of Theorem 1 comes directly from a previous result of Katona and Tarján [6] from 1983 on  $\mathcal{V}$ -free families. The poset  $\mathcal{V}$  consists of three elements  $X, Y, Z$  such that  $Y \subset X$  and  $Y \subset Z$ . It is clear that  $\text{La}(n, \mathcal{V}) \leq \text{La}(n, \mathcal{N})$  because any  $\mathcal{V}$ -free family is also  $\mathcal{N}$ -free.

To establish the lower bound, Katona and Tarján used a constant-weight code construction due to Graham and Sloane [3] from 1980. In the proof of Theorem 4, we obtain a lower bound that appears to be larger than the current known bound. However, whether it is an improvement depends on the behavior of some functions well-known in coding theory. In order to discuss our results we need some brief coding theory background.

## 1.1 Coding Theory Background

Let  $A(n, 2\delta, k)$  denote the size of the largest family of  $\{0, 1\}$ -vectors of length  $n$  such that each vector has exactly  $k$  ones and the Hamming distance between any pair of distinct vectors is at least  $2\delta$ . This is the same as the size of the largest family of subsets of  $[n]$  such that each subset has size exactly  $k$  and the symmetric difference of any pair of distinct sets is at least  $2\delta$ .

The quantity  $A(n, 2\delta, k)$  is important in the field of error-correcting codes. In fact,  $A(n, 4, k)$  computes the size of a single-error-correcting code with constant weight  $k$ . Henceforth, we will use “SEC code” as shorthand for “single-error-correcting code.”

The first nontrivial value of  $\delta$  for  $A(n, 2\delta, k)$  is  $\delta = 2$ . Graham and Sloane [3] give a lower bound construction for  $A(n, 4, k)$ .

**Theorem 2** (Graham and Sloane [3]).  $A(n, 4, k) \geq \frac{1}{n} \binom{n}{k}$ .

## 1.2 Main Result

Katona and Tarján [6] estimated the following lower bound for  $\mathcal{N}$ -free families.

**Theorem 3.** *Let  $k = \lfloor n/2 \rfloor$ . Then,*

$$\text{La}(n, \mathcal{N}) \geq \binom{n}{k} + A(n, 4, k+1).$$

The following theorem is our main result of the note.

**Theorem 4.** *Let  $n$  be even and let  $k = n/2$ . Then,*

$$\text{La}(n, \mathcal{N}) \geq \binom{n}{k} + A(n, 4, k). \quad (1)$$

*Remark 5.* This is potentially an improvement when  $n$  is even. We note that the same 3-level construction works for  $n$  odd and  $k = (n-1)/2$ . This gives a family of size  $\binom{n}{k} + A(n, 4, k)$  nontrivially in three layers. However, since  $A(n, 4, k) = A(n, 4, k+1)$  in the odd case, this does not provide an improvement to the known bounds.

We believe that, for  $n \geq 6$ , the quantity  $A(n, 4, k)$  is strictly unimodal as a function of  $k$  as long as  $3 \leq k \leq n-3$ . This strict unimodality has been established [1] for  $6 \leq n \leq 12$  and known bounds suggest that it is the case for larger values of  $n$  as well. If unimodality holds, then  $A(n, 4, k)$  would achieve its maximum uniquely at  $k = \lfloor n/2 \rfloor$  or  $k = \lceil n/2 \rceil$ . Therefore, we expect (1) to also be a strict improvement over Theorem 3 in the case where  $n$  is even. However, to our knowledge, the unimodality of  $A(n, 4, k)$  has never been established and seems to be a highly nontrivial problem.

### Proof of Theorem 4.

Given  $k = n/2$ , let  $C$  be a constant weight SEC code of size  $A(n, 4, k)$ . Define  $C_{\text{up}} := \{c \cup \{i\} : c \in C, i \notin c\}$  and  $C_{\text{down}} := \{c - \{i\} : c \in C, i \in c\}$ . Claim 6 gives some important properties of  $C_{\text{up}} \cup C_{\text{down}}$ .

### Claim 6.

(i) *Both  $C_{\text{up}}$  and  $C_{\text{down}}$  are SEC codes with constant weight  $k+1$  and  $k-1$ , respectively.*

(ii) *If  $c'' \in C_{\text{up}}$  and  $c' \in C_{\text{down}}$ ,  $c' \not\subseteq c''$ .*

*Proof.* (i). Let  $c_1, c_2 \in C_{\text{up}}$ . Then  $|c_1 \Delta c_2| = |(c_1 - \{i\}) \Delta (c_2 - \{i\})| \geq 4$  since  $(c_1 - \{i\}), (c_2 - \{i\}) \in C$  and their symmetric difference must be at least 4 in order for  $C$  to be a 1-EC code. Thus,  $C_{\text{up}}$  is a SEC code. By a similar argument,  $C_{\text{down}}$  is a SEC code.

(ii). Let  $c'' \in C_{\text{up}}$ ,  $c' \in C_{\text{down}}$ , and  $c' \subset c''$ . Then,  $(c' \cup \{i\}), (c'' - \{i\}) \in C$ . So,  $|(c'' - \{i\}) \Delta (c' \cup \{i\})| \geq 4$ . This implies that there are two members of  $[n]$  that are in  $(c' \cup \{i\}) - (c'' - \{i\})$ . One is  $i$  and the other is some  $j \in c' - c''$ , which contradicts the assumption that  $c' \subset c''$ . This concludes the proof of Claim 6.  $\square$

In order to finish the proof, we just need to show that the family  $\mathcal{F} := \binom{[n]}{k} \cup C_{\text{up}} \cup C_{\text{down}}$  is  $\mathcal{N}$ -free.

To that end, suppose there is a subposet  $\mathcal{N}$  with elements  $W, X, Y, Z$  where  $W \subset X$ ,  $Y \subset X$  and  $Y \subset Z$  (see Figure 1). Where is the element  $X$ ?

We know that  $X \notin C_{\text{down}}$  because it has to have elements below it and the elements of  $C_{\text{down}}$  are all minimal in  $\mathcal{F}$ . We know that  $X \notin \binom{[n]}{k}$  because that would force  $W, Y \in C_{\text{down}}$  and, being subsets of  $X$  would require  $|W \triangle Y| = 2$ , a contradiction to  $C_{\text{down}}$  being a SEC code. Therefore,  $X \in C_{\text{up}}$ .

Now, where is  $Y$ ? We know that  $Y \notin C_{\text{up}}$  because  $Y \subset X$ . We know  $Y \notin \binom{[n]}{k}$  because that would force  $X, Z \in C_{\text{up}}$  and thus would force  $|X \triangle Z| = 2$ , this is a contradiction to the fact that  $C_{\text{up}}$  is a SEC code. Therefore,  $Y \in C_{\text{down}}$ .

In order for the copy of  $\mathcal{N}$  to exist,  $Y \subset X$ , which implies  $Y \subset X - \{i\}$  and so  $|(Y \cup \{i\}) \triangle (X - \{i\})| = 2$ . Recall, however, that  $Y \cup \{i\}$  and  $X - \{i\}$  are distinct members of  $C$  and so have symmetric difference at least 4, a contradiction.  $\square$

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