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Some Equilibrium Finite Element Methods for Two-Dimensional Elasticity Problems

C. Johnson and B. Mercier*

Department of Computer Science, Chalmers University of Technology, Fack, S-40220 Göteborg 5, Sweden

Summary. We consider some equilibrium finite element methods for two-dimensional elasticity problems. The stresses and the displacements are approximated by using piecewise linear functions. We establish L_2-estimates of order $O(h^2)$ for both stresses and displacements.

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1. Introduction

The linear elasticity problem in two dimensions (plane stress or plane strain) can be formulated as follows: Given $f=(f_1,f_2)$ find a symmetric stress tensor $\sigma=\{\sigma_{ij}\}$, $i,j=1,2$, and a displacement $u=(u_1,u_2)$ such that

(1.1a) $\varepsilon(u)=\lambda\,\mathrm{tr}(\sigma)\,\delta+\mu\,\sigma \quad$ in Ω,

(1.1b) $\operatorname{div}\sigma+f=0 \quad$ in Ω,

(1.1c) $u=0 \quad$ on Γ,

where

$$\varepsilon(u)=\{\varepsilon_{ij}(u)\}, \qquad \varepsilon_{ij}=\tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \qquad i,j=1,2,$$

is the deformation tensor,

$$\mathrm{tr}(\sigma)=(\sigma_{11}+\sigma_{22}),$$

$$\operatorname{div}\sigma=\left\{\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2},\ \frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}\right\},$$

$$\delta\ =\{\delta_{ij}\}, \qquad \delta_{ij}=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases} \qquad i,j=1,2,$$

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* *Present address:* Centre de Mathématiques Appliquées, Ecole Polytechnique, Route de Saclay, F-91120 Palaiseau, France

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λ and μ are constants satisfying $\mu>0$ and $2\lambda+\mu>0$, and Ω is a bounded region in the plane with boundary Γ.

For simplicity we have restricted ourselves to the case of an isotropic elastic body fixed along its boundary (cf. Remark 3 below). Introducing the spaces

$$\begin{aligned} V &= V(\Omega)=[L^2(\Omega)]^2,\\ Y &= Y(\Omega)=\{\tau=\{\tau_{ij}\}:\ \tau_{ij}\in L^2(\Omega),\ \tau_{ij}=\tau_{ji},\ i,j=1,2\},\\ H &= H(\operatorname{div};\Omega)=\{\tau\in Y(\Omega):\ \operatorname{div}\tau\in V(\Omega)\},\end{aligned}$$

the elasticity problem can be given the following variational (weak) formulation: Find $(\sigma,u)\in H\times V$ such that

(1.2a) $a(\sigma,\tau)+(u,\operatorname{div}\tau)=0, \qquad \tau\in H,$

(1.2b) $(\operatorname{div}\sigma,v)+(f,v)=0, \qquad v\in V,$

where $(\cdot,\cdot)$ denotes the scalar product in V and

$$a(\sigma,\tau)=\int(\lambda\operatorname{tr}(\sigma)\operatorname{tr}(\tau)+\mu\,\sigma\cdot\tau)\,dx,$$

where

$$\sigma\cdot\tau=\sum_{j=1}^{2}\sigma_{ij}\cdot\tau_{ij}.$$

Using Korn's inequality and the fact that since $\mu>0$ and $2\lambda+\mu>0$,

(1.3) $\|\tau\|_0^2\leqq C a(\tau,\tau), \qquad \tau\in H,$

where $\|\tau\|_0$ is the $L^2(\Omega)$-norm, it is easy to see that if $f\in V$ then there exist a unique pair $(\sigma,u)\in H\times V$ satisfying (1.2) (see e.g. Duvaut-Lions [6]).

The purpose of this note is to develop some finite element methods for the elasticity problem based on the above variational formulation, i.e., methods of the form: Find $(\sigma_h,u_h)\in H_h\times V_h$ such that

(1.4a) $a(\sigma_h,\tau)+(u_h,\operatorname{div}\tau)=0, \qquad \tau\in H_h,$

(1.4b) $(\operatorname{div}\sigma_h,v)+(f,v)=0, \qquad v\in V_h,$

where H_h and V_h are finite dimensional subspaces of H and V, respectively.

In order to be able to derive optimal error estimates, quite specific requirements have to be put on the spaces H_h and V_h. We shall consider so called *equilibrium methods*, i.e. the spaces H_h and V_h will be chosen so that the following condition is satisfied:

(1.5) If $\tau\in H_h$ and $(\operatorname{div}\tau,v)=0$ for all $v\in V_h$, then $\operatorname{div}\tau=0$ in Ω.

In addition, it will be necessary to be able to construct a suitable interpolation operator π_h: $H\to H_h$ satisfying

(1.6) $(\operatorname{div}\pi_h\tau,v)=(\operatorname{div}\tau,v), \qquad v\in V_h.$

We have only found two different choices of the spaces H_h and V_h using low degree polynomials which meet all these requirements. In both cases we use composite piecewise linear finite elements (macro elements) for the stresses, one triangular and one quadrilateral, together with piecewise linear discontinuous displacements. There seems to be a connection between these stress elements and the composite piecewise cubic plate bending elements of Hsieh-Clough-Tocher and Fraeijs de Veubeke-Sander (see [4]) via the Airy's stress function (see Sect. 3 below). The quadrilateral stress element has been first introduced by Fraeijs de Vebeuke (see [8]) for problems where $\operatorname{div}\sigma=0$.

Similar methods for plate bending problems have been introduced by Hellan and Herrmann and analyzed by Johnson [9] and Brezzi-Raviart [3]. Further, Raviart-Thomas [12] and Thomas [13] have developed equilibrium methods for Poisson's equation (for another method of this type, see Remark 5 below). Due to the symmetry of the stress tensor their methods do not seem to generalize to elasticity problems.

The main application of the proposed equilibrium finite element method is to plasticity problems. For elasticity problems one can as well use a conventional conforming displacement method, but this is not possible for plasticity problems (in the case of perfect plasticity) due to lack of smoothness of the displacements (see e.g. [10]). To simplify the exposition we have chosen here to treat the elasticity problem. The analysis of the proposed method when applied to plasticity problems is parallel to that of [2], where the Hellan-Herrmann equilibrium method for elasto-plastic plates was analyzed. Another possible application is to incompressible materials and Stokes problem (see Remark 4 below). It would be interesting to see if these elements could be used also for the Navier-Stokes equation, the main difficulty in this case being how to handle the nonlinear term. Let us also mention that using a displacement method and an equilibrium method for the elasticity problem one can for particular loads f obtain a posteriori error bounds.

An outline of the paper is as follows: In Section 2 we define the finite element spaces H_h and V_h which will be used in the discrete problem (1.4). In Section 3 we prove unisolvence and the equilibrium property (1.5) for the triangular stress element. In Section 4 we carry out the same program for the quadrilateral stress element of Fraeijs de Veubeke. In Section 5 we derive error estimates for the finite element approximation (1.4) using by now standard techniques (see e.g. [1, 3, 9] and [13]); we obtain L_2-estimates of order $O(h^2)$ for both stresses and displacements.

We shall use the following notation: $\|\cdot\|_{s,T}$ and $|\cdot|_{s,T}$ will denote the natural norm and seminorm in $|H^s(T)|^\alpha$, where s and α are integers and T an open set in $\mathbb{R}^2$ (the subcript T may be omitted when $T=\Omega$).

By $\|\cdot\|_H$ we will denote the norm of H, i.e.,

$$\|\tau\|_H=(\|\tau\|_0^2+\|\operatorname{div}\tau\|_0^2)^{\frac{1}{2}}.$$

For k a positive integer, let $P_k(T)$ be the set of functions defined on T agreeing with a polynomial of degree less than or equal to k. Finally, by C we shall denote a positive constant, independent of the parameter h, not necessarily the same at each occurence.

2. The Finite Element Spaces H_h and V_h

For simplicity we shall assume that the domain Ω is polygonal. Let C_h be a family of triangulations of Ω (see [4]),

$$\Omega = \bigcup_{K \in C_h} K,$$

indexed by a parameter h representing the maximum diameter of the elements K. We shall consider both the case of triangular elements (Sect. 3) and convex quadrilateral elements (Sect. 4). Each element $K \in C_h$ will itself be divided into ν subtriangles T_i ($\nu = 3$ for the composite triangle and $\nu = 4$ for the composite quadrilateral, see Fig. 1 and 2). We shall assume that C_h is regular in the following sense: All angles of the elements $K \in C_h$ are bounded away from zero and π uniformly in h and there is a positive constant α such that the length of any side of any $K \in C_h$ is at least αh.

For each $K \in C_h$ we shall define below a finite dimensional space $H_K \subset H(\mathrm{div}; K)$ of piecewise linear stress tensors and we shall then define

$$H_h = \{\tau \in \tilde{H}_h: \ \mathrm{div}\,\tau \in V\},$$

where

$$\tilde{H}_h = \{\tau \in Y: \ \tau|_K \in H_K, \ K \in C_h\}.$$

Further we introduce, in both cases, the displacement space

$$V_h = \{v: \ v|_K \in [P_1(K)]^2, \ K \in C_h\}.$$

To choose the degrees of freedom for the stress space H_h in a suitable way, we note:

Lemma 1. Let $\tau \in \tilde{H}_h$. Then $\tau \in H_h$ (i.e. $\mathrm{div}\,\tau \in V$) if and only if $\tau \cdot n$ is continuous at interelement boundaries, i.e. for any side S common to two triangles (quadrilaterals) K and $K' \in C_h$ the following relation holds

$$\tau|_K \cdot n = \tau|_{K'} \cdot n \qquad \text{on } S,$$

where $n = (n_1, n_2)$ is a normal to S and

$$\tau \cdot n = (\tau_{11} n_1 + \tau_{12} n_2, \tau_{21} n_1 + \tau_{22} n_2).$$

Proof. This follows easily (see e.g. [13]) by using, on each element K, the following Green's formula:

$$(3.1) \quad \int_T \tau \cdot \varepsilon(v)\, dx = \int_{\partial T} v \cdot (\tau \cdot n)\, ds - \int_T v \cdot \mathrm{div}\,\tau\, dx,$$

where n denotes the outward unit normal to ∂T, the boundary of T.

3. The Triangular Composite Equilibrium Element

In the case of the triangular element, the three subtriangles T_i are simply obtained by connecting an interior point of a triangle K to the three vertices (see Fig. 1).

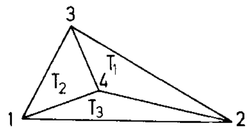

Fig. 1. Composite triangle

For each $K \in C_h$ the space H_K is defined as follows:

$H_K = \{\tau \in \tilde{H}_K \colon \tau \cdot n$ is continuous across the subtriangle boundaries 1-4, 2-4 and 3-4$\}$,

where

$$\tilde{H}_K = \{\tau \in Y(K) \colon \tau|_{T_i} \in [P_1(T_i)]^4, \; i=1,2,3\}.$$

We have

Proposition 1. An element $\tau \in H_K$ is uniquely determined by the following 15 degrees of freedom:

(i) the values of $\tau \cdot n$ at two points of each side of K,

(ii) $\int_K \tau_{ij}\, dx, \; i,j=1,2.$

Proof. Since $\dim \tilde{H}_K = 27$ and for an element $\tau \in \tilde{H}_K$ at most 12 conditions are required to guarantee the continuity of $\tau \cdot n$ across the subtriangle boundaries 1-4, 2-4 and 3-4, it follows that $\dim H_K$ is at least 15. Therefore it is sufficient to prove that if all degrees of freedom corresponding to an element $\tau \in H_K$ are zero, then $\tau \equiv 0$. This will follow from the following lemma, which will also be used to establish the equilibrium property (1.5).

Lemma 2. If $\tau \in H_K$ satisfies

$$(3.2) \quad \int_K v \cdot \operatorname{div} \tau \, dx = 0, \qquad \text{for } v \in [P_1(K)]^2,$$

then $\operatorname{div} \tau = 0$ on K.

Proof. If $\tau \in H_K$ and $v \in [P_1(K)]^2$, then

$$\int_K v \cdot \operatorname{div} \tau \, dx = \sum_{i=1}^{3} \int_{T_i} v \cdot \operatorname{div} \tau \, dx = \sum_{i=1}^{3} v(G_i)(\operatorname{div} \tau)^i \operatorname{area}(T_i),$$

where G_i denotes the gravity center of the triangle T_i and $(\operatorname{div} \tau)^i$ the (constant) value of $\operatorname{div} \tau$ on T_i. Choosing here $v=(1,0)$ (and $v=(0,1)$) at G_i and $v=0$ at G_j, $j \neq i$, and using (3.2), we see that $(\operatorname{div})^i = 0$, $i=1,2,3$, which proves the lemma.

End of Proof of Proposition 1. Assume that all the degrees of freedom for the function $\tau \in H_K$ are zero. By Green's formula, we then have for $v \in [P_1(K)]^2$,

$$\int_K v \operatorname{div} \tau = \int_{\partial K} v \cdot (\tau \cdot n)\, ds - \int_K \tau \cdot \varepsilon(v)\, dx = 0,$$

since $\tau \cdot n \equiv 0$ on ∂K, $\int_K \tau\, dx = 0$ and $\varepsilon(v)$ is constant on K.

Thus, by Lemma 2 it follows that $\operatorname{div} \tau = 0$ on K. But then (see e.g. [10]) there exists an Airy's function $\psi \in C^1(K)$ with $\psi|_{T_i} \in P_3(T_i)$ such that

$$\tau_{11} = \frac{\partial^2 \psi}{\partial x_2}, \quad \tau_{12} = \tau_{21} = -\frac{\partial^2 \psi}{\partial x_1\, \partial x_2}, \quad \tau_{22} = \frac{\partial^2 \psi}{\partial x_1^2}. \tag{3.3}$$

Since ψ is only determined up to a linear function, we may assume that ψ vanishes at the three vertices of K. The fact that $\tau \cdot n = 0$ on ∂K translates via (3.3) into

$$\frac{\partial}{\partial s}\left(\frac{\partial \psi}{\partial x_i}\right) = 0 \quad \text{on } \partial K, \quad i = 1, 2,$$

where $\frac{\partial}{\partial s}$ denotes the derivative in the direction tangential to ∂K. Hence $\frac{\partial \psi}{\partial x_1}$ and $\frac{\partial \psi}{\partial x_2}$ are both constant on ∂K. But then $\frac{\partial \psi}{\partial s}$ is constant on each side of K and since ψ vanishes at the vertices of K, we conclude that in fact $\frac{\partial \psi}{\partial s} = 0$ on ∂K. This implies that $\frac{\partial \psi}{\partial x_1}$ and $\frac{\partial \psi}{\partial x_2}$ vanishes on ∂K and therefore also $\frac{\partial \psi}{\partial n}$ vanishes on ∂K, where $\frac{\partial}{\partial n}$ denotes the outward normal derivative to ∂K. Finally it follows that ψ and thus also τ vanishes on K by using the unisolvence of the Hsieh-Clough-Tocher plate element: If $\psi \in C^1(K)$ with $\psi|_{T_i} \in P_3(T_i)$ vanishes on ∂K together with $\frac{\partial \psi}{\partial n}$, then $\psi = 0$ on K.

4. The Quadrilateral Composite Equilibrium Element

In the case of the quadrilateral element the four subtriangles T_i are generated by the two diagonals of K (see Fig. 2). To define H_K we first introduce

$$R_K = \{\tau \in Y(K) \colon \tau \in [P_1(T)]^4,\ \tau \cdot n = 0 \text{ on the diagonal 1-3 and } \tau = 0 \text{ on } K \setminus T\},$$

where n is a normal to the diagonal 1-3 and $T = T_1 \cup T_2$. In a similar way we define

$$R'_K = \{\tau \in Y(K) \colon \tau \in [P_1(T')]^4,\ \tau \cdot n' = 0 \text{ on the diagonal 2-4 and } \tau = 0 \text{ on } K \setminus T'\},$$

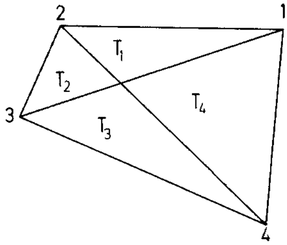

Fig. 2. Composite quadrilateral

where $T' = T_1 \cup T_4$. We note that $\dim R_K = \dim R'_K = 5$. Setting

$$S_K = Y(K) \cap [P_1(K)]^4,$$

we can now define

$$H_K = S_K \oplus R_K \oplus R'_K.$$

Corresponding to Lemma 2 and Proposition 1, we have the following two results:

Lemma 3. If $\tau \in H_K$ satisfies

$$(4.1) \quad \int_K v \cdot \operatorname{div} \tau \, dx = 0 \qquad \text{for } v \in [P_1(K)]^2,$$

then $\operatorname{div} \tau = 0$ on K.

Proof. Since $\tau \in H_K$ has the form $\tau = \tau^1 + \tau^2 + \tau^3$, where $\tau^1 \in S_K$, $\tau^2 \in R_K$ and $\tau^3 \in R_K$, we have by (4.1)

$$0 = \int_K v \cdot \operatorname{div} \tau \, dx = \operatorname{div} \tau^1 \int_K v \, dx + \operatorname{div} \tau^2 \int_T v \, dx + \operatorname{div} \tau^3 \int_{T'} v \, dx,$$

for $v \in [P_1(K)]^2$.

Introducing the gravity centers H, G and G' of K, T and T', respectively, this relation can be written

$$\operatorname{div} \tau^1 \, v(H) \operatorname{area}(K) + \operatorname{div} \tau^2 \, v(G) \operatorname{area}(T) + \operatorname{div} \tau^3 \, v(G') \operatorname{area}(T) = 0,$$

for $v \in [P_1(K)]^2$.

Since the quadrilateral K is convex and non degenerated, the points H, G and G' are not colinear and therefore we may again choose $v = (1, 0)$ (and $v = (0, 1)$) at one of these three points and $v = 0$ at the others. This proves that $\operatorname{div} \tau^i = 0$, $i = 1, 2, 3$, and thus $\operatorname{div} \tau = 0$.

Proposition 2. An element $\tau \in H_K$ is uniquely determined by the following 19 degrees of freedom:

(i) the values of $\tau \cdot n$ at two points at each side of K,

(ii) $\int_K \tau_{ij} \, dx$, $i, j = 1, 2$.

Proof. Since obviously $\dim H_K=19$, it is sufficient to prove that if all degrees of freedom for an element $\tau\in H_K$ are zero, then $\tau=0$. But this follows by an argument similar to that used in the proof of Proposition 2. In this case we use the unisolvence of the Fraeijs de Veubeke-Sander plate element established in [5] (see also [4, Exercise 6.1-5]).

Remark. It is also possible to define a cubic triangular equilibrium element by choosing

$$H_K=\{\tau\in Y(K)\colon \tau\in[P_3(K)]^4\} \quad \text{and} \quad V_h=\{v\colon v|_K\in[P_2(K)]^2,\ K\in C_h\}.$$

The degrees of freedom for such a stress element can be chosen as follows:

(i) the value of τ at each vertex,

(ii) the value of $\tau\cdot n$ at two points at each side,

(iii) $\int_K x_1^{\alpha_1}x_2^{\alpha_2}\tau_{ij}\,dx$, $\alpha_1+\alpha_2\leqq 1$, $i,j=1,2$.

However, we have not been able to derive any error estimates for this element because with degrees of freedom of type (i), we have not been able to construct an interpolation operator satisfying (1.5).

5. Error Estimates

We recall the definition of H_h and V_h and the finite element problem (1.4). It is clear that the degrees of freedom for an element $\tau\in H_h$ can be chosen as follows:

(i) the value of $\tau\cdot n$ at two points at each side S of the triangulation C_h.

(ii) *the value of* $\int_K \tau_{ij}\,dx$, $i,j=1,2$, for each $K\in C_h$.

Introducing the sets

$$E(f)=\{\tau\in H\colon (\operatorname{div}\tau,v)+(f,v)=0,\ \forall v\in V\},$$

$$E_h(f)=\{\tau\in H_h\colon (\operatorname{div}\tau,v)+(f,v)=0,\ \forall v\in V_h\},$$

we note that the equilibrium property (1.5) can be written

(5.1) $\quad E_h(0)\subset E(0)$.

Using property, we easily obtain the following estimate:

Theorem 1. There is a constant C depending on λ and μ such that

$$\|\sigma-\sigma_h\|_0\leqq C\inf_{\tau_h\in E_h(f)}\|\sigma-\tau_h\|_0.$$

Proof. Let $\tau_h\in E_h(f)$. Recalling (1.3) and using (1.2a) and (1.4a) with $\tau=\sigma_h-\tau_h$, we see that

$$\begin{aligned}\frac{1}{C}\|\sigma_h-\tau_h\|_0^2&\leqq a(\sigma_h-\tau_h,\sigma_h-\tau_h)\\&=a(\sigma-\tau_h,\sigma_h-\tau_h)+a(\sigma_h-\sigma,\sigma_h-\tau_h)\\&=a(\sigma-\tau_h,\sigma_h-\tau_h)+(u-u_h,\operatorname{div}(\sigma_h-\tau_h)).\end{aligned}$$

But since $\sigma_h - \tau_h \in E_h(0)$, we see using (5.1) that the last term vanishes and therefore we obtain by Cauchy's inequality

$$\|\sigma_h - \tau_h\|_0 \leqq C \|\sigma - \tau_h\|_0, \quad \tau_h \in E_h(f).$$

Thus,

$$\|\sigma - \sigma_h\|_0 \leqq \|\sigma - \tau_h\|_0 + \|\sigma_h - \tau_h\|_0 \leqq C \|\sigma - \tau_h\|_0,$$

which proves the result.

An estimate for $u - u_h$ will follow if we can prove that the following condition is satisfied (see Brezzi [1]): There is a constant β independent of h such that

$$(5.2)\quad \sup_{\tau \in H_h} \frac{(v, \operatorname{div} \tau)}{\|\tau\|_H} \geqq \beta \|v\|_0.$$

we have:

Theorem 2. If (5.2) holds, then

$$\|u - u_h\|_0 \leqq C(\|\sigma - \sigma_h\|_0 + \inf_{v \in V_h} \|u - v\|_0).$$

Proof. By (1.2a) and (1.4a), we have for any $v \in V_h$ and $\tau \in H_h$,

$$(u_h - v, \operatorname{div} \tau) = (u_h - u, \operatorname{div} \tau) + (u - v, \operatorname{div} \tau) = a(\sigma - \sigma_h, \tau) + (u - v, \operatorname{div} \tau),$$

so that by (5.2),

$$\beta \|u_h - v\|_0 \leqq \sup_{\tau \in H_h} \frac{(u_h - v, \operatorname{div} \tau)}{\|\tau\|_H} \leqq C \|\sigma - \sigma_h\|_0 + \|u - v\|_0,$$

for any $v \in v_h$. Therefore

$$\|u - u_h\|_0 \leqq \|u - v_h\|_0 + \|u_h - v\|_0 \leqq C(\|\sigma - \sigma_h\|_0 + \|u - v\|_0),$$

which completes the proof.

To evaluate the infimum in Theorem 1 and prove that (5.2) holds, we shall use the interpolation operator π_h defined as follows: For $\sigma \in [H^1(\Omega)]^4$ we define $\pi_h \sigma$ to be the element of H_h satisfying

$$(5.3)\quad \int_S v \cdot ((\sigma - \pi_h \sigma) \cdot n)\, ds = 0, \qquad v \in [P_1(S)]^2,$$

for any side S of C_h, n being the normal of S, and

$$(5.4)\quad \int_K (\sigma - \pi_h \sigma)\, dx = 0,$$

for $K \in C_h$. By Propositions 1 and 2 it follows that $\pi_h \sigma$ is uniquely determined by these requirements. Let us note that by Green's formula, (5.3) and (5.4), we have

$$(5.5)\quad (v, \operatorname{div} \sigma) = (v, \operatorname{div} \pi_h \sigma), \qquad v \in V_h,$$

i.e., if $\sigma \in E(f) \cap [H^1(\Omega)]^4$, then $\pi_h \sigma \in E_h(f)$.

In the proofs of the following two lemmas we shall restrict ourselves to the case of the composite triangular element of Section 3 and for simplicity we shall then assume that the internal point is the gravity center (cf. Remark 1 below). The same results hold in the case of the quadrilateral element with analogous proofs, see Remark 2 below for modifications needed in the case.

Lemma 4. There is a constant C, independent of K, such that

(5.6) $\|\pi_h \sigma\|_{0,K} \leqq C\|\sigma\|_{1,K}$,

(5.7) $\|\sigma - \pi_h \sigma\|_{0,K} \leqq C h_K^2 |\sigma|_{2,K}$,

where h_K denotes the diameter of K.

Proof. As reference element $\hat{K}$ we choose the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ and the internal point $(\frac{1}{3}, \frac{1}{3})$ (the center of gravity): We introduce the affine one-to-one function F mapping $\hat{K}$ onto K given by $F(\hat{x}) = B\hat{x} + b$, where B is a 2×2 matrix and $b \in \mathbb{R}^2$. We note that the gravity center of $\hat{K}$ is mapped onto the gravity center of K. Given the stress tensor τ defined on K we define the stress tensor $\hat{\tau}$ defined on $\hat{K}$ by

(5.8) $\hat{\tau}(\hat{x}) = B^{-1}\tau(F(\hat{x}))B^{-T}, \quad \hat{x} \in \hat{K}$,

where T denotes the transpose and $B^{-T} = (B^{-1})^T$.

Further, let $\hat{\pi}$: $H^1(\hat{K})^4 \to H_{\hat{K}}$ be the interpolation operator defined in the same way as π_h, i.e.,

(5.9) $\int_{\hat{S}} v \cdot ((\hat{\tau} - \hat{\pi}\hat{\tau}) \cdot \hat{n})\, d\hat{s} = 0, \quad v \in [P_1(\hat{S})]^2$,

(5.10) $\int_{\hat{K}} (\hat{\tau} - \hat{\pi}\hat{\tau})\, d\hat{x} = 0$,

for any side $\hat{S}$ of $\hat{K}$ with normal $\hat{n}$.

With the correspondence (5.8), we have $\widehat{\pi_h \tau} = \hat{\pi}\hat{\tau}$, i.e. (5.9) and (5.10) hold with $\hat{\pi}\hat{\tau}$ replaced by $\widehat{\pi_h \tau}$. This follows easily by changing coordinates in (5.3) and (5.4). In fact, (5.10) follows directly and if $\hat{n}$ is a normal to the side $\hat{S}$ of $\hat{K}$, then $n = B^{-T}\hat{n}$ is a normal to the corresponding side $S = F(\hat{S})$ of K so that for any $v \in [P_1(S)]^2$ using the notation $\hat{v}(\hat{x}) = v(F(\hat{x}))$ and $\delta =$ length of S,

$$0 = \int_S n^T(\tau - \pi_h \tau) v\, ds = \delta \int_{\hat{S}} (B^{-T}\hat{n})^T B(\hat{\tau} - \widehat{\pi_h \tau}) B^T \hat{v}\, d\hat{s}$$
$$= \delta \int_S \hat{n}^T(\hat{\tau} - \widehat{\pi_h \tau}) B^T \hat{v}\, d\hat{s},$$

which proves (5.9).

Using the trace theorem

$$\|w\|_{0,\partial\hat{K}} \leqq C\|w\|_{1,\hat{K}}, \quad w \in H^1(\hat{K}),$$

we see that

(5.11) $\|\hat{\pi}\hat{\tau}\|_{0,\hat{K}} \leqq C\|\hat{\tau}\|_{1,\hat{K}}$, so that in particular

$$\|\hat{\tau}-\hat{\pi}\hat{\tau}\|_{0,\hat{K}} \leqq C\|\hat{\tau}\|_{2,\hat{K}}.$$

Since $\hat{\tau}=\hat{\pi}\hat{\tau}$ if $\hat{\tau}\in[P_1(\hat{K})]^4 \cap Y(\hat{K})$, it follows (see e.g. [4], Theorem 3.1-1) that

$$\|\hat{\tau}-\hat{\pi}\hat{\tau}\|_{0,\hat{K}} \leqq C|\tau|_{2,\hat{K}}. \tag{5.12}$$

We now obtain the estimates (5.6) and (5.7) by changing coordinates in (5.11) and (5.12) using the correspondance (5.8) and the fact that for a regular family of triangulations

$$\|B\| \leqq Ch, \qquad \|B^{-1}\| \leqq Ch^{-1},$$

where $\|B\|=\sup\{|Bx|\colon\ |x|=1\}$ (cf. [4, Chapter 3]).

Lemma 5. There is a constant C such that

$$\|\operatorname{div}\pi_h\tau\| \leqq C\|\operatorname{div}\tau\|, \qquad \tau\in H\cap[H^1(\Omega)]^4.$$

Proof. By (6.3) we have for any $v\in V_h$,

$$\begin{aligned}\int_\Omega v\cdot\operatorname{div}\tau\,dx &= \int_\Omega v\cdot\operatorname{div}\pi_h\tau = \sum_K \int_K v\cdot\operatorname{div}\pi_h\tau\,dx \\ &= \sum_{K\in C_h} \frac{\operatorname{area}(K)}{3}\sum_{i=1}^{3} v(G_i)(\operatorname{div}\pi_h\tau)^i,\end{aligned} \tag{5.13}$$

where for each K, the G_i are the gravity centers of the triangles T_i and $(\cdot)^i$ denotes the value on T_i. Choosing now v on each K so that $v(G_i)=(\operatorname{div}\pi_h\tau)^i$ we obtain from (5.13) using Cauchy's inequality

$$\|\operatorname{div}\pi_h\tau\|^2 = \sum_{K\in C_h}\frac{\operatorname{area}(K)}{3}\sum_{i=1}^{3}[(\operatorname{div}\pi_h\tau)^i]^2 \leqq \|v\|_0\|\operatorname{div}\|_0,$$

where the first equality follows from the fact that $\operatorname{div}\pi_h\tau$ is constant on each T_i. The lemma will now follow if we prove that

$$\|v\|_0 \leqq C\left(\sum_{K\in C_h}\frac{\operatorname{area}(K)}{3}\sum_{i=1}^{3}|v(G_i)|^2\right)^{\frac{1}{2}}, \qquad v\in V_h.$$

But this follows changing coordinates on each K from the fact that

$$\left(\frac{1}{3}\sum_{i=1}^{3}|v(\hat{G}_i)|^2\right)^{\frac{1}{2}},$$

is a norm equivalent to the L_2-norm on $[P_1(\hat{K})]^2$, where the $\hat{G}_i$ denote the gravity centers of the subtriangles of $\hat{K}$ (note that $G_i=F(\hat{G}_i)$).

We can now prove:

Lemma 6. With our choices of the spaces H_h and V_h, condition (5.2) holds.

Proof. Given $v_h\in V_h$ let ψ be the unique solution of the elastic problem

$$\operatorname{div}(\varepsilon(\psi)) = v_h \quad \text{in } \tilde{\Omega},$$
$$\psi = 0 \quad \text{on } \tilde{\Gamma},$$

where $\tilde{\Omega}$ is a bounded domain with smooth boundary $\tilde{\Gamma}$ such that $\Omega \subset \tilde{\Omega}$ and v_h is extended by zero outside $\tilde{\Omega}$. Setting $\sigma_e = \varepsilon(\psi)$ and using elliptic regularity (see [11]), we have

$$(5.14) \quad \|\sigma_e\|_{1,\Omega} \leqq C \|v_h\|_{0,\Omega},$$

so that in particular, using Lemmas 4 and 5,

$$\|\pi_h \sigma_e\|_H \leqq C \|v_h\|_{0,\Omega}.$$

Together with (6.3), this gives

$$\|v_h\| = \frac{(v_h, \operatorname{div} \sigma_e)}{\|v_h\|} = \frac{(v_h, \operatorname{div} \pi_h \sigma_e)}{\|v_h\|} \leqq C \frac{(v_h, \operatorname{div} \pi_h \sigma_e)}{\|\pi_h \sigma_e\|_H},$$

which proves (5.2).

Remark. The idea of using the interpolation operator π_h to prove Brezzi's condition has been used by Fortin [7] (see also [9]).

Combining Theorems 1 and 2 and Lemma 4 recalling the fact that $\pi_h \sigma \in E_h(f)$, we now finally get the following error estimate:

Theorem 3. There is a constant C independent of h such that

$$(5.15) \quad \|\sigma - \sigma_h\|_0 \leqq C h^2 |\sigma|_2,$$

$$(5.16) \quad \|u - u_h\|_0 \leqq C h^2 (|\sigma|_2 + |u|_2).$$

Remark 1. If the internal points P of the composite triangular elements K are no longer the gravity centers, Lemma 4 and thus also Theorem 3 still hold as long as $P \in F(\hat{K}_1)$, where $\hat{K}_1$ is a compact set included in the interior of $\hat{K}$ (see [4, Theorem 6.1–3]).

Remark 2. In the case of the quadrilateral elements, the reference element is no more fixed, but depends on two parameters a and b (see Fig. 3).

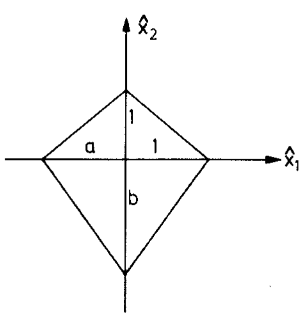

Fig. 3. The reference elements in the case of quadrilateral elements

As long as a and b (which depend on K) belong to some fixed interval α_1, α_2 with $\alpha_1, \alpha_2 > 0$, Lemma 4 and thus also Theorem 3 still hold (cf. [5]).

Remark 3. The case of mixed boundary conditions,

$$u=0 \quad \text{on } \Gamma_1,$$
$$\sigma \cdot n = 0 \quad \text{on } \Gamma_2,$$

where $\Gamma_2 \cup \Gamma_1 = \Gamma$, $\Gamma_1 \cup \Gamma_2 = \emptyset$, can be analyzed in essentially the same way as above with one important exception: With mixed boundary conditions the regularity result (5.14) may not hold and therefore the proof of Lemma 6 has to be modified in this case. In [13] one can find in a situation similar to ours a proof of (5.2) without using this regularity result.

Remark 4. Let us briefly discuss the modifications needed in the case of an incompressible material corresponding to taking $2\lambda + \mu = 0$ in (1.1a): In this case the estimate (5.15) is replaced by

$$\|\sigma^D - \sigma_h^D\|_0 < Ch^2 |\sigma|_2,$$

where $\tau^D = \tau - \frac{1}{2}\operatorname{tr}(\tau)\delta$ is the stress deviatoric. For the pressure $\operatorname{tr}(\sigma - \sigma_h)$ we obtain the estimate

$$\|\operatorname{tr}(\sigma - \sigma_h)\|_{L^2|\mathbb{R}} \leqq Ch,$$

and for the displacements we have the same estimate as before. We note that the functions v in V_h do not have to satisfy the incompressibility condition $\operatorname{div} v = 0$. Since Stokes equations in the stationary case are of the form (1.1) with $2\lambda + \mu = 0$, our method could be applied also to these equations.

Remark 5. The Dirichlet problem

$$(1.17) \quad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

can be given the following variational formulation:

Find $(p, u) \in H(\operatorname{div}; \Omega) \times L^2(\Omega)$ such that

$$(5.18) \quad \begin{aligned} (p, q) + (u, \operatorname{div} q) &= 0, \quad q \in H(\operatorname{div}; \Omega), \\ (\operatorname{div} p, v) + (f, v) &= 0, \quad v \in L^2(\Omega), \end{aligned}$$

where

$$H(\operatorname{div}; \Omega) = \{q \in [L^2(\Omega)]^2 : \operatorname{div} q \in L^2(\Omega)\};$$

if u is the solution of (5.17), then $(\nabla u, u)$ is the solution of (5.18). We want to mention an equilibrium finite element method for (5.17) based on the formulation (5.18) which is not contained in the family of such methods considered by Raviart-Thomas [12] and Thomas [13]. In this method the finite dimensional spaces $H_h \subset H(\operatorname{div}; \Omega)$ and $V_h \subset L^2(\Omega)$ are chosen as follows:

$$H_h = \{q \in H(\mathrm{div}; \Omega):\ q|_K \in [P_1(K)]^2,\ K \in C_h\},$$
$$V_h = \{v \in L^2(\Omega):\ v|_K \in P_0(K),\ K \in C_h\},$$

where C_h is a triangulation of the two-dimensional domain Ω. As degrees of freedom for H_h one can choose the value of $q \cdot n$ at two points at each side S of C_h, where n denotes a unit normal to S. The properties and analysis of this method is analogous to what has been presented above in the case of the elasticity problem above.

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