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A Family of Mixed Finite Elements for the Elasticity Problem

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Summary. A new mixed finite element formulation for the equations of linear elasticity is considered. In the formulation the variables approximated are the displacement, the unsymmetric stress tensor and the rotation. The rotation act as a Lagrange multiplier introduced in order to enforce the symmetry of the stress tensor. Based on this formulation a new family of both two- and three-dimensional mixed methods is defined. Optimal error estimates, which are valid uniformly with respect to the Poisson ratio, are derived. Finally, a new postprocessing scheme for improving the displacement is introduced and analyzed.

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1. Introduction

The purpose of this paper is to continue our discussion [22, 24] on mixed finite element methods in elasticity. By mixed methods we here mean methods where both the displacement vector and the stress tensor are approximated simultaneously using the Hellinger-Reissner principle. This approach has a number of advantages compared with traditional displacement methods. First, it allows a direct approximation of the stresses which usually are the variables of primary interest. Second, a correctly designed mixed method (i.e. satisfying the equilibrium condition, see below) gives a good modelling of incompressible and nearly incompressible materials for which standard displacement methods break down. Third, the approach has been shown [4, 16] to be efficient for plasticity problems, for which the elimination of the stresses from the equilibrium and constitutive equations is difficult.

It has, however, turned out to be difficult to find good mixed methods. A direct approach of using e.g. continuous piecewise polynomials for both the stresses and the displacement does not usually work. For this there is two different reasons. The main difficulty is that usually these methods are unstable. In practical calculations this is seen in the appearance of displacement modes which are nearly unphysical "mechanisms" (often also referred to as "zero energy modes"). In fact, mechanisms are only avoided if displacement boundary condition are applied on a part of the boundary. In addition, even if one uses higher order polynomials for the stresses than for the displacement the desired goal of obtaining a good stress approximation is not achieved. The reason for this is that the equations couple the stresses and displacement and the error in the displacement will also influence the error in the stresses. As a consequence, the results are not substantially better than what one would obtain using a displacement method with the same displacement approximations as those used in the mixed method. Let us also point out that the good results reported [19, 27] with some methods of this type are apparently due to the rectangular meshes used. For rectangular meshes it is known that certain "superapproximation" phenomenon will occur, cf. [18, 22].

The difficulties sketched above can, however, be avoided by using methods of the so called "equilibrium type" [2, 4, 16]. For the sake of improving the stability, in these methods usually only the normal component of the stress tensor is assumed to be continuous along the interelement boundaries. In addition a discontinuous approximation is used for the displacement. By this it is possible to decouple the error in the stresses from the displacement error.

These equilibrium methods have however some serious shortcomings. First, the equilibrium methods usually imply more degrees of freedom per node than a displacement method with the same order of accuracy (for the stress tensor). Secondly, the elements are rather complicated to construct; cf. [2, 16].

In our papers we have tried to overcome these difficulties. In [22] we showed that it is possible to impose stronger continuity requirements on the stress field without loss of accuracy, and that this leads to methods which, in fact, have less degrees of freedom than displacement methods with the same accuracy. In [24] we considered the second problem with mixed methods, i.e. the complicated construction of the elements. We introduced a general method of constructing stable methods with optimal convergence rates. Our construction was based on a systematic use of the "bubble function technique", which is a widely used method for stabilizing mixed methods [8, 13].

In two recent papers Brezzi et al. [11] and Arnold et al. [4] discussed another method (originally proposed by Fraijs de Veubeke [15]) for constructing mixed elements. The idea consists of using unsymmetric approximations for the stress tensor and imposing a weakened symmetry condition through the use of a Lagrange multiplier, which has the physical meaning as an approximation to the rotation. The approximation for the stress tensor is built upon mixed methods for scalar valued second order elliptic problems. Therefore the approach appears to be very promising since there exist excellent mixed methods for scalar elliptic equations in both two [10, 21] and three dimensions [20]. Let us also mention that the same idea has been applied in connection with dual hybrid equilibrium methods by Amara and Thomas [1].

In this paper we will elaborate this idea. In the papers [4] and [11] two methods of this type were introduced and analyzed. The approximations for the stress tensor were based on the lowest order Raviart – Thomas [21] and Brezzi-Douglas-Marini elements [10] whereas a continuous piecewise linear approximation was used for the rotation. The analysis of [4, 11] was, however, restricted to Dirichlet boundary condition and it seems that the generalization to more general boundary conditions is non-trivial. Below we will generalize and extend the analysis of mixed finite elements of this type. We show how the methods should be analyzed for general boundary conditions and we show that discontinuous approximations can be used for the approximation of the rotation. This means that the rotational Lagrange multiplier can be eliminated at the element level. Therefore this approach leads to mixed methods that very much resemble traditional equilibrium methods. The construction of these new elements is, however, considerably easier and more systematic than the construction of the equilibrium methods given in e.g. [2, 16]. We also note that we by this approach get methods for three-dimensional problems for which no useful stable equilibrium methods have been known.

Finally, we also introduce a new postprocessing method by which it is possible to obtain a considerably better approximation for the displacement.

2. Notation and Preliminaries

Let Ω be a bounded domain in $\mathbf{R}^N$, $N=2,3$. The linear elasticity problem is described by the following partial differential equations and boundary conditions:

$$\begin{aligned} A\sigma-\varepsilon(u)&=0 && \text{in } \Omega,\\ \operatorname{div}\sigma+f&=0 && \text{in } \Omega,\\ u&=0 && \text{on } \Gamma_1,\\ \sigma\cdot n&=g && \text{on } \Gamma_2, \end{aligned} \tag{2.1}$$

where $u\colon \Omega\to\mathbf{R}^N$ is the displacement, $\sigma\colon\Omega\to\mathbf{R}^N\times\mathbf{R}^N$ is the symmetric stress tensor and $\varepsilon(u)$ is the strain tensor given by

$$\varepsilon(u)_{ij}=\tfrac{1}{2}(\partial_i u_j+\partial_j u_i), \qquad i,j=1,\ldots,N.$$

f denotes the given body load and g is the surface traction on Γ_2, with $\Gamma_1\cap\Gamma_2=\emptyset$, $\bar\Gamma_1\cup\bar\Gamma_2=\partial\Omega$ and $\Gamma_i\neq\emptyset$, $i=1,2$. For simplicity we assume that Ω is a polygonal (polyhedral) domain in $\mathbf{R}^2(\mathbf{R}^3)$. Further, we define

$$(\sigma\cdot n)_i=\sum_{j=1}^N \sigma_{ij}n_j, \qquad i=1,\ldots,N,$$

$$(\operatorname{div}\sigma)_i=\sum_{j=1}^N \partial_j\sigma_{ij}, \qquad i=1,\ldots,N,$$

where n is the unit outward normal to $\partial\Omega$.

A is the fourth order compliance tensor (the inverse of the elasticity tensor). We will assume that the material is homogeneous and isotropic, so that A

depends on two physical constants, e.g. the Young modulus E and the Poisson ratio ν. Hence we have

$$A\sigma = \alpha\sigma + \beta \operatorname{tr}(\sigma)\delta,$$

with

$$\alpha = \frac{(1+\nu)}{E}, \qquad \beta = -\frac{\nu(1+\nu)}{E}$$

for the plane strain problem and

$$\alpha = \frac{(1+\nu)}{E}, \qquad \beta = -\frac{\nu}{E}$$

for the plane stress and three-dimensional problem. Above δ denotes the unit tensor and $\operatorname{tr}(\sigma)$ stands for the trace of σ.

Let us suppose that $f \in [L^2(\Omega)]^N$ and $g \in [L^2(\Gamma_2)]^N$. Then (2.1) has a unique solution satisfying (cf. [5])

$$\|u\|_1 + \|\sigma\|_0 \leqq C(\|f\|_0 + \|g\|_{0,\Gamma_2}).$$

Here and below $\|\cdot\|_{k,\Sigma}$ denotes the norms in the Sobolev spaces $[H^k(\Sigma)]^N$ and $[H^k(\Sigma)]^{N\times N}$, with the subscript dropped if $\Sigma = \Omega$; C stands for a generic positive constant independent of the Poisson ratio. We emphasize that this means that all estimates of the paper are uniformly valid independent of the Poisson ratio. The constants C are also independent of the mesh parameter h.

To obtain L^2-estimates for the displacement we use standard duality arguments for which the following regularity assumption is needed

$$\|u\|_2 + \|\sigma\|_1 \leqq C\|f\|_0, \tag{2.2}$$

for the solution of (2.1) with $g = 0$. We remark, however, that in practice (2.2) is usually not valid since the corner singularities are rather severe; cf. [17, 26].

The mixed methods we are going to discuss are based on a modification of (2.1) obtained by writing

$$\varepsilon(u) = \nabla u - \omega(u),$$

where

$$(\nabla u)_{ij} = \partial_j u_i, \qquad i, j = 1, \ldots, N,$$

and ω denotes the skew symmetric rotation tensor

$$\omega(u)_{ij} = \tfrac{1}{2}(\partial_j u_i - \partial_i u_j), \qquad i, j = 1, \ldots, N.$$

Furthermore, since the stress tensor is not longer a priori assumed to be symmetric, we have to introduce the symmetry condition

$$\sigma - \sigma^T = 0, \qquad i, j = 1, \ldots, N,$$

where σ^T denotes the transpose of σ. Upon introducing the skew symmetric tensor γ as a new unknown, the elasticity equations (2.1) can be stated as

$$\begin{aligned} A\sigma - \nabla u + \gamma &= 0 \quad \text{in } \Omega, \\ \sigma - \sigma^T &= 0 \quad \text{in } \Omega, \\ \operatorname{div} \sigma + f &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ \sigma \cdot n &= g \quad \text{on } \Gamma_2. \end{aligned} \tag{2.3}$$

The solution to the above problem clearly coincides with the solution of (2.1) with $\gamma = \omega(u)$.

The variational form of (2.2) reads as follows: Find $(\sigma, u, \gamma) \in H_g \times V \times W$ such that

$$\begin{aligned} &a(\sigma, \tau) + b(\tau; u, \gamma) = 0, \\ &b(\sigma; \eta, v) + (f, v) = 0, \qquad (\tau, v, \eta) \in H_0 \times V \times W, \end{aligned} \tag{2.4}$$

where

$$b(\sigma; u, \gamma) = (\operatorname{div} \sigma, u) + (\sigma, \gamma),$$

$$a(\sigma, \tau) = \sum_{i,j=1}^{N} \int_\Omega (A\sigma)_{ij} \tau_{ij} \, dx,$$

and

$$\begin{aligned} &V = [L^2(\Omega)]^N, \\ &H_t = \{\sigma \in [L^2(\Omega)]^{N \times N} \mid \operatorname{div} \sigma \in V, \sigma \cdot n = t \text{ on } \Gamma_2\}, \\ &W = \{\gamma \in [L^2(\Omega)]^{N \times N} \mid \gamma + \gamma^T = 0\}, \end{aligned}$$

where $t = 0$ or $t = g$. As usual $(\cdot, \cdot)$ stands for the inner product in $[L^2(\Omega)]^N$ or $[L^2(\Omega)]^{N \times N}$.

Remark. Contrary to common practice we will not assume homogeneous boundary conditions. This since some of our results would be weaker if this would be assumed, cf. the remarks after Theorems 4.1 and 4.2. □

In the finite element approximation of (2.3) we seek the approximate solution $(\sigma_h, u_h, \gamma_h) \in H_{g,h} \times V_h \times W_h$ such that

$$\begin{aligned} &a(\sigma_h, \tau) + b(\tau; u_h, \tau_h) = 0, \\ &b(\sigma_h; v, \eta) + (f, v) = 0, \qquad (\tau, v, \eta) \in H_{0,h} \times V_h \times W_h, \end{aligned} \tag{2.5}$$

where $H_{t,h}$ $(t = 0, g)$, V_h and W_h are some typical finite element spaces approximating H_t, V and W, respectively.

In the analysis we will first consider higher order methods since the analysis is in this case more transparent. Next, we will give the analysis of the PEERS element introduced in [4], and then we show how an optimal low order method can be constructed.

3. Higher Order Methods

In order to fix our ideas we will carry out the construction of the finite element spaces starting from the Brezzi-Douglas-Marini family of triangular and tetrahedral elements. (For the triangular case these elements were introduced and analyzed in [10]. The corresponding tetrahedral spaces are, however, also easily seen to be stable; cf. below). It will become clear that the same construction can also be done for other spaces such as e.g. the rectangular elements of [10] and the Raviart-Thomas-Nedelec spaces [20, 21].

The finite element spaces will be defined on a subdivision $\mathscr{C}_h$ of $\bar{\Omega}$ into closed triangles or tetrahedrons. The elements of $\mathscr{C}_h$ are assumed to be regular in the usual sense (cf. [12]), and any two disjoint elements are assumed to meet either in a common vertex or side (here and in the sequel an edge of a two-dimensional element will also be refered to as a side). Let $K \in \mathscr{C}_h$ and define the "bubble function" b_K through

$$b_K(x) = \prod_{i=1}^{N+1} \lambda_i(x), \qquad x \in K,$$

where $\lambda_1, \ldots, \lambda_{N+1}$ are the barycentric coordinates in K. Let $P_k(K)$ be the space of polynomials of degree k on K. For $K \in \mathscr{C}_h$, $l \geqq 0$, we define

$$B_l(K) = \{\{\sigma_{ij}\}, i, j = 1, \ldots, N \,|\, (\sigma_{i1}, \ldots, \sigma_{iN}) = \operatorname{curl}(b_K w),\ i = 1, \ldots, N\}, \tag{3.1}$$

where for $N = 3$

$$w \in [P_l(K)]^3, \qquad b_K w = (b_K w_1, \ldots, b_K w_N), \qquad \operatorname{curl} z = \nabla \times z,$$

and for $N = 2$

$$w \in P_l(K), \qquad \operatorname{curl} z = (\partial_2 z, -\partial_1 z).$$

The finite element spaces are now defined for $k \geqq 2$ through

$$V_h = \{v \in V \,|\, v_{|K} \in [P_{k-1}(K)]^N,\ K \in \mathscr{C}_h\}, \tag{3.2a}$$

$$W_h = \{\gamma \in W \,|\, \gamma_{|K} \in [P_k(K)]^{N \times N},\ K \in \mathscr{C}_h\}, \tag{3.2b}$$

$$\begin{aligned} H_{t,h} = \{&\sigma \in [L^2(\Omega)]^{N \times N} \,|\, \operatorname{div} \sigma \in V,\ \sigma_{|K} \in [P_k(K)]^{N \times N} \oplus B_{k-1}(K), \\ &K \in \mathscr{C}_h,\ \sigma \cdot n = \tilde{t}, \text{ on } \Gamma_2\}, \end{aligned} \tag{3.2c}$$

where for each element side $T \subset \Gamma_2$, $\tilde{t}_{|T}$ is the L^2-projection of t onto the space $[P_k(T)]^N$ (it is easily seen that the degrees of freedom for σ can be chosen so that we can assign $\sigma \cdot n = \tilde{t}$ *on* Γ_2).

We note that we have $\operatorname{div} \sigma \in V_h$ for each $\sigma \in H_{t,h}$ and hence the following "equilibrium condition"

$$(\operatorname{div} \sigma, u) = 0 \qquad \forall\, u \in V_h \Rightarrow \operatorname{div} \sigma = 0 \tag{3.3}$$

is satisfied.

Generally, if for two finite element spaces the above equilibrium condition is satisfied, then there exist a mapping $P_h\colon V \to V_h$ such that

$$(\operatorname{div}\sigma, u - P_h u) = 0, \qquad \sigma \in H_{t,h}, \qquad u \in V. \tag{3.4}$$

For the method (3.2) the mapping P_h coincides with the L^2-projection from V onto V_h and hence we have

$$\|u - P_h u\|_0 \leqq C h^k |u|_k. \tag{3.5}$$

We can now state the following optimal error estimates for the method:

Theorem 3.1. *Let* $H_{t,h} \times V_h \times W_h$ *be defined according to* (3.2). *Then there is a unique solution* $(\sigma_h, u_h, \gamma_h) \in H_{g,h} \times V_h \times W_h$ *to* (2.5) *such that*

$$\|\sigma - \sigma_h\|_0 \leqq C h^{k+1} (|\sigma|_{k+1} + |\gamma|_{k+1}) \tag{3.6}$$

and

$$\|u - u_h\|_0 \leqq C h^k (|\sigma|_k + |\gamma|_k + |u|_k). \tag{3.7}$$

Moreover, if the regularity estimate (2.2) *is valid we have*

$$\|u - u_h\|_0 \leqq C h^k (|\sigma|_{k-1} + |\gamma|_{k-1} + |u|_k) \tag{3.8}$$

and

$$\|u_h - P_h u\|_0 \leqq C h^{k+2} (|\sigma|_{k+1} + |\gamma|_{k+1}). \quad \square \tag{3.9}$$

Remark. The estimate (3.9) will be needed for the analysis of the postprocessing scheme, cf. Sect. 5 below. $\square$

The proof of Theorem 3.1 will be based on a number of lemmata to be proved below. In them we will use the following non-standard mesh dependent norms defined in [22] and [24]:

$$\|\sigma\|_{0,h}^2 = \|\sigma\|_0^2 + \sum_{T \in \Gamma_h,\, T \subset \Gamma} h_T \int_T |\sigma \cdot n|^2 \, ds, \qquad \sigma \in \mathscr{H}_h, \tag{3.10}$$

where

$$\mathscr{H}_h = \{\sigma \in [L^2(\Omega)]^{N \times N} \mid \sigma \cdot n_{|T} \in [L^2(T)]^N,\ T \in \Gamma_h,\ T \subset \Gamma\}, \tag{3.11}$$

$$\|u\|_{1,h}^2 = \sum_{K \in \mathscr{C}_h} \|\varepsilon(u)\|_{0,K}^2 + \sum_{T \in \Gamma_h} h_T^{-1} \int_T |[u]^2| \, ds + \sum_{T \subset \Gamma_1} h_T^{-1} \int_T |u^2| \, ds, \qquad u \in V_h, \tag{3.12}$$

and

$$\|(u, \gamma)\|_h^2 = \|u\|_{1,h}^2 + \sum_{K \in \mathscr{C}_h} \|\gamma - \omega(u)\|_{0,K}^2, \qquad u \in V_h,\ \gamma \in W_h. \tag{3.13}$$

Here T denotes a side of an element of $\mathscr{C}_h$ and h_T stands for the diameter of T. Γ_h denotes the collection of sides in the interior of Ω and $[u]$ denotes the value of the jump discontinuity in u at the interelement boundaries. We note that since $\Gamma_1 \neq \emptyset$, $\|\cdot\|_{1,h}$ and $\|\cdot\|_h$ are norms in V_h and $V_h \times W_h$, respectively. The norms have the following properties:

Lemma 3.1. *There exist positive constants C such that*

$$\|\sigma\|_0 \leqq \|\sigma\|_{0,h} \leqq C\|\sigma\|_0, \qquad \sigma \in H_{t,h},$$

$$\inf_{\tau\in H_{t,h}} \|\sigma-\tau\|_{0,h} \leqq C h^{k+1}|\sigma|_{k+1}, \qquad \sigma\in[H^{k+1}(\Omega)]^{N\times N}\cap H_t$$

and

$$b(\sigma; u, \gamma) \leqq \|\sigma\|_{0,h}\|(u,\gamma)\|_h, \quad \sigma\in\mathscr{H}_h\cap H_0, \quad u\in V_h, \quad \gamma\in W_h.$$

Proof. The two first estimates are proved using normal scaling arguments, cf. [6]. To prove the last estimate we first integrate by parts on each $K\in\mathscr{C}_h$ in order to obtain

$$(\operatorname{div}\sigma, u) = -\sum_{K\in\mathscr{C}_h}(\sigma, \nabla u)_K + \sum_{T\in\Gamma_h}\int_T(\sigma\cdot n)\cdot([u])\,ds + \sum_{T\subset\Gamma_1}\int_T(\sigma\cdot n)\cdot u\,ds.$$

Next, we write on each $K\in\mathscr{C}_h$

$$\nabla u = \varepsilon(u) + \omega(u),$$

which gives

$$\begin{aligned} b(\sigma; u, \gamma) = &-\sum_{K\in\mathscr{C}_h}(\sigma, \varepsilon(u))_K + \sum_{T\in\Gamma_h}\int_T(\sigma\cdot n)\cdot([u])\,ds + \sum_{T\subset\Gamma_1}\int_T(\sigma\cdot n)\cdot u\,ds \\ &+ \sum_{K\in\mathscr{C}_h}(\sigma, \gamma-\omega(u))_K. \end{aligned}$$

The estimate then follows upon applying the Schwarz inequality. □

Next, we recall the following

Lemma 3.2. *There is a constant C such that*

$$a(\sigma,\sigma) \geqq C\|\sigma\|_{0,h}^2, \qquad \sigma\in Z_h,$$

where

$$Z_h = \{\sigma\in H_{t,h} \mid b(\sigma; u, \gamma) = 0, \ u\in V_h, \ \gamma\in W_h\}.$$

Proof. The claim follows from the equilibrium condition (3.3), cf. [4, Lemma 4.3] and [5, Lemma 3.2]. □

The next lemma shows that the method is stable with respect to the mesh dependent norms.

Lemma 3.3. *The space $H_{0,h}\times V_h\times W_h$ satisfies the following inequality*

$$\sup_{0\neq\sigma\in H_{0,h}} \frac{b(\sigma; u, \gamma)}{\|\sigma\|_{0,h}} \geqq C\|(u,\gamma)\|_h, \qquad (u,\gamma)\in V_h\times W_h. \quad \square \tag{3.14}$$

Let us postpone the proof of the stability estimate and first use it for the

Proof of Theorem 3.1. Let $\tilde{\sigma}$ and $\tilde{\gamma}$ be the interpolants to σ and γ, respectively. We then have $\sigma_h - \tilde{\sigma} \in H_{0,h}$ and by the general theory of saddle point problems given in [9], Lemmata 3.2 and 3.3 imply the existence of $(\tau, v, \eta) \in H_{0,h} \times V_h \times W_h$ such that

$$\|\tau\|_{0,h} + \|(v, \eta)\|_h \leqq C \tag{3.15}$$

and

$$\begin{aligned} \|\sigma_h - \tilde{\sigma}\|_{0,h} &+ \|(u_h - P_h u, \gamma_h - \tilde{\gamma})\|_h \\ &\leqq a(\sigma - \tilde{\sigma}, \tau) + b(\sigma - \tilde{\sigma}; v, \eta) + b(\tau; u - P_h u, \gamma - \tilde{\gamma}) \\ &= a(\sigma - \tilde{\sigma}, \tau) + (\operatorname{div}(\sigma - \tilde{\sigma})\, v) + (\sigma - \tilde{\sigma}, \eta) \\ &\quad + (\operatorname{div} \tau, u - P_h u) + (\tau, \gamma - \tilde{\gamma}). \end{aligned} \tag{3.16}$$

Due to the way $\tilde{\sigma} \in H_{g,h}$ is defined on Γ_2, we have $\int_{\Gamma_2} (\sigma \cdot n - \tilde{\sigma} \cdot n) \cdot v \, ds = 0$. Hence Lemma 3.1 and standard interpolation theory gives

$$a(\sigma - \tilde{\sigma}, \tau) \leqq C \|\sigma - \tilde{\sigma}\|_0 \|\tau\|_0 \leqq C h^{k+1} |\sigma|_{k+1}, \tag{3.17}$$

$$(\operatorname{div}(\sigma - \tilde{\sigma}), v) + (\sigma - \tilde{\sigma}, \eta) \leqq \|\sigma - \tilde{\sigma}\|_{0,h} \|(v, \eta)\|_h \leqq C h^{k+1} |\sigma|_{k+1} \tag{3.18}$$

and

$$(\tau, \gamma - \tilde{\gamma}) \leqq \|\tau\|_0 \|\gamma - \tilde{\gamma}\|_0 \leqq C h^{k+1} |\gamma|_{k+1}. \tag{3.19}$$

Further, recalling (3.4) we have

$$(\operatorname{div} \tau, u - P_h u) = 0. \tag{3.20}$$

From (3.15) through (3.20) we thus get

$$\|\sigma_h - \tilde{\sigma}\|_{0,h} + \|(u_h - P_h u, \gamma_h - \tilde{\gamma})\|_h \leqq C h^{k+1} (|\sigma|_{k+1} + |\gamma|_{k+1}). \tag{3.21}$$

The estimate for $\|\sigma - \sigma_h\|_0$ now follows from above, Lemma 3.1 and the triangle inequality.

Finally, let us prove the L^2-estimates for the displacement. Consider the solution $(\pi, z, \mu) \in H_0 \times V \times W$ to the problem

$$\begin{aligned} &a(\pi, \tau) + b(\tau; z, \mu) = 0, \\ &\qquad b(\pi; v, \xi) = (P_h u - u_h, v), \qquad (\tau, v, \xi) \in H_0 \times V \times W. \end{aligned} \tag{3.22}$$

Let $\tilde{\pi} \in H_{0,h}$ and $\tilde{\mu} \in W_h$ be the interpolants to π and μ, and let $P_h z \in V_h$ be the L^2-projection of z. Choosing $\tau = \sigma - \sigma_h$, $\xi = \gamma - \gamma_h$ and $v = P_h u - u_h$ we get in the normal way

$$\begin{aligned} \|P_h u - u_h\|_0^2 &= a(\sigma - \sigma_h, \pi - \tilde{\pi}) + b(\sigma - \sigma_h; z - P_h z, \mu - \tilde{\mu}) \\ &\quad + b(\pi; P_h u - u_h, \gamma - \gamma_h) - b(\tilde{\pi}; u - u_h, \gamma - \gamma_h) \\ &= a(\sigma - \sigma_h, \pi - \tilde{\pi}) + b(\sigma - \sigma_h; z - P_h z, \mu - \tilde{\mu}) \\ &\quad + b(\pi - \tilde{\pi}; P_h u - u_h, \tilde{\gamma} - \gamma_h) + (\pi - \tilde{\pi}, \gamma - \tilde{\gamma}) + (\operatorname{div} \tilde{\pi}, P_h u - u). \end{aligned} \tag{3.23}$$

Due to the equilibrium condition the last term above vanishes and we obtain

$$\begin{aligned}\|P_h u-u_h\|_0^2 &\leqq C(\|\sigma-\sigma_h\|_{0,h}+\|(P_h u-u_h,\tilde{\gamma}-\gamma_h)\|_h+\|\gamma-\tilde{\gamma}\|_0)\\ &\quad\cdot(\|\pi-\tilde{\pi}\|_{0,h}+\|(z-P_h z,\mu-\tilde{\mu})\|_h)\\ &\leqq Ch(|\pi|_1+|\mu|_1+|z|_2)\\ &\quad\cdot(\|\sigma-\sigma_h\|_{0,h}+\|(P_h u-u_h,\tilde{\gamma}-\gamma_h)\|_h+\|\gamma-\tilde{\gamma}\|_0). \qquad (3.24)\end{aligned}$$

If the estimate (2.2) is valid we have

$$|z|_2+|\mu|_1+|\pi|_1\leqq C\,\|P_h u-u_h\|_0$$

and hence (3.24) together with (3.21) proves (3.9).

The estimates for $\|u-u_h\|_0$ also follows from above and the estimates corresponding to (3.21) when less regularity of the solution to (2.4) is assumed. For the estimate (3.7) the regularity estimate (2.2) is not needed. □

Remark. If we in addition to the regularity estimate (2.2) assume that the mesh is quasi-uniform (i.e. $h_K\geqq Ch\ \forall\, K\in\mathscr{C}_h$), then (3.21) and (3.9) yield the following estimate for the rotation

$$\|\gamma-\gamma_h\|_0\leqq Ch^{k+1}(|\sigma|_{k+1}+|\gamma|_{k+1}). \quad \square$$

It remains to verify the stability estimate of Lemma 3.3. The proof will be based on on the macroelement principle introduced in [22–24].

It will be sufficient to use macroelements M consisting of two neighboring elements with one common side. The elements of M are assumed to satisfy the usual regularity assumptions. For a macroelement M we define

$$\begin{aligned}H_{M,L}=\{\sigma\in[L^2(M)]^{N\times N}\,|\,\operatorname{div}\sigma\in[L^2(M)]^N,\ \sigma_{|K}\in[P_k(K)]^{N\times N}\oplus B_{k-1}(K),\\ K\subset M,\ \sigma\cdot n=0 \text{ on } \partial M\setminus L,\ L\subset\partial M\}, \qquad (3.25\text{a})\end{aligned}$$

where L is either empty or the union of one or more of the edges or sides of the elements of M,

$$V_M=\{v\in[L^2(M)]^N\,|\,v_{|K}\in[P_{k-1}(K)]^N,\ K\subset M\} \qquad (3.25\text{b})$$

and

$$W_M=\{\gamma\in[L^2(M)]^{N\times N}\,|\,\gamma+\gamma^T=0,\ \gamma_{|K}\in[P_k(K)]^{N\times N},\ K\subset M\}. \qquad (3.25\text{c})$$

The spaces we supply with the (semi)norms

$$\|\sigma\|_M^2=\|\sigma\|_{0,M}^2+h_S\int_{S\cup L}|\sigma\cdot n|^2\,ds, \qquad \sigma\in H_{M,L}$$

and

$$\begin{aligned}|(u,\gamma)|_M^2=&\sum_{K\subset M}(\|\varepsilon(u)\|_{0,K}^2+\|\gamma-\omega(u)\|_{0,K}^2)\\ &+h_S^{-1}\Big(\int_S|[u]|^2\,ds+\int_L|u|^2\,ds\Big), \qquad (u,\gamma)\in V_M\times W_M,\end{aligned}$$

where S denotes the common side of the two elements of M.

Let $\mathcal{M}_h$ be a macroelement partitioning obtained from $\mathcal{C}_h$ in such a way that each $M \in \mathcal{M}_h$ consist of two elements and each $T \in \Gamma_h$ is contained in the interior of exactly one macroelement. Further, we define

$$b_M(\sigma; u, \gamma) = (\operatorname{div} \sigma, u)_M + (\sigma, \gamma)_M.$$

Next, we recall that in order to prove the stability of the method it is sufficient to verify the corresponding local stability estimate.

Lemma 3.4. *Suppose that there is a constant C independent of M such that*

$$\sup_{0 \neq \sigma \in H_{M,L}} \frac{b_M(\sigma; u, \gamma)}{\|\sigma\|_M} \geqq C|(u, \gamma)|_M, \qquad (u, \gamma) \in V_M \times W_M, \tag{3.26}$$

for every $M \in \mathcal{M}_h$. Then the stability inequality (3.15) *is valid.*

Proof. Cf. [22, Lemma 4.4]. □

We thus have to prove

Lemma 3.5. *There is a positive constant C independent of M such that the local condition* (3.26) *is valid for an arbitrary regular macroelement M.*

Proof. We first note that a necessary condition for (3.26) to be valid is that the left hand side of (3.26) defines a (semi)norm on $V_M \times W_M$ equivalent to $|\cdot|_M$, i.e. the following condition has to be satisfied

$$N_M = \begin{cases} \{(0, 0)\}, & \text{if } L \neq \emptyset, \\ \{(r, \omega(r)) \mid r \in R_M\}, & \text{if } L = \emptyset, \end{cases} \tag{3.27}$$

where

$$N_M = \{(u, \gamma) \in V_M \times W_M \mid b_M(\sigma; u, \gamma) = 0, \sigma \in H_{M,L}\}$$

and R_Σ is the space of rigid body modes on Σ:

$$R_\Sigma = \begin{cases} \{v \in [L^2(\Sigma)]^3 \mid v = a + b \times x, \ a, b \in \mathbf{R}^3\} & \text{in } \mathbf{R}^3, \\ \{v \in [L^2(\Sigma)]^2 \mid v = (a, b) + c(-x_2, x_1), \ a, b, c \in \mathbf{R}\}, & \text{in } \mathbf{R}^2. \end{cases}$$

Now, it is not difficult to show that since the elements of M are assumed to be regular the condition is also sufficient for the local stability condition with a constant independent of M; cf. [23, Lemma 3.1] for the idea of the proof.

Let us therefore prove the condition (3.27). In the proof it will be convenient to use the subspace $\tilde{H}_{M,L}$ of $H_{M,L}$ constructed from the Raviart-Thomas-Nedelec spaces [20, 21], i.e.

$$\begin{aligned} \tilde{H}_{M,L} = \{H_{M,L} \mid (\sigma_{i1}, \ldots, \sigma_{iN})_{|K} \in [P_{k-1}(K)]^N \oplus \tilde{P}_{k-1}(K) \cdot x, \\ K \subset M, i = 1, \ldots, N\}, \end{aligned} \tag{3.28}$$

where $\tilde{P}_{k-1}(K)$ is the space of homogeneous polynomials of degree $k-1$. In [20, 21] it is shown that the elements of $\tilde{H}_{M,L}$ are uniquely defined on each K through the following degrees of freedom

$$(\sigma, q)_K, \quad q \in [P_{k-2}(K)]^{N \times N} \tag{3.29a}$$

and

$$\int_T (\sigma \cdot n) \cdot q \, ds, \quad q \in [P_{k-1}(T)]^N, \quad T \subset \partial K, \tag{3.29b}$$

where T denotes a side of the element K.

The proof will be slightly different in the two-dimensional and three-dimensional case. Let us start by the proof in $\mathbf{R}^2$.

Let $(u, \gamma) \in V_M \times W_M$ be arbitrary. Denote $\gamma_{12} = -\gamma_{21} = z$ and define $\sigma \in H_{M,L}$ through

$$(\sigma_{i1}, \sigma_{i2})_{|K} = \operatorname{curl}(b_K \, \partial_i z), \quad i = 1, 2,$$

on each $K \subset M$.

Since $\sigma_{|K} \in B_{k-1}(K)$, $K \subset M$, we have $\operatorname{div} \sigma = 0$ on M. Integrating by parts, and noting that the boundary integral vanishes, we obtain

$$\begin{aligned} b_M(\sigma; u, \gamma) = (\sigma, \gamma)_M &= \sum_{K \subset M} \int_K [-\partial_1(b_K \, \partial_1 z) - \partial_2(b_K \, \partial_2 z)] \, z \, dx \\ &= \sum_{K \subset M} \int_K b_K |\nabla z|^2 \, dx. \end{aligned}$$

We thus conclude that the condition

$$b_M(\sigma; u, \gamma) = 0, \quad \sigma \in H_{M,L},$$

implies that γ is a constant tensor on each $K \subset M$. Hence, for each $K \subset M$ there is a $w_K \in R_K$ such that

$$\gamma_{|K} = \omega(w_K) = \nabla w_K.$$

An integration by parts then gives

$$\begin{aligned} b_M(\sigma; u, \gamma) &= (\operatorname{div} \sigma, u)_M + (\sigma, \gamma)_M \\ &= \sum_{K \subset M} (\sigma, \nabla(w_K - u))_K + \int_S (\sigma \cdot n) \cdot ([u]) \, ds + \int_L (\sigma \cdot n) \cdot u \, ds. \end{aligned} \tag{3.30}$$

Now, since $w_K \in V_{h|K}$ (3.29) and (3.30) show that we can choose $\sigma \in \tilde{H}_{M,L}$ such that $b_M(\sigma; u, \gamma) = 0$ implies

(i) $\nabla(w_K - u) = 0$ on each $K \subset M$,
(ii) u is continuous along S,
(iii) $u = 0$ on L.

The conditions (i)–(iii) now show that u is a rigid body mode on M, which vanishes if L is nonempty.

The claim is thus proved in $\mathbf{R}^2$.

Let us turn to the three-dimensional case.

Let $\gamma \in W_M$ and denote by γ^i the i-th row of γ. Define

$$z^i = \operatorname{curl} \gamma^i, \qquad i = 1, 2, 3,$$

and the i-th row σ^i of $\sigma \in H_{M,L}$ through

$$\sigma^i_{|K} = \operatorname{curl}(b_K z^i), \qquad K \subset M.$$

Integrating by parts we obtain

$$(\sigma, \gamma)_K = \sum_{i=1}^{3} (\operatorname{curl}(b_K z^i), \gamma^i)_K = \sum_{i=1}^{3} \int_K b_K |\operatorname{curl} \gamma^i|^2 \, dx, \qquad K \subset M.$$

Since $\operatorname{div} \sigma = 0$ on M, we conclude that the condition

$$b_M(\sigma; u, \gamma) = 0, \qquad \sigma \in H_{M,L},$$

implies that $\operatorname{curl} \gamma^i = 0$, $i = 1, 2, 3$, on each $K \subset M$. Since γ is skew symmetric, this implies that γ has constant components on each $K \subset M$.

The rest of the proof is as in the two-dimensional case. □

4. Lower Order Methods

4.1. An Analysis of the PEERS

In this section we will give a refined analysis of the PEERS introduced in [4]. The method is defined as follows:

$$V_h = \{v \in V \mid v_{|K} \in [P_0(K)]^N, \; K \in \mathscr{C}_h\}, \tag{4.1 a}$$

$$W_h = \{\gamma \in W \cap [C(\Omega)]^{N \times N} \mid \gamma_{|K} \in [P_1(K)]^{N \times N}, \; K \in \mathscr{C}_h\} \tag{4.1 b}$$

and

$$\begin{aligned} H_{t,h} = \{\sigma \in [L^2(\Omega)]^{N \times N} \mid & \operatorname{div} \sigma \in V \;\; \sigma_{|K} \in RT(K) \oplus B_0(K), \\ & K \in \mathscr{C}_h, \;\; \sigma \cdot n = \tilde{t}, \;\; t = 0, g, \;\; \text{on } \Gamma_2\}, \end{aligned} \tag{4.1 c}$$

where

$$\begin{aligned} RT(K) = \{\sigma \in [L^2(K)]^{N \times N} \mid & (\sigma_{i1}, \ldots, \sigma_{iN}) \\ & = a^i + b^i x, \; a^i \in \mathbf{R}^N, \; b^i \in \mathbf{R}, \; i = 1, \ldots, N\}, \end{aligned}$$

$B_0(K)$ is the space (3.1) for $l = 0$ and for each $T \subset \Gamma_2$ $\tilde{t}_{|T}$ is the L^2-projection of t onto $[P_0(T)]^N$.

In order to prove the stability of the method we have to slightly modify the mesh dependent norms.

Starting from $\mathscr{C}_h$ we first introduce a coarser subdivision $\mathscr{K}_h$ of $\bar{\Omega}$ where each $M \in \mathscr{K}_h$ is a macroelement consisting of at least N and not more than κ elements of $\mathscr{C}_h$, where κ is a fixed positive integer. Further we assume that:

(i) Two distinct macroelements in $\mathscr{K}_h$ are either disjoint or have one common side.

(ii) If $\partial M \cap \Gamma_1 \neq \emptyset$ then there is N distinct sides of the elements of M that are contained in Γ_1. For $N=3$ we further assume that the midpoints of these sides are not colinear.

For h small enough it is clearly possible to finds such a partitioning $\mathscr{K}_h$ for some κ.

Next, let r^i, $i=1,2,\ldots,p=3N-3$, be the L^2-orthonormal basis functions of R_Ω. We now supply the spaces $V_{h|_M}$, $M\in\mathscr{K}_h$ with a basis $\{\xi_M^i\}_{i=1}^{q_M}$ $(q_M>p)$ such that $\xi_M^i=(P_h r^i)_{|_M}$ for $i=1,2,\ldots,p$, where P_h denotes the L^2-projection from V onto V_h (a straightforward calculation shows that $\xi_M^1, \xi_M^2, \ldots, \xi_M^p$ are linearly independent).

Writing

$$u=\sum_{M\in\mathscr{K}_h}\sum_{i=1}^{q_M} c_M^i\,\xi_M^i,\qquad u\in V_h,$$

we define the norms $\|\cdot\|_{1,h}$ and $\|(\cdot,\cdot)\|_h$ as

$$\begin{aligned}\|u\|_{1,h}^2=&\sum_{M\in\mathscr{K}_h} h_M^{-2}\left\|\sum_{i=p+1}^{q_M} c_M^i\,\xi_M^i\right\|_{0,M}^2+\sum_{\substack{M,R\in\mathscr{K}_h\\ \partial M\cap\partial R\neq\emptyset}} h_M^{N-2}\left(\sum_{i=1}^{p}|c_M^i-c_R^i|^2\right)\\&+\sum_{\substack{M\in\mathscr{K}_h\\ \partial M\cap\Gamma_1\neq\emptyset}} h_M^{N-2}\sum_{i=1}^{p}|c_M^i|^2,\qquad u\in V_h,\end{aligned}$$

and

$$\|(u,\gamma)\|_h^2=\|u\|_{1,h}^2+\sum_{M\in\mathscr{K}_h}\left\|\gamma-\sum_{i=1}^{p}c_M^i\,\omega(r^i)\right\|_{0,M},\qquad (u,\gamma)\in V_h\times W_h.$$

For the stresses we use the norm $\|\cdot\|_{0,h}$ defined in (3.10) for the space (3.11). An easy calculation shows that

$$b(\sigma;u,\gamma)\leqq C\|\sigma\|_{0,h}\|(u,\gamma)\|_h,\qquad \sigma\in\mathscr{H}_h\cap H_0,\qquad (u,\gamma)\in V_h\times W_h,$$

and hence the stability inequality to be proved is (3.14) with the above spaces and norms.

For the analysis we need still another macroelement partitioning $\mathscr{M}_h$ obtained from $\mathscr{K}_h$ in such a way that each $M\in\mathscr{M}_h$ consist of two neighboring macroelements of $\mathscr{K}_h$, and for any two macroelements $R, M\in\mathscr{K}_h$, which $\partial R\cap\partial M\neq\emptyset$, $\partial R\cap\partial M$ is contained in the interior of exactly one macroelement of $\mathscr{M}_h$.

The analysis of PEERS is now analoguous to the analysis given in Sect. 3 when the partitioning $\mathscr{K}_h$ now take the role there taken by $\mathscr{C}_h$.

Next, let M be an arbitrary macroelement and define N_M according to (3.27) with $H_{M,L}\times V_M\times W_M$ defined in analogy with (3.25) for the PEERS. For the spaces $H_{M,L}$ we again suppose that if $L \neq \emptyset$ then L contains at least N sides of the elements and that the midpoints of these sides are not colinear if $N=3$.

From the analysis of Sect. 3 it is now clear that the stability follows from the following

Lemma 4.1. *Let M be a macroelement consisting of at least N neighboring elements. Then we have*

$$N_M = \begin{cases} \{(0,0)\}, & \text{if } L \neq \emptyset, \\ \{(u,\gamma) \mid \gamma = \omega(r),\ u = P_M r,\ r \in R_M\}, & \text{if } L = \emptyset, \end{cases}$$

where P_M denotes the L^2-projection from $[L^2(M)]^N$ onto $V_{h|_M}$.

Proof. First, exactly as in the proof of Lemma 3.5 we conclude that if $(u,\gamma) \in N_M$ then γ has to be a constant tensor on each $K \subset M$. Since γ is by definition assumed to be continuous in M, it is constant on the whole of M. Hence we have (i) $\gamma = \omega(w_M) = \nabla w_M$, for some $w_M \in R_M$. This gives

$$\begin{aligned} b_M(\sigma; u, \gamma) &= (\operatorname{div} \sigma, u)_M + (\sigma, \gamma)_M = (\operatorname{div} \sigma, u)_M + (\sigma, \nabla w_M)_M \\ &= (\operatorname{div} \sigma, u)_M - (\operatorname{div} \sigma, w_M)_M + \int_L (\sigma \cdot n) \cdot w_M \, ds \\ &= (\operatorname{div} \sigma, u - P_M w_M)_M + \int_L (\sigma \cdot n) \cdot w_M \, ds \\ &= \sum_{T \subset \operatorname{Int} M} \int_T (\sigma \cdot n) \cdot ([u - P_M w_M]) \, ds \\ &\quad + \int_L (\sigma \cdot n) \cdot (u - P_M w_M + w_M) \, ds. \end{aligned} \tag{4.2}$$

Now, let $\tilde{H}_{M,L} \subset H_{M,L}$ be as in (3.28) for $k=1$. The degrees of freedom for $\sigma \in \tilde{H}_{M,L}$ are given in (3.29b) for $k=1$, i.e. they are the values of $\sigma \cdot n$ at the midpoints of each side of the elements. Hence, (4.2) implies that $u - P_M w_M$ is continuous in M. We thus have

$$u = P_M w_M + d,$$

where d is a constant. Without loss of generality we can put $d=0$ and write

$$\gamma = \omega(w_M), \qquad u = P_M w_M.$$

Further, if $L \neq \emptyset$ then (4.2) also shows that w_M have to vanish at the midpoints of the sides contained in L. Due to our assumptions on L it follows that $w_M = 0$. □

For the PEERS Lemma 3.2 is proved in [4, Lemma 4.3] and hence we get the following

Theorem 4.1. *Let H_h, V_h and W_h be defined according to* (4.1). *Then there is a unique solution to* (2.5) *such that*

$$\|\sigma - \sigma_h\|_0 \leqq C h (|\sigma|_1 + |\gamma|_1)$$

and

$$\|u-u_h\|_0 \leqq Ch(|u|_1+|\sigma|_1+|\gamma|_1).$$

Furthermore, if the regularity estimate (2.2) *is valid and if* $f \in V_h$, *then we have*

$$\|u-u_h\|_0 \leqq Ch(|u|_1+|\sigma|_r+|\gamma|_0), \qquad r>1/2,$$

and

$$\|P_h u-u_h\|_0 \leqq Ch^2(|\sigma|_1+|\gamma|_1),$$

where P_h *is the* L^2*-projection from* V *onto* V_h.

Proof. The proof of the two first estimates is identical to that given in Theorem 3.1. To prove the two latter we write out (3.23) as

$$\begin{aligned}\|P_h u-u_h\|_0^2 &= a(\sigma-\sigma_h, \pi-\tilde{\pi})+b(\sigma-\sigma_h; z-P_h z, \mu-\tilde{\mu})\\ &\quad +b(\pi-\tilde{\pi}; P_h u-u_h, \tilde{\gamma}-\gamma_h)+(\pi-\tilde{\pi}, \gamma-\tilde{\gamma})\\ &= a(\sigma-\sigma_h, \pi-\tilde{\pi})+(\operatorname{div}(\sigma-\sigma_h), z-P_h z)+(\pi-\tilde{\pi}, \gamma-\tilde{\gamma})\\ &\quad +b(\pi-\tilde{\pi}; P_h u-u_h, \tilde{\gamma}-\gamma_h)+(\pi-\tilde{\pi}, \gamma-\tilde{\gamma})\end{aligned}$$

and we note that the term

$$(\operatorname{div}(\sigma-\sigma_h), z-P_h z)$$

vanishes when $\operatorname{div}\sigma=-f \in V_h$. The estimates then follow as in the proof of Theorem 3.1. The extra regularity assumption $\sigma \in [H^r(\Omega)]^{N\times N}$, $r>1/2$, is needed in the estimate corresponding to (3.18). □

Remark. In the elasticity problem the body force f is usually a constant, or a piecewise constant, and hence the assumption $f \in V_h$ does not mean a restriction. Without the assumption $f \in V_h$ we have only been able to prove

$$\|P_h u-u_h\|_0 \leqq Ch^2(|u|_2+|\gamma|_1+|\sigma|_1+|f|_1). \quad \square$$

4.2. *An Optimal Method*

The method will be based on the lowest order Brezzi-Douglas-Marini spaces which will be modified using the idea presented in [24]. Define

$$V_h=\{v \in V \mid v_{|K} \in R_K,\ K \in \mathscr{C}_h\}, \tag{4.5a}$$

$$W_h=\{\gamma \in W \mid \gamma_{|K} \in [P_1(K)]^{N\times N},\ K \in \mathscr{C}_h\} \tag{4.5b}$$

and

$$\begin{aligned}H_{t,h}=\{\sigma \in [L^2(\Omega)]^{N\times N} \mid &\operatorname{div}\sigma \in V\ \sigma=\sigma^1+\sigma^2+\sigma^3,\\ &\sigma^1_{|K} \in [P_1(K)]^{N\times N},\ \sigma^2_{|K}=b_K \nabla r,\ r \in R_K,\ \sigma^3_{|K} \in B_0(K),\\ &K \in \mathscr{C}_h,\ \sigma\cdot n=\tilde{t} \text{ on } \Gamma_2\},\end{aligned} \tag{4.5c}$$

where $B_0(K)$ is defined according to (3.1) for $l=0$. The mesh dependent norms and spaces to be used are (3.10) through (3.13). For each $T \subset \Gamma_2 \tilde{t}_{|T}$ is the L^2-projection of t onto $[P_1(T)]^N$.

Lemma 4.2. *There is a positive constant C such that*

$$a(\sigma, \sigma) \geqq C \|\sigma\|_{0,h}^2 \qquad \sigma \in Z_h,$$

where

$$Z_h = \{\sigma \in H_{t,h} | b(\sigma; u, \gamma) = 0, (u, \gamma) \in V_h \times W_h\}.$$

Proof. It is sufficient to verify the equilibrium condition (3.3) for the above spaces, cf. [4, 5]. For this let $\sigma \in Z_h$ and write $\sigma = \sigma^1 + \sigma^2 + \sigma^3$ according to (4.5c). From the definition of $B_0(K)$ we immediatly get

$$\operatorname{div} \sigma^3 = 0.$$

Next, choose $u = \operatorname{div} \sigma^1 (\in V_h)$. We then obtain

$$\begin{aligned}(\operatorname{div} \sigma, u) &= (\operatorname{div} \sigma^1, \operatorname{div} \sigma^1) + (\operatorname{div} \sigma^2, \operatorname{div} \sigma^1)\\ &= \|\operatorname{div} \sigma^1\|_0^2 + \sum_{K \in \mathscr{C}_h} \{-(\sigma^2, \nabla(\operatorname{div} \sigma^1))_K + \int_{\partial K} (\sigma^2 \cdot n) \cdot (\operatorname{div} \sigma^1)\, ds\}\\ &= \|\operatorname{div} \sigma^1\|_0^2.\end{aligned}$$

Hence the condition $(\operatorname{div} \sigma, u) = 0$ implies

$$\operatorname{div} \sigma^1 = 0.$$

Finally, choose $u \in V_h$ such that $\sigma^2_{|K} = b_K \nabla u$, $K \subset M$. This gives

$$(\operatorname{div} \sigma^2, u) = -\sum_{K \in \mathscr{C}_h} (\sigma^2, \nabla u)_K = \sum_{K \in \mathscr{C}_h} \int_K b_K |\nabla u|^2\, dx,$$

and $(\operatorname{div} \sigma, u) = 0$ implies that

$$\sigma^2 = 0.$$

The assertion is thus proved. □

Further, the following result is easily proved.

Lemma 4.3. *The operator P_h defined through the equilibrium condition is bounded and we have*

$$\|u - P_h u\|_0 \leqq C h |u|_1. \quad \square$$

Lemma 4.4. *Let $H_{t,h} \times V_h \times W_h$ be defined according to* (4.5). *Then the stability inequality* (3.14) *is valid.*

Proof. Let us indicate how the condition (3.27) is verified for a macroelement consisting of two elements.

We again conclude that if $(u, \gamma) \in N_M$, then we can write

$$\gamma_{|K} = \omega(w_K) = \nabla w_K, \qquad w_K \in R_K, \qquad K \subset M.$$

Next, choose $\sigma \in H_{M,L}$ such that

$$\sigma^1 = \sigma^3 = 0, \qquad \sigma^2_{|K} = b_K \nabla(w_K - u), \qquad K \subset M.$$

Since σ^2 vanishes along the interelement boundaries we get

$$\begin{aligned} b_M(\sigma; u, \gamma) &= (\operatorname{div} \sigma^2, u)_M + (\sigma^2, \gamma)_M \\ &= \sum_{K \subset M} \{-(\sigma^2, \nabla u)_K + (\sigma^2, \nabla w_K)_K\} \\ &= \sum_{K \subset M} \int_K b_K |\nabla(w_K - u)|^2 \, dx, \end{aligned}$$

and hence we conclude that

(i) $\nabla(w_K - u) = 0$ on each $K \subset M$.

Finally, choose $\sigma \in H_{M,L}$ such that $\sigma^2 = \sigma^3 = 0$. The degrees of freedom for σ^1 can be taken as [16, 10]

$$\int_T (\sigma^1 \cdot n) \cdot q \, ds, \qquad q \in [P_1(T)]^N,$$

for each side T of the elements in M. Hence, we conclude as in the proof of Lemma 3.5, that

(ii) u is continuous in M,
(iii) $u = 0$ on L.

The conditions (i)–(iii) proves the claim. □

As in the proofs of Theorems 3.1 and 4.1 Lemmata 4.2 to 4.4 now imply

Theorem 4.2. *Let $H_{t,h} \times V_h \times W_h$ be defined according to* (4.5). *Then there is a unique solution to* (2.5) *such that*

$$\|\sigma - \sigma_h\|_0 \leqq C h^2 (|\sigma|_2 + |\gamma|_2)$$

and

$$\|u - u_h\|_0 \leqq C h (|u|_1 + |\sigma|_1 + |\gamma|_1).$$

Moreover, if the regularity estimate (2.2) *is valid and if $f \in V_h$, then we have*

$$\|u - u_h\|_0 \leqq C h (|u|_1 + |\sigma|_r + |\gamma|_0), \qquad r > 1/2,$$

and

$$\|P_h u - u_h\|_0 \leqq C h^3 (|\sigma|_2 + |\gamma|_2), \tag{4.6}$$

where P_h is the projection operator defined through the equilibrium condition. □

Remark. Without the assumption $f \in V_h$ we only get

$$\|P_h u - u_h\|_0 \leqq C h^2 (|\sigma|_2 + |\gamma|_2).$$

This is also the estimate one gets for the lowest order method in the Brezzi-Douglas-Marini family (cf. [10]) for the Poisson equation when the assumption $f \in V_h$ cannot be made. However, we would like to point out that in that case it is possible to use the technique introduced in [24] in order to get a method with the same accuracy for the gradient of the displacement and a cubic L^2-convergence rate for the displacement without the assumption $f \in V_h$. Furthermore, the method so constructed is not substantially more costly than the Brezzi-Douglas-Marini element. We refer directly to [24] from where the idea should be clear. □

5. Remarks on the Implementation and Postprocessing

For the solution of the discretized equations (2.5) one can use a number of different techniques, of which we will here shortly discuss the penalty function method and a method introduced by Fraijs de Veubeke [14] and further developed by Arnold and Brezzi [3].

In the approach of [3, 14] the normal component of the stress tensor is not a priori assumed to be continuous across interelement boundaries. Instead, the continuity is enforced by introducing yet another Lagrange multiplier. More precisely, denote by $\hat{H}_{t,h} \times V_h \times W_h$ the spaces (3.2), (4.1) or (4.5) modified in such a way that the assumption $\operatorname{div} \sigma \in V$, in the definition of $H_{t,h}$, is dropped. Further, let

$$M_h = \{\lambda \mid \lambda_{|T} \in [P_k(T)]^N,\ T \subset \partial K,\ K \in \mathscr{C}_h \text{ and } \lambda_{|T} = 0 \text{ if } T \subset \Gamma_1\},$$

where T denotes a side of an element, and $k=0$ for the PEERS, $k=1$ for the method of Sect. 4.2 and $k \geqq 2$ for the family (3.2). The modified method can now be stated as: Find $(\sigma_h, u_h, \gamma_h, \lambda_h) \in \hat{H}_{g,h} \times V_h \times W_h \times M_h$ such that

$$\begin{aligned}
a(\sigma_h, \tau) + (\tau, \gamma_h) + \sum_{K \in \mathscr{C}_h} \{(\operatorname{div} \tau, u_h)_K + \int_{\partial K} (\tau \cdot n) \cdot \lambda_h \, ds\} &= 0,\\
(\sigma_h, \eta) &= 0,\\
\sum_{K \in \mathscr{C}_h} (\operatorname{div} \sigma_h, v)_K + (f, v) &= 0, \qquad (5.1)\\
\sum_{K \in \mathscr{C}_h} \int_{\partial K} (\sigma_h \cdot n) \cdot \mu \, ds, \qquad (\tau, v, \eta, \mu) &\in \hat{H}_{0,h} \times V_h \times W_h \times M_h.
\end{aligned}$$

It is clear that the three first components σ_h, u_h and γ_h of the solution to the above problem coincides with the solution to (2.5), whereas λ_h constitutes an approximation to the displacement u along the interelement boundaries.

The algebraic equations generated by the above discretization are of the form

$$
\begin{aligned}
\mathscr{A}\Sigma+\mathscr{B}\Gamma+\mathscr{C}U+\mathscr{D}\Lambda&=0,\\
\mathscr{B}^T\Sigma&=0,\\
\mathscr{C}^T\Sigma&=F,\\
\mathscr{D}^T\Sigma&=0,
\end{aligned}
\tag{5.2}
$$

where Σ, Γ, U, and Λ now denote the column vectors for the degrees of freedom for σ_h, γ_h, u_h and λ_h, respectively. $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ are matrices of appropriate dimensions, where $\mathscr{A}$ is symmetric and positively definite (let us assume that in the incompressible case the standard trick (cf. e.g. [4, 7]), of slightly perturbating the problem in order to obtain a compressible one, is used). Now it is possible to successively eliminate the unknowns Σ, Γ and U in order to obtain a system for Λ alone:

$$\mathscr{G}\Lambda=\tilde{F}, \tag{5.3}$$

where $\mathscr{G}$ is symmetric and positive definite; cf. [3, 10, 11]. Furthermore, since in the methods considered the basis functions of $\hat{H}_{t,h}$, V_h and W_h, respectively, are discontinuous across the element boundaries, this elimination can be done separately on each element in the assembling phase of the calculations, and hence the cost of the elimination is relatively small (more precisely, this applies for all methods except for the PEERS in which continuous approximations are used for the rotation and hence this variable cannot be locally eliminated, cf. [4]).

For the solution of the final system (5.3) there is a wide choice of direct and iterative methods. It is, however, worth to point out that in the nearly incompressible case the matrix $\mathscr{A}$ in (5.2) is nearly singular and hence the condition number of $\mathscr{G}$ in (5.3) will be large. This will clearly slow down the convergence rate when iterative methods are used for solving the system.

The problem of an illconditioned matrix to invert also occures when the finite element equations are solved using penalty techniques. In the penalty method the discretized equations (2.5) are replaced by

$$
\begin{aligned}
a(\sigma_h,\tau)+(\operatorname{div}\tau,u_h)+(\tau,\gamma_h)&=0,\\
-\varepsilon_1(\gamma_h,\eta)+(\sigma_h,\eta)&=0,\\
-\varepsilon_2(u_h,v)+(\operatorname{div}\sigma_h,v)+(f,v)&=0,\qquad (\tau,v,\eta)\in H_{0,h}\times V_h\times W_h,
\end{aligned}
\tag{5.4}
$$

where ε_i, $i=1,2$, are (sufficiently small, cf. [7]) perturbation parameters. Denote by Q_h, Π_h the orthogonal projections from W onto W_h, and V onto V_h, respectively. Then the system (5.4) can solved in the following way:

$$a(\sigma_h,\tau)+\frac{1}{\varepsilon_1}(Q_h\sigma_h,Q_h\tau)+\frac{1}{\varepsilon_2}(\Pi_h(\operatorname{div}\sigma_h+f),\Pi_h\operatorname{div}\tau)=0,\qquad \tau\in H_{0,h}, \tag{5.5a}$$

$$\gamma_h=\frac{1}{\varepsilon_1}Q_h\sigma_h, \tag{5.5b}$$

$$u_h=\frac{1}{\varepsilon_2}\Pi_h(\operatorname{div}\sigma_h). \tag{5.5c}$$

We note that since the functions of V_h and W_h (again excluding the PEERS) are discontinuous, the projections Π_h and Q_h are computed locally on each element. Furthermore, when solving (5.5a) local condensation can be used to eliminate all degrees of freedom except the normal components of the stress tensor along the element boundaries. Hence, the final system to be solved has exactly the same number of unknowns as the system (5.3) obtained from the formulation (5.1). Let us also note that in the penalty formulation it is possible to further reduce the number of degrees of freedom by imposing stronger continuity requirements on the stress tensor; cf. [22, 24] where this idea is advocated. In order to reduce the effect of the bad conditioning of the matrix in (5.5a) (and also in (5.3) for nearly incompressible materials) one can e.g. use the augmented Lagrangian method (i.e. the multiplier method of Hestenes and Powell).

In the papers [3] and [10] it was shown that the formulation (5.1) not only leads to an efficient implementation of the mixed method, but that the new unknown λ_h contains additional information which can be utilized to construct a considerably more accurate approximation to the displacement. The postprocessing schemes of [3] and [10] are, however, slightly ad hoc in the sense that the same construction is not used for all mixed methods considered. Also, these postprocessing methods are not available when the penalty formulation is used.

Let us close this paper by presenting a new postprocessing method (a similar procedure was announced in [25]) which can be used for both in connection with the formulation (5.1) and the penalty function approach (5.5). Our postprocessing method has the advantages of beeing applicable for all equilibrium methods and simplicity of implementation.

Now, let us consider the methods (3.2) and (4.5) (for the PEERS both the implementation and postprocessing is most efficiently done as in [3]), and let P_h be the operator defined through the equilibrium condition (3.4). Define

$$V_h^* = \{v \in V \mid v_{|K} \in [P_{k+1}(K)]^N,\ K \in \mathscr{C}_h\}$$

and denote by B the elasticity tensor, i.e. the inverse of the compliance tensor A (let us again assume that an incompressible material is approximated by a slightly compressible one).

We can now define the

Postprocessing Method. *Construct the approximation $u_h^* \in V_h^*$ to u separately on each $K \in \mathscr{C}_h$ by solving the system*

$$\begin{aligned} &T_K\, u^*_{h|K} = T_K\, u_{h|K}, \\ &(B\varepsilon(u_h^*), \varepsilon(v))_K = (f, v)_K + \int_{\partial K} (\sigma_h \cdot n)\cdot v\, ds, \qquad v \in V^*_{h|K}, \end{aligned} \tag{5.6}$$

where $T_K = P_{h|K}$ for the method (4.5), *and $T_K\colon V_{|K} \to R_K$ is the L^2-projection for the family* (3.2). □

For this new approximation we obtain the following error estimate:

Theorem 5.1. *Suppose that the estimate* (2.2) *is valid and that for the lowest order method we have* $f \in V_h$. *Then we have*

$$\|u-u_h^*\|_0 \leq C h^{k+2}(|u|_{k+2}+|\sigma|_{k+1}+|\gamma|_{k+1}).$$

Proof. Since $R_K \subset V_{h|K}$ for all methods considered, we conclude from (2.4) that

$$(f, v)_K + \int_{\partial K} (\sigma_h \cdot n) \cdot v \, ds = 0, \qquad v \in R_K, \qquad K \in \mathscr{C}_h.$$

Hence (5.6) has a unique solution.

Next, let us prove the assertion for the plain strain problem or the three-dimensional problem and show that the constant in the estimate in these cases is independent of the Poisson ratio (the proof of course also covers the plain stress problem). Denote by

$$\mu = \frac{E}{2(1+\nu)}$$

and

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

the Lamé coefficients. We then have

$$B\varepsilon = 2\mu\varepsilon + \lambda\delta \operatorname{tr}(\varepsilon),$$

where δ is the unit tensor and

$$\operatorname{tr}(\varepsilon) = \sum_{i=1}^{N} \varepsilon_{ii}.$$

The problem (5.6) can now equivalently be stated as (this claim will be proved below): Find $(u_h^*, p_h^*) \in V_h^* \times W_h^*$ such that

$$\begin{aligned} &T_K u_{h|K}^* = T_K u_{h|K}, \\ &2\mu(\varepsilon(u_h^*), \varepsilon(v))_K + (p_h^*, \operatorname{div} v)_K = (f, v)_K + \int_{\partial K} (\sigma_h \cdot n) \cdot v \, ds, \qquad v \in V_{h|K}^* \\ &-\frac{1}{\lambda}(p_h^*, q)_K + (q, \operatorname{div} u_h^*)_K = 0, \qquad q \in W_{h|K}^*, \qquad K \in \mathscr{C}_h, \end{aligned} \tag{5.7}$$

where

$$W_h^* = \{q \in L^2(\Omega) \mid q_{|K} \in P_k(K), K \in \mathscr{C}_h\}.$$

Clearly, the solution to (5.6) satisfies (5.7) with $p_h^* = \lambda \operatorname{div} u_h^*$.

Now, (5.7) has the form of a perturbated saddle point problem and hence it can be analyzed by the general theory of Babuška-Brezzi [9]. Since the functions of V_h^* are discontinuous along the interelement boundaries and only natural boundary conditions are imposed, the unperturbated problem (obtained from (5.7) for $\lambda=\infty$) is easily seen to be stable, i.e. we have

$$\sup_{0 \neq v \in V_h^*|K} \frac{(\operatorname{div} v, q)_K}{\|\varepsilon(v)\|_{0,K}} \geqq C\|q\|_{0,K}, \qquad q \in W_h^*, \qquad K \in \mathscr{C}_h. \tag{5.8}$$

Now, one can show (cf. e.g. [2, Theorem 2.1]) that (5.8) also implies the stability of (5.7), and that the stability is uniform with respect to the second Lamé coefficient. Hence, there is a pair $(v, q) \in V_h^* \times W_h^*$ and a constant C, depending only on μ, such that

$$\begin{aligned}\|\varepsilon(u_h^* - \tilde{u})\|_{0,K} + \|p_h^* - \tilde{p}\|_{0,K} &\leqq 2\mu(\varepsilon(u_h^* - \tilde{u}), \varepsilon(v))_K \\ &+ (p_h^* - \tilde{p}, \operatorname{div} v)_K - \frac{1}{\lambda}(p_h^* - \tilde{p}, q)_K + (q, \operatorname{div}(u_h^* - \tilde{u}))_K\end{aligned} \tag{5.9}$$

and

$$\|\varepsilon(v)\|_{0,K} + \|q\|_{0,K} \leqq C, \qquad K \in \mathscr{C}_h,$$

where $\tilde{u} \in V_h^*$ and $\tilde{p} \in W_h^*$ are the L^2-projections of u and p, respectively. Denote

$$p = \lambda \operatorname{div} u.$$

From (5.6), (2.1) and (5.9) we then obtain

$$\begin{aligned}&\|\varepsilon(u_h^* - \tilde{u})\|_{0,K} + \|p_h^* - \tilde{p}\|_{0,K} \\ &\quad \leqq 2\mu(\varepsilon(u - \tilde{u}), \varepsilon(v))_K + (p - \tilde{p}, \operatorname{div} v)_K \\ &\qquad -\frac{1}{\lambda}(p - \tilde{p}, q)_K + (q, \operatorname{div}(u - \tilde{u}))_K - \int_{\partial K} ((\sigma - \sigma_h) \cdot n) \cdot v \, ds \\ &\quad \leqq 2\mu |u - \tilde{u}|_{1,K} \|\varepsilon(v)\|_{0,K} + |u - \tilde{u}|_{1,K} \|q\|_{0,K} \\ &\qquad + \Big(h_K \int_{\partial K} |(\sigma - \sigma_h) \cdot n|^2 \, ds\Big)^{1/2} \Big(h_K^{-1} \int_{\partial K} |v|^2 \, ds\Big)^{1/2}.\end{aligned} \tag{5.10}$$

Now, since v can be chosen such that $T_{K^{v}|_K} = 0$ for $K \in C_h$, a scaling argument gives

$$\Big(h_K^{-1} \int_{\partial K} |v|^2 \, ds\Big)^{1/2} \leqq C\|\varepsilon(v)\|_{0,K},$$

and hence (5.10) gives

$$\begin{aligned}&\|\varepsilon(u_h^* - \tilde{u})\|_{0,K} + \|p_h^* - \tilde{p}\|_{0,K} \\ &\quad \leqq C\Big\{|u - \tilde{u}|_{1,K} + \Big(h_K \int_{\partial K} |(\sigma - \sigma_h) \cdot n|^2 \, ds\Big)^{1/2}\Big\},\end{aligned} \tag{5.11}$$

where C only depends on μ.

By scaling one can also show that

$$\|w\|_{0,K} \leqq C h_K \|\varepsilon(w)\|_{0,K}$$

for each $w \in (I - T_K) V^*_{h|K}$. Hence, since

$$\varepsilon(T_K(u^*_{h|K} - \tilde{u}_{|K})) = 0,$$

(5.11) gives

$$\|(I - T_K)(u^*_h - \tilde{u})\|_{0,K} \leqq C h_K \{|u - \tilde{u}|_{1,K} + (h_K \int_{\partial K} |(\sigma - \sigma_h) \cdot n|^2 \, ds)^{1/2}\}. \tag{5.12}$$

Next, let us estimate $\|T_K(u^*_h - \tilde{u})\|_{0,K}$. Since T_K is bounded we get

$$\begin{aligned} \|T_K(u^*_h - \tilde{u})\|_{0,K} &\leqq \|T_K(u^*_h - u)\|_{0,K} + \|T_K(u - \tilde{u})\|_{0,K} \\ &\leqq \|T_K(u^*_h - u)\|_{0,K} + C\|u - \tilde{u}\|_{0,K}. \end{aligned} \tag{5.13}$$

From the definition of T_K we have $T_K P_{h|K} = T_K$ and hence

$$\begin{aligned} \|T_K(u^*_h - u)\|_{0,K} = \|T_K(u_h - u)\|_{0,K} &= \|T_K(u_h - P_h u)\|_{0,K} \\ &\leqq C\|u_h - P_h u\|_{0,K}. \end{aligned} \tag{5.14}$$

Combining (5.12)–(5.14) gives

$$\begin{aligned} \|u^*_h - \tilde{u}\|_{0,K} \leqq C\{&\|u_h - P_h u\|_{0,K} + \|u - \tilde{u}\|_{0,K} \\ &+ h_K(|u - \tilde{u}|_{1,K} + (h_K \int_{\partial K} |(\sigma - \sigma_h) \cdot n|^2 \, ds)^{1/2})\}, \end{aligned} \tag{5.15}$$

where C only depends on μ. Summing over all $K \in \mathscr{C}_h$ we then obtain

$$\|u - u^*_h\|_0 \leqq C(h^{k+2}|u|_{k+2} + \|u_h - P_h u\|_0 + h\|\sigma - \sigma_h\|_{0,h}). \tag{5.16}$$

In the proofs of Theorems 3.1 and 4.1 the estimate

$$\|\sigma - \sigma_h\|_{0,h} \leqq C h^{k+1}(|\sigma|_{k+1} + |\gamma|_{k+1})$$

is derived. Thus, (5.16), (3.8) and (4.6) give

$$\|u - u^*_h\|_0 \leqq C h^{k+2}(|u|_{k+2} + |\sigma|_{k+1} + |\gamma|_{k+1}),$$

where C depends only on μ. Hence, since $0 < \nu < 1/2$, the constant can be chosen to be independent of the Poisson ratio. $\square$

Remark. A new approximation σ_h^* for the stress tensor can, naturally, also obtained from u_h^*. In the above proof it is implicitely shown that

$$\|\sigma-\sigma_h^*\|_0 \leqq C h^{k+1}(|\sigma|_{k+1}+|\gamma|_{k+1}+|u|_{k+2}),$$

i.e. the convergence rate is the same as that of the original approximation σ_h. In particular, the constant C does not depend on the Poisson ratio. □

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