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An extended Hu–Washizu formulation for elasticity

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Abstract

A class of new mixed formulations for elasticity is developed and analysed. The formulations are based on the discrete evss method, introduced in the context of incompressible viscoelastic flows by A. Fortin, M. Fortin and co-workers. A key feature is a stabilization term that renders coercive a problem that might not otherwise be so. The focus in this work is on behaviour in the incompressible limit and the goal is that of obtaining formulations that are uniformly stable and convergent. Concrete examples are presented of element choices that lead to unstable formulations in the classical formulation, and which are stable for the formulations introduced here. A selection of numerical results illustrates in a comparative way the behaviour of the elements introduced.

Keywords: Mixed finite elements; Elastic-viscous-split stress (evss); Hu–Washizu formulation; Ellipticity; Low-order elements

1. Introduction

Over the past two decades extensive research has been directed towards improving the performance of low-order quadrilateral finite elements, which exhibit poor accuracy for certain classes of problems in solid mechanics, and which lock in the incompressible limit. A comprehensive overview of the many contributions in this area may be found, for example, in the recent work [5]. Many of these developments have centred on the use of mixed and enhanced methods [5,6,16,17,21] and stabilization approaches [1,4,7,15], including those using Galerkin least squares [10–12].

The approaches referred to all essentially address basic deficiencies in the standard formulation based on the four-noded quadrilateral element in two dimensions, and the eight-noded hexahedral element in three, which arise from the fact that these elements do not satisfy the inf–sup condition for stability when viewed as mixed problems. Alternatively, it can be shown that the bounds that occur in the analyses of well-posedness depend on the Lamé parameter λ, which becomes unbounded in the incompressible limit.

The purpose of the present contribution is to explore alternative formulations that are designed to overcome a *lack of coercivity* or ellipticity in the basic formulation. The point of departure is the classical evss (elastic-visco-split-stress) formulation introduced within the context of incompressible viscoelastic fluids by Rajagopalan et al. [20]. In its original form this is a four-field problem with unknowns the velocity $\boldsymbol{u}$, pressure p, extra stress $\boldsymbol{\sigma}$, and rate of deformation tensor $\boldsymbol{d}$. Fortin and coworkers [8,9] have introduced a modification of the evss formulation, known as the discrete evss method (devss). In this method a stabilizing term of the form $2\alpha(\boldsymbol{d},\boldsymbol{\epsilon}(\boldsymbol{v})) - 2\alpha(\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{\epsilon}(\boldsymbol{v}))$, where α is an arbitrary scalar, is introduced in the equilibrium equation, where $\boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}})$. In the continuous case the extra term is trivial, but in a discrete setting it plays a non-trivial role, in that the positive parameter α may be chosen in such a way as to enhance stability and ensure the well-posedness of finite element discretizations. In [8] the discrete spaces of stresses and rate of deformation coincide, whereas in the later version [9] this constraint is relaxed.

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In the present work a related approach is explored in the context of elasticity, with a particular focus on obtaining formulations that are uniformly stable and convergent in the incompressible limit. The goal is to include a stabilization term of the kind described above, and to use this term to render stable a problem that may not otherwise be so. The resulting formulations are three-field ones involving displacement, strain, and stress, and can be regarded as extensions of the Hu–Washizu problem in the sense that when $\alpha = 0$, the Hu–Washizu formulation is recovered.

The problems that result from the extension described take the form most generally of non-standard mixed variational problems of the form

$$a(\phi,\psi) + b_1(\psi,m) = \langle f,\psi\rangle,$$
$$b_2(\phi,n) - c(m,n) = \langle g,n\rangle,$$

in which $a(\cdot,\cdot)$, $b_1(\cdot,\cdot)$, $b_2(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are bilinear forms and f and g are linear functionals. In its standard form (see, for example, [3]) $b_2 = b_1$ and $c = 0$. The analysis of the above form with $c = 0$ has been carried out by Nicolaides [18] and Bernardi et al. [2]. These analyses are extended here to the case in which the bilinear form $c(\cdot,\cdot)$ is present.

The remainder of this work is organized as follows. Section 2 is concerned with the presentation of the governing equations and relevant details concerning the abstract mixed problems. Here the non-standard mixed formulation originally considered in [2,18] is extended to include the case $c(\cdot,\cdot) = 0$. Section 3 is devoted to the presentation and analysis of the continuous versions of the two extended mixed problems. Here it is necessary to introduce the pressure p as an additional variable, in order to obtain results that hold independently of the Lamé parameter λ. This is simply a mathematical artifice, though; the problem that is solved in practice remains the three-field formulation.

Finite element approximations of the extended problems are introduced in Section 4. Conditions on the spaces that are sufficient for stability are established. In addition, examples of families of spaces that satisfy these conditions are presented. These examples are of spaces for which the problem without the stabilization term is not coercive. As in the case of the Hu–Washizu formulation based on continuous piecewise-bilinear approximations of displacements in two dimensions, it is shown here that the stresses contain checkerboard modes. This phenomenon, which is well known and has been analysed in detail in [6] and illustrated in [5], is present in the classical enhanced assumed strain method [21], which is a particular case of the Hu–Washizu formulation. The displacement is nevertheless not affected by the presence of these zero-energy modes, which can if required be filtered out.

Finally, in Section 5 we present a selection of numerical results that illustrate the performance of the new formulations.

2. The boundary value problem

In the context of elasticity, vector- and tensor- or matrix-valued functions will be written in boldface form. The scalar product of two tensors or matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ will be denoted by $\boldsymbol{\sigma} : \boldsymbol{\tau}$, and is given by $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, the summation convention on repeated indices being invoked.

Consider a homogeneous isotropic linear elastic material body which occupies a bounded domain Ω in $\mathbb{R}^2$ with Lipschitz boundary Γ. For a prescribed body force $\boldsymbol{f}$, the equilibrium equation is given by

$$\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0}, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the symmetric stress tensor or matrix. The infinitesimal strain $\boldsymbol{d}$ is defined as a function of the displacement $\boldsymbol{u}$ by

$$\boldsymbol{d} = \boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + [\nabla\boldsymbol{u}]^t) \tag{2.2}$$

and the constitutive equation is given by

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{d} + \lambda(\operatorname{tr}\boldsymbol{d})\mathbf{1}. \tag{2.3}$$

Here $\mathbf{1}$ is the identity tensor and μ and λ are Lamé parameters, which are assumed positive, and which are constant in view of the assumption of material homogeneity. Of particular interest is the incompressible limit, which corresponds to $\lambda \to \infty$.

2.1. The modified equilibrium equation

We add to (2.1) the term

$$2\alpha\operatorname{div}(\boldsymbol{d} - \boldsymbol{\epsilon}(\boldsymbol{u}))$$

to give the modified equilibrium equation

$$\operatorname{div}\boldsymbol{\sigma} + 2\alpha\operatorname{div}(\boldsymbol{d} - \boldsymbol{\epsilon}(\boldsymbol{u})) + \boldsymbol{f} = \mathbf{0}. \tag{2.4}$$

This modification is trivial for the continuous problem, in view of (2.2), but will be of significance in the discrete version.

To the governing equations we add the boundary condition, which for simplicity is taken to be the homogeneous Dirichlet condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma. \tag{2.5}$$

2.2. Function spaces

We will make use of the space $L^2(\Omega)$ of square-integrable functions defined on Ω with the standard inner product and norm being denoted by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$, respectively. The inner product and norm on the vector-valued space $[L^2(\Omega)]^2$ are defined componentwise in the standard way, and are also denoted by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$.

We will also make use of the Sobolev space $H^1(\Omega)$. This is a Hilbert space with inner product and associated norm

$$(u,v)_1 := (u,v)_0 + (\nabla u, \nabla v)_0 \quad \text{and} \quad \|v\|_1 := (v,v)_1^{1/2}. \tag{2.6}$$

The semi-norm $|\cdot|_1$ on $H^1(\Omega)$ is defined by

$$|v|_1 := \|\nabla v\|_0. \tag{2.7}$$

The space $H_0^1(\Omega)$ consists of functions in $H^1(\Omega)$ which vanish on the boundary in the sense of traces.

For the weak or variational formulations we will require the space $V := [H_0^1(\Omega)]^2$ of displacements. This is a Hilbert space with inner product $(\cdot,\cdot)_1$ and norm $\|\cdot\|_1$ defined in the standard way; that is, $(\boldsymbol{u},\boldsymbol{v})_1^2 = \sum_{i=1}^2 (u_i, v_i)_1$, with the norm being induced by this inner product.

The space of stresses is denoted by S, while the space of strains is denoted by D. For the continuous case these spaces are equal, and $D := \{e | e_{ji} = e_{ij},\ e_{ij} \in L^2(\Omega)\} =: S$, with norm $\|\cdot\|_0$ generated in the standard way by the L^2-norm.

2.3. An abstract non-standard mixed problem

Let Φ, Ψ, M and N be Hilbert spaces and introduce the continuous bilinear forms

$$\begin{aligned} &a : \Phi \times \Psi \to \mathbb{R}, \\ &b_1 : \Psi \times M \to \mathbb{R}, \\ &b_2 : \Phi \times N \to \mathbb{R}, \\ &c : M \times N \to \mathbb{R}. \end{aligned} \tag{2.8}$$

Consider the problem of finding $(\phi, m) \in \Phi \times M$ that satisfy, for $(f, g) \in \Psi' \times N'$,

$$\begin{aligned} &a(\phi,\psi) + b_1(\psi, m) = \langle f, \psi \rangle \quad \text{for all } \psi \in \Psi, \\ &b_2(\phi, n) - c(m,n) = \langle g, n \rangle \quad \text{for all } n \in N. \end{aligned} \tag{2.9}$$

The standard form of this problem, for which a complete analysis may be found in [3], is recovered by setting $b_2 = b_1$.

We associate with the bilinear forms $a(\cdot,\cdot)$, $c(\cdot,\cdot)$ and $b_i(\cdot,\cdot)$ the bounded linear operators $A : \Phi \to \Psi'$, $B_1 : \Psi \to M'$, $B_2 : \Phi \to N'$, $C : M \to N'$ and the transposes $B_1^t : M \to \Psi'$ and $B_2^t : N \to \Phi'$, according to

$$\begin{aligned} &a(\phi,\psi) = \langle A\phi, \psi \rangle, \quad c(m,n) = \langle Cm, n \rangle, \\ &b_1(\psi, m) = \langle B_1 \psi, m \rangle = \langle B_1^t m, \psi \rangle, \quad b_2(\phi, n) = \langle B_2 \phi, n \rangle = \langle B_2^t n, \phi \rangle. \end{aligned} \tag{2.10}$$

We require conditions on $a(\cdot,\cdot)$, $b_i(\cdot,\cdot)$, and $c(\cdot,\cdot)$ under which the system (2.9) has a unique solution which is bounded independently of the properties of $c(\cdot,\cdot)$. The last requirement will be relevant later in the context of the elasticity problem.

For any $(h_1, h_2) \in M' \times N'$, we define the subspaces

$$\begin{aligned} &Z_1(h_1) = \{\phi \in \Phi | b_1(\phi, m) = \langle h_1, m \rangle \text{ for all } m \in M\}, \\ &Z_2(h_2) = \{\psi \in \Psi | b_2(\psi, n) = \langle h_2, n \rangle \text{ for all } n \in N\}, \end{aligned} \tag{2.11}$$

and denote by $Z_i = Z_i(0)$ the kernel of the operator B_i. We also introduce the subspaces Z_1^t and Z_2^t defined by

$$\begin{aligned} &Z_1^t = \{m \in M | b_1(\psi, m) = 0 \text{ for all } \psi \in \Psi\}, \\ &Z_2^t = \{n \in N | b_2(\phi, n) = 0 \text{ for all } \phi \in \Phi\}, \end{aligned} \tag{2.12}$$

and define

$$\ker C = \{m \in M | c(m,n) = 0 \text{ for all } n \in N\}. \tag{2.13}$$

With problem (2.9) we associate the following problem: find $(\phi, m) \in Z_2(g) \times M$ that satisfies

$$\begin{aligned} a(\phi,\psi) &= \langle f,\psi\rangle \quad \text{for all } \psi \in Z_1, \\ c(m,n) &= \langle g,n\rangle \quad \text{for all } n \in Z_2^t. \end{aligned} \tag{2.14}$$

Clearly, any solution (ϕ, m) of (2.14) also satisfies (2.9). We want sufficient conditions which ensure that the converse of this statement holds. For this purpose we have the following.

Theorem 2.1. *Suppose that the bilinear forms $a(\cdot,\cdot)$, $b_i(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are given in such a way that*

(a) *there exists a constant $\alpha_1 > 0$ such that*

$$\sup_{\psi\in Z_1} \frac{a(\phi,\psi)}{\|\psi\|_\Psi} \geqslant \alpha_1\|\phi\|_\Phi \quad \forall\phi \in Z_2 \quad \textit{and} \quad \sup_{\phi\in Z_2} a(\phi,\psi) > 0 \quad \forall\psi \in Z_1 \setminus \{0\};$$

(b) *there exist two positive constants β_1, β_2 such that for all $(m,n) \in M \times N$,*

$$\sup_{\psi\in\Psi} \frac{b_1(\psi,m)}{\|\psi\|_\Psi} \geqslant \beta_1\|m\|_M \quad \textit{and} \quad \sup_{\phi\in\Phi} \frac{b_2(\phi,n)}{\|\phi\|_\Phi} \geqslant \beta_2\|n\|_N;$$

(c) *given $\widetilde{m} \in (Z_1^t)^\perp$, and the continuous bilinear form $c(\cdot,\cdot)$, one can find $m_0 \in Z_1^t$ such that*

$$c(m_0,n) = -c(\widetilde{m},n) \quad \forall n \in Z_2^t. \tag{2.15}$$

Then for $(f,g) \in \Psi' \times \operatorname{Im} B_2$, the problem (2.9) has a unique solution (ϕ,m) in $\Phi \times M/Z_1^t \cap \ker C$, which moreover satisfies the bounds

$$\|(m,\phi)\|_{M\times\Phi} \leqslant d_1\|(f,g)\|_{\Psi'\times N'}, \tag{2.16}$$

where $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ (or one of its equivalents), and d_1 is a positive constant depending on $\|a\|$, $\|c\|$, α_1, β_i (i = 1, 2), and γ_0.

The bounds (2.16) are obtained in the standard way by assuming that the solution m_0 of (2.15) is such that

$$\gamma_0\|m_0\|_M \leqslant \|\tilde{m}\|_M,$$

where γ_0 is a positive constant. Of course, that condition is a consequence of the well-posedness of (2.15).

Remark 2.2. If $(\phi,m) \in \Phi \times M$ solves (2.9), then for $\tilde{m} \in Z_1^t \cap \ker C$, $(\phi, m+\tilde{m})$ will also solves (2.9). Thus we do not have uniqueness of the solution in $\Phi \times M$ but in $\Phi \times M/Z_1^t \cap \ker C$.

Bernardi et al. [2] have studied this problem for the special case $c(\cdot,\cdot) = 0$, and have established sufficient conditions for well-posedness and convergence of finite element approximations. Nicolaides [18] has shown the conditions to be necessary.

Proof of Theorem 2.1. From the sup conditions (b) on $b_2(\cdot,\cdot)$, B_2^t is injective. Hence B_2 is surjective and the restrictions of B_2 and B_2^t defined by: $B_2 : Z_2^\perp \to N'$, and $B_2^t : N \to (Z_2^\perp)'$ are isomorphic and

$$\|B_2^{-1}\|_{\mathscr{L}(N',Z_2^\perp)} = \|(B_2^t)^{-1}\|_{\mathscr{L}((Z_2^\perp)',N)} \leqslant \beta_2^{-1}. \tag{2.17}$$

Since $g \in \operatorname{Im}B_2$, there exists $\phi_1 \in Z_2^\perp$ such that

$$B_2\phi_1 = g, \quad \text{and} \quad \beta_2\|\phi_1\|_\Phi \leqslant \|g\|_{N'}. \tag{2.18}$$

Let $\Pi : \Psi' \to Z_1'$ be defined as follows: for each $f \in \Psi'$,

$$\langle \Pi f, v\rangle = \langle f, v\rangle \quad \text{for all } v \in Z_1.$$

Then by condition (a), $A_0 = \Pi A : Z_2 \to Z_1'$ is an isomorphism satisfying

$$\|A_0^{-1}\| = \alpha_1^{-1}.$$

By means of the isomorphic properties on A_0, knowing that $\Pi f - A_0\phi_1 \in Z_1'$, there exists a unique $\phi_0 \in Z_2$ such that

$$A_0\phi_0 = \Pi f - A_0\phi_1, \quad \text{and} \quad \alpha_1\|\phi_0\|_\Phi \leqslant \|f\|_{\Psi'} + \|a\|\|\phi_1\|_\Phi. \tag{2.19}$$

We have $\phi = \phi_0 + \phi_1 \in Z_2(g)$ and $(2.14)_1$ is satisfied. The next task is to find $m \in M$ such that $(2.14)_2$ is satisfied. From the sup condition (b) on $b_1(\cdot,\cdot)$, B_1^t is injective. Hence B_1 is surjective. Therefore the restrictions of B_1 and B_1^t defined by: $B_1 : Z_1^\perp \to M'$, $B_1^t : M \to (Z_1^\perp)'$ are isomorphic and

$$\|B_1^{-1}\|_{\mathscr{L}(M',Z_1^{\perp})} = \|(B_1^t)^{-1}\|_{\mathscr{L}((Z_1^{\perp})',M)} \leqslant \beta_1^{-1}. \tag{2.20}$$

Now, from the Closed Range Theorem and the sup condition on $b_1(\cdot,\cdot)$, $\operatorname{Im} B_1^t = \Psi'$. Thus, B_1^t is bijective and there exists a unique $\tilde{m} \in (Z_1^t)^{\perp}$ such that

$$B_1^t\tilde{m} = f - A\phi_0 - A\phi_1 = f - A\phi, \quad \text{and} \quad \beta_1\|\tilde{m}\|_M \leqslant \|f\|_{\Psi'} + \|a\|\|\phi\|_{\Phi}. \tag{2.21}$$

Finally, since $g \in \operatorname{Im} B_2$, Eq. (2.14)$_2$ can be equivalently stated in the following way: find $m_0 \in (Z_1^t)^{\perp}$ satisfying

$$c(m_0, n) = -c(\tilde{m}, n) \quad \text{for all } n \in Z_2^t, \tag{2.22}$$

which is exactly the condition (c). So, it suffices to take $(\phi, m) = (\phi_0 + \phi_1, m_0 + \tilde{m})$. $\square$

3. The extended problems

3.1. The first formulation

We derive the first of the two three-field formulations based on the modified equilibrium Eq. (2.4), and for this purpose introduce the bilinear forms

$$\begin{aligned}
&a_1(\boldsymbol{\sigma}, \boldsymbol{d}; \boldsymbol{\tau}, \boldsymbol{v}) := (\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\boldsymbol{v}))_0 - 2\alpha(\boldsymbol{d}, \boldsymbol{\epsilon}(\boldsymbol{v}))_0 - (\boldsymbol{d}, \boldsymbol{\tau})_0,\\
&b_1(\boldsymbol{\tau}, \boldsymbol{v}; \boldsymbol{u}) := (2\alpha\boldsymbol{\epsilon}(\boldsymbol{v}) + \boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{u}))_0,\\
&b_2(\boldsymbol{\sigma}, \boldsymbol{d}; \boldsymbol{e}) := (\boldsymbol{\sigma} - \mathscr{C}\boldsymbol{d}, \boldsymbol{e})_0.
\end{aligned} \tag{3.1}$$

Then by analogy with [8] we introduce the following problem.

Problem I. Given $\ell \in V'$, find $[\boldsymbol{\sigma}, \boldsymbol{d}, \boldsymbol{u}] \in S \times D \times V$ that satisfy

$$\begin{aligned}
&a_1(\boldsymbol{\sigma}, \boldsymbol{d}; \boldsymbol{\tau}, \boldsymbol{v}) + b_1(\boldsymbol{\tau}, \boldsymbol{v}; \boldsymbol{u}) = \ell(\boldsymbol{v}) \quad \text{for all } [\boldsymbol{\tau}, \boldsymbol{v}] \in S \times V,\\
&b_2(\boldsymbol{\sigma}, \boldsymbol{d}; \boldsymbol{e}) = 0 \quad \text{for all } \boldsymbol{m} \in D.
\end{aligned} \tag{3.2}$$

It is straightforward to show the equivalence between Problem I and (2.2)–(2.5). We also observe that for $\alpha = 0$, Problem I reduces to the classical Hu–Washizu problem [14,24]. Eq. (3.2) constitute a non-standard mixed method of the form (2.9) with $c = 0$ and $g = 0$. Referring to Theorem 2.1 we see that the a priori bounds on the solution are directly related to α_1, which in turn depends on λ^{-1}. In the incompressible limit, therefore, we are unable to show well-posedness by this approach. Instead, we pose the problem as an equivalent four-field mixed formulation by introducing the pressure $p := \lambda \operatorname{tr} \boldsymbol{d}$. For this we need the space Q of pressures defined by

$$Q = L_0^2(\Omega) = \left\{ v \in L^2(\Omega) \middle| \int_\Omega v \, \mathrm{d}x = 0 \right\}, \tag{3.3}$$

and we introduce the bilinear forms

$$\begin{aligned}
&a_{1p}(\boldsymbol{d}, \boldsymbol{u}; \boldsymbol{e}, \boldsymbol{v}) := -2\alpha(\boldsymbol{d}, \boldsymbol{\epsilon}(\boldsymbol{v})) + 2\alpha(\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v}))_0 + 2\mu(\boldsymbol{d}, \boldsymbol{e})_0,\\
&b_{1p}(\boldsymbol{e}, \boldsymbol{v}; \boldsymbol{\sigma}, p) := -(\boldsymbol{\sigma}, \boldsymbol{e})_0 + (\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\boldsymbol{v}))_0 + (p, \operatorname{tr} \boldsymbol{e})_0,\\
&c(\boldsymbol{\sigma}, p; \boldsymbol{\tau}, q) := \lambda^{-1}(p, q)_0.
\end{aligned} \tag{3.4}$$

Then the following problem is equivalent to Problem I.

Problem Ip. Given $\ell \in V'$, find $(\boldsymbol{\sigma}, \boldsymbol{d}, \boldsymbol{u}, p) \in S \times D \times V \times Q$ that satisfy

$$\begin{aligned}
&a_{1p}(\boldsymbol{d}, \boldsymbol{u}; \boldsymbol{e}, \boldsymbol{v}) + b_{1p}(\boldsymbol{e}, \boldsymbol{v}; \boldsymbol{\sigma}, p) = \ell(\boldsymbol{v}) \quad \text{for all } [\boldsymbol{\tau}, \boldsymbol{v}] \in S \times V,\\
&b_{1p}(\boldsymbol{d}, \boldsymbol{u}; \boldsymbol{\tau}, q) - c(\boldsymbol{\sigma}, p; \boldsymbol{\tau}, q) = 0 \quad \text{for all } [\boldsymbol{\tau}, q] \in S \times Q.
\end{aligned} \tag{3.5}$$

This is a *standard* mixed formulation. The conditions of Theorem 2.1 can nevertheless be used to establish well-posedness of this problem. Indeed, for $(\boldsymbol{d}, \boldsymbol{u}) \in \ker B_{1p}$, that is, $\boldsymbol{d} = \boldsymbol{\epsilon}(\boldsymbol{u})$ with $\operatorname{div} \boldsymbol{u} = 0$, and using the continuity of $\boldsymbol{\epsilon}$, we easily get

$$\|(\boldsymbol{d}, \boldsymbol{u})\|_{D\times V} \leqslant \|\boldsymbol{\epsilon}(\boldsymbol{u})\|_0 + \|\boldsymbol{u}\|_V \leqslant \|\boldsymbol{u}\|_1 + \|\boldsymbol{u}\|_1 = 2\|\boldsymbol{u}\|_1, \tag{3.6}$$

so that using Korn's inequality $\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_0 \geqslant c_K\|\boldsymbol{u}\|_1$ we have

$$a_{1p}(\boldsymbol{d}, \boldsymbol{u}; \boldsymbol{d}, \boldsymbol{u}) = 2\mu\|\boldsymbol{d}\|_0^2 + 2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_0^2 - 2\alpha(\boldsymbol{d}, \boldsymbol{\epsilon}(\boldsymbol{u}))_0 = 2\mu\|\boldsymbol{\epsilon}(\boldsymbol{u})\|^2 + 2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u})\|^2 - 2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_0^2 \geqslant 2\mu c_K^2\|\boldsymbol{u}\|_1^2 \geqslant \frac{\mu c_K^2}{2}\|(\boldsymbol{d}, \boldsymbol{u})\|_{D\times V}^2. \tag{3.7}$$

Thus, the bilinear form $a_{1p}(\cdot,\cdot)$ is coercive on $\ker B_{1p}$.

It is trivial to show that a_{1p} and b_{1p} are continuous. For the sup condition on $b_{1p}(\cdot,\cdot)$, we note first that, for given $p \in Q$, there exists a unique $\boldsymbol{w} \in V$ and a positive constant c_1 depending on Ω such that [13]

$$\operatorname{div} \boldsymbol{w} = p \quad \text{and} \quad \|\boldsymbol{w}\|_1 \leqslant c_1 \|p\|_0. \tag{3.8}$$

Take $\boldsymbol{e} = \boldsymbol{\epsilon}(\boldsymbol{w}) - \gamma\boldsymbol{\sigma}$ where $0 < \gamma < 2$. For any δ that satisfies $1/\sqrt{2} < \delta < \sqrt{2}/\gamma$, we have

$$\begin{aligned}\sup_{(\boldsymbol{e},\boldsymbol{v})\in D\times V} \frac{b_{1p}(\boldsymbol{e},\boldsymbol{v};p,\boldsymbol{\sigma})}{\|[\boldsymbol{e},\boldsymbol{v}]\|_{D\times V}} &\geqslant \frac{b_{1p}(\boldsymbol{\epsilon}(\boldsymbol{w})-\gamma\boldsymbol{\sigma},\boldsymbol{w};p,\boldsymbol{\sigma})}{\|[\boldsymbol{\epsilon}(\boldsymbol{w})-\gamma\boldsymbol{\sigma},\boldsymbol{w}]\|_{D\times V}} = \frac{(p,\operatorname{div}\boldsymbol{w}) + \gamma\|\boldsymbol{\sigma}\|_S^2 - \gamma(p,\operatorname{tr}\boldsymbol{\sigma})}{\|[\boldsymbol{\epsilon}(\boldsymbol{w})-\gamma\boldsymbol{\sigma},\boldsymbol{w}]\|_{D\times V}} = \frac{\|p\|_0^2 + \gamma\|\boldsymbol{\sigma}\|_0^2 - \gamma(p,\operatorname{tr}\boldsymbol{\sigma})}{\|[\boldsymbol{\epsilon}(\boldsymbol{w})-\gamma\boldsymbol{\sigma},\boldsymbol{w}]\|_{D\times V}} \\ &\geqslant \frac{\|p\|_0^2 + \gamma\|\boldsymbol{\sigma}\|_0^2 - \gamma\sqrt{2}\|p\|_0\|\boldsymbol{\sigma}\|_0}{\|[\boldsymbol{\epsilon}(\boldsymbol{w})-\gamma\boldsymbol{\sigma},\boldsymbol{w}]\|_{D\times V}} \geqslant \frac{(1-\gamma\delta/\sqrt{2})\|p\|_0^2 + \gamma\left(1-\frac{1}{\sqrt{2}\delta}\right)\|\boldsymbol{\sigma}\|_0^2}{\|[\boldsymbol{\epsilon}(\boldsymbol{w})-\gamma\boldsymbol{\sigma},\boldsymbol{w}]\|_{D\times V}}.\end{aligned} \tag{3.9}$$

Here we have used the fact the $\|\operatorname{tr}\| \leqslant \sqrt{2}$, and Young's inequality. Turning to the denominator of (3.9), using the continuity of $\boldsymbol{\epsilon}$ and $(3.8)_2$ we have

$$\|[\boldsymbol{\epsilon}(\boldsymbol{w})-\gamma\boldsymbol{\sigma},\boldsymbol{w}]\|_{D\times V} \leqslant \|\boldsymbol{\epsilon}(\boldsymbol{w})\|_0 + \gamma\|\boldsymbol{\sigma}\|_0 + \|\boldsymbol{w}\|_1 = 2\|\boldsymbol{w}\|_1 + \gamma\|\boldsymbol{\sigma}\|_0 \leqslant \frac{2}{c_1}\|p\|_0 + \gamma\|\boldsymbol{\sigma}\|_0 \leqslant \left(\gamma + \frac{2}{c_1}\right)\|[p,\boldsymbol{\sigma}]\|_{Q\times S}. \tag{3.10}$$

Thus, putting (3.9) and (3.10) together we obtain

$$\sup_{(\boldsymbol{e},\boldsymbol{v})\in D\times V} \frac{b_{1p}(\boldsymbol{e},\boldsymbol{v};p,\boldsymbol{\sigma})}{\|(\boldsymbol{e},\boldsymbol{v})\|_{D\times V}} \geqslant \frac{(1-\gamma\delta/\sqrt{2})\|p\|_0^2 + \gamma\left(1-\frac{1}{\sqrt{2}\delta}\right)\|\boldsymbol{\sigma}\|_0^2}{\left(\gamma+\frac{2}{c_1}\right)\|(p,\boldsymbol{\sigma})\|_{Q\times S}} \geqslant \frac{\min\left(1-\gamma\delta/\sqrt{2},\gamma\left(1-\frac{1}{\sqrt{2}\delta}\right)\right)}{\gamma+\frac{2}{c_1}}\|(p,\boldsymbol{\sigma})\|_{Q\times S} \geqslant \beta\|(\boldsymbol{\sigma},p)\|_{S\times Q}, \tag{3.11}$$

where

$$\beta = \frac{\min\left(1-\gamma\delta/\sqrt{2},\gamma\left(1-\frac{1}{\sqrt{2}\delta}\right)\right)}{\gamma+\frac{2}{c_1}}.$$

Finally for every $[\widetilde{\boldsymbol{\sigma}},\widetilde{p}] \in (Z^t)^\perp$, we have to find $[\boldsymbol{\sigma}_0,p_0] \in Z^t$ satisfying

$$c(\boldsymbol{\sigma}_0,p_0;\widetilde{\boldsymbol{\tau}},\widetilde{q}) = -c(\widetilde{\boldsymbol{\sigma}},\widetilde{p};\widetilde{\boldsymbol{\tau}},\widetilde{q}) \quad \text{for all } [\widetilde{\boldsymbol{\tau}},\widetilde{q}] \in Z^t. \tag{3.12}$$

But $c(\widetilde{\boldsymbol{\sigma}},\widetilde{p};\widetilde{\boldsymbol{\tau}},\widetilde{q}) = \lambda^{-1}(\widetilde{p},\widetilde{q})_0$, and $[\widetilde{\boldsymbol{\sigma}},\widetilde{p};\widetilde{\boldsymbol{\tau}},\widetilde{q}] \in (Z^t)^\perp \times Z^t$, so $c(\widetilde{\boldsymbol{\sigma}},\widetilde{p};\widetilde{\boldsymbol{\tau}},\widetilde{q}) = 0$. Thus $[\boldsymbol{\sigma}_0,p_0]$ should satisfy $c(\boldsymbol{\sigma}_0,p_0;\widetilde{\boldsymbol{\tau}},\widetilde{q}) = 0$ for all $(\widetilde{\boldsymbol{\tau}},\widetilde{q}) \in Z^t$, hence we can take $(\boldsymbol{\sigma}_0,p_0) = (\boldsymbol{0},\operatorname{tr}\boldsymbol{\sigma}_0) = (\boldsymbol{0},0)$.

Lemma 3.1. *Problems* I *and* Ip *have a unique solution* $[\boldsymbol{d},\boldsymbol{u},\boldsymbol{\sigma},p]$ *in* $(D\times V)\times(S\times Q)$ *that satisfies the bound*

$$\|\boldsymbol{d}\|_0 + \|\boldsymbol{u}\|_1 + \|\boldsymbol{\sigma}\|_0 + \|p\|_0 \leqslant \eta_1\|\ell\|_{V'}, \tag{3.13}$$

in which the constant η_1 *is independent of* λ.

Remark. The λ-independence of the constant follows from the fact that the constants appearing in (3.13) depend inter alia on $\|c\|$, which in turn depends on λ^{-1}, and so is bounded.

3.2. The second formulation

We turn next to an alternative formulation based on that presented in [9]. We introduce the bilinear forms

$$\begin{aligned}&a_2(\boldsymbol{\sigma},\boldsymbol{d};\boldsymbol{e},\boldsymbol{v}) := (\boldsymbol{\sigma},\boldsymbol{\epsilon}(\boldsymbol{v}))_0 - (2\alpha\boldsymbol{\epsilon}(\boldsymbol{v}),\boldsymbol{d})_0 + (\boldsymbol{e},\boldsymbol{d})_0,\\ &b_3(\boldsymbol{e},\boldsymbol{v};\boldsymbol{u}) := (2\alpha\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{e},\boldsymbol{\epsilon}(\boldsymbol{u}))_0,\\ &b_4(\boldsymbol{\sigma},\boldsymbol{d};\boldsymbol{\tau}) := (\boldsymbol{\sigma} - \mathscr{C}\boldsymbol{d},\boldsymbol{\tau})_0,\end{aligned} \tag{3.14}$$

and define the following problem.

Problem II. Given $\ell \in V'$, find $[\boldsymbol{\sigma},\boldsymbol{d},\boldsymbol{u}] \in S\times D\times V$ that satisfy

$$\begin{aligned}&a_2(\boldsymbol{\sigma},\boldsymbol{d};\boldsymbol{e},\boldsymbol{v}) + b_3(\boldsymbol{e},\boldsymbol{v};\boldsymbol{u}) = \ell(\boldsymbol{v}) \quad \text{for all } [\boldsymbol{e},\boldsymbol{v}] \in D\times V,\\ &b_4(\boldsymbol{\sigma},\boldsymbol{d};\boldsymbol{\tau}) = 0 \quad \text{for all } \boldsymbol{\tau} \in S.\end{aligned} \tag{3.15}$$

Again we construct an equivalent four-field formulation of Problem II. To this end we introduce the bilinear forms

$$
\begin{aligned}
&a_{2p}(\boldsymbol{d},\boldsymbol{u};\boldsymbol{\tau},\boldsymbol{v}) = -2\alpha(\boldsymbol{d},\epsilon(\boldsymbol{v}))_0 + 2\alpha(\epsilon(\boldsymbol{v}),\epsilon(\boldsymbol{u}))_0 + 2\mu(\boldsymbol{d},\boldsymbol{\tau})_0,\\
&b_5(\boldsymbol{\tau},\boldsymbol{v};\boldsymbol{\sigma},p) = (\boldsymbol{\sigma},\epsilon(\boldsymbol{v}))_0 - (\boldsymbol{\sigma},\boldsymbol{\tau})_0 + (p,\operatorname{tr}\boldsymbol{\tau})_0,\\
&b_6(\boldsymbol{d},\boldsymbol{u};q,\boldsymbol{e}) = (q,\operatorname{tr}\boldsymbol{d})_0 - (\boldsymbol{d},\boldsymbol{e})_0 + (\epsilon(\boldsymbol{u}),\boldsymbol{e})_0,\\
&c(\boldsymbol{\sigma},p;q,\boldsymbol{e}) = \lambda^{-1}(p,q)_0.
\end{aligned}
\tag{3.16}
$$

The following problem is equivalent to Problem II.

Problem IIp. Given $\ell \in V'$, find $[\boldsymbol{\sigma},\boldsymbol{d},\boldsymbol{u},p] \in S\times D\times V\times Q$ that satisfy

$$
\begin{aligned}
&a_{2p}(\boldsymbol{d},\boldsymbol{u};\boldsymbol{\tau},\boldsymbol{v}) + b_5(\boldsymbol{\tau},\boldsymbol{v};\boldsymbol{\sigma},p) = \ell(\boldsymbol{v}) \quad \text{for all } [\boldsymbol{\tau},\boldsymbol{v}] \in S\times V,\\
&b_6(\boldsymbol{d},\boldsymbol{u};q,\boldsymbol{e}) - c(\boldsymbol{\sigma},p;q,\boldsymbol{e}) = 0 \quad \text{for all } [\boldsymbol{e},q] \in D\times Q.
\end{aligned}
\tag{3.17}
$$

In contrast to Problem Ip, Problem IIp is a non-standard mixed formulation. We turn again to the conditions for well-posedness as set out in Theorem 2.1.

It is easy to show that all the bilinear forms are continuous with constants independent of λ.

Second, we note that uniqueness of the solution should be in $(D\times V)\times(Q\times S)/Z_5^t \cap \ker C$, where $\ker C$ and Z_5^t are the kernels of the linear operators C and B_5^t. But $\ker C \cap Z_5^t = \{0\}$, thus the solution is unique in $(D\times V)\times(Q\times S)$.

Next, considering the sup condition on $b_5(\cdot,\cdot)$, let $[\boldsymbol{\sigma},p] \in S\times Q$. Choosing $\boldsymbol{w}$ as in (3.8) and setting $[\boldsymbol{\tau},\boldsymbol{v}] = [\epsilon(\boldsymbol{w}) - \gamma\boldsymbol{\sigma},\boldsymbol{w}]$ where $0<\gamma<2$, we have

$$
\begin{aligned}
\sup_{[\boldsymbol{\tau},\boldsymbol{v}]\in S\times V} \frac{b_5(\boldsymbol{\tau},\boldsymbol{v};\boldsymbol{\sigma},p)}{\|[\boldsymbol{\tau},\boldsymbol{v}]\|_{S\times V}} &= \sup_{[\boldsymbol{\tau},\boldsymbol{v}]\in S\times V} \frac{(\boldsymbol{\sigma},\epsilon(\boldsymbol{v}))_0 + (-\boldsymbol{\sigma}+p\mathbf{1},\boldsymbol{\tau})_0}{\|[\boldsymbol{\tau},\boldsymbol{v}]\|_{S\times V}} \geqslant \frac{(\boldsymbol{\sigma},\epsilon(\boldsymbol{w}))_0 + (-\boldsymbol{\sigma}+p\mathbf{1},\epsilon(\boldsymbol{w})-\gamma\boldsymbol{\sigma})_0}{\|[\epsilon(\boldsymbol{w})-\gamma\boldsymbol{\sigma},\boldsymbol{w}]\|_{S\times V}}\\
&\geqslant \frac{\gamma\|\boldsymbol{\sigma}\|_0^2 + \|p\|_0^2 - \gamma(p,\operatorname{tr}\boldsymbol{\sigma})_0}{2\|\boldsymbol{w}\|_V + \gamma\|\boldsymbol{\sigma}\|_0} \geqslant \beta_1\|[\boldsymbol{\sigma},p]\|_{S\times Q}.
\end{aligned}
\tag{3.18}
$$

Similarly we can show that $b_6(\cdot,\cdot)$ is inf–sup stable.

For the stability condition on $a_{2p}(\cdot,\cdot)$ let $[\boldsymbol{d},\boldsymbol{u}] \in \ker B_6$, that is $\boldsymbol{d} = \epsilon(\boldsymbol{u})$. Thus for $\boldsymbol{v}=\boldsymbol{u}$ we have

$$
\sup_{[\boldsymbol{\tau},\boldsymbol{v}]\in Z_5} \frac{a_{2p}(\boldsymbol{d},\boldsymbol{u};\boldsymbol{\tau},\boldsymbol{v})}{\|[\boldsymbol{\tau},\boldsymbol{v}]\|_{S\times V}} \geqslant \frac{a_{2p}(\epsilon(\boldsymbol{u}),\boldsymbol{u};\epsilon(\boldsymbol{u}),\boldsymbol{u})}{\|[\epsilon(\boldsymbol{u}),\boldsymbol{u}]\|_{S\times V}} \geqslant \mu c_K^2\|\boldsymbol{u}\|_1.
\tag{3.19}
$$

But since $[\boldsymbol{d},\boldsymbol{u}] \in Z_6$, we have $\|[\boldsymbol{d},\boldsymbol{u}]\| \leqslant \|\boldsymbol{d}\|_0 + \|\boldsymbol{u}\|_1 \leqslant \|\epsilon(\boldsymbol{u})\|_0 + \|\boldsymbol{u}\|_1 \leqslant 2\|\boldsymbol{u}\|_1$. Thus

$$
\sup_{[\boldsymbol{\tau},\boldsymbol{v}]\in Z_5} \frac{a_{2p}(\boldsymbol{d},\boldsymbol{u};\boldsymbol{\tau},\boldsymbol{v})}{\|[\boldsymbol{\tau},\boldsymbol{v}]\|_{S\times V}} \geqslant \frac{\mu c_K^2}{4}\|[\boldsymbol{d},\boldsymbol{u}]\|_{D\times V}.
\tag{3.20}
$$

Turning to the second condition on $a_{2p}(\cdot,\cdot)$, take $[\boldsymbol{\tau},\boldsymbol{v}] \in Z_5\backslash\{\mathbf{0}\}$; then

$$
\begin{aligned}
\sup_{[\boldsymbol{d},\boldsymbol{u}]\in Z_6} a_{2p}(\boldsymbol{\tau},\boldsymbol{v};\boldsymbol{d},\boldsymbol{u}) &= \sup_{\boldsymbol{u}\in V} a_{2p}(\epsilon(\boldsymbol{v}),\boldsymbol{v};\epsilon(\boldsymbol{u}),\boldsymbol{u}) \geqslant a_{2p}(\epsilon(\boldsymbol{v}),\boldsymbol{v};\epsilon(\boldsymbol{v}),\boldsymbol{v}) \quad (\text{taking } \boldsymbol{u}=\boldsymbol{v})\\
&= 2\mu\|\epsilon(\boldsymbol{v})\|_0^2 > 2\mu c_K^2\|\boldsymbol{v}\|_1^2 > 0 \quad (\text{since } \boldsymbol{v}\neq\mathbf{0}).
\end{aligned}
\tag{3.21}
$$

Finally, for the verification of condition (c) in Theorem 2.1, let $[\widetilde{\boldsymbol{\sigma}},\widetilde{p}] \in (\ker B_5^t)^{\perp}$, and $[q,\boldsymbol{e}] \in \ker B_6^t$, we have $(\widetilde{p},q)_0 = 0$ and $c(\widetilde{\boldsymbol{\sigma}},\widetilde{p};q,\boldsymbol{e}) = \lambda^{-1}(\widetilde{p},q) = 0$, thus it suffices to take $[\boldsymbol{\sigma}_0,p_0] = [\mathbf{0},\operatorname{tr}\sigma_0] = [\mathbf{0},0]$.

Lemma 3.2. *Problems* II *and* IIp *have a unique solution* $[\boldsymbol{d},\boldsymbol{u},p,\boldsymbol{\sigma}]$ *in* $(D\times V)\times(S\times Q)$ *that satisfies the bound*

$$
\|\boldsymbol{d}\|_0 + \|\boldsymbol{u}\|_1 + \|\boldsymbol{\sigma}\|_0 + \|p\|_0 \leqslant \eta_2\|\ell\|_{V'},
\tag{3.22}
$$

in which the constant η_2 *is independent of* λ.

4. Finite element approximations

In this section we analyse the discrete formulations of Problems I and II, referred to respectively as Problems Ih and IIh, via the discrete counterparts of Problems Iph and IIph. The discrete spaces are chosen in such a way that Problems Iph and IIph are equivalent to Ih and IIh, respectively. The goal is to establish well-posedness and convergence uniformly in the incompressible limit. The emphasis is on low-order quadrilateral elements, with the displacement approximated by continuous piecewise-bilinear functions.

4.1. Finite element spaces

We assume that $\mathscr{T}_h$ is a regular partition of Ω, in which the elements are generated from a reference square $\hat{K} = (-1,1)^2$ through a family of affine maps. The diameter of an element $K \in \mathscr{T}_h$ is denoted by h_K, and the mesh size h is defined by $h = \max_{K\in\mathscr{T}_h} h_K$. For non-negative integer k, let $\mathscr{P}_k$ be the space of polynomials in two variables of total degree less than or equal to k, and $\mathscr{Q}_k$ the space of polynomials in two variables of total degree less than or equal to k in each variable. Then a typical element $K \in \mathscr{T}_h$ in generated by the affine map F_K from the reference element $\hat{K}$; that is,

$$F_K(\hat{\mathbf{x}}) = B_K\hat{\mathbf{x}} + b_K \tag{4.1}$$

where B_K is a constant, invertible matrix and b_K is a vector. It is clear that if $\hat{\mathbf{v}} \in [\mathscr{Q}_1(\hat{K})]^2$, then $\hat{\mathbf{v}} \circ F_K^{-1} \in [\mathscr{Q}_1(K)]^2$.

We consider the following discrete version of the abstract problem (2.9). With the bilinear forms defined as before, we introduce the finite-dimensional subspaces $\Phi_h \subset \Phi$, $\Psi_h \subset \Psi$, $M_h \subset M$, and $N_h \subset N$, and consider the problem of finding $(\phi_h, m_h) \in \Phi_h \times M_h$ that satisfy, for $(f,g) \in \Psi' \times N'$,

$$\begin{aligned} &a(\phi_h,\psi_h) + b_1(\psi_h,m_h) = \langle f,\psi_h\rangle \quad \text{for all } \psi_h \in \Psi_h,\\ &b_2(\phi_h,n_h) - c(m_h,n_h) = \langle g,n_h\rangle \quad \text{for all } n_h \in N_h. \end{aligned} \tag{4.2}$$

By analogy with Theorem 2.1 we have the following result on the discrete problem. As in the continuous case the results in [2,18] have been extended to the case $c(\cdot,\cdot) \neq 0$. We omit the details of the proof, which follows that in [2,18].

Theorem 4.1. *Define the kernels Z_h, Z_h^t and K_h according to*

$$\begin{aligned} &Z_h = \{\psi_h \in \Phi_h | b(\psi_h,n_h) = 0 \textit{ for all } n_h \in M_h\},\\ &Z_h^t = \{n_h \in M_h | b(\psi_h,n_h) = 0 \textit{ for all } \psi_h \in \Phi_h\},\\ &K_h = \{m_h \in M_h | c(m_h,n_h) = 0 \textit{ for all } n_h \in N_h\}, \end{aligned} \tag{4.3}$$

and suppose that the bilinear forms $a(\cdot,\cdot)$, $b_i(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are such that

(a) *there exists a constant $\alpha_{1h} > 0$ such that*

$$\sup_{\psi_h\in Z_{1h}} \frac{a(\phi_h,\psi_h)}{\|\psi_h\|_\Psi} \geqslant \alpha_{1h}\|\phi_h\|_\Phi, \quad \forall \phi_h \in Z_{2h},$$
$$\sup_{\phi_h\in Z_{2h}} a(\phi_h,\psi_h) > 0, \quad \forall \psi_h \in Z_{1h}\setminus\{0\};$$

(b) *there exist two positive constants β_{1h}, β_{2h} such that for all $(m_h,n_h) \in M_h \times N_h$,*

$$\sup_{\psi_h\in\Psi_h} \frac{b_1(\psi_h,m_h)}{\|\psi_h\|_\Psi} \geqslant \beta_{1h}\|m_h\|_M \quad \textit{and} \quad \sup_{\phi_h\in\Phi_h} \frac{b_2(\phi_h,n_h)}{\|\phi_h\|_\Phi} \geqslant \beta_{2h}\|n_h\|_N;$$

(c) *the continuous bilinear form $c(\cdot,\cdot)$ is positive semidefinite and symmetric, and there exists a positive constant γ_{0h} such that for all $\tilde{m}_h \in (Z_{2h}^t)^\perp$, one can find $m_{0h} \in Z_{2h}^t$ satisfying*

$$\begin{aligned} &\gamma_{0h}\|m_{0h}\|_M \leqslant \|\tilde{m}_h\|_M,\\ &c(m_{0h},n_h) = -c(\tilde{m}_h,n_h), \quad \textit{for all } n_h \in Z_{2h}^t. \end{aligned} \tag{4.4}$$

Then for $[f,g] \in \Psi' \times \operatorname{Im} B_2'$, the problem (4.2) has a unique solution $[\phi_h, m_h]$ in $\Phi_h \times M_h/Z_{1h}^t \cap K_h$ which moreover satisfies the bounds

$$\|[m_h,\phi_h]\|_{M\times\Phi} \leqslant d_{1h}\|[f,g]\|_{\Psi'\times N'}. \tag{4.5}$$

If $(\phi,m) \in \Phi \times M$ is the solution of (2.9) then

$$\|[\phi-\phi_h, m-m_h]\|_{\Phi\times M} \leqslant K_3\left\{\inf_{\overline{\phi}_h\in\Phi_h}\|\phi-\overline{\phi}_h\|_\Phi + \inf_{\overline{m}_h\in M_h}\|m-\overline{m}_h\|_M\right\}. \tag{4.6}$$

Returning to the problems at hand, we define the space V_h of discrete displacements to be the subspace of V comprising continuous functions whose restriction to an element are bilinear polynomials; that is,

$$V_h = \{\mathbf{v}_h | \mathbf{v}_h|_K \in [\mathscr{Q}_1(K)]^2\}. \tag{4.7}$$

The finite element space of pressures Q_h is defined by

$$Q_h = \{q_h \in L_0^2(\Omega);\ q_h|_K = \hat{q}_h \circ F_K^{-1},\ \hat{q}_h \in \mathscr{P}_0(\hat{K}) \text{ for all } K \in \mathscr{T}_h\}. \tag{4.8}$$

The finite element spaces of strains and stresses are denoted respectively by D_h and S_h, and are subspaces of D and S, respectively. These spaces will be defined shortly, in the context of Problems Ih and IIh.

4.2. Finite element analysis of Problem I

An assumption made in [8], and adopted here, is that the discrete spaces of stresses and strains are equal; that is, for Problem I

$$S_h = D_h. \tag{4.9}$$

We now introduce discrete variants of Problems I and Ip.

Problem Ih. Given $\ell \in V'$, find $[\boldsymbol{\sigma}_h, \boldsymbol{d}_h, \boldsymbol{u}_h] \in S_h \times D_h \times V_h$ that satisfy

$$\begin{aligned} &a_1(\boldsymbol{\sigma}_h, \boldsymbol{d}_h; \boldsymbol{\tau}_h, \boldsymbol{v}_h) + b_1(\boldsymbol{\tau}_h, \boldsymbol{v}_h; \boldsymbol{u}_h) = \ell(\boldsymbol{v}_h) \quad \text{for all } [\boldsymbol{\tau}_h, \boldsymbol{v}_h] \in S_h \times V_h, \\ &b_2(\boldsymbol{\sigma}_h, \boldsymbol{d}_h; \boldsymbol{e}_h) = 0 \quad \text{for all } \boldsymbol{e}_h \in D_h. \end{aligned} \tag{4.10}$$

Problem Iph. Given $\ell \in V'$, find $[\boldsymbol{\sigma}_h, \boldsymbol{d}_h, p_h, \boldsymbol{u}_h] \in S_h \times D_h \times Q_h \times V_h$ that satisfy

$$\begin{aligned} &a_{1p}(\boldsymbol{d}_h, \boldsymbol{u}_h; \boldsymbol{e}_h, \boldsymbol{v}_h) + b_{1p}(\boldsymbol{e}_h, \boldsymbol{v}_h; p_h, \boldsymbol{\sigma}_h) = \ell(\boldsymbol{v}_h) \quad \text{for all } [\boldsymbol{e}_h, \boldsymbol{v}_h] \in D_h \times V_h, \\ &b_{1p}(\boldsymbol{d}_h, \boldsymbol{u}_h; q_h, \boldsymbol{\tau}_h) - c(p_h, \boldsymbol{\sigma}_h; q_h, \boldsymbol{\tau}_h) = 0 \quad \text{for all } [\boldsymbol{\tau}_h, q_h] \in S_h \times Q_h. \end{aligned} \tag{4.11}$$

Assumption. We assume that

$$\operatorname{tr} D_h \subseteq Q_h. \tag{4.12}$$

Under assumption (4.12) Problems Ih and Iph are *equivalent*. We focus attention henceforth on Problem Iph.

Lemma 4.2. *Assume that (4.9) and (4.12) hold, and further, that $\alpha > 0$. If*

(a) (V_h, Q_h) *is stable for the classical Stokes problem,*
(b) $Q_h\mathbf{1} \subseteq S_h$,

then Problems Ih and Iph have a unique solution $[\boldsymbol{d}_h, \boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h] \in D_h \times V_h \times Q_h \times S_h$ which satisfies the bound

$$\|\boldsymbol{d}_h\|_0 + \|\boldsymbol{u}_h\|_1 + \|\boldsymbol{\sigma}_h\|_0 + \|p_h\|_0 \leqslant c\|\ell\|_{V'}. \tag{4.13}$$

If $[\boldsymbol{d}, \boldsymbol{u}, \boldsymbol{\sigma}, p] \in D \times V \times S \times Q$ is the solution to Problem Ip, then the discretization error

$$\eta_h^2 = \|\boldsymbol{d} - \boldsymbol{d}_h\|_0^2 + \|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 + \|p - p_h\|_0^2$$

is bounded by

$$\eta_h \leqslant c\left\{\inf_{\boldsymbol{e}_h \in D_h} \|\boldsymbol{d}_h - \boldsymbol{e}_h\|_0 + \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \inf_{\boldsymbol{\tau}_h \in S_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_0 + \inf_{q_h \in Q_h} \|p - q_h\|_0\right\}, \tag{4.14}$$

where c is a positive constant independent of λ and h.

Remark. The lemma asserts that the solution $[\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h, p_h]$ of Problems Ih and Iph converges uniformly to the solution $[\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}, p]$ of Problems I and Ip under the given set of conditions.

Proof. To prove this result, we need to verify the conditions of Theorem 4.1. We first show that $b_{1p}(\cdot,\cdot)$ is λ-independent inf–sup stable. Let $[q_h, \boldsymbol{\tau}_h] \in Q_h \times S_h$. Then [13] there exists $c_2 > 0$, independent of h and $\boldsymbol{w}_h \in V_h$, such that

$$(q_h, \operatorname{div} \boldsymbol{w}_h)_0 = \|q_h\|_0^2 \quad \text{and} \quad \|\boldsymbol{w}_h\|_1 \leqslant c_2\|q_h\|_0. \tag{4.15}$$

Let P_{S_h} be the orthogonal projection onto the subspace S_h. Since $S_h = D_h$ we can take $\boldsymbol{e}_h = P_{S_h}\epsilon(\boldsymbol{w}_h) - \boldsymbol{\tau}_h$. From the Projection Theorem and (4.15) we obtain

$$\sup_{[\boldsymbol{e}_h,\boldsymbol{v}_h]\in D_h\times V_h}\frac{b_{1p}(\boldsymbol{e}_h,\boldsymbol{v}_h;q_h,\boldsymbol{\tau}_h)}{\|[\boldsymbol{e}_h,\boldsymbol{v}_h]\|_{D\times V}} \geqslant \frac{b_{1p}(P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{w}_h)-\boldsymbol{\tau}_h,\boldsymbol{w}_h;q_h,\boldsymbol{\tau}_h)_0}{\|[P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{w}_h)-\boldsymbol{\tau}_h,\boldsymbol{w}_h]\|_{D\times V}} \geqslant \frac{\|q_h\|_0^2+\|\boldsymbol{\tau}_h\|_0^2-(q_h,\operatorname{tr}\boldsymbol{\tau}_h)_0}{2\|\boldsymbol{w}_h\|_1+\|\boldsymbol{\tau}_h\|_0}$$
$$\geqslant \frac{\left(1-\frac{\sqrt{2}}{2}\right)\|[q_h,\boldsymbol{\tau}_h]\|_{Q\times S}^2}{2c_2\|q_h\|_0+\|\boldsymbol{\tau}_h\|_0} \geqslant \frac{2-\sqrt{2}}{2\sqrt{4c_2^2+1}}\|[q_h,\boldsymbol{\tau}_h]\|_{Q\times S}. \tag{4.16}$$

Next, to establish the coercivity of $a_{1p}(\cdot,\cdot)$ on the discrete kernel Z_h of $b_{1p}(\cdot,\cdot)$, defined by

$$Z_h=\left\{[P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h),\boldsymbol{v}_h]\mid \operatorname{tr}(P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h))=0 \text{ for all } \boldsymbol{v}_h\in V_h\right\},$$

we take $[\boldsymbol{d}_h,\boldsymbol{u}_h]\in Z_h$, that is, $\boldsymbol{d}_h=P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)$, and let $s\in\mathbb{R}$. Then

$$\begin{aligned}a_{1p}(\boldsymbol{d}_h,\boldsymbol{u}_h;\boldsymbol{d}_h,\boldsymbol{u}_h)&=2\mu\|\boldsymbol{d}_h\|_0^2+2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2-2\alpha\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2\\&=2\mu\|\boldsymbol{d}_h\|_0^2+2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2-2s\alpha\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2-2\alpha(1-s)\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h)\|_0^2\\&=2\mu\|\boldsymbol{d}_h\|_0^2+2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2-2s\alpha\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2-2\alpha(1-s)\|\boldsymbol{d}_h\|_0^2\\&=2(\mu-\alpha(1-s))\|\boldsymbol{d}_h\|_0^2+2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2-2s\alpha\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2.\end{aligned}$$

We now consider two cases.

(a) $1-\frac{\mu}{\alpha}<s<0$: for this case we have $0<\alpha<\mu$, and $2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2-2s\alpha\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2\geqslant 2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_0^2$. Thus

$$\begin{aligned}a_{1p}(\boldsymbol{d}_h,\boldsymbol{u}_h;\boldsymbol{d}_h,\boldsymbol{u}_h)&\geqslant 2(\mu-\alpha(1-s))\|\boldsymbol{d}_h\|_0^2+2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_0^2\geqslant 2(\mu-\alpha(1-s))\|\boldsymbol{d}_h\|_0^2+2\alpha c_K^2\|\boldsymbol{u}\|_1^2\\&\geqslant 2\min\left(\mu-\alpha(1-s),\alpha c_K^2\right)\|[\boldsymbol{d}_h,\boldsymbol{u}_h]\|_{D\times V}^2,\end{aligned}\tag{4.17}$$

where as before c_K is the constant in Korn's inequality.

(b) $0\leqslant 1-\frac{\mu}{\alpha}<s<1$: for this case we have $\alpha\geqslant\mu$, and from $\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0\leqslant\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0$, we obtain

$$2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2-2s\alpha\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2\geqslant 2\alpha(1-s)\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_0^2.$$

Thus

$$\begin{aligned}a_{1p}(\boldsymbol{d}_h,\boldsymbol{u}_h;\boldsymbol{d}_h,\boldsymbol{u}_h)&=2(\mu-\alpha(1-s))\|\boldsymbol{d}_h\|_0^2+2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2-2s\alpha\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2\\&\geqslant 2(\mu-\alpha(1-s))\|\boldsymbol{d}_h\|_0^2+2\alpha(1-s)\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2\\&\geqslant 2(\mu-\alpha(1-s))\|\boldsymbol{d}_h\|_0^2+2\alpha(1-s)c_K^2\|\boldsymbol{u}_h\|_1^2\\&\geqslant 2\min\left(\mu-\alpha(1-s),\alpha(1-s)c_K^2\right)\|[\boldsymbol{d}_h,\boldsymbol{u}_h]\|_{D\times V}^2.\end{aligned}$$

Finally, we check condition (c) of Theorem 2.1 for $Z_1=Z_2=Z_h$. From the definition of $c(\cdot,\cdot)$, and the fact that $[\widetilde{\boldsymbol{\sigma}}_h,\widetilde{p}_h;\widetilde{\boldsymbol{\tau}}_h,\widetilde{q}_h]\in(Z_h^t)^\perp\times Z_h^t$, we have $c(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{p}_h;\widetilde{\boldsymbol{\tau}}_h,\widetilde{q}_h)=0$. Thus $c(\boldsymbol{\sigma}_0,p_0;\widetilde{\boldsymbol{\tau}}_h,\widetilde{q}_h)=0$ for all $[\widetilde{\boldsymbol{\tau}}_h,\widetilde{q}_h]\in Z_h^t$, and we can take $[\boldsymbol{\sigma}_0,p_0]=[\boldsymbol{0},\operatorname{tr}\boldsymbol{\sigma}_0]=[\boldsymbol{0},0]$, with γ_0 any positive number.

This completes the proof of the lemma. □

Remarks

(a) It is important to note that the stability of the displacement–pressure pair is needed for the inf–sup condition on $b_{1p}(\cdot,\cdot)$.

(b) The admissible range of α varies according to the sign of s, which can be chosen a priori.

(c) If $\alpha=0$, then the bilinear form a_{1p} is coercive if a discrete Korn's inequality which has been recently discussed in length in [6] is satisfied. That condition requires that there exists a positive constant $c<1$, independent of h, such that for all $(\boldsymbol{v}_h,K)\in V_h\times\mathscr{T}_h$,

$$\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h)\|_K\geqslant c\|\boldsymbol{\epsilon}(\boldsymbol{v}_h)\|_K. \tag{4.18}$$

Later, we shall choose bases for which (4.18) is *not* satisfied for the case $\alpha=0$.

4.3. Finite element analysis of Problem II

We assume here that

$$D_h\subseteq S_h. \tag{4.19}$$

As in the case of Problem I we begin by defining the discrete versions of Problems II and IIp.

Problem IIh. Given $\ell \in V'$, find $[\boldsymbol{\sigma}_h, \boldsymbol{d}_h, \boldsymbol{u}_h] \in S_h \times D_h \times V_h$ that satisfy

$$\begin{aligned} &a_2(\boldsymbol{\sigma}_h, \boldsymbol{d}_h; \boldsymbol{m}_h, \boldsymbol{v}_h) + b_3(\boldsymbol{m}_h, \boldsymbol{v}_h; \boldsymbol{u}_h) = \ell(\boldsymbol{v}_h) \quad \text{for all } [\boldsymbol{m}_h, \boldsymbol{v}_h] \in D_h \times V_h, \\ &b_4(\boldsymbol{\sigma}_h, \boldsymbol{d}_h; \boldsymbol{\tau}_h) = 0 \quad \text{for all } \boldsymbol{\tau}_h \in D_h. \end{aligned} \tag{4.20}$$

Problem IIph. Given $\ell \in V'$, find $[\boldsymbol{\sigma}_h, \boldsymbol{d}_h, \boldsymbol{u}_h, p_h] \in S_h \times D_h \times V_h \times Q_h$ that satisfy

$$\begin{aligned} &a_{2p}(\boldsymbol{d}_h, \boldsymbol{u}_h; \boldsymbol{\tau}_h, \boldsymbol{v}_h) + b_5(\boldsymbol{\tau}_h, \boldsymbol{v}_h; \boldsymbol{\sigma}_h, p_h) = \ell(\boldsymbol{v}_h) \quad \text{for all } [\boldsymbol{\tau}_h, \boldsymbol{v}_h] \in S_h \times V_h, \\ &b_6(\boldsymbol{d}_h, \boldsymbol{u}_h; q_h, \boldsymbol{e}_h) - c(\boldsymbol{\sigma}_h, p_h; q_h, \boldsymbol{e}_h) = 0 \quad \text{for all } [\boldsymbol{e}_h, q_h] \in D_h \times Q_h. \end{aligned} \tag{4.21}$$

We once again adopt assumption (4.12), so that Problems IIh and IIph are equivalent, and we proceed by analyzing Problem IIph.

Remark. It is easy to show that when $S_h = D_h$, Problems Ih and IIh are equivalent.

For the well-posedness and uniform convergence of Problems IIh and IIph we have

Lemma 4.3. *If $0 < \alpha \leqslant \mu$, $\operatorname{tr} D_h \subseteq Q_h$, the pair (V_h, Q_h) is Stokes stable, and $Q_h\mathbf{1} \subseteq D_h$, then Problems IIph and IIh have a solution $(\boldsymbol{d}_h, \boldsymbol{u}_h, \boldsymbol{\sigma}_h, p_h)$ which is unique in $(D_h \times V_h) \times (S_h \times Q_h)$ and which satisfy the bound*

$$\|\boldsymbol{d}_h\|_0 + \|\boldsymbol{u}_h\|_1 + \|\boldsymbol{\sigma}_h\|_0 + \|p_h\|_0 \leqslant c\|\ell\|_{V'}. \tag{4.22}$$

If $(\boldsymbol{d}, \boldsymbol{u}; p, \boldsymbol{\sigma}) \in (D \times V) \times (Q \times S)$ is the solution to Problem IIp, then the error $\eta_h^2 = \|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 + \|\boldsymbol{d} - \boldsymbol{d}_h\|_0^2 + \|p - p_h\|_0^2$ is bounded by

$$\eta_h \leqslant c\left\{\inf_{\tilde{\boldsymbol{e}}_h \in D_h} \|\boldsymbol{d}_h - \tilde{\boldsymbol{e}}_h\|_0 + \inf_{\tilde{\boldsymbol{v}}_h \in V_h} \|\boldsymbol{u} - \tilde{\boldsymbol{v}}_h\|_1 + \inf_{\tilde{\boldsymbol{\tau}}_h \in S_h} \|\boldsymbol{\sigma} - \tilde{\boldsymbol{\tau}}_h\|_0 + \inf_{\tilde{q}_h \in Q_h} \|p - \tilde{q}_h\|_0\right\}$$

in which the constant c is independent of λ.

Proof. Again here, we need to verify the conditions of Theorem 4.1. The bilinear forms $a_{2p}(\cdot,\cdot)$, $b_5(\cdot,\cdot)$ and $b_6(\cdot,\cdot)$, and $c(\cdot,\cdot)$ are continuous since $V_h \hookrightarrow V$, $D_h \hookrightarrow D$, $S_h \hookrightarrow S, Q_h \hookrightarrow Q$, and we have $\|a_{2p}\| \leqslant 4\alpha + 2\mu$, $\|b_i\| \leqslant 2\sqrt{2}$ and $\|c\| \leqslant \frac{1}{\lambda}$.

The sup conditions on b_5 and b_6 can be established exactly as in the case of Problem Iph, and the details will therefore will not be presented here.

In order to verify the stability of $a_{2p}(\cdot,\cdot)$, knowing that $Q_h\mathbf{1} \subseteq D_h \subset S_h$, we first observe that the kernels Z_{5h} and Z_{6h} are given by

$$\begin{aligned} Z_{5h} &= \{[P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h), \boldsymbol{v}_h] \mid (p_h\mathbf{1}, \boldsymbol{\epsilon}(\boldsymbol{v}_h))_0 = 0 \text{ for all } p_h \in Q_h\}, \\ Z_{6h} &= \{[P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h), \boldsymbol{u}_h] \mid (q_h\mathbf{1}, \boldsymbol{\epsilon}(\boldsymbol{u}_h)) = 0 \text{ for all } q_h \in Q_h\}. \end{aligned} \tag{4.23}$$

For $[\boldsymbol{d}_h, \boldsymbol{u}_h] \in Z_{6h}$, that is $\boldsymbol{d}_h = P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)$, and taking $\boldsymbol{v}_h = \boldsymbol{u}_h$, we have

$$\begin{aligned} \sup_{[\boldsymbol{\tau}_h, \boldsymbol{v}_h] \in Z_{1h}} \frac{a_{2p}(\boldsymbol{d}_h, \boldsymbol{u}_h; \boldsymbol{\tau}_h, \boldsymbol{v}_h)}{\|[\boldsymbol{\tau}_h, \boldsymbol{v}_h]\|_{S\times V}} &= \sup_{\boldsymbol{v}_h \in V_h} \frac{a_{2p}(P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h), \boldsymbol{u}_h; P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h), \boldsymbol{v}_h)}{\|[P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h), \boldsymbol{v}_h]\|_{S\times V}} \\ &\geqslant \frac{2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2 + (2\mu - 2\alpha)\|P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0^2}{2\|\boldsymbol{u}_h\|_1} + \frac{2\mu(P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h), P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h) - \boldsymbol{\epsilon}(\boldsymbol{u}_h))_0}{2\|\boldsymbol{u}_h\|_1}. \end{aligned}$$

The satisfaction of the stability condition depends on the sizes of D_h, S_h and α. We clearly see that we should take $0 < \alpha \leqslant \mu$. Now, for $D_h \subset S_h$, and using the identity $(P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h), P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h) - \boldsymbol{\epsilon}(\boldsymbol{u}_h))_0 = 0$, we have

$$\sup_{(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in Z_{1h}} \frac{a_{2p}(\boldsymbol{d}_h, \boldsymbol{u}_h; \boldsymbol{\tau}_h, \boldsymbol{v}_h)}{\|[\boldsymbol{\tau}_h, \boldsymbol{v}_h]\|_{S\times V}} \geqslant \alpha c_K^2 \|\boldsymbol{u}_h\|_1. \tag{4.24}$$

But since $[\boldsymbol{d}_h, \boldsymbol{u}_h] \in Z_{2h}$, we have $\|[\boldsymbol{d}_h, \boldsymbol{u}_h]\| \leqslant \|\boldsymbol{d}_h\|_0 + \|\boldsymbol{u}_h\|_1 \leqslant \|P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h)\|_0 + \|\boldsymbol{u}_h\|_1 \leqslant 2\|\boldsymbol{u}_h\|_1$. That is, $\|\boldsymbol{u}_h\|_1 \geqslant \frac{1}{2}\|(\boldsymbol{d}, \boldsymbol{u})\|_{D\times V}$. Finally,

$$\sup_{[\boldsymbol{\tau}_h, \boldsymbol{v}_h] \in Z_{1h}} \frac{a_{2p}(\boldsymbol{d}_h, \boldsymbol{u}_h; \boldsymbol{\tau}_h, \boldsymbol{v}_h)}{\|[\boldsymbol{\tau}_h, \boldsymbol{v}_h]\|_{S\times V}} \geqslant \frac{\alpha c_K^2}{2}\|[\boldsymbol{d}_h, \boldsymbol{u}_h]\|_{D\times V}. \tag{4.25}$$

Turning to the second sup condition on $a_{2p}(\cdot,\cdot)$, for $[\boldsymbol{\tau}_h, \boldsymbol{v}_h] \in Z_{5h}\backslash\{\boldsymbol{0}\}$ we have

$$\begin{aligned}\sup_{[\boldsymbol{d}_h,\boldsymbol{u}_h]\in Z_{2h}} a_{2p}(\boldsymbol{d}_h, \boldsymbol{u}_h; \boldsymbol{\tau}_h, \boldsymbol{v}_h) &= \sup_{\boldsymbol{u}_h\in V_h} a_{2p}(P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{v}), \boldsymbol{v}; P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h), \boldsymbol{v}_h) \geqslant a_{2p}(P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h), \boldsymbol{v}_h; P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h), \boldsymbol{v}_h) \quad (\text{taking } \boldsymbol{u}_h = \boldsymbol{v}_h)\\ &= 2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{v}_h)\|_0^2 + (2\mu - 2\alpha)\|P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h)\|_0^2 + 2\mu(P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h), P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h) - \boldsymbol{\epsilon}(\boldsymbol{v}_h))_0\\ &= 2\alpha\|\boldsymbol{\epsilon}(\boldsymbol{v}_h)\|_0^2 + 2(\mu - \alpha)\|P_{D_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h)\|_0^2 > 0.\end{aligned}$$

The second condition on $a_{2p}(\cdot,\cdot)$ is obtained in the same way. The condition (c) is checked exactly as in the continuous case and therefore will not be repeated here. Thus the lemma is proven. □

4.4. Some examples of spaces

In this section we construct finite-dimensional spaces V_h, D_h, S_h and Q_h that satisfy the conditions specified in Lemmas 4.2 and 4.3.

Before giving some concrete examples of bases for S_h and D_h, we introduce the Voigt vectorial representation of the tensorial quantities stress and strain. These are written as

$$\boldsymbol{\sigma} = \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} \quad \text{and} \quad \boldsymbol{d} = \begin{bmatrix} d_{11}\\ d_{22}\\ 2d_{12}\end{bmatrix}. \tag{4.26}$$

The function $\boldsymbol{\epsilon}(\boldsymbol{u})$ representing the strain as a function of displacement is written in vectorial form in the same way as $\boldsymbol{d}$.

For convenience we work with a uniform mesh of square elements. Starting with Lemma 4.2, and knowing that the pair $Q_1 - P_0$ is not Stokes stable [3,13], we first partition $\mathscr{T}_h$ into a mesh of 2×2 macro-elements, as shown in Fig. 1.

Next, from standard arguments [3,4,13] we define the subspace Q_h by

$$Q_h = \{q_h \in L^2(\Omega); q_h|_K = \hat{q}_h \circ F_K^{-1}, q_h \in \mathscr{P}_0(\hat{K}),\ q_h|_{\mathscr{K}} = \alpha_1\phi^1_{\mathscr{K}} + \alpha_2\phi^2_{\mathscr{K}} + \alpha_3\phi^3_{\mathscr{K}} \text{ for all } \mathscr{K} \in \mathscr{T}_{2h}\},$$

where α_i are real numbers and $\phi^i_{\mathscr{K}}$ are defined by

$$\phi^1_{\mathscr{K}} = \begin{cases} 1 & \text{on } K_1,\\ 1 & \text{on } K_2,\\ 1 & \text{on } K_3,\\ 1 & \text{on } K_4,\end{cases} \quad \phi^2_{\mathscr{K}} = \begin{cases} -1 & \text{on } K_1,\\ 1 & \text{on } K_2,\\ 1 & \text{on } K_3,\\ -1 & \text{on } K_4,\end{cases} \quad \phi^3_{\mathscr{K}} = \begin{cases} -1 & \text{on } K_1,\\ -1 & \text{on } K_2,\\ 1 & \text{on } K_3,\\ 1 & \text{on } K_4.\end{cases} \tag{4.27}$$

We introduce the checkerboard space

$$CB_h = \{q_h \in L^2(\Omega); q_h|_K = \hat{q}_h \circ F_K^{-1}, q_h \in \mathscr{P}_0(\hat{K}),\ q_h|_{\mathscr{K}} = \alpha_4 CB_{\mathscr{K}} \text{ for all } \mathscr{K} \in \mathscr{T}_{2h}\},$$

where

$$CB_{\mathscr{K}} = \begin{cases} 1 & \text{on } K_1,\\ -1 & \text{on } K_2,\\ 1 & \text{on } K_3,\\ -1 & \text{on } K_4.\end{cases} \tag{4.28}$$

Finally, we define the space

$$V_h = \{\boldsymbol{v}_h | \boldsymbol{v}_h \in \widetilde{V}_h | (\operatorname{div} \boldsymbol{v}_h, q_h)_0 = 0 \text{ for all } q_h \in CB_h\}, \tag{4.29}$$

where

$$\widetilde{V}_h = \{\boldsymbol{v}_h \in V, \boldsymbol{v}_h|_K = \hat{\boldsymbol{v}} \circ F_K^{-1}, \hat{\boldsymbol{v}}_h \in \mathscr{Q}_1(\hat{K}) \text{ for all } K \in \mathscr{T}_h\}. \tag{4.30}$$

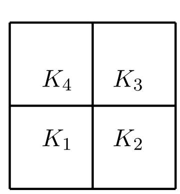

Fig. 1. Macro element $\mathscr{K}$.

From [3,13], the pair $(\widetilde{V}_h, Q_h)$ is stable for the Stokes problem and equivalent in terms of degrees of freedom to the $Q_2 - P_1$ discretization on the macro-element $\mathscr{K}$.

An alternative approach would be to adopt the "overstabilization" procedure proposed in [22,23], which amounts to taking

$$Q_h = \{q_h \in L^2(\Omega) | q_h|_K = \hat{q}_h \circ F_K^{-1}, q_h \in \mathscr{P}_0(\hat{K}),\ q_h|_{\mathscr{K}} = \alpha_1 \phi^1_{\mathscr{K}} \text{ for all } \mathscr{K} \in \mathscr{T}_{2h}\}.$$

In this last approximation the couple (V_h, Q_h) is equivalent to the $Q_2 - P_0$ element.

Now for the stress approximation we must first impose the condition (4.12), and taking into account the fact that the pressure is approximated on macro-elements, the same will apply to the stress. Therefore the direct components of the stress should be spanned by $(\phi^i_{\mathscr{K}})^3_{i=1}$. For Problem Ih we consider

$$D_h = S_h = \{\boldsymbol{\tau}_h \in S | (\boldsymbol{\tau}_h|_{\mathscr{K}})_{kl} = (\hat{\boldsymbol{\tau}}_h)_{kl} \circ F^{-1}_{\mathscr{K}}, \hat{\boldsymbol{\tau}}_h \in S_{\hat{K}} \text{ for all } \mathscr{K} \in \mathscr{T}_{2h}\}, \tag{4.31}$$

where $S_{\hat{K}} = \sum_{i=1}^3 S^1 \phi^i_{\mathscr{K}} + S^2 CB_{\mathscr{K}}$ and

$$S^1 = \text{span}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S^2 = \text{span}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \hat{x} & \hat{y} \end{bmatrix}. \tag{4.32}$$

We see that because of the choice of S^2, the checkerboard mode has been eliminated, and the trace of such stress and strain is constant on each macro-element. With this choice we see that (4.18) is not satisfied, since for that inequality to hold we should require at least that $\dim S_h \geqslant \dim \epsilon(V_h) = 15$ on each macro-element. But here the dimension of S_h restricted to each macro-element is 12. Therefore for that choice of element, the corresponding Hu–Washizu formulation for $\alpha = 0$ is not well-posed.

We conclude the study of Problems Iph and Ih with the following result.

Theorem 4.4. *Let* $[\boldsymbol{d}, \boldsymbol{u}; p, \boldsymbol{\sigma}] \in (D \times V) \times (Q \times S)$ *be the solution to Problem* I *or* Ip *such that* $[\boldsymbol{d}, \boldsymbol{u}, p, \boldsymbol{\sigma}] \in [H^1(\Omega)]^4 \times [H^2(\Omega)]^2 \times H^1(\Omega) \times [H^1(\Omega)]^4$. *Then for* $\alpha > 0$ *and with the discrete spaces chosen according to (4.29)–(4.31), there exists a unique solution* $[\boldsymbol{d}_h, \boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h] \in D_h \times V_h \times Q_h \times S_h$ *of Problems* Iph *and* Ih *such that*

$$\|\boldsymbol{d} - \boldsymbol{d}_h\|_0 + \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|p - p_h\|_0 \leqslant ch, \tag{4.33}$$

for some positive constant c independent of h and λ.

Turning next to Problem II, for the spaces satisfying Lemma 4.3 we take (V_h, Q_h) as before. Again the direct components of stresses and strains are spanned by $(\phi^i_{\mathscr{K}})^3_{i=1}$. Since $D_h \subset S_h$ we can choose

$$\begin{aligned} S_h &= \{\boldsymbol{\tau}_h \in S | (\boldsymbol{\tau}_h|_{\mathscr{K}})_{kl} = (\hat{\boldsymbol{\tau}}_h)_{kl} \circ F^{-1}_{\mathscr{K}}, \hat{\boldsymbol{\tau}}_h \in S_{\hat{K}} \text{ for all } \mathscr{K} \in \mathscr{T}_{2h}\}, \\ D_h &= \{\boldsymbol{e}_h \in D | (\boldsymbol{e}_h|_{\mathscr{K}})_{kl} = (\hat{\boldsymbol{e}}_h)_{kl} \circ F^{-1}_{\mathscr{K}}, \hat{\boldsymbol{e}}_h \in D_{\hat{K}} \text{ for all } \mathscr{K} \in \mathscr{T}_{2h}\}, \end{aligned} \tag{4.34}$$

where $D_{\hat{K}} = \sum_{i=1}^3 S^1 \phi^i_{\mathscr{K}} + S^2 CB_{\mathscr{K}}$, $S_{\hat{K}} = \sum_{i=1}^3 S^1 \phi^i_{\mathscr{K}} + S^3 CB_{\mathscr{K}}$, and

$$S^3 = \text{span}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \hat{x} & \hat{y} & \hat{x}\hat{y} \end{bmatrix}.$$

Moreover, on each macro-element $\mathscr{K} \in \mathscr{T}_{2h}$ we have $\dim S_h = 13$, $\dim D_h = 12$. Again for this choice condition (4.18) is not satisfied, so the corresponding Hu–Washizu formulation for $\alpha = 0$ is not well-posed. With this choice of spaces we have the following convergence result for Problem IIph.

Theorem 4.5. *Let* $[\boldsymbol{d}, \boldsymbol{u}; p, \boldsymbol{\sigma}] \in (D \times V) \times (Q \times S)$ *be the weak solution to Problem* IIp *such that* $[\boldsymbol{d}, \boldsymbol{u}, p, \boldsymbol{\sigma}] \in [H^1(\Omega)]^{2\times 2} \times [H^2(\Omega)]^2 \times H^1(\Omega) \times [H^1(\Omega)]^{2\times 2}$. *Then for* V_h, S_h, Q_h *and* D_h *as chosen, there exists a unique solution* $[\boldsymbol{d}_h, \boldsymbol{u}_h; p_h, \boldsymbol{\sigma}_h]$ *in* $(D_h \times V_h) \times (Q_h \times S_h)$ *to Problems* IIph *and* IIh, *and a positive constant c independent of h and* λ, *such that*

$$\|\boldsymbol{d} - \boldsymbol{d}_h\|_0 + \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|p - p_h\|_0 \leqslant ch. \tag{4.35}$$

5. Numerical experiments

In this section we present a selection of numerical results in which the features of the formulations introduced earlier are compared with existing formulations. All examples are assumed to be under conditions of plane strain and for quasi-incompressible materials, with Poisson's ratio $\nu = 0.4999$. Four-noded quadrilateral elements with standard bilinear interpolation of the displacement field are used.

There is a minor difference between the bases (4.32) for stress and strain associated with Problem Ih, and those in (4.34) that are associated with Problem IIh. In fact the difference lies in the bilinear term $\hat{x}\hat{y}$ for the shear stress in Problem IIh. The numerical results were found to be essentially unaffected by the presence of this term, without which we have $S_h = D_h$, so that Problems IIh and Ih are equivalent. In the examples that are presented, attention is therefore confined to Problem Ih with the basis (4.32).

If $\alpha = 0$ then Problem Ih is ill-posed for this choice of bases since the corresponding bilinear form will not be coercive, as pointed out earlier.

We are interested in showing computationally the stability of Problem Ih when the stress and strain are approximated element-wise by S^1, and also in showing the poor behaviour of these formulations for $\alpha = 0$, using the same bases.

It has been shown in [5,6] for an extended version of the Hu–Washizu formulation that even when the spaces of stresses and strains are chosen to be simply constant element-wise, in which case they contain a checkerboard mode, the displacement satisfies an a priori bound that is λ-independent. This will be demonstrated numerically in the examples that follow.

The formulations referred to in the examples that follow are:

Ih0 Problem Ih with S^h and D^h spanned by S^1, with $\alpha = 0$
Ih1 As Ih, but with $\alpha = \mu/10$
Ih2 As Ih, but with $\alpha = \mu$
EAS Method of enhanced assumed strains [21]
Q1 Standard displacement formulation using the four-noded quadrilateral
MHW Modified Hu–Washizu formulation [5,6] with stress spanned by the Pian–Sumihara basis [19]
MES Method of mixed enhanced strains [16,17]

Example 1 (*Cook's membrane problem*). In this standard benchmark problem simulations are carried out for progressive uniform refinements of the mesh shown in Fig. 2. This figure also shows results for the top corner vertical displacement, for various formulations. Convergence is achieved for all formulations except the Q_1 element. For formulation Ih0, for which we expect lack of convergence, it is found that convergence is achieved from above, while Ih1, that is, the case for which $\alpha/\mu = 0.1$, gives the most rapid convergence. Of course the results on error estimates that have been established relate to errors in the norms of the spaces on which the problem is defined. We examine these errors in the following example.

Example 2 (*Cantilever beam*). We consider a beam of unit thickness, subjected to a couple at one end, as shown in Fig. 3. Along the edge $x = 0$, the horizontal displacement and vertical surface traction are zero. At point $(0,0)$, the vertical displacement is also zero. The exact solution is given by

$$u(x,y) = \frac{2f(1-\nu^2)}{El}x\left(\frac{l}{2}-y\right), \quad \text{and} \quad v(x,y) = \frac{f}{El}\left[x^2 + \frac{\nu}{1-\nu}(y^2 - ly)\right].$$

We set $L = 10$, $l = 2$, $E = 1500$, $\nu = 0.4999$, and $f = 3000$.

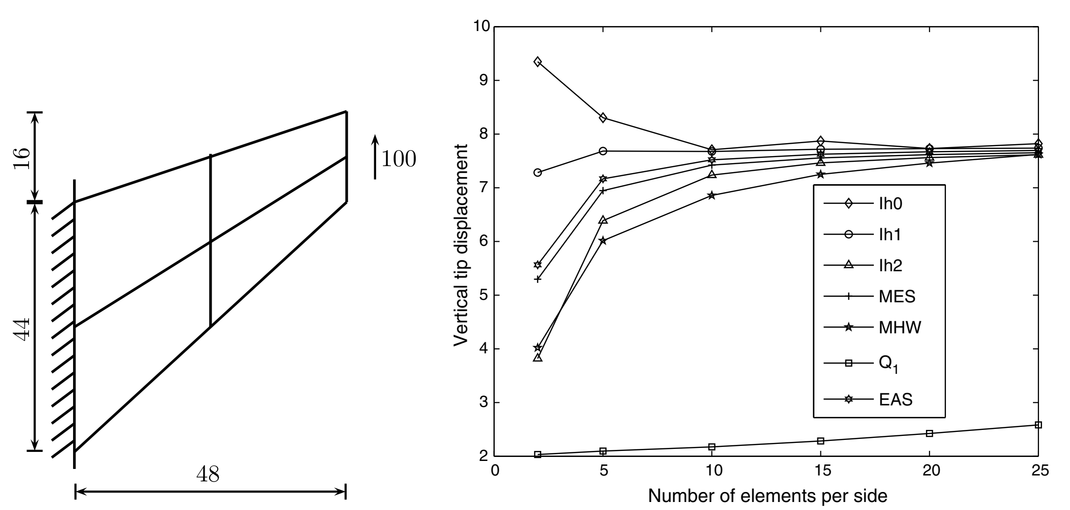

Fig. 2. Cook's membrane problem.

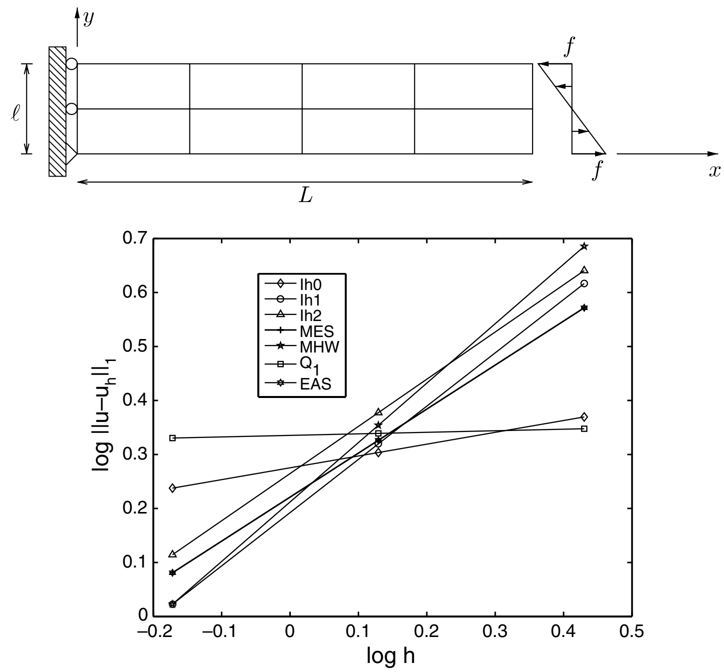

Fig. 3. Cantilever problem: behaviour of displacement error, measured in the H^1 norm, with mesh size, using regular refinements of a rectangular mesh.

Fig. 3 shows the behaviour of the displacement error

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1^2 = \|\boldsymbol{u}-\boldsymbol{u}_h\|_0^2 + \|\nabla\boldsymbol{u}-\nabla\boldsymbol{u}_h\|_0^2$$

with regular meshes of 4×2, 8×4, and 16×8 elements. It is manifest that formulations Ih0 and Q_1 behave poorly, while the other formulations show approximately linear convergence. The same behaviour was observed for distorted meshes, and these results are accordingly not presented here.

Example 3 (*Influence of* α). Next, we investigate the dependence on α in the displacement error. For this purpose we compute solutions to the cantilever problem in Example 2 for a mesh of 200 elements, for varying α. While the analysis in Section 4 is not amenable to determining optimal values of α, it is seen in Fig. 4(a), at least for this example problem, that an optimal value exists in the region between $\mu/4$ and $\mu/8$. The same results are shown in Fig. 4(b), this time as a log-log plot.

Example 4 (*Driven cavity*). We consider the problem of a unit square subjected to a unit horizontal displacement along the upper boundary. The material properties used are Young's modulus $E = 0.1$, Poisson's ratio $\nu = 0.4999$, and $\lambda/\mu = 10^7$. The aim here is to demonstrate numerically the existence of the checkerboard modes predicted in our analysis.

The hydrostatic pressure p is computed from $\mathrm{tr}\,\boldsymbol{\sigma}_h = \sigma_{h1} + \sigma_{h2}$, for uniform meshes of square elements with 20 and 40 elements per side. The pressure distribution along the line $y = 0.22$ is shown for Ih0 and Ih1 in Fig. 5. The checkerboard

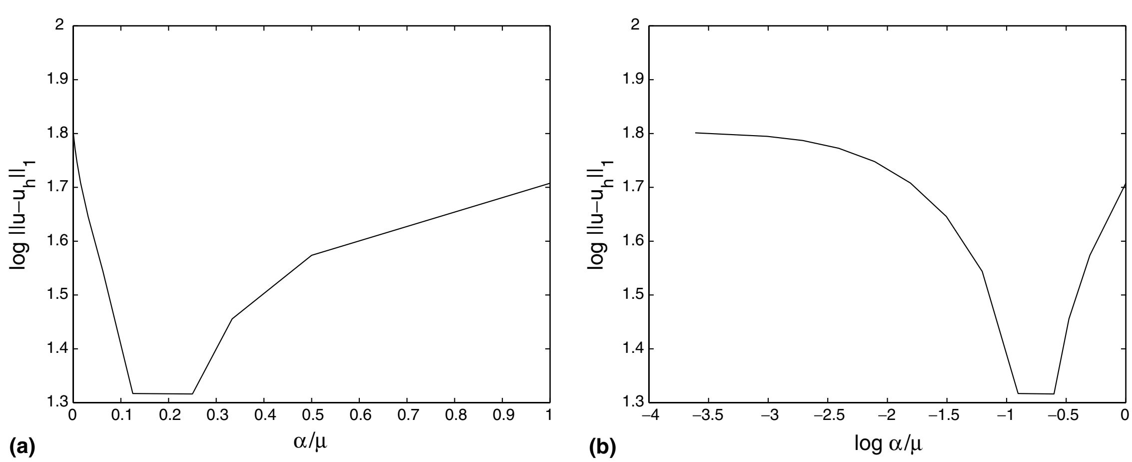

Fig. 4. Log-normal plot of norm error in displacement vs α for the cantilever problem.

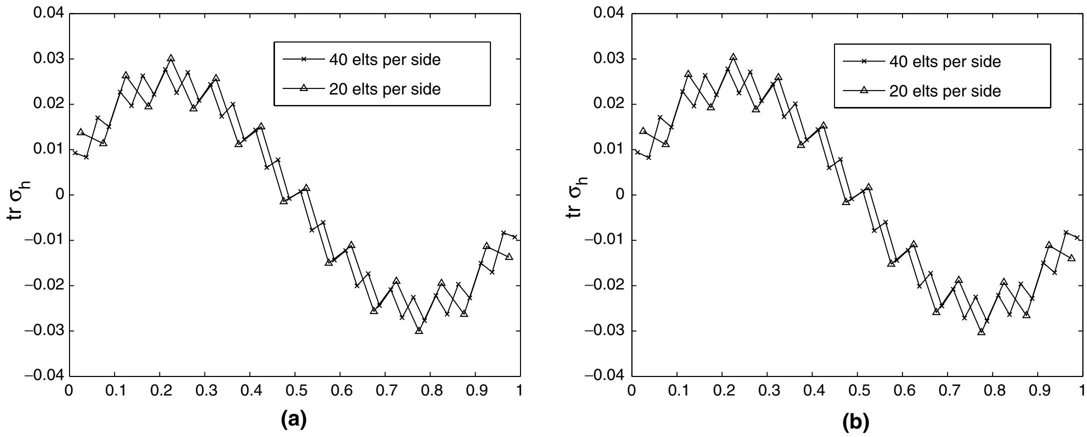

Fig. 5. The checkerboard mode associated with the trace of the stress, or the pressure: (a) problem 1h0; (b) problem 1h1.

mode is easily seen for both problems, as expected; the stabilization that arises in problem Ih1 from a choice of a non-zero value of α has no effect on the presence of the checkerboard mode, as the theory in Section 4 makes clear.

6. Closure

Extended Hu–Washizu formulations have been presented based on the elastic-visco-split-stress method proposed in [8,9]. These formulations extend the classical Hu–Washizu formulation in the sense that for $\alpha = 0$, one recovers the classical formulation. These formulations relax the system since the stabilization term introduced allows the use of very simple elements while still achieving a stabilized formulation, and uniform convergence in the incompressible limit. Error estimates obtained are first order accurate and the numerical results reflect this feature. Extensions to three-dimensional problems are under investigation. In addition, it will be useful to obtain theoretical results that explain the dependence of the solutions on the parameter α.

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