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From the Hu–Washizu formulation to the average nodal strain formulation

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We present a stabilized finite element method for the Hu–Washizu formulation of linear elasticity based on simplicial meshes leading to the stabilized nodal strain formulation or node-based uniform strain elements. We show that the finite element approximation converges uniformly to the exact solution for the nearly incompressible case.

1. Introduction

There are many different mixed formulations for linear elasticity. Among the most popular mixed formulations are Hellinger–Reissner and Hu–Washizu formulations. There are mainly two reasons to use a mixed formulation in elasticity. One reason is to compute the variables of interest: stress, strain or pressure more accurately and the other reason is to alleviate the locking effect in the nearly incompressible regime [10,6].

In this paper, we present a variational approach to obtain the stabilized nodally integrated simplicial elements presented in [22]. Recently, the average nodal formulation for different quantities of interest in linear and non-linear elasticity has been of particular interest [4,13,5,21,12]. We have shown that the average nodal pressure formulation originally presented in [4] can be analyzed within the framework of mixed finite elements [18]. The mixed finite element scheme in [4] is based on displacement–pressure formulation, where the displacement is discretized by using a standard linear finite element space and the pressure is discretized by using a piecewise constant finite element space on the dual mesh. Using a similar approach for the Hu–Washizu formulation of linear elasticity, we show that the average nodal strain formulation can also be analyzed in the framework of a mixed finite element method. However, in contrast to the average nodal pressure formulation we have to stabilize the scheme in the average nodal strain formulation to obtain the coercivity. This yields a consistent variational framework for the nodally integrated tetrahedral presented in [22].

2. The boundary value problem of linear elasticity

In this section, we introduce the boundary value problem of linear elasticity. Let $\Omega \subset \mathbb{R}^d, d = \{2,3\}$, be a bounded domain with Lipschitz boundary Γ, and let Ω be occupied with a homogeneous isotropic linear elastic material body. For a prescribed body force $\boldsymbol{f} \in L^2(\Omega)^d$, the governing equilibrium equation is written as

$$-\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f}, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor. The stress tensor $\boldsymbol{\sigma}$ is a function of the displacement $\boldsymbol{u}$ defined as

$$\boldsymbol{\sigma} = \mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \tag{2.2}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + [\nabla \boldsymbol{u}]^t)$ is the strain tensor, and $\mathscr{C}$ is the fourth-order elasticity tensor, which acts on a tensor $\boldsymbol{d}$ as

$$\boldsymbol{\sigma} = \mathscr{C}\boldsymbol{d} := \lambda(\operatorname{tr}\boldsymbol{d})\mathbf{1} + 2\mu\boldsymbol{d}. \tag{2.3}$$

Here $\mathbf{1}$ is the identity tensor, and λ and μ are the Lamé parameters, which are positive and constant in view of the assumption of a

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homogeneous body. For simplicity, we assume homogeneous Dirichlet boundary condition on Γ:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma. \tag{2.4}$$

We are also interested in the nearly incompressible regime, which corresponds to $\lambda \to \infty$.

2.1. The standard and a stabilized Hu–Washizu formulation

As usual, $L^2(\Omega)$ denotes the space of square-integrable functions defined on Ω with the inner product and norm being denoted by $(\cdot,\cdot)_{0,\Omega}$ and $\|\cdot\|_{0,\Omega}$, respectively. We denote the set of symmetric tensors in Ω by $\boldsymbol{S} := \{\boldsymbol{d} \in L^2(\Omega)^{d\times d} | \boldsymbol{d} \text{ is symmetric}\}$ having each component being square-integrable. We will use the notation for the dot product between two tensors $\boldsymbol{\tau}$, $\boldsymbol{d} \in \boldsymbol{S}$ as $\boldsymbol{\tau} : \boldsymbol{d} = \sum_{i=1}^d \sum_{j=1}^d \tau_{ij} d_{ij}$. The space $H_0^1(\Omega)$ consists of functions in $H^1(\Omega)$ which vanish on the boundary in the sense of traces. To write the weak or variational formulation of the boundary value problem, we introduce the space $\boldsymbol{V} := [H_0^1(\Omega)]^d$ of displacements with inner product $(\cdot,\cdot)_{1,\Omega}$ and norm $\|\cdot\|_{1,\Omega}$ defined in the standard way, see [11,9].

We define the bilinear form $A(\cdot,\cdot)$ and the linear functional $\ell(\cdot)$ by

$$A : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}, \quad A(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \mathscr{C} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx,$$

$$\ell : V \to \mathbb{R}, \quad \ell(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

Let $\boldsymbol{V}^*$ be the space of continuous linear functionals defined on $\boldsymbol{V}$. Then the standard weak form of linear elasticity problem is as follows: given $\ell \in \boldsymbol{V}^*$, find $\boldsymbol{u} \in V$ that satisfies

$$A(\boldsymbol{u}, \boldsymbol{v}) = \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{2.5}$$

The assumptions on $\mathscr{C}$ guarantee that $A(\cdot,\cdot)$ is symmetric, continuous, and V-elliptic. Hence by using standard arguments it can be shown that (2.5) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}$. Furthermore, we assume that the domain Ω is convex polygonal or polyhedral so that $\boldsymbol{u} \in [H^2(\Omega)]^d \cap \boldsymbol{V}$, and there exists a constant C independent of λ such that

$$\|\boldsymbol{u}\|_2 + \lambda \|\operatorname{div} \boldsymbol{u}\|_1 \leqslant C \|\boldsymbol{f}\|_0. \tag{2.6}$$

We refer to [9,17] for the proof of this regularity estimate. Since we have imposed homogeneous Dirichlet boundary condition, Cauchy stress $\boldsymbol{\sigma}$ satisfies

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{1}\, dx = \int_\Omega (2\mu + d\lambda) \operatorname{div} \boldsymbol{u}\, dx = (2\mu + d\lambda) \int_{\partial\Omega} \boldsymbol{u}\, d\sigma = 0.$$

Thus we restrict the space of stress to $\boldsymbol{S}_0$ with

$$\boldsymbol{S}_0 := \{\boldsymbol{\tau} \in \boldsymbol{S} | \int_\Omega \boldsymbol{\tau} : \boldsymbol{1}\, dx = 0\}, \tag{2.7}$$

see also [19]. The standard Hu–Washizu formulation [16,23] is to find $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in \boldsymbol{V} \times \boldsymbol{S} \times \boldsymbol{S}_0$ such that

$$\begin{aligned} &\tilde{a}((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + b((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) = \ell(\boldsymbol{v}), \quad (\boldsymbol{v}, \boldsymbol{e}) \in \boldsymbol{V} \times \boldsymbol{S}, \\ &b((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in \boldsymbol{S}_0, \end{aligned} \tag{2.8}$$

where

$$\tilde{a}((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) = \int_\Omega \boldsymbol{d} : \mathscr{C} \boldsymbol{e}\, dx, \quad \text{and} \quad b((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) = \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{d}) : \boldsymbol{\tau}\, dx.$$

The well-posedness of the saddle point problem (2.8) is analyzed by using the standard saddle point theory, see [10,6]. The main difficulty in the discrete setting is to show that the bilinear form $b(\cdot,\cdot)$ satisfies a uniform inf–sup condition and the bilinear form $\tilde{a}(\cdot,\cdot)$ is coercive on a suitable kernel space. Using some simple finite element spaces, it is not possible to satisfy these two conditions simultaneously as the bilinear form $\tilde{a}(\cdot,\cdot)$ is not elliptic on the whole space $\boldsymbol{V} \times \boldsymbol{S}$ already in the continuous setting. This problem is well-known in the context of Mindlin–Reissner plate theory and Darcy equation, see [1,20,2].

This gives us a motivation to modify the bilinear form $\tilde{a}(\cdot,\cdot)$ consistently by adding a stabilization term so that we obtain the ellipticity on the whole space $\boldsymbol{V} \times \boldsymbol{S}$. The modification of the bilinear form $\tilde{a}(\cdot,\cdot)$ is obtained by adding an additional term

$$\int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{d}) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{e})\, dx$$

to the bilinear form $\tilde{a}(\cdot,\cdot)$ so that our modified saddle point problem is to find $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in \boldsymbol{V} \times \boldsymbol{S} \times \boldsymbol{S}_0$ such that

$$\begin{aligned} &a((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + b((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) = \ell(\boldsymbol{v}), \quad (\boldsymbol{v}, \boldsymbol{e}) \in \boldsymbol{V} \times \boldsymbol{S}, \\ &b((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in \boldsymbol{S}_0, \end{aligned} \tag{2.9}$$

where

$$a((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) = \int_\Omega \boldsymbol{d} : \mathscr{C} \boldsymbol{e}\, dx + \alpha \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{d}) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{e})\, dx,$$

$b(\cdot,\cdot)$ is defined as above, and $\alpha > 0$ is a parameter.

3. Finite element discretization

We consider a quasi-uniform triangulation $\mathscr{T}_h$ of the polygonal domain Ω, where $\mathscr{T}_h$ consists of triangles in $d = 2$ or tetrahedra in $d = 3$. Let $\mathscr{N}_h$ be the set of all vertices of $\mathscr{T}_h$, defined as

$$\mathscr{N}_h := \{i : i \text{ is a vertex of element } T \in \mathscr{T}_h\}, \text{ and } N := \#\mathscr{N}_h.$$

Each vertex i will also be identified with its co-ordinate vector x_i. A dual mesh $\mathscr{T}_h^*$ is constructed based on the primal mesh $\mathscr{T}_h$ so that the elements of $\mathscr{T}_h^*$ are called control volumes. The dual mesh for triangular meshes is introduced in the following way. Let x_i, x_j and x_k be three vertices of an element $T \in \mathscr{T}_h$, and x_{ij}, x_{jk} and x_{ki} be middle points of the three edges of T. Let c_T be the centroid of the triangle T. We connect c_T to the three middle points of the edges by straight lines to divide the triangle into three quadrilaterals Q_i, Q_j and Q_k, where each quadrilateral Q_i shares only one vertex x_i of the triangle T, and hence corresponds to the vertex x_i of the triangle T. Let $\mathscr{T}_i$ be the set of all triangles in $\mathscr{T}_h$ having the common vertex i.

For each vertex $i \in \mathscr{N}_h$, we select a set of quadrilaterals

$$\mathscr{Q}_i := \{Q : Q \text{ corresponds to the vertex } i \text{ of the triangle } T, T \in \mathscr{T}_i\}.$$

The control volume V_i corresponding to the vertex i is defined as

$$\overline{V}_i := \cup_{Q \in \mathscr{Q}_i} \overline{Q},$$

see also [14]. The collection of these control volumes then defines the dual mesh, see Fig. 1. We can follow a similar construction for a tetrahedron T selecting centroids of four faces, middle points of four edges and a centroid c_T of T.

We call the control volume mesh $\mathscr{T}_h^*$ regular or quasi-uniform if there exists a positive constant $C > 0$ such that

$$Ch^d \leqslant |V_i| \leqslant h^d, \quad V_i \in \mathscr{T}_h^*,$$

where h is the maximum diameter of all elements $T \in \mathscr{T}_h$.

It can be shown that, if $\mathscr{T}_h$ is locally regular, i.e., there is a constant C such that

$$Ch_T^d \leqslant |T| \leqslant h_T^d, \quad T \in \mathscr{T}_h$$

with $\operatorname{diam}(T) = h_T$ for all elements $T \in \mathscr{T}_h$, then this dual mesh $\mathscr{T}_h^*$ is also locally regular. However, the analysis can be extended to other dual meshes. Let S_h be the standard linear finite element space defined on the triangulation $\mathscr{T}_h$,

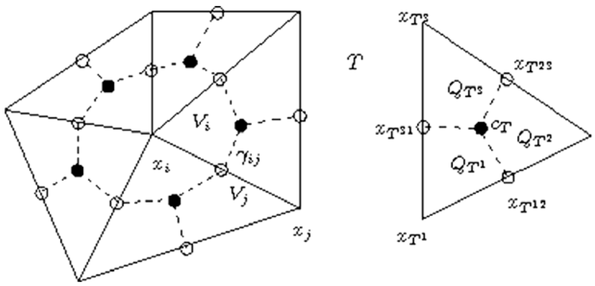

Fig. 1. Primal and dual meshes with vertices x_i and x_j and the triangle T divided into three quadrilaterals in the two-dimensional case.

$$S_h := \{v \in H^1(\Omega) | \, v_{|_T} \in \mathscr{P}_1(T), \; T \in \mathscr{T}_h\},$$

and its dual volume element space M_h,

$$M_h := \{p \in L^2(\Omega) | \, p_{|_V} \in \mathscr{P}_0(V), \; V \in \mathscr{T}_h^*\}.$$

Let $p_h \in M_h$ and $u_h \in S_h$ such that $u_h = \sum_{i=1}^N u_i \phi_i$ and $p_h = \sum_{i=1}^N p_i \chi_i$, where ϕ_i is the standard nodal basis function associated with the node i, and χ_i is the characteristic function of the volume set V_i defined by

$$\chi_i(x) = \begin{cases} 1 & \text{if } x \in V_i, \\ 0 & \text{else}. \end{cases}$$

Defining the space of bubble functions

$$B_h := \{b_T \in H^1(T) : b_{T|_{\partial T}} = 0, \text{ and } \int_T b_T \, dx > 0, T \in \mathscr{T}_h\},$$

we introduce our finite element space for the displacement as $\boldsymbol{V}_h = [S_h \oplus B_h]^d$. Denoting the $d+1$ barycentric co-ordinates of an element $T \in \mathscr{T}_h$ by $\lambda_i, 1 \leqslant i \leqslant d+1$, the cubic bubble function b_T associated with the element T can be written as $b_T(x) = c_b \Pi_{i=1}^{d+1} \lambda_{T^i}(x)$, where the constant c_b is determined by using the fact that b_T evaluated at the centroid of the element T is one.

Then the finite element approximation of (2.8) is defined as a solution to the following problem: find $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h) \in \boldsymbol{V}_h \times \boldsymbol{S}_h \times \boldsymbol{M}_h$ such that

$$\begin{aligned} & a((\boldsymbol{u}_h, \boldsymbol{d}_h), (\boldsymbol{v}_h, \boldsymbol{e}_h)) + b((\boldsymbol{v}_h, \boldsymbol{e}_h), \boldsymbol{\sigma}_h) = \ell(\boldsymbol{v}_h), \quad (\boldsymbol{v}_h, \boldsymbol{e}_h) \in \boldsymbol{V}_h \times \boldsymbol{S}_h, \\ & b((\boldsymbol{u}_h, \boldsymbol{d}_h), \boldsymbol{\tau}_h) = 0, \quad \boldsymbol{\tau}_h \in \boldsymbol{M}_h, \end{aligned} \tag{3.1}$$

where

$$\boldsymbol{M}_h := \{\boldsymbol{\tau}_h \in [M_h]^{d \times d} | \, \boldsymbol{\tau}_h \text{ is symmetric, and } \int_\Omega \boldsymbol{\tau}_h : \mathbf{1} \, dx = 0\}.$$

4. An a priori error estimate

In order to show that the finite element approximation converges uniformly to the continuous solution, we introduce an orthogonal projection operator: $\Pi_h : L^2(\Omega) \to M_h$. Using the definition of Π_h, we write the strain as

$$\boldsymbol{d}_h = \Pi_h(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)),$$

where operator Π_h is applied to tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}_h)$ componentwise. The definition of dual meshes allows us to write the action of operator Π_h on a function $v \in L^2(\Omega)$ as

$$\Pi_h v = \sum_{i=1}^n c_i \chi_i, \tag{4.1}$$

where

$$c_i = \frac{1}{|V_i|} \int_{V_i} v \, dx.$$

It is easy to see that operator Π_h is local. Using this expression of $\boldsymbol{d}_h$, we can now eliminate the stress $\boldsymbol{\sigma}_h$ and the strain $\boldsymbol{d}_h$ from our discrete problem to obtain the reduced problem of finding $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

where

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = & \int_\Omega \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h) : \mathscr{C} \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h) \, dx \\ & + \alpha \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h)) : (\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)) \, dx. \end{aligned} \tag{4.2}$$

In the following, we will use a generic constant C, which will take different values at different places but will be always independent of the mesh-size h and Lamé parameter λ. Since Π_h is stable in the L^2-norm, we have

$$\|\Pi_h v\|_{0,\Omega} \leqslant C \|v\|_{0,\Omega}, \quad v \in L^2(\Omega). \tag{4.3}$$

Furthermore, the following approximation property follows by using the fact that M_h contains piecewise constant functions with respect to the dual mesh $\mathscr{T}_h^*$.

$$\|v - \Pi_h v\|_{0,\Omega} \leqslant Ch \|v\|_{1,\Omega}, \quad v \in H^1(\Omega). \tag{4.4}$$

Using the fact that Π_h is a projection operator, and $\Pi_h \mathscr{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)) = \mathscr{C} \Pi_h(\boldsymbol{\varepsilon}(\boldsymbol{u}_h))$, we can simplify the expression for $A_h(\cdot, \cdot)$:

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h) : \mathscr{C} \boldsymbol{\varepsilon}(\boldsymbol{v}_h) \, dx + \alpha \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h)) : \boldsymbol{\varepsilon}(\boldsymbol{v}_h) \, dx. \tag{4.5}$$

Remark 4.1. When $\alpha = 0$, we recover the node-based uniform strain elements for simplices introduced in [13]. The nodally integrated tetrahedral element analyzed in [22] is obtained by replacing α by some constant fourth-order tensor. However, the consistent derivation shows that it is not necessary to use a fourth-order tensor.

Lemma 4.2. *The bilinear from $A_h(\cdot, \cdot)$ is coercive on $\boldsymbol{V}_h \times \boldsymbol{V}_h$ uniformly with respect to λ, i.e., there exists a constant C independent of λ such that*

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geqslant C \|\boldsymbol{u}_h\|_{1,\Omega}^2. \tag{4.6}$$

Proof. In the first step, we apply the Korn's inequality to obtain

$$\|\boldsymbol{u}_h\|_{1,\Omega}^2 \leqslant C_K \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}^2.$$

An application of the triangle inequality gives

$$\begin{aligned} \|\boldsymbol{u}_h\|_{1,\Omega}^2 & \leqslant C_K \left(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}^2 + \|\Pi_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}^2 \right) \\ & \leqslant \max \left(\frac{C_K}{\alpha}, \frac{C_K}{2\mu} \right) \left(\alpha \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}^2 + 2\mu \|\Pi_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}^2 \right. \\ & \qquad \left. + \lambda \|\mathrm{tr} \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}^2 \right) = \max \left(\frac{C_K}{\alpha}, \frac{C_K}{2\mu} \right) A_h(\boldsymbol{u}_h, \boldsymbol{u}_h). \end{aligned}$$

Hence

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geqslant C \|\boldsymbol{u}_h\|_{1,\Omega}^2$$

with

$$C = \frac{1}{\max \left(\frac{C_K}{\alpha}, \frac{C_K}{2\mu} \right)}.$$

Thus the coercivity constant C depends on μ and α but does not depend on λ. $\square$

Remark 4.3. Here the parameter $\alpha > 0$ can be arbitrary as we have a consistent stabilization. However, the smaller value of α decreases the coercivity constant in the previous lemma, and the larger value of α can pollute the approximation. The choice of the parameter $\alpha > 0$ can be utilized for accelerating the solver as in an augmented Lagrangian formulation [3]. As suggested by the previous lemma, one natural choice of the parameter α is 2μ [22]. Since we do not focus on this aspect of the problem, we simply put $\alpha = 1$ in the rest of the paper.

The projection property of Π_h also allows us to write $A_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$ as

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega (\Pi_h \mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \Pi_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h)) : \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\, dx \quad \text{or}$$

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega (\Pi_h \mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \Pi_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h)) : \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\, dx. \tag{4.7}$$

Thus we pose the positive definite formulation: find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.8}$$

Remark 4.4. As Π_h is not a uniformly coercive operator with respect to the L^2-norm even on the space of piecewise constant functions, we cannot show the coercivity of $A_h(\cdot,\cdot)$ without adding the stabilization term. This is closely related to the instability of P_1–P_1 discretization of Stokes problem, see [15,18].

Remark 4.5. We have to use $\boldsymbol{V}_h = [S_h \oplus B_h]^d$ for the finite element space for the displacement field to obtain the uniform convergence of the finite element approximation in the incompressible limit. However, we obtain the coercivity of $A_h(\cdot,\cdot)$ also on the space $[S_h]^d$ as can be seen from Lemma 4.2. Hence the formulation is stable in the compressible regime even if we use $[S_h]^d$ to discretize displacement in the Hu–Washizu formulation.

Now we focus to show the uniform convergence of the finite element solutions of the statically condensed formulation (4.8) in the incompressible limit. The main ingredients of the proof are Fortin interpolation operator and Strang's first lemma. A similar idea has been used in [19] to prove the uniform approximation of the Hu–Washizu formulation in the nearly incompressible regime using a class of quadrilateral meshes.

We know that the pair of spaces $(\boldsymbol{V}_h, M_h)$ satisfies the uniform inf–sup condition [18]. There exists a constant $\beta > 0$ independent of the mesh-size such that

$$\inf_{q_h \in M_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}_h q_h}{\|\boldsymbol{v}_h\|_{1,\Omega} \|q_h\|_{0,\Omega}} \geqslant \beta.$$

Therefore, the Fortin's interpolation operator [10, Section II] or [7, 4.8 and 4.9] $\boldsymbol{I}_h^F : [H^2(\Omega) \cap H_0^1(\Omega)]^d \to \boldsymbol{V}_h$ can be defined for the problem of finding $(\boldsymbol{I}_h^F \boldsymbol{w}, p_h) \in \boldsymbol{V}_h \times M_h$ such that

$$\begin{aligned} &\int_\Omega \nabla \boldsymbol{I}_h^F \boldsymbol{w} : \nabla \boldsymbol{z}_h\, dx + \int_\Omega \operatorname{div} \boldsymbol{z}_h\, p_h\, dx = \int_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{z}_h\, dx, \quad \boldsymbol{z}_h \in \boldsymbol{V}_h, \\ &\int_\Omega \operatorname{div} \boldsymbol{I}_h^F \boldsymbol{w}\, q_h\, dx = \int_\Omega \operatorname{div} \boldsymbol{w}\, q_h\, dx, \quad q_h \in M_h. \end{aligned} \tag{4.9}$$

The approximation property of the Fortin's interpolation operator is shown in the following lemma. For proofs, see [8,19].

Lemma 4.6. *Under the regularity assumption (2.6), the operator $\boldsymbol{I}_h^F$ satisfies the approximation property*

$$\|\boldsymbol{u} - \boldsymbol{I}_h^F \boldsymbol{u}\|_1 + \lambda \|div\, \boldsymbol{u} - \Pi_h(div\, \boldsymbol{I}_h^F \boldsymbol{u})\|_0 \leqslant Ch\|\boldsymbol{f}\|_0.$$

Lemma 4.7. *Assume that $\boldsymbol{u}$ is the solution of (2.5), and regularity estimate (2.6) holds. Then there exists a $\boldsymbol{v}_h \in \boldsymbol{V}_h$ so that*

$$|A(\boldsymbol{w}_h, \boldsymbol{u}) - A_h(\boldsymbol{w}_h, \boldsymbol{v}_h)| \leqslant C|\boldsymbol{w}_h|_{1,\Omega} h \|\boldsymbol{f}\|_{0,\Omega}, \quad \boldsymbol{w}_h \in \boldsymbol{V}_h.$$

Proof. Using the simplified expression of the bilinear form $A_h(\cdot,\cdot)$ from (4.7), a combination of Cauchy–Schwartz and triangle inequality yields

$$\begin{aligned} &|A(\boldsymbol{w}_h, \boldsymbol{u}) - A_h(\boldsymbol{w}_h, \boldsymbol{v}_h)| \\ &\quad = \left| \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{w}_h) : (\mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathscr{C}\Pi_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h))\, dx \right| \\ &\quad \leqslant C|\boldsymbol{w}_h|_{1,\Omega} \Big(\|\mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \Pi_h \mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} + \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} \Big) \qquad (4.10) \\ &\quad \leqslant C|\boldsymbol{w}_h|_{1,\Omega} G(\boldsymbol{u}, \boldsymbol{v}_h), \qquad (4.11) \end{aligned}$$

where

$$G(\boldsymbol{u}, \boldsymbol{v}_h) = \|\mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \Pi_h \mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} + \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}. \tag{4.12}$$

We use the expression (2.3) of the action of $\mathscr{C}$ on a tensor and the fact that $\Pi_h \mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \mathscr{C}\Pi_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)$ as well as a triangle inequality to write

$$\begin{aligned} G(\boldsymbol{u}, \boldsymbol{v}_h) \leqslant\ & 2\mu \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} + \lambda \|\operatorname{div} \boldsymbol{u} - \Pi_h \operatorname{div} \boldsymbol{v}_h\|_{0,\Omega} \\ & + \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \Pi_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}. \end{aligned}$$

Using again a triangle inequality, the approximation property of Π_h given by (4.4) and stability of Π_h in L^2-norm from (4.3), we obtain

$$G(\boldsymbol{u}, \boldsymbol{v}_h) \leqslant C(h\|\boldsymbol{u}\|_{2,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,\Omega} + \lambda \|\operatorname{div} \boldsymbol{u} - \Pi_h \operatorname{div} \boldsymbol{v}_h\|_{0,\Omega}). \tag{4.13}$$

The result now follows by choosing $\boldsymbol{v}_h = \boldsymbol{I}_h^F \boldsymbol{u}$ and using Lemma 4.6. □

Now we formulate the main result of this paper.

Theorem 4.8. *Assume that $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of (2.5) and (4.8), respectively, and regularity estimate (2.6) holds. Then, we obtain an optimal a priori estimate for the discretization error in the displacement*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \leqslant Ch\|\boldsymbol{f}\|_{0,\Omega}. \tag{4.14}$$

where $C < \infty$ is independent of λ and h.

Proof. We use the first lemma of Strang, see [6], to prove this theorem. Using the coercivity of $A_h(\cdot,\cdot)$, Lemma 4.7 and (2.5), we find that for $\boldsymbol{v}_h := I_h^F \boldsymbol{u}$

$$\begin{aligned} \|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{1,\Omega}^2 &\leqslant C|A_h(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{v}_h)| \\ &= C|A_h(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{u}_h) - A_h(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{v}_h)| \\ &= C|A(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{u}) - A_h(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{v}_h)| \\ &\leqslant C\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{1,\Omega} h \|\boldsymbol{f}\|_{0,\Omega}. \end{aligned}$$

In terms of the triangle inequality, we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \leqslant C(\|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,\Omega} + \|\boldsymbol{v}_h - \boldsymbol{u}_h\|_{1,\Omega}) \leqslant Ch\|\boldsymbol{f}\|_{0,\Omega}. \quad \square$$

5. Conclusion

We have presented a finite element method for a stabilized Hu–Washizu formulation using primal and dual meshes. The approach yields a consistent variational framework for recently introduced average nodal strain formulation [13,5,21,12,22] explaining the necessity of stabilization. However, we show that it is necessary to enrich the space of displacement with the bubble functions in order to achieve the uniform convergence of the finite element approximation in the incompressible regime.

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