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Convergence in the incompressible limit of finite element approximations based on the Hu-Washizu formulation

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Abstract The classical Hu–Washizu mixed formulation for plane problems in elasticity is examined afresh, with the emphasis on behavior in the incompressible limit. The classical continuous problem is embedded in a family of Hu–Washizu problems parametrized by a scalar α for which $\alpha = \lambda/\mu$ corresponds to the classical formulation, with λ and μ being the Lamé parameters. Uniform well-posedness in the incompressible limit of the continuous problem is established for $\alpha \neq -1$. Finite element approximations are based on the choice of piecewise bilinear approximations for the displacements on quadrilateral meshes. Conditions for uniform convergence are made explicit. These conditions are shown to be met by particular choices of bases for stresses and strains, and include bases that are well known, as well as newly constructed bases. Though a discrete version of the spherical part of the stress exhibits checkerboard modes, it is shown that a λ-independent a priori error estimate for the displacement can be established. Furthermore, a λ-independent estimate is established for the post-processed stress. The theoretical results are explored further through selected numerical examples.

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1 Introduction

There now exists a large number of publications devoted to the task of constructing and analyzing finite element approximations for problems in solid mechanics, in which it is necessary to circumvent volumetric locking, or alternatively expressed, to ensure uniform convergence in the incompressible limit. A further consideration is that of obtaining approximations of high quality using low-order elements, and for coarse meshes: when quadrilateral elements are used in two-dimensional problems, and hexahedral in three, it is well-known that the standard piecewise bi- and trilinear approximations in two and three dimensions lead to poor approximations when coarse meshes are used.

There is a growing literature dealing with the well-posedness of stabilization methods, and Brezzi and Fortin [9] have undertaken a detailed abstract analysis that is applicable to a wide range of such approaches. Methods associated with the enrichment or enhancement of the strain or stress field by the addition of carefully chosen basis functions have proved to be highly effective and popular. The key work dealing with enhanced assumed strain formulations is that of Simo and Rifai [20]. Reddy and Simo [19] have carried out a detailed analysis of the convergence of enhanced assumed strain methods, for affine-equivalent meshes, and for the compressible and incompressible cases. They established an a priori error estimate for displacements that confirms convergence at the standard linear rate. Braess [5] has re-examined the sufficient conditions for convergence, in particular relating the stability condition to a strengthened Cauchy inequality, and elucidating the influence of the Lamé constant λ. The case of limiting compressibility has been the subject of a recent analysis by Braess, Carstensen and Reddy [6], in which λ-independent asymptotic convergence of the displacement error is obtained, for a class of meshes. The assumed stress approach [18] leads to a formulation very similar to that based on enhanced strains, and in fact the two are equivalent under certain conditions [1,6].

Three-field mixed formulations, in which the unknown variables are displacement, stress and strain, are a popular approach to overcoming the problems referred to above. The corresponding weak formulation is known as the Hu–Washizu problem [14,22], and incorporates weak statements of the equation of equilibrium, the strain-displacement equation, and the elasticity equation. Historically, the formulation was first introduced by Fraeijs de Veubeke [12]. The method of enhanced assumed strains is a special case of the Hu–Washizu formulation, which also serves as the generating formulation for approaches such as the method of mixed enhanced strains [15].

Despite its importance and wide usage in computational situations, the question as to the conditions under which the Hu–Washizu formulation is stable and convergent in the incompressible limit remains open. The purpose of this work is to address that question.

It will be shown that the classical Hu–Washizu formulation is not amenable to an analysis in which λ-independent well-posedness and convergence are sought. This problem is circumvented by embedding the classical problem in a one-parameter family of problems parametrized by a scalar α, in which the classical problem corresponds to the choice $\alpha = \lambda/\mu$. It is this more general problem that is analyzed, with a view to establishing conditions under which uniform well-posedness can be established.

In a companion work [10], particular attention is paid to the relationship between the classical and modified Hu–Washizu formulations, and a range of mixed and enhanced formulations.

The structure of the rest of this work is as follows. Section 2 is devoted to the continuous Hu–Washizu problem, in its original and extended settings, and to the presentation of relevant results on well-posedness. In Sect. 3 the discrete formulations, based on finite element approximations, are presented. Section 4 focuses on the analysis of the modified formulation, and on the conditions for its uniform well-posedness in the incompressible limit. For this purpose it is necessary to introduce discrete deviatoric and spherical operators on the spaces of discrete stresses and strains. An abstract requirement for well-posedness is then that the trace of the space of discrete spherical stresses forms, with the space of displacements, a stable pairing in the sense of the classical Stokes problem.

Of exclusive interest here are situations corresponding to low-order approximations, with these being built on a space of displacements corresponding to piecewise continuous bilinear approximations on quadrilaterals. As with the Stokes problem, in which the pressure space of piecewise constants has to be modified to extract from it a space of so-called checkerboard modes, it can be shown that the lack of stability resulting from the presence of a checkerboard mode is confined to the stress, and does not affect the displacement. This result is important in the computational context, especially in situations in which the displacement is the primary variable of interest. In Sect. 5, an a priori error estimate for the displacement is derived, making clear that the finite element solution converges with optimal order to the exact solution uniformly in the incompressible limit. Similarly in Sect. 6, an a priori error estimate for the postprocessed stress is presented. Finally, in Sect. 7, the results of Sect. 5 are explored through two numerical examples. The numerical results reflect the good performance of the modified formulation.

2 The boundary value problem of elasticity

In the context of elasticity, vector- and tensor- or matrix-valued functions will be written in boldface form. The scalar product of two tensors or matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ will be denoted by $\boldsymbol{\sigma} : \boldsymbol{\tau}$, and is given by $\boldsymbol{a} : \boldsymbol{b} = a_{ij} b_{ij}$, the summation convention on repeated indices being invoked.

Consider a homogeneous isotropic linear elastic material body which occupies a bounded domain Ω in $\mathbb{R}^2$ with Lipschitz boundary Γ. For a prescribed body force $\boldsymbol{f} \in L^2(\Omega)^2$, the governing equilibrium equation in Ω reads

$$-\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor. The infinitesimal strain tensor $\boldsymbol{d}$ is defined as a function of the displacement $\boldsymbol{u}$ by

$$\boldsymbol{d} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \tfrac{1}{2}(\nabla \boldsymbol{u} + [\nabla \boldsymbol{u}]^t). \tag{2}$$

The displacement is assumed to satisfy the homogeneous Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma. \tag{3}$$

With the fourth-order elasticity tensor denoted by $\mathcal{C}$, the constitutive equation reads

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{d} := \lambda(\operatorname{tr}\boldsymbol{d})\mathbf{1} + 2\mu\,\boldsymbol{d}. \tag{4}$$

Here, $\mathbf{1}$ is the identity tensor, and λ and μ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body, and which are assumed positive. Of particular interest is the incompressible limit, which corresponds to $\lambda \to \infty$.

The inverse $\mathcal{C}^{-1}$ of $\mathcal{C}$ is given by

$$\boldsymbol{d} = \mathcal{C}^{-1}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \gamma(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1}\right), \quad \gamma := \frac{\lambda}{\kappa}, \;\; \kappa := 2(\mu + \lambda).$$

Standard weak formulation. We will make use of the space $L^2(\Omega)$ of square-integrable functions defined on Ω with the inner product and norm being denoted by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$, respectively. We will also make use of the Sobolev spaces $H^m(\Omega)$, for nonnegative integers m. The space $H_0^1(\Omega)$ consists of functions in $H^1(\Omega)$ which vanish on the boundary in the sense of traces.

For the weak or variational formulations we will require the space $V := [H_0^1(\Omega)]^2$ of displacements; this is a Hilbert space with inner product $(\cdot,\cdot)_1$ and norm $\|\cdot\|_1$ defined in the standard way; that is, $(\boldsymbol{u},\boldsymbol{v})_1 := \sum_{i=1}^2 (u_i, v_i)_1$, with the norm being induced by this inner product. The space of stresses is denoted by S, while the space of strains is denoted by D. For the continuous case these spaces are equal, and $D := \{\boldsymbol{e} \mid e_{ji} = e_{ij},\ e_{ij} \in L^2(\Omega)\} =: S$, with the norm $\|\cdot\|_0$ generated in the standard way by the L^2-norm. We also introduce the space S_0 defined by

$$S_0 := \{\boldsymbol{\tau} \in S \mid (\boldsymbol{\tau}, \mathbf{1})_0 = 0\};$$

this is a closed subspace of S.

Remark 2.1 It is crucial that the stresses be sought in S_0 rather than in S. A similar consideration appears in the conditions for well-posedness of the Hellinger-Reissner problem [6]. The subspace S_0 is, however, a natural subspace in which to work, since for the continuous problem the solution $(\boldsymbol{u},\ \boldsymbol{d},\ \boldsymbol{\sigma})$ satisfies $\operatorname{tr}\boldsymbol{\sigma} = \kappa \operatorname{div}\boldsymbol{u}$, and using the Dirichlet boundary condition it is seen that $\int_\Omega \operatorname{tr}\boldsymbol{\sigma}\,\mathrm{d}x = 0$.

Define the bilinear form $A(\cdot,\cdot)$ and linear functional $\ell(\cdot)$ by

$$\begin{aligned} &A : V \times V \to \mathbb{R}, \quad &&A(\boldsymbol{u},\boldsymbol{v}) := \textstyle\int_\Omega \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x,\\ &\ell : V \to \mathbb{R}, &&\ell(\boldsymbol{v}) \quad := \textstyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x. \end{aligned}$$

Then the standard form of the weak problem for elasticity is as follows: given $\ell \in V'$, find $\boldsymbol{u} \in V$ that satisfies

$$A(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in V. \tag{5}$$

The assumptions on $\mathcal{C}$ guarantee that $A(\cdot,\cdot)$ is symmetric, continuous, and V-elliptic.

If Ω is a convex domain with polygonal boundary it is known [7,21] that (5) has a unique solution $\boldsymbol{u} \in [H^2(\Omega)]^2$ and that there exists a constant C, independent of λ, such that

$$\|\boldsymbol{u}\|_2 + \lambda\|\operatorname{div}\,\boldsymbol{u}\|_1 \le C\,\|\boldsymbol{f}\|_0. \tag{6}$$

Mixed formulations. The problem described above may be cast in a variety of alternative mixed forms, the term ‘mixed’ carrying in this context the connotation that the resulting weak formulation has a link to a saddlepoint problem. We focus on the Hu–Washizu formulation, in which the displacement, strain, and stress are unknown variables. The standard Hu–Washizu formulation is obtained by considering the constitutive equation, the strain-displacement equation and the equation of equilibrium in weak form, and is the problem of finding $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in V \times D \times S_0$ such that

$$\begin{aligned} a((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + b((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) &= \ell(\boldsymbol{v}), \quad (\boldsymbol{v}, \boldsymbol{e}) \in V \times D, \\ b((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) &= 0, \quad \boldsymbol{\tau} \in S_0, \end{aligned} \tag{7}$$

where the bilinear forms are defined by

$$\begin{aligned} a((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) &:= (\mathcal{C}\boldsymbol{d}, \boldsymbol{e})_0 = 2\mu(\boldsymbol{d}, \boldsymbol{e})_0 + \lambda(\operatorname{tr} \boldsymbol{d}, \operatorname{tr} \boldsymbol{e})_0, \\ b((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) &:= (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{e}, \boldsymbol{\sigma})_0. \end{aligned}$$

It is readily shown, using the theory of Babuška and Brezzi (see, for example, [8, 13]) that this problem has a unique solution. But the continuity constant of $a(\cdot, \cdot)$ depends on λ, so that an analysis aimed at establishing λ-independent well-posedness cannot be based on the classical formulation.

We therefore consider a more general form depending on a parameter $\alpha \in \mathbb{R}$. This problem takes the form of finding $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in V \times D \times S_0$ such that

$$\begin{aligned} a_\alpha((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + b_\alpha((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) &= \ell(\boldsymbol{v}), \quad (\boldsymbol{v}, \boldsymbol{e}) \in V \times D, \\ b_\alpha((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) - \tfrac{(\lambda - \alpha\mu)}{\kappa^2}\, c(\boldsymbol{\sigma}, \boldsymbol{\tau}) &= 0, \quad \boldsymbol{\tau} \in S_0, \end{aligned} \tag{8}$$

where the bilinear forms are defined by

$$\begin{aligned} a_\alpha((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) &:= 2\mu(\boldsymbol{d}, \boldsymbol{e})_0 + \alpha\mu(\operatorname{tr} \boldsymbol{d}, \operatorname{tr} \boldsymbol{e})_0, \\ b_\alpha((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) &:= (\boldsymbol{\varepsilon}(\boldsymbol{v}) - 2\mu\, \mathcal{C}^{-1}\boldsymbol{e}, \boldsymbol{\sigma})_0 - \frac{\alpha\mu}{\kappa}(\operatorname{tr} \boldsymbol{\sigma}, \operatorname{tr} \boldsymbol{e})_0, \\ c(\boldsymbol{\sigma}, \boldsymbol{\tau}) &:= (\operatorname{tr} \boldsymbol{\sigma}, \operatorname{tr} \boldsymbol{\tau})_0. \end{aligned}$$

The standard Hu–Washizu formulation is given by $\alpha = \lambda/\mu$.

It is shown in [10] that the modified Hu–Washizu formulation may be obtained formally by imposing on the original saddle-point functional the constraint $\operatorname{tr} \boldsymbol{\sigma} = \kappa \operatorname{tr} \boldsymbol{d}$ which corresponds to the volumetric part of the elasticity equation, then by linearly interpolating between this and the classical formulation through the addition of a term involving the volumetric equation with α as a coefficient.

The spherical part of the stresses plays a crucial role in the analysis of the incompressible limit (see also [6]). Therefore, in the following, we will make extensive use of the L^2-orthogonal decomposition of S into its deviatoric and spherical parts. We define the L^2-orthogonal projections sph and dev on S by $\operatorname{sph} \boldsymbol{\tau} := (1/2)\,(\operatorname{tr} \boldsymbol{\tau})\mathbf{1}$, and $\operatorname{dev} \boldsymbol{\tau} := \boldsymbol{\tau} - \operatorname{sph} \boldsymbol{\tau}$. We note that $\operatorname{dev} S$ is a proper subset of S_0.

Lemma 2.1 *For $\alpha \neq -1$, there exists a unique solution $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in V \times D \times S_0$ of the modified Hu–Washizu formulation* (8). *Moreover, the solution does not depend on α and satisfies the bound*

$$\|\boldsymbol{u}\|_1 + \|\boldsymbol{d}\|_0 + \|\boldsymbol{\sigma}\|_0 \leq C\|\ell\|_{V'},$$

in which the constant C is independent of λ.

Proof In a first step, we show that the solution of (8) does not depend on α. The first equation in (8) with $\boldsymbol{v} = 0$ yields

$$0 = \left(\mathcal{C}\boldsymbol{d} - \boldsymbol{\sigma}, \boldsymbol{e} + \frac{\alpha\mu - \lambda}{\kappa}(\operatorname{tr}\boldsymbol{e})\,\mathbf{1}\right)_0 =: (\mathcal{C}\boldsymbol{d} - \boldsymbol{\sigma}, \mathcal{B}_\alpha \boldsymbol{e})_0, \quad \boldsymbol{e} \in D. \tag{9}$$

The inverse of $\mathcal{B}_\alpha$ exists for $\alpha \neq -1$ and satisfies $\mathcal{B}_\alpha^{-1}\boldsymbol{e} = \boldsymbol{e} - (\alpha\mu - \lambda)/2\mu(1+\alpha)$ $(\operatorname{tr}\boldsymbol{e})\,\mathbf{1}$. As a consequence, we find $\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{d}$ and thus $\operatorname{tr}\boldsymbol{\sigma} = \kappa \operatorname{tr}\boldsymbol{d}$. The bilinear forms $a_\alpha(\cdot,\cdot)$ and $b_\alpha(\cdot,\cdot)$ can be equivalently written as

$$\begin{aligned} a_\alpha((\boldsymbol{w}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{e})) &= a_{\lambda/\mu}((\boldsymbol{w}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{e})) + (\alpha\mu - \lambda)c(\boldsymbol{g}, \boldsymbol{e}), \\ b_\alpha((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}) &= b_{\lambda/\mu}((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}) - \tfrac{\alpha\mu - \lambda}{\kappa} c(\boldsymbol{\tau}, \boldsymbol{e}). \end{aligned} \tag{10}$$

Using $\lambda c(\boldsymbol{d}, \boldsymbol{e}) = \gamma c(\boldsymbol{\sigma}, \boldsymbol{e})$ and (10), it is trivial to see that the solution does not depend on α.

The proof of uniform stability is carried out by showing that the bilinear forms for $\alpha = 0$ satisfy the conditions for well-posedness for the extended case in which $c(\cdot,\cdot) \neq 0$ (see [8], Sect. II.1.2 and [4]). First, we note that $c(\cdot,\cdot)$ is symmetric and positive semi-definite, and that $\ker B^t := \{\boldsymbol{\tau} \in S_0 \mid b_0((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}) = 0,$ $(\boldsymbol{v}, \boldsymbol{e}) \in V \times D\} = \{0\}$, so that $c(\cdot,\cdot)$ plays no further role in determining the well-posedness of the problem.

Next, it is easy to see that all bilinear forms for $\alpha = 0$ are continuous, and that the continuity constants do not depend on λ. It remains therefore to verify that the coercivity and inf-sup conditions are satisfied. We establish the coercivity of the bilinear form $a_0(\cdot,\cdot)$ on the kernel Z of $b_0(\cdot,\cdot)$, which is given by

$$\begin{aligned} Z &= \{(\boldsymbol{v}, \boldsymbol{e}) \in V \times D \mid b_0((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}) = 0,\ \boldsymbol{\tau} \in S_0\} \\ &= \{(\boldsymbol{v}, \boldsymbol{e}) \in V \times D \mid \boldsymbol{\varepsilon}(\boldsymbol{v}) - 2\mu\,\mathcal{C}^{-1}\boldsymbol{e} \in S_0^{\perp}\}. \end{aligned}$$

Let $(\boldsymbol{v}, \boldsymbol{e})$ be in Z; using Korn's inequality and the uniform continuity of $\mathcal{C}^{-1}$, and noting that $(\boldsymbol{\varepsilon}(\boldsymbol{v}), \mathbf{1})_0 = 0$, we find that

$$a_0((\boldsymbol{v}, \boldsymbol{e}), (\boldsymbol{v}, \boldsymbol{e})) = 2\mu\|\boldsymbol{e}\|_0^2 \geq c\left(\|\boldsymbol{e}\|_0^2 + \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2\right) \geq c\left(\|\boldsymbol{e}\|_0^2 + \|\boldsymbol{v}\|_1^2\right).$$

We note that the coercivity constant depends on μ but not on λ.

Next, to establish the inf–sup condition we note that there exists a constant $C < \infty$ such that for each $q \in M := \{q \in L^2(\Omega) | \int_\Omega q \, \mathrm{d}x = 0\}$, there exists $\boldsymbol{v}_q \in V$ satisfying

$$\operatorname{div}\boldsymbol{v}_q = q, \quad \|\boldsymbol{v}_q\|_1 \leq C\|q\|_0, \tag{11}$$

(see, for example, [13]). For $\boldsymbol{\tau} \in S_0$, we have $\operatorname{tr}\boldsymbol{\tau} \in M$, and we define $\boldsymbol{e}_\tau := \operatorname{dev}(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\operatorname{tr}\tau}) - \boldsymbol{\tau}) \in D$. Then using (11), the norm of $(\boldsymbol{v}_{\operatorname{tr}\tau}, \boldsymbol{e}_\tau)$ is bounded by

$$\|\boldsymbol{v}_{\operatorname{tr}\tau}\|_1^2 + \|\boldsymbol{e}_\tau\|_0^2 \leq C(\|\operatorname{tr}\boldsymbol{\tau}\|_0^2 + \|\operatorname{dev}\boldsymbol{\tau}\|_0^2) \leq C\|\boldsymbol{\tau}\|_0^2.$$

Since $b_0((\boldsymbol{v}_{\operatorname{tr}\tau}, \boldsymbol{e}_\tau), \boldsymbol{\tau}) = (\boldsymbol{\varepsilon}(\boldsymbol{v}_{\operatorname{tr}\tau}) - 2\mu\mathcal{C}^{-1}\boldsymbol{e}_\tau, \boldsymbol{\tau})_0$, we can use $2\mu\mathcal{C}^{-1}\boldsymbol{e}_\tau = \boldsymbol{e}_\tau$ to write

$$b_0((\boldsymbol{v}_{\operatorname{tr}\tau}, \boldsymbol{e}_\tau), \boldsymbol{\tau}) = (\operatorname{sph}\boldsymbol{\varepsilon}(\boldsymbol{v}_{\operatorname{tr}\tau}) + \operatorname{dev}\boldsymbol{\tau}, \boldsymbol{\tau})_0.$$

The inf–sup condition now results from the orthogonality of the decomposition of $\boldsymbol{\tau}$ into its spherical and deviatoric parts; using also (11), we find that

$$\begin{aligned} b_0((\boldsymbol{v}_{\mathrm{tr}\,\boldsymbol{\tau}}, \boldsymbol{e}_{\boldsymbol{\tau}}), \boldsymbol{\tau}) &= (\mathrm{sph}\,\boldsymbol{\varepsilon}(\boldsymbol{v}_{\mathrm{tr}\,\boldsymbol{\tau}}), \mathrm{sph}\,\boldsymbol{\tau})_0 + (\mathrm{dev}\,\boldsymbol{\tau}, \mathrm{dev}\,\boldsymbol{\tau})_0 \\ &= \left(\frac{1}{2}\right)(\mathrm{div}\,\boldsymbol{v}_{\mathrm{tr}\,\boldsymbol{\tau}}, \mathrm{tr}\,\boldsymbol{\tau})_0 + (\mathrm{dev}\,\boldsymbol{\tau}, \mathrm{dev}\,\boldsymbol{\tau})_0 \\ &= \left(\frac{1}{2}\right)\|\,\mathrm{tr}\,\boldsymbol{\tau}\|_0^2 + \|\,\mathrm{dev}\,\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|_0^2. \end{aligned}$$

Remark 2.2 The result of Lemma 2.1 can be generalized to the case $\alpha = -1$ for which the operator $\mathcal{B}_\alpha$ is singular. The kernel of $\mathcal{B}_\alpha$ is then given by $\{\boldsymbol{e} \in D \mid \mathrm{dev}\,\boldsymbol{e} = 0\}$. To obtain a unique solution of (8), we therefore have to seek a solution in the subspace $\{(\boldsymbol{v}, \boldsymbol{e}, \boldsymbol{\tau}) \in V \times D \times S_0 \mid \mathrm{tr}\,\boldsymbol{\tau} = \kappa\,\mathrm{tr}\,\boldsymbol{e}\}$.

3 Finite element formulations

Let $\mathcal{T}_h$ be a quasi-uniform, shape-regular quadrilateral triangulation of the polygonal domain Ω. The diameter of an element K in $\mathcal{T}_h$ is denoted by h_K. Finite element spaces are defined by maps from a reference square $\hat{K} = (-1, 1)^2$. For nonnegative integer k, let $\mathcal{P}_k(\cdot)$ denote the space of polynomials in two variables of total degree less than or equal to k, and $\mathcal{Q}_k(\cdot)$ the space of polynomials in two variables of total degree less than or equal to k in each variable. A typical element $K \in \mathcal{T}_h$ is generated by an isoparametric map F_K from the reference element $\hat{K}$. It is clear that if $\hat{v} \in \mathcal{Q}_1(\hat{K})$, then $\hat{v} \circ F_K^{-1}$ is in general not a polynomial on the quadrilateral K.

The finite element space for the displacement is taken to be the space of continuous functions whose restrictions to an element K are obtained by maps of bilinear functions from the reference element; that is,

$$V_h := \left\{\boldsymbol{v}_h \in V,\ \boldsymbol{v}_h|_K = \hat{\boldsymbol{v}}_h \circ F_K^{-1},\quad \hat{\boldsymbol{v}}_h \in \mathcal{Q}_1(\hat{K})^2,\ K \in \mathcal{T}_h\right\}.$$

The spaces of stresses and strains are discretized by defining the finite-dimensional spaces

$$\begin{aligned} S_h &:= \left\{\boldsymbol{\tau}_h \in S_0 \mid (\boldsymbol{\tau}_h|_K)_{ij} = \left(\hat{\boldsymbol{\tau}}_h\right)_{ij} \circ F_K^{-1},\quad \hat{\boldsymbol{\tau}}_h \in S_\square,\ K \in \mathcal{T}_h\right\}, \\ D_h &:= \left\{\boldsymbol{e}_h \in S_0 \mid (\boldsymbol{e}_h|_K)_{ij} = \left(\hat{\boldsymbol{e}}_h\right)_{ij} \circ F_K^{-1},\quad \hat{\boldsymbol{e}}_h \in D_\square,\ K \in \mathcal{T}_h\right\}, \end{aligned}$$

where $D_\square$ and $S_\square$ are the reference bases of strains and stresses, defined on $\hat{K}$. These two variables are defined locally on each element and no continuity conditions apply at the element boundaries. Moreover, we define the space M_h by

$$M_h := \mathrm{tr}\,S_h.$$

Before giving some concrete examples, we recall the Voigt representation of the tensorial quantities stress and strain in vectorial form, in two dimensions. These are written as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{d} = \begin{bmatrix} d_{11} \\ d_{22} \\ 2d_{12} \end{bmatrix}, \quad \mathcal{I} := \operatorname{span} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that, in vectorial form, $\boldsymbol{\sigma}^T \boldsymbol{d} = \sigma_{ij} d_{ij}$.

The spaces S_h and D_h will be generated from bases defined on $\hat{K}$. We will make use of the following bases on $\hat{K}$:

$$A := \operatorname{span} \begin{bmatrix} \hat{y} & 0 \\ 0 & \hat{x} \\ 0 & 0 \end{bmatrix}, \quad B := \operatorname{span} \begin{bmatrix} \hat{x} & 0 \\ 0 & \hat{y} \\ 0 & 0 \end{bmatrix}, \quad C := \operatorname{span} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \hat{x} & \hat{y} \end{bmatrix}. \tag{12}$$

Of special interest will be the following cases:

Case II corresponds to the method of mixed enhanced strains [15,16] while Case V corresponds to the method of enhanced assumed strains [20]. We remark that $\operatorname{tr}(\mathcal{I} + A) = \operatorname{tr}(\mathcal{I} + A + C) = P_1(\hat{K})$ and $\operatorname{tr}(\mathcal{I} + C) = P_0(\hat{K})$.

In the following, all constants are generic, do not depend on the mesh size, and do not degenerate in the limit case $\lambda \longrightarrow \infty$. We restrict our analysis to the case of meshes for which F_K is an affine mapping for each element $K \in \mathcal{T}_h$. However, numerical results will also be given for the more general case of meshes of arbitrary quadrilaterals in Sect. 7.

From (8), the discrete modified Hu-Washizu formulation is as follows: find $(\boldsymbol{u}_h^\alpha, \boldsymbol{d}_h^\alpha, \boldsymbol{\sigma}_h^\alpha) \in V_h \times D_h \times S_h$ such that

$$\begin{aligned} a_\alpha((\boldsymbol{u}_h^\alpha, \boldsymbol{d}_h^\alpha), (\boldsymbol{v}_h, \boldsymbol{e}_h)) + b_\alpha((\boldsymbol{v}_h, \boldsymbol{e}_h), \boldsymbol{\sigma}_h^\alpha) &= \ell(\boldsymbol{v}_h), \quad (\boldsymbol{v}_h, \boldsymbol{e}_h) \in V_h \times D_h, \\ b_\alpha((\boldsymbol{u}_h^\alpha, \boldsymbol{d}_h^\alpha), \boldsymbol{\tau}_h) - \tfrac{(\lambda - \alpha\mu)}{\kappa^2} c(\boldsymbol{\sigma}_h^\alpha, \boldsymbol{\tau}_h) &= 0, \quad \boldsymbol{\tau}_h \in S_h. \end{aligned} \tag{13}$$

In contrast to the continuous setting (8), the discrete solution can depend on α. However for simplicity of notation, we suppress from now on the additional index α in the solution and replace $(\boldsymbol{u}_h^\alpha, \boldsymbol{d}_h^\alpha, \boldsymbol{\sigma}_h^\alpha)$ by $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h)$.

Next, we introduce two assumptions that are key to the study of well-posedness of the problem (13).

Assumption 3.1

3.1(i) $S_h \subset D_h$
3.1(ii) $\operatorname{tr} D_h \mathbf{1} \subset D_h$

Lemma 3.1 *Under the Assumptions* 3.1(i) *and* 3.1(ii)*, the solution of* (13) *does not depend on* $\alpha \neq -1$.

Proof Let $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h) \in V_h \times D_h \times S_h$ be the solution of (13). Using (9) and Assumptions 3.1(i) and 3.1(ii), we find that $\mathcal{B}_\alpha \boldsymbol{e}_h, \mathcal{C}\boldsymbol{d}_h, \boldsymbol{\sigma}_h$ all belong to D_h and thus $\boldsymbol{\sigma}_h = \mathcal{C}\boldsymbol{d}_h$. The rest of the proof follows the same lines as in the continuous setting.

Remark 3.1 For Lemma 3.1 the Assumption 3.1(ii) is essential, as this implies that $\mathcal{C}D_h = D_h$. In particular, for the choice $S_h = D_h$ but $\operatorname{tr} D_h \mathbf{1} \not\subset D_h$, the Hu-Washizu formulation with $\alpha = \lambda/\mu$ can yield bad numerical results in the limit case whereas the formulation with, for example, $\alpha = 0$ gives good results (see also

Table 1 Different cases for the discrete spaces

Case	I	II	III	IV	V
$S_\square$	$\mathcal{I}+A$	$\mathcal{I}+A$	$\mathcal{I}+C$	$\mathcal{I}+A+C$	$\mathcal{I}+A+C$
$D_\square$	$\mathcal{I}+A$	$\mathcal{I}+A+B$	$\mathcal{I}+C$	$\mathcal{I}+A+C$	$\mathcal{I}+A+B+C$
	$S_h^I = D_h^I$	$S_h^{II} \subset D_h^{II}$	$S_h^{III} = D_h^{III}$	$S_h^{IV} = D_h^{IV}$	$S_h^V \subset D_h^V$

Section 7). We note that Cases II, III and V in Table 1 satisfy both Assumptions 3.1(i) and 3.1(ii), but that the Cases I and IV do not satisfy 3.1(ii). Furthermore, Case IV in combination with the classical Hu-Washizu formulation ($\alpha = \lambda/\mu$) leads to the standard Q_1-approach.

We note that there exists a strong link between the modified Hu-Washizu formulation (13), the Hellinger–Reissner, mixed enhanced strain, enhanced assumed strain and the classical Q_1–P_0 formulation. A detailed discussion on these equivalences can be found in [10].

4 Analysis of the modified Hu-Washizu formulation

In this section we turn to the task of establishing conditions under which the discrete saddle point problem (13) is uniformly stable in the limit case. For the inf-sup condition, we need some preliminary results. First we note that dev S_h and sph S_h are in general not subspaces of S_h. This is remedied by introducing the space $\widetilde{M}_h := \{\tau_h, \tau_h\mathbf{1} \in S_h\}$, and defining the discrete spherical and deviatoric parts of $\boldsymbol{\tau}_h \in S_h$ as

$$\mathrm{sph_h}\, \boldsymbol{\tau}_h := \frac{1}{2} P_{\widetilde{M}_h} \operatorname{tr} \boldsymbol{\tau}_h \mathbf{1} \quad \text{and} \quad \mathrm{dev_h}\, \boldsymbol{\tau}_h := \boldsymbol{\tau}_h - \frac{1}{2} P_{\widetilde{M}_h} \operatorname{tr} \boldsymbol{\tau}_h \mathbf{1},$$

where $P_{\widetilde{M}_h}$ is the orthogonal projection onto $\widetilde{M}_h$. Denoting the orthogonal projection onto S_h by P_{S_h}, we can easily verify that $\mathrm{dev_h}\, \boldsymbol{\tau}_h = P_{S_h} \operatorname{dev} \boldsymbol{\tau}_h$. This leads to an orthogonal decomposition of S_h with

$$S_h = \mathrm{dev_h}\, S_h \oplus \mathrm{sph_h}\, S_h. \tag{14}$$

As a result, we find that not only the elements $\boldsymbol{\tau}_h \in \mathrm{sph_h}\, S_h$ and $\hat{\boldsymbol{\tau}}_h \in \mathrm{dev_h}\, S_h$ are orthogonal with respect to the L^2-inner product, but also their traces: that is,

$$2(\operatorname{tr} \boldsymbol{\tau}_h, \operatorname{tr} \hat{\boldsymbol{\tau}}_h)_0 = (\operatorname{tr} \boldsymbol{\tau}_h \mathbf{1}, \operatorname{tr} \hat{\boldsymbol{\tau}}_h \mathbf{1})_0 = 2(\boldsymbol{\tau}_h, \hat{\boldsymbol{\tau}}_h)_0 = 0. \tag{15}$$

To get a better feeling for the discrete spherical and deviatoric parts, we consider Case I in more detail. We have

$$\operatorname{dev}(\mathcal{I}+A) = \operatorname{span} \begin{bmatrix} 1 & 0 & \hat{x} & \hat{y} \\ -1 & 0 & -\hat{x} & -\hat{y} \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \operatorname{sph}(\mathcal{I}+A) = \operatorname{span} \begin{bmatrix} 1 & \hat{x} & \hat{y} \\ 1 & \hat{x} & \hat{y} \\ 0 & 0 & 0 \end{bmatrix},$$

and we find that $\dim(\mathcal{I}+A) = 5$, $\dim \operatorname{dev}(\mathcal{I}+A) = 4$, and $\dim \operatorname{sph}(\mathcal{I}+A) = 3$. We point out that in general $\mathrm{sph_h}\, S_h \neq P_{S_h} \operatorname{sph} S_h$; for example

$$\mathrm{dev_h}(\mathcal{I}+A) = \operatorname{span}\begin{bmatrix} 1 & 0 & \hat{y} & 0 \\ -1 & 0 & 0 & \hat{x} \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad P_{S_h} \operatorname{sph}(\mathcal{I}+A) = \operatorname{span}\begin{bmatrix} 1 & \hat{y} & 0 \\ 1 & 0 & \hat{x} \\ 0 & 0 & 0 \end{bmatrix},$$

whereas $\mathrm{sph_h}(\mathcal{I}+A) = \operatorname{span}\,[1, 1, 0]^{\mathrm{T}}$. In the continuous setting, the definition of the deviatoric part yields that the trace of the deviatoric part is equal to zero. This is not the case for an element in $\mathrm{dev_h}\, S_h$.

For $\boldsymbol{\tau} \in S$, the norm of the trace is bounded by $\|\operatorname{tr}\boldsymbol{\tau}\|_0^2 \leq 2\|\boldsymbol{\tau}\|_0^2$. The following lemma shows that a stronger bound holds for the norm of the trace of an element in $\mathrm{dev_h}\, S_h$.

Lemma 4.1 *There exists a* $\omega < 1$ *such that*

$$\|\operatorname{tr}\boldsymbol{\tau}_h\|_0^2 \leq 2\omega\|\boldsymbol{\tau}_h\|_0^2, \quad \boldsymbol{\tau}_h \in \mathrm{dev_h}\, S_h,$$

and thus $(\mathcal{C}^{-1}\boldsymbol{\tau}_h, \boldsymbol{\tau}_h)_0 \geq (1-\omega)/2\mu\|\boldsymbol{\tau}_h\|_0^2$, $\boldsymbol{\tau}_h \in \mathrm{dev_h}\, S_h$.

Proof The proof is based on a discrete norm equivalence on the reference element. To start, we define $\|\boldsymbol{\tau}\|_{*;\hat{K}} := \|\operatorname{dev}\boldsymbol{\tau}\|_{0;\hat{K}}$. It is trivial to see that $\|\cdot\|_*$ defines a seminorm on $(L^2(\hat{K}))^{2\times 2}$ and that it is zero if and only if $\operatorname{dev}\boldsymbol{\tau}$ is zero. Given $\boldsymbol{\tau}_h \in \mathrm{dev_h}\, S_\square$ and $\operatorname{dev}\boldsymbol{\tau}_h = 0$, we find by means of (14) and definitions of $\mathrm{sph_h}$ and $\mathrm{dev_h}$ that $\boldsymbol{\tau}_h \in \mathrm{dev_h}\, S_\square \cap \mathrm{sph_h}\, S_\square = \{0\}$. Thus the seminorm $\|\cdot\|_{*;K}$ restricted to $\mathrm{dev_h}\, S_\square$ is a norm. The dimension of the space $\mathrm{dev_h}\, S_\square$ is finite and thus $\|\cdot\|_{*;\hat{K}}$ is equivalent to $\|\cdot\|_{0;\hat{K}}$. Using the definition of S_h and the special structure of the isoparametric element map F_K, the equivalence of $\|\cdot\|_*^2 := \sum_{K\in\mathcal{T}_h}\|\cdot\|_{*;K}^2$ and $\|\cdot\|_0^2$ on $\mathrm{dev_h}\, S_h$ is established. Now let $0 < c < 1$ be such that $\|\boldsymbol{\tau}_h\|_*^2 \geq c\|\boldsymbol{\tau}_h\|_0^2$, $\boldsymbol{\tau}_h \in \mathrm{dev_h}\, S_h$; then a straightforward computation reveals that

$$\begin{aligned} 2(1-c)\|\boldsymbol{\tau}_h\|_0^2 &\geq \|\operatorname{tr}\boldsymbol{\tau}_h\|_0^2, \\ (\mathcal{C}^{-1}\boldsymbol{\tau}_h, \boldsymbol{\tau}_h)_0 &\geq \frac{1}{2\mu}\left(\|\boldsymbol{\tau}_h\|_0^2 - 2\gamma(1-c)\|\boldsymbol{\tau}_h\|_0^2\right). \end{aligned}$$

We can now use these preliminary results to establish a uniform inf–sup condition.

Assumption 4.1 *The following assumptions are essential for uniform stability in the discrete case:*

4.1(i) *There exists a constant* $0 < c_0 \leq 1$ *such that*

$$\|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \geq c_0\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0;$$

4.1(ii) *There exist constants* c_l *and* c_u *with* $0 < c_l < 1$ *and* $0 < c_u < \infty$*, such that* α *satisfies the bound*

$$\max\left(-c_l, \frac{\lambda}{\mu}\left(1 - \frac{c_l}{\omega}\right)\right) \leq \alpha \leq \min\left(c_u, \frac{\lambda}{\mu}\right);$$

4.1(iii) $(V_h, \widetilde{M}_h)$ *forms a stable Stokes pairing; that is, there exists a constant* $\beta^* > 0$, *independent of h, such that*

$$\sup_{v_h \in V_h} \frac{(\operatorname{div} \boldsymbol{v}_h, q_h)_0}{\|\boldsymbol{v}_h\|_V} \geq \beta^* \|q_h\|_0 \quad \textit{for all } q_h \in \widetilde{M}_h.$$

Here ω is the bound from Lemma 4.1.

In the following, the generic constants depend on c_l and c_u. Therefore, care has to be exercised in choosing these constants. It is easy to verify that $\alpha = 0$ is a choice that satisfies Assumption 4.1(ii), as is $\alpha = 1$ when $\lambda/\mu \geq 1$, for some suitable c_l and c_u. As we will see later, of special interest for Cases I and IV will be a negative value of α. However, the choice of negative α can lead to the loss of coercivity.

Roughly speaking, Assumption 4.1 (i) says that S_h has to be large enough and Assumption 4.1 (iii), that S_h has to be small enough. We point out that Assumption 4.1 (iii) is weaker than the assumption that (V_h, M_h) forms a stable Stokes pairing. For example, in Case I we find that $\operatorname{tr}(\mathcal{I} + A) = P_1(\hat{K})$, whereas $\operatorname{tr} \operatorname{sph_h}(\mathcal{I} + A) = P_0(\hat{K})$.

The discrete kernel associated with the bilinear form $b_\alpha(\cdot, \cdot)$ is given by

$$Z_h := \left\{ (\boldsymbol{v}_h, \boldsymbol{e}_h) \in V_h \times D_h \mid P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h) = P_{S_h} \left(2\mu \, \mathcal{C}^{-1} \boldsymbol{e}_h + \frac{\alpha \mu}{\kappa} \operatorname{tr} \boldsymbol{e}_h \mathbf{1} \right) \right\}.$$

Lemma 4.2 *Under the Assumptions* 4.1 (i) *and* 4.1 (ii), *the bilinear form* $a_\alpha(\cdot, \cdot)$ *is uniformly coercive on* $Z_h \times Z_h$ *and uniformly continuous on* $(V_h \times D_h) \times (V_h \times D_h)$.

Proof Let $(\boldsymbol{v}_h, \boldsymbol{e}_h) \in Z_h$. Assuming 4.1 (i) and 4.1 (ii), we find that

$$\begin{aligned} \|\boldsymbol{v}_h\|_1^2 &\leq C \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 \leq C \|P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 = C \|P_{S_h} (2\mu \, \mathcal{C}^{-1} \boldsymbol{e}_h + \frac{\alpha \mu}{\kappa} (\operatorname{tr} \boldsymbol{e}_h) \mathbf{1})\|_0^2 \\ &\leq C \|\boldsymbol{e}_h\|_0^2 \leq C a_\alpha((\boldsymbol{v}_h, \boldsymbol{e}_h), (\boldsymbol{v}_h, \boldsymbol{e}_h)). \end{aligned}$$

Lemma 4.3 *Under the Assumptions* 3.1 (i), 4.1 (ii) *and* 4.1 (iii), *the bilinear form* $b_\alpha(\cdot, \cdot)$ *satisfies a uniform inf-sup condition.*

Proof The proof is based on the decomposition of $\boldsymbol{\tau}_h$ into its discrete deviatoric and spherical parts, that is,

$$\boldsymbol{\tau}_h = \operatorname{dev_h} \boldsymbol{\tau}_h + \operatorname{sph_h} \boldsymbol{\tau}_h = \operatorname{dev_h} \boldsymbol{\tau}_h + q_h \mathbf{1},$$

where $\operatorname{dev_h} \boldsymbol{\tau}_h \in \operatorname{dev_h} S_h$, $\operatorname{sph_h} \boldsymbol{\tau}_h \in \operatorname{sph_h} S_h$ and $q_h \in \widetilde{M}_h$. Under the Assumption 4.1 (iii) we can find, for each $q_h \in \widetilde{M}_h$, a displacement $\boldsymbol{v}_{q_h} \in V_h$ such that

$$(\operatorname{div} \boldsymbol{v}_{q_h}, q_h)_0 = \|q_h\|_0^2, \quad \|\boldsymbol{v}_{q_h}\|_1 \leq C \|q_h\|_0. \tag{16}$$

Using the unique decomposition of $P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})$ into its discrete spherical and deviatoric parts, so that $P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) = \operatorname{sph_h} P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) + \operatorname{dev_h} P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})$, we define $\boldsymbol{e}_{\boldsymbol{\tau}_h} \in D_h$ by

$$\boldsymbol{e}_{\boldsymbol{\tau}_h} := \operatorname{dev_h} P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) - \beta \operatorname{dev_h} \boldsymbol{\tau}_h, \quad \beta > 0.$$

By means of (16) and the fact that $\| \operatorname{dev_h} P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})\|_0 \leq \|\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})\|_0 \leq C\|\boldsymbol{v}_{q_h}\|_1$, we can bound the norm of $(\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\tau_h})$ by

$$\|\boldsymbol{v}_{q_h}\|_1^2 + \|\boldsymbol{e}_{\tau_h}\|_0^2 \leq C(\|q_h\|_0^2 + \beta^2 \| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2) \leq C(1+\beta^2)\|\boldsymbol{\tau}_h\|_0^2.$$

To obtain the inf-sup condition, we use the equivalence (10) and consider the two terms of $b_\alpha(\cdot,\cdot)$ separately. We start by obtaining a lower bound for $b_{\lambda/\mu}((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\tau_h}), \boldsymbol{\tau}_h)$:

$$\begin{aligned}
b_{\lambda/\mu}((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\tau_h}), \boldsymbol{\tau}_h) &= (P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) - \boldsymbol{e}_{\tau_h}, \boldsymbol{\tau}_h)_0 \\
&= (\operatorname{sph_h} P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) + \beta \operatorname{dev_h} \boldsymbol{\tau}_h, \boldsymbol{\tau}_h)_0 \\
&= (\operatorname{sph_h} P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}), q_h\mathbf{1})_0 + \beta\| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2 \\
&= (\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}), q_h\mathbf{1})_0 + \beta\| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2 \\
&= (\operatorname{div} \boldsymbol{v}_{q_h}, q_h)_0 + \beta\| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2 \\
&= \|q_h\|_0^2 + \beta\| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2.
\end{aligned}$$

Next, we bound $c(\boldsymbol{e}_{\tau_h}, \boldsymbol{\tau}_h)$ by applying Lemma 4.1, (15) and (16):

$$\begin{aligned}
c(\boldsymbol{e}_{\tau_h}, \boldsymbol{\tau}_h) &= (\operatorname{tr} \boldsymbol{e}_{\tau_h}, \operatorname{tr} \boldsymbol{\tau}_h)_0 \\
&= (\operatorname{tr} (\operatorname{dev_h} P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})) - \beta \operatorname{tr} \operatorname{dev_h} \boldsymbol{\tau}_h, \operatorname{tr} \operatorname{dev_h} \boldsymbol{\tau}_h)_0 \\
&= (\operatorname{tr} P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}), \operatorname{tr} \operatorname{dev_h} \boldsymbol{\tau}_h)_0 - \beta\| \operatorname{tr} \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2 \\
&\geq -C\|q_h\|_0\| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0 - 2\beta\omega\| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2.
\end{aligned}$$

Defining $g_\alpha := 1 - \omega(1-(\alpha\mu/\lambda))$ we combine the lower bounds for $b_{\lambda/\mu}((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\boldsymbol{\tau}_h}), \boldsymbol{\tau}_h)$ and $c(\boldsymbol{e}_{\tau_h}, \boldsymbol{\tau}_h)$ and use $2\gamma \leq 1$ to obtain

$$b_\alpha((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\tau_h}), \boldsymbol{\tau}_h) \geq \|q_h\|_0^2 - 2C\|q_h\|_0\| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0 + \beta g_\alpha\| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2.$$

Assumption 4.1 (ii) gives $g_\alpha \geq 1 - c_l > 0$, and we can apply Young's inequality to find

$$b_\alpha((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\tau_h}), \boldsymbol{\tau}_h) \geq (1 - C\epsilon)\|q_h\|_0^2 + (\beta(1-c_l) - C\epsilon^{-1})\| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2.$$

For $0 < \epsilon$ small enough and $\beta < \infty$ large enough, we obtain

$$b_\alpha((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\tau_h}), \boldsymbol{\tau}_h) \geq c(\|q_h\|_0^2 + \| \operatorname{dev_h} \boldsymbol{\tau}_h\|_0^2) \geq c\|\boldsymbol{\tau}_h\|_0^2.$$

We now formulate our main result. The following theorem provides optimal a priori estimates for the displacement, strain, and stress.

Theorem 4.2 *Under the Assumptions* 3.1 (i), 4.1 (i)–4.1 (iii)*, the discretization error* $\eta_h^2 := \|\boldsymbol{u}-\boldsymbol{u}_h\|_1^2 + \|\boldsymbol{d}-\boldsymbol{d}_h\|_0^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0^2$*, where* $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h)$ *is the solution of* (13)*, is bounded by the best approximation error*

$$\eta_h^2 \leq C\left(\inf_{\boldsymbol{v}_h\in V_h} \|\boldsymbol{u}-\boldsymbol{v}_h\|_1^2 + \inf_{\boldsymbol{e}_h\in D_h} \|\boldsymbol{d}-\boldsymbol{e}_h\|_0^2 + \inf_{\boldsymbol{\tau}_h\in S_h} \|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_0^2\right).$$

Proof The a priori estimate results from the continuity of the bilinear forms and Lemmas 2.1, 4.2 and 4.3 (see, for example, [8]). In particular, it is seen that $\ker B_h^T = \{\mathbf{0}\}$, as in the continuous case, and $\|c(\cdot,\cdot)\|$ is bounded, independently of λ. These conditions suffice to establish the uniqueness of $\boldsymbol{\sigma}_h$ and the uniform error estimate (see [8], Sect. II and [4]).

In the rest of this section, we consider the Assumptions 4.1 (i) and 4.1 (iii) in more detail. To verify Assumption 4.1 (i) we consider the reference element and observe that

$$\boldsymbol{\varepsilon}(V_\square) := \operatorname{span}\begin{bmatrix} 1 & 0 & 0 & \hat{y} & 0 \\ 0 & 1 & 0 & 0 & \hat{x} \\ 0 & 0 & 1 & \hat{x} & \hat{y} \end{bmatrix},$$

where $V_\square$ is the space spanned by the standard bilinear polynomials on the reference element. A straightforward computation shows that on the reference element and thus on all elements $K \in \mathcal{T}_h$, Assumption 4.1 (i) is satisfied for all choices of $S_\square$ in Table 1.

We recall that Assumption 4.1 (iii) is on $\widetilde{M}_h$, which must be such that it forms with V_h a stable Stokes pair. Now for all our five choices of pairings (D_h^i, S_h^i), $1 \le i \le 5$, we find that $\widetilde{M}_h^i$ is given by

$$\widetilde{M}_h^i = \{q \in L_0^2(\Omega) \mid q|_K \in P_0(K),\ K \in \mathcal{T}_h\}, \quad 1 \le i \le 5. \tag{17}$$

Using bilinear elements for the displacements, it is well known that the pairing $(V_h, \widetilde{M}_h^i)$ does not satisfy Assumption 4.1 (iii) when $\widetilde{M}_h^i$ is given by (17), and that checkerboard modes might be observed (see, for example, [13]). Thus it is necessary that $\widetilde{M}_h$ be a proper subset of $\widetilde{M}_h^i$.

There are different ways in which to overcome this difficulty. One option is to work with macro-elements and to extract from $\widetilde{M}_h^i$ the checkerboard mode on each macro-element, as in [13] (Sect. II.3). The restrictions of functions in $\widetilde{M}_h^i$ to a macro-element are spanned by the four functions depicted in Fig. 1. The functions having the signs indicated in Fig. 1d are the local checkerboard modes. To obtain a stable pairing, we have to modify the space $\mathrm{sph_h}\, S_h^i$ but not $\mathrm{dev_h}\, S_h^i$.

A reduction in the dimension of $\widetilde{M}_h^i$ results in a smaller dimension of S_h^i, and thus for our choices Assumption 4.1 (i) cannot be verified locally on each element, but must be done on macro-elements. Using the fact that V_h comprises continuous displacements, and taking into account the rigid body motions, we find that the

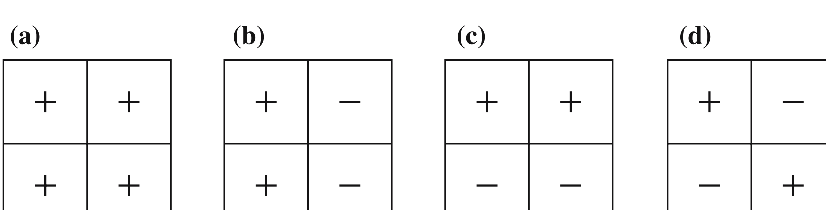

Fig. 1 Restrictions of the basis functions of $\widetilde{M}_h^i$ to a macro-element with $\pm$ indicating the sign inside the elements

dimension of $\boldsymbol{\varepsilon}(V_h)$ restricted to one macro-element is 15 and not 4×5. The modification of $\widetilde{M}_h^i$ reduces the dimension of S_h^i on each macro-element by one, and thus for all our examples the dimension of S_h^i restricted to one macro-element is not less than 19. Moreover, a straightforward computation on the macro-element shows that for all S_h^i, $1 \leq i \leq 5$, Assumption 4.1 (i) is satisfied even if we work on macro-elements and reduce the dimension of S_h^i restricted to a macro-element by one.

The other option is to enrich the displacement field. Assumption 4.1 (i) and (iii) are automatically satisfied if the displacement field is discretized by biquadratic finite elements.

A third possibility is to work with the macro-triangulation itself and replace M_h^i by M_{2h}^i. In all these three situations our new pairings satisfy a uniform inf-sup condition.

5 A priori estimate for the displacement

In this section we show that, despite the presence of checkerboard modes in the stress, as elucidated in the previous section, it is nevertheless possible, under certain conditions, to obtain optimal λ-independent a priori estimates of the displacement error. We will show furthermore that the conditions are verified by all five cases presented in Sect. 3.

The problem is approached by eliminating the discrete stress and strain from the formulation. This static condensation can be performed as S_h and D_h are defined locally on each element. The process applied to (8) yields a displacement-based formulation of the following form: find $\boldsymbol{u}_h \in V_h$ such that

$$(Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h, \tag{18}$$

where Q_h is a suitable local projection and $\mathcal{C}_h$ is some positive-definite symmetric operator. The symmetry results from the symmetry in (8). We note that the operators as well as the displacement depend on S_h, D_h and α. For simplicity of notation, we do not indicate this dependence.

The derivation of the optimal a priori estimate is based on the construction of a Fortin interpolation operator [11], which is shown to satisfy an approximation property. Then a set of assumptions on the operator $\mathcal{C}_h$, together with the use of the approximation property of the Fortin interpolation operator, permit the a priori result to be obtained.

By analogy with the situation pertaining to the Stokes problem and the use of the Q_1–P_0 element pair, we assume that the triangulation $\mathcal{T}_h$ has a macro-element structure and that there is a local checkerboard free subspace $\widetilde{M}_h^s \subset \widetilde{M}_h$ (cf. [13]). The space $\widetilde{M}_h^s$ is the orthogonal complement of $\widetilde{M}_h^u$, which is spanned by the local checkerboard modes (Figure 1 (d)). Moreover, we assume that the pairing $(V_h^s, \widetilde{M}_h^s)$ satisfies a uniform inf-sup condition, where

$$V_h^s := \{\boldsymbol{v}_h \in V_h \mid (\operatorname{div} \boldsymbol{v}_h, q_h)_0 = 0, \quad q_h \in \widetilde{M}_h^u\},$$

and that both subspaces $\widetilde{M}_h^s$ and V_h^s satisfy suitable approximation properties [6, 13]. In the following, we assume that $\boldsymbol{u}$ is the solution of (1), (2), (3) and (4), $\boldsymbol{u}_h$ is the first component of the solution of (13), and we set $W := [H^2(\Omega) \cap H_0^1(\Omega)]^2$.

We recall that Fortin's interpolation operator $\boldsymbol{I}_h^F : W \longrightarrow V_h^s$ for the stable Stokes pairing $(V_h^s, \widetilde{M}_h^s)$ is given by the problem of finding $(\boldsymbol{I}_h^F \boldsymbol{w}, p_h) \in V_h^s \times \widetilde{M}_h^s$ such that

$$\begin{aligned} (\nabla \boldsymbol{I}_h^F \boldsymbol{w}, \nabla z_h)_0 + (\operatorname{div} z_h, p_h)_0 &= (\nabla \boldsymbol{w}, \nabla z_h)_0, \quad z_h \in V_h^s, \\ (\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{w}, q_h)_0 &= (\operatorname{div} \boldsymbol{w}, q_h)_0, \quad q_h \in \widetilde{M}_h^s. \end{aligned} \tag{19}$$

Lemma 5.1 *Under the regularity assumption* (6)*, the operator* $\boldsymbol{I}_h^F$ *satisfies the approximation property*

$$\|\boldsymbol{u} - \boldsymbol{I}_h^F \boldsymbol{u}\|_1 + \lambda \| \operatorname{div} \boldsymbol{u} - \Pi_h(\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{u})\|_0 \leq Ch\|f\|_0,$$

where Π_h *is the* L^2*-projection onto* $\widetilde{M}_h$.

Proof Using the stability of the saddlepoint problem (19) and (6), the upper bound for $\|\boldsymbol{u} - \boldsymbol{I}_h^F \boldsymbol{u}\|_1$ is standard. There exists a $\tilde{\boldsymbol{u}} \in W$ such that $\operatorname{div} \boldsymbol{u} = \operatorname{div} \tilde{\boldsymbol{u}}$ and $\|\tilde{\boldsymbol{u}}\|_2 \leq C\| \operatorname{div} \boldsymbol{u}\|_1$, see, for example, [2]. Observing the definitions of V_h^s and of I_h^F, we find that

$$(\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{u}, q_h)_0 = (\operatorname{div} \boldsymbol{I}_h^F \tilde{\boldsymbol{u}}, q_h)_0, \quad q_h \in \widetilde{M}_h,$$

and thus $\Pi_h(\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{u}) = \Pi_h(\operatorname{div} \boldsymbol{I}_h^F \tilde{\boldsymbol{u}})$. In terms of (6), we obtain

$$\begin{aligned} \| \operatorname{div} \boldsymbol{u} - \Pi_h(\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{u})\|_0 &= \| \operatorname{div} \tilde{\boldsymbol{u}} - \Pi_h(\operatorname{div} \boldsymbol{I}_h^F \tilde{\boldsymbol{u}})\|_0 \\ &\leq \| \operatorname{div}(\tilde{\boldsymbol{u}} - \boldsymbol{I}_h^F \tilde{\boldsymbol{u}})\|_0 + \| \operatorname{div} \tilde{\boldsymbol{u}} - \Pi_h \operatorname{div} \tilde{\boldsymbol{u}}\|_0 \\ &\leq C(\|\tilde{\boldsymbol{u}} - \boldsymbol{I}_h^F \tilde{\boldsymbol{u}}\|_1 + h\|\tilde{\boldsymbol{u}}\|_2) \\ &\leq Ch\| \operatorname{div} \boldsymbol{u}\|_1 \leq \frac{C}{\lambda} h\|f\|_0. \end{aligned}$$

In this section, we provide some assumptions on Q_h and $\mathcal{C}_h$ under which optimal a priori estimates for (18) can be established. These assumptions can be easily verified for all given examples.

Assumption 5.1 *The following assumptions are essential for* λ*-independent constants in the a priori estimates: for* $\boldsymbol{v}_h \in V_h$,

5.1(i) $\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 \leq C(Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0$;

5.1(ii) $(Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{w}_h))_0 = (\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{w}_h))_0$, $\boldsymbol{w}_h \in V_h$;

5.1(iii) $\|(\mathcal{C}_h Q_h - \boldsymbol{\Pi}_h \mathcal{C})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \leq C\|(\mathbf{Id} - \boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_\mathrm{h})\|_0$, *where* $\boldsymbol{\Pi}_h$ *is the component-wise and element-wise* L^2*-projector onto piecewise constant functions.*

We note that Assumption 5.1(iii) is not met by the standard displacement-based formulation.

Theorem 5.2 *Under the Assumptions* 5.1(i)-(iii) *and* (6)*, the upper bound*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \leq Ch\|\boldsymbol{f}\|_0$$

for the discretization error holds, where $C < \infty$ *is independent of* λ *and* h.

Proof In a first step, we bound $\|\boldsymbol{v}_h - \boldsymbol{u}_h\|_1$, $\boldsymbol{v}_h \in V_h$, in terms of $\|\boldsymbol{v}_h - \boldsymbol{u}\|_1$. Using Korn's inequality and Assumptions 5.1(i)–(ii) and (5), we find that

$$\begin{aligned}\|\boldsymbol{v}_h - \boldsymbol{u}_h\|_1^2 &\le C\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}_h)\|_0^2 \le C(Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}_h))_0 \\ &= C\left((\boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0 - (\boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}_h), \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))_0\right) \\ &\le C\|\boldsymbol{v}_h - \boldsymbol{u}_h\|_1 \|\mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0.\end{aligned}$$

In terms of Assumption 5.1(iii), the triangle inequality, the properties of $\boldsymbol{I}_h^F$, and Lemma 5.1, we obtain

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 &\le C(\|(\mathcal{C}_h Q_h - \boldsymbol{\Pi}_h \mathcal{C})\boldsymbol{\varepsilon}(\boldsymbol{I}_h^F \boldsymbol{u})\|_0 \\ &\quad + \|\mathcal{C}(\boldsymbol{\Pi}_h \boldsymbol{\varepsilon}(\boldsymbol{I}_h^F \boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\|_0 + \|\boldsymbol{u} - \boldsymbol{I}_h^F \boldsymbol{u}\|_1) \\ &\le C(\lambda\|\Pi_h(\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{u}) - \operatorname{div} \boldsymbol{u}\|_0 + \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\Pi}_h \boldsymbol{\varepsilon}(\boldsymbol{u})\|_0 \\ &\quad + \|\boldsymbol{u} - \boldsymbol{I}_h^F \boldsymbol{u}\|_1) \le Ch\|f\|_0.\end{aligned}$$

Next, we consider the operators $\mathcal{C}_h$ and Q_h in more detail. We start by constructing an orthogonal decomposition of S_h in the form

$$S_h = S_h^c \oplus S_h^t,$$

where

$$S_h^c := \{\boldsymbol{\tau} \in S_h \mid \mathcal{C}\boldsymbol{\tau} \in S_h\}$$

and S_h^t is the orthogonal complement of S_h^c.We note that $S_h^t \subset \mathrm{dev_h}\, S_h$, $\mathrm{sph_h}\, S_h \subset S_h^c$, and $\mathrm{tr}(\mathrm{sph_h}\, S_h) = \mathrm{tr}\, S_h^c$.

Assumption 5.3 *For any* $\boldsymbol{\tau}_h^t, \boldsymbol{\omega}_h^t \in S_h^t$,

$$(\boldsymbol{\tau}_h^t, \boldsymbol{\omega}_h^t)_0 = (\mathrm{tr}\, \boldsymbol{\tau}_h^t, \mathrm{tr}\, \boldsymbol{\omega}_h^t)_0.$$

It is straightforward to verify this assumption for all our Cases I–V. For Case III, we find $S_h^t = \{\mathbf{0}\}$, and for all other cases, S_h^t is spanned locally by the basis A in (12). Focusing on this basis, we let $\hat{\boldsymbol{\tau}}_h, \hat{\boldsymbol{\omega}}_h \in S_\square^t$ be given by $\hat{\tau}_{h_{11}} = a\hat{y}$, $\hat{\tau}_{h_{22}} = b\hat{x}$, $\hat{\tau}_{h_{12}} = \hat{\tau}_{h_{21}} = 0$, $\hat{\omega}_{h_{11}} = c\hat{y}$, $\hat{\omega}_{h_{22}} = d\hat{x}$, $\hat{\omega}_{h_{12}} = \hat{\omega}_{h_{21}} = 0$. Then $\mathrm{tr}\, \hat{\boldsymbol{\tau}}_h = a\hat{y} + b\hat{x}$ and $\mathrm{tr}\, \hat{\boldsymbol{\omega}}_h = c\hat{y} + d\hat{x}$. Using the definition of the reference element we find that

$$(\hat{\boldsymbol{\tau}}_h, \hat{\boldsymbol{\omega}}_h)_{0;\hat{K}} = \tfrac{4}{3}(ac + bd) = (\mathrm{tr}\, \hat{\boldsymbol{\tau}}_h, \mathrm{tr}\, \hat{\boldsymbol{\omega}}_h)_{0;\hat{K}},$$

so that Assumption 5.3 is satisfied for all Cases I–V.

Next, we characterize the discrete elasticity operator $\mathcal{C}_h$.

Lemma 5.2

(a) *Under the Assumptions* 3.1(i)–(ii) *and* 5.3, *and for* $\alpha \neq -1$, *the operator* $\mathcal{C}_h$ *is given by*

$$\mathcal{C}_h \boldsymbol{e}_h = \mathcal{C} P_{S_h^c} \boldsymbol{e}_h + \frac{2\mu}{1-\gamma} P_{S_h^t} \boldsymbol{e}_h.$$

(b) *Under the assumptions* $S_h = D_h$ *and* 5.3, *and for* $\alpha \notin \{-1, -2\}$,

$$\mathcal{C}_h \boldsymbol{e}_h = \mathcal{C} P_{S_h^c} \boldsymbol{e}_h + \frac{4\mu(\mu+\lambda)^2(2+\alpha)}{\lambda^2 + \mu(2\mu+3\lambda)(2+\alpha)} P_{S_h^t} \boldsymbol{e}_h,$$

where $P_{S_h^c}$ *and* $P_{S_h^t}$ *are orthogonal projections onto the spaces* S_h^c *and* S_h^t, *respectively.*

(c) *The operator* Q_h *is given by* $Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h) = P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h)$ *for both cases* (a) *and* (b), *and the displacement-based formulation* (18) *is equivalent to the three field formulation* (13).

Proof The main idea of the proof is to use the orthogonal decomposition of S_h as given above and to apply static condensation. In the case that Assumptions 3.1(i)–(ii) hold, Lemma 3.1 yields $\boldsymbol{\sigma}_h = \mathcal{C}\boldsymbol{d}_h$. In terms of the decomposition given above, we can write $\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_h^c + \boldsymbol{\sigma}_h^t$, $\boldsymbol{\sigma}_h^c \in S_h^c$, $\boldsymbol{\sigma}_h^t \in S_h^t$. Using $\boldsymbol{d}_h = \mathcal{C}^{-1}\boldsymbol{\sigma}_h$, in the second equation of (13), we get $(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{d}_h, \boldsymbol{\tau}_h)_0 = 0$. The orthogonal decomposition of $\boldsymbol{\tau}_h \in S_h$ with $\boldsymbol{\tau}_h = \boldsymbol{\tau}_h^c + \boldsymbol{\tau}_h^t$ yields $(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{d}_h, \boldsymbol{\tau}_h^c)_0 = 0$ and $(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{d}_h, \boldsymbol{\tau}_h^t)_0 = 0$ from which using Assumption 5.3, we obtain

$$\boldsymbol{\sigma}_h^c = \mathcal{C} P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{u}_h), \quad \text{and} \quad \boldsymbol{\sigma}_h^t = \frac{2\mu}{1-\gamma} P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{u}_h). \tag{20}$$

We point out that $0 < 2\gamma \le 1$ independently of λ. If $S_h = D_h$, the first equation in (13) yields $P_{S_h}(\mathcal{B}_\alpha(\mathcal{C}\boldsymbol{d}_h - \boldsymbol{\sigma}_h)) = 0$. Using the decomposition for $\boldsymbol{d}_h$, we find that $\boldsymbol{d}_h = \boldsymbol{d}_h^c + \boldsymbol{d}_h^t$, $\boldsymbol{d}_h^c \in S_h^c$, $\boldsymbol{d}_h^t \in S_h^t$, and

$$P_{S_h^c}(\mathcal{B}_\alpha(\mathcal{C}\boldsymbol{d}_h^c - \boldsymbol{\sigma}_h^c)) = \mathbf{0}, \quad P_{S_h^t}(\mathcal{B}_\alpha(\mathcal{C}\boldsymbol{d}_h^t - \boldsymbol{\sigma}_h^t)) = \mathbf{0}.$$

Using the fact that $\mathcal{C}\boldsymbol{\tau}_h^c \in S_h^c$ in $P_{S_h^c}(\mathcal{B}_\alpha(\mathcal{C}\boldsymbol{d}_h^c - \boldsymbol{\sigma}_h^c)) = \mathbf{0}$, we have

$$\boldsymbol{d}_h^c = \mathcal{C}^{-1}\boldsymbol{\sigma}_h^c. \tag{21}$$

Assumption 5.3 in $P_{S_h^t}(\mathcal{B}_\alpha(\mathcal{C}\boldsymbol{d}_h^t - \boldsymbol{\sigma}_h^t)) = \mathbf{0}$ yields

$$\boldsymbol{d}_h^t = \frac{\lambda + \gamma(\alpha\mu - \lambda)}{\lambda\mu(\alpha+2)} \boldsymbol{\sigma}_h^t. \tag{22}$$

In terms of (21) and (22), we can eliminate $\boldsymbol{d}_h^c$ and $\boldsymbol{d}_h^t$ from the second equation of (13). The elimination of $\boldsymbol{d}_h^c$ gives $\boldsymbol{\sigma}_h^c = \mathcal{C} P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{u}_h)$, and by means of (22), we obtain

$$\boldsymbol{\sigma}_h^t = \frac{4\mu(\mu+\lambda)^2(2+\alpha)}{\lambda^2 + \mu(2\mu+3\lambda)(2+\alpha)} P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{u}_h). \tag{23}$$

Remark 5.1 The theoretical analysis requires that the discrete space S_h be a subset of S_0. This requirement does not present computational difficulties, since by using the bigger space $\hat{S}_h := S_h \oplus \operatorname{span}\{\mathbf{1}\}$, $\dim \hat{S}_h = \dim S_h + 1$, we find that $(P_{\hat{S}_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathbf{1})_0 = (\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathbf{1})_0 = (\operatorname{div} \boldsymbol{v}_h, 1)_0 = 0$, $\boldsymbol{v}_h \in V_h$. Thus Q_h can be locally computed on each element $K \in \mathcal{T}_h$ according to

$$Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h) = P_{\hat{S}_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h) \in S_h, \quad \boldsymbol{v}_h \in V_h.$$

Remark 5.2 The operators specified in Lemma 5.2 are also well defined for the cases $\alpha = -1$ and $\alpha = -2$. Thus the displacement based formulation (18) can be extended to these cases. The well-posedness of (18) for $\alpha = -2$ depends on S_h (see also Sect. 7).

In a next step, we provide sufficient conditions such that Assumptions 5.1(i)–(iii) are satisfied.

Assumption 5.4 $\mathrm{sph}_h\, S_{h|_K} \subset \mathcal{I} \subset S_{h|_K},\ K \in \mathcal{T}_h$.

By construction all our cases satisfy $\mathcal{I} \subset S_{h|_K}$ and $\mathrm{sph}_h\, S_{h|_K} = P_0(K)\mathbf{1} \subset \mathcal{I}$. Thus Assumption 5.4 is satisfied for all our cases. We note that Assumption 5.4 guarantees that the best approximation error in the L^2-norm of the space $S_h^c \subset S_h$ is of order h.

Lemma 5.3 *Assume that Assumptions* 3.1(i), 4.1(i), 5.3 *and* 5.4 *are satisfied. If either the Assumption* 3.1(ii) *or* $S_h = D_h$ *and* $0 < c_d \le \mu(2+\alpha) \le C_d < \infty$ *hold, then the Assumptions* 5.1(i)–(iii) *are satisfied.*

Proof For both cases, we have the displacement based formulation (18) with $\mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \mathcal{C} P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \theta(\mu,\lambda,\alpha) P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{v}_h)$, where

$$\theta(\mu,\lambda,\alpha) := \frac{2\mu}{1-\gamma} \quad \text{in the first case,}$$

and

$$\theta(\mu,\lambda,\alpha) := \frac{4\mu(\mu+\lambda)^2(2+\alpha)}{\lambda^2 + \mu(2\mu+3\lambda)(2+\alpha)} \quad \text{in the second case.}$$

Since in both cases θ is positive and bounded independently of λ from below, Assumption 4.1(i) yields

$$\begin{aligned}(Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0 &= (P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C} P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0 \\ &\quad + \theta(\mu,\lambda,\alpha)(P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{v}_h),\ P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0 \\ &\ge C(P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h),\ P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0 \ge C\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2\end{aligned}$$

with a λ-independent constant C. Assumption 5.1 (ii) is satisfied as a result of $Q_h = P_{S_h}$, and $\mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h) \in S_h$. Now, we turn our attention to 5.1 (iii). Assumption 5.4 yields $\mathrm{tr}(P_{S_h^c} - \boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = 0$, and thus

$$\begin{aligned}(\mathcal{C}_h Q_h - \mathcal{C}\boldsymbol{\Pi}_h)\boldsymbol{\varepsilon}(\boldsymbol{v}_h) &= \mathcal{C} P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \theta(\mu,\lambda,\alpha) P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \mathcal{C}\boldsymbol{\Pi}_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h) \\ &= 2\mu(P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \boldsymbol{\Pi}_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)) + \theta(\mu,\lambda,\alpha) P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{v}_h).\end{aligned}$$

Since $\theta(\mu,\lambda,\alpha)$ is bounded independently of λ from above, it suffices to show that

$$\begin{aligned}\|(P_{S_h^c} - \boldsymbol{\Pi}_h)\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 &\le C\|(\mathbf{Id} - \boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_\mathrm{h})\|_0, \\ \|P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 &\le C\|(\mathbf{Id} - \boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_\mathrm{h})\|_0.\end{aligned} \tag{24}$$

To prove the first inequality in (24), we start with the identity

$$\|(P_{S_h^c} - \mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 = ((P_{S_h^c} - \mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h), (P_{S_h^c} - \boldsymbol{\Pi}_h + \boldsymbol{\Pi}_h - \mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0.$$

By definition of $P_{S_h^c}$ and Assumption 5.4, we find that

$$(P_{S_h^c} - \boldsymbol{\Pi}_h)\boldsymbol{\varepsilon}(\boldsymbol{v}_h) \in S_h^c, \text{ and } \|(P_{S_h^c} - \mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \leq \|(\boldsymbol{\Pi}_h - \mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0.$$

Hence the first inequality in (24) results from the triangle inequality.

Using $\boldsymbol{\Pi}_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h) \in S_h^c$, and the orthogonality of S_h^c and S_h^t, the following upper bound is obtained:

$$\begin{aligned}\|P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 &= (-P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h), (\mathbf{Id} - P_{S_h^t} - \mathbf{Id} + \boldsymbol{\Pi}_h)\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0\\ &\leq \|P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0\|(\mathbf{Id} - \boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_\mathrm{h})\|_0.\end{aligned}$$

Now, we show that Assumptions 5.1(i)–(iii) are satisfied for all our cases. To do so we use Lemma 5.3. We recall that Assumptions 3.1(i), 4.1(i), 5.3 and 5.4 are satisfied for all our cases. Moreover Assumption 3.1(ii) holds for Cases II, III and V and $D_h = S_h$ holds for Cases I and IV.

Remark 5.3 Lemma 5.3 provides sufficient conditions such that the displacement based formulation (18) is well defined and yields λ-independent optimal a priori estimates. We recall that for Cases I and IV, the choice of α is crucial. Setting $\alpha = \lambda/\mu$ will result in volumetric locking whereas the case $\alpha = 0$, for example, will give good numerical results. For the λ-independent estimate, the assumption $0 < c_d \leq (2+\alpha) \leq C_d < \infty$ is essential. We remark that this condition on α is weaker than Assumption 4.1 (ii).

To conclude this section, we show that the well known Q_1–P_0 saddlepoint problem can be analyzed as a special case of the extended Hu-Washizu formulation.

Lemma 5.4 *Case IV with* $\alpha = -((3\lambda+2\mu)/(2\lambda+\mu))$ *is equivalent to the penalized* Q_1–P_0 *saddlepoint problem (Stokes)*

$$\begin{aligned}2\mu(\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 + (\operatorname{div}\boldsymbol{v}_h, p_h)_0 &= \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h,\\ (\operatorname{div}\boldsymbol{u}_h, q_h)_0 - \frac{1}{\lambda}(p_h, q_h)_0 &= 0, \quad q_h \in \widetilde{M}_h.\end{aligned}$$

Proof By static condensation, we can eliminate the pressure from the saddle point problem and arrive at

$$2\mu(\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 + \lambda(\Pi_h \operatorname{div}\boldsymbol{v}_h, \Pi_h \operatorname{div}\boldsymbol{u}_h)_0 = \ell(\boldsymbol{v}_h), \ \boldsymbol{v}_h \in V_h.$$

Using the given α, we show that (18) is equivalent to the above equation. With the given α, we get $\mathcal{C}_h\boldsymbol{\tau}_h = \mathcal{C}P_{S_h^c}\boldsymbol{\tau}_h + 2\mu P_{S_h^t}\boldsymbol{\tau}_h$. Hence

$$\begin{aligned}(Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 &= (P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 + (P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h), 2\mu P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0\\ &= 2\mu(P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h), P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0\\ &\quad + \lambda(\operatorname{tr} P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \operatorname{tr} P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0\\ &= 2\mu(P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h), P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 + \lambda(\Pi_h \operatorname{div}\boldsymbol{v}_h, \Pi_h \operatorname{div}\boldsymbol{u}_h)_0.\end{aligned}$$

Finally, the result follows by using the fact that $P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \boldsymbol{\varepsilon}(\boldsymbol{v}_h)$ for Case IV.

Remark 5.4 The choice $\alpha = -((3\lambda + 2\mu)/(2\lambda + \mu))$ does not satisfy Assumption 4.1(ii), but it does satisfy the bound $0 < c_d \leq (2 + \alpha) \leq C_d < \infty$.

6 A priori results for the stress

The results of the previous sections show that the lack of stability resulting from the presence of a checkerboard mode is confined to the stress, and does not affect the displacement. In this section, we establish an optimal a priori result for a post-processed stress.

Let $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h)$ be the unique solution of the modified Hu-Washizu formulation (13), and recall that the discrete stress can be written in the form

$$\boldsymbol{\sigma}_h = \mathrm{dev_h}\, \boldsymbol{\sigma}_h + \mathrm{sph_h}\, \boldsymbol{\sigma}_h = \mathrm{dev_h}\, \boldsymbol{\sigma}_h + p_h \mathbf{1}, \quad p_h \in \widetilde{M}_h.$$

Define the post-processed stress $\boldsymbol{\sigma}_h^s$ by

$$\boldsymbol{\sigma}_h^s := \mathrm{dev_h}\, \boldsymbol{\sigma}_h + p_h^s \mathbf{1}, \quad p_h^s := \Pi_h^s p_h,$$

where Π_h^s is the L^2-projection onto the checkerboard-free subspace $\widetilde{M}_h^s$.

Theorem 6.1 *Under the Assumptions* 4.1(ii)*,* 5.1(i)–(iii)*,* 5.3*,* 5.4 *and* (6)*, the λ-independent error bound*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^s\|_0 \leq Ch\|\boldsymbol{f}\|_0$$

holds for stress.

Proof We start by eliminating the strain from the saddlepoint problem (13), to obtain a modified Hellinger-Reissner two-field formulation. Using (20), (23) and the definition of $\theta(\mu, \lambda, \alpha)$, we find that

$$\begin{aligned} \hat{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}_h)_0 &= 0, \quad \boldsymbol{\tau}_h \in S_h, \\ (\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \boldsymbol{\sigma}_h)_0 &= \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h, \end{aligned}$$

where

$$\hat{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) := (\mathcal{C}_h^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_0 \text{ with } \mathcal{C}_h^{-1}\boldsymbol{\tau}_h = \mathcal{C}^{-1} P_{S_h^c}\boldsymbol{\tau}_h + \frac{1}{\theta(\mu, \lambda, \alpha)} P_{S_h^t}\boldsymbol{\tau}_h.$$

We define S_h^s and $V_h^\perp$ such that the orthogonal decomposition of $S_h = S_h^s \oplus \widetilde{M}_h^u \mathbf{1}$, and $V_h = V_h^s \oplus V_h^\perp$ holds, respectively. Then, we have $(\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s), q_h^u \mathbf{1})_0 = (\mathrm{div}\, \boldsymbol{v}_h^s, q_h^u)_0 = 0$, $\boldsymbol{v}_h^s \in V_h^s$, $q_h^u \in \widetilde{M}_h^u$. Moreover, since $p_h^u \in \widetilde{M}_h$, $\hat{a}(p_h^u \mathbf{1}, \boldsymbol{\tau}_h^s) = (\mathcal{C}_h^{-1}(p_h^u \mathbf{1}), \boldsymbol{\tau}_h^s)_0 = (\mathcal{C}^{-1} P_{S_h^c}(p_h^u \mathbf{1}), \boldsymbol{\tau}_h^s)_0 = 0$, and hence $\hat{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h^s) = \hat{a}(\boldsymbol{\sigma}_h^s, \boldsymbol{\tau}_h^s)$.

Using the orthogonality between S_h^s and $\widetilde{M}_h^u \mathbf{1}$ and observing that $(\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s), p_h \mathbf{1})_0 = (\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s), p_h^s \mathbf{1})_0$ and $\hat{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h^s) = \hat{a}(\boldsymbol{\sigma}_h^s, \boldsymbol{\tau}_h^s)$, we find that $(\boldsymbol{\sigma}_h^s, \boldsymbol{u}_h) \in (S_h^s \times V_h)$ satisfies

$$\begin{aligned} \hat{a}(\boldsymbol{\sigma}_h^s, \boldsymbol{\tau}_h^s) - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}_h^s)_0 &= 0, \quad \boldsymbol{\tau}_h^s \in S_h^s, \\ (\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s), \boldsymbol{\sigma}_h^s)_0 &= \ell(\boldsymbol{v}_h^s), \quad \boldsymbol{v}_h^s \in V_h^s. \end{aligned} \tag{25}$$

The bilinear form $\hat{a}(\cdot,\cdot)$ is not uniformly in λ coercive on $S_h^s \times S_h^s$. We recall that $S_h^t \subset \mathrm{dev}_h\, S_h$, and decompose $\boldsymbol{\tau}_h^s$ according to $\boldsymbol{\tau}_h^s = \boldsymbol{\tau}_h^c + \boldsymbol{\tau}_h^t$, $\boldsymbol{\tau}_h^c \in S_h^c$, $\boldsymbol{\tau}_h^t \in S_h^t$, and $\boldsymbol{\tau}_h^c$ according to $\boldsymbol{\tau}_h^c = \mathrm{dev}_h\, \boldsymbol{\tau}_h^c + \mathrm{sph}_h\, \boldsymbol{\tau}_h^c$. Due to Lemma 4.1 and using the fact that $\theta(\mu,\lambda,\alpha)$ is uniformly bounded from above, we obtain

$$\begin{aligned}
\hat{a}(\boldsymbol{\tau}_h^s, \boldsymbol{\tau}_h^s) &= (\mathcal{C}^{-1}\boldsymbol{\tau}_h^c, \boldsymbol{\tau}_h^c)_0 + \frac{1}{\theta(\mu,\lambda,\alpha)}(\boldsymbol{\tau}_h^t, \boldsymbol{\tau}_h^t)_0 \\
&\geq (\mathcal{C}^{-1}\,\mathrm{dev_h}\,\boldsymbol{\tau}_h^c, \mathrm{dev_h}\,\boldsymbol{\tau}_h^c)_0 + \frac{1}{\theta(\mu,\lambda,\alpha)}\|\boldsymbol{\tau}_h^t\|_0^2 \\
&\geq c(\|\,\mathrm{dev_h}\,\boldsymbol{\tau}_h^c\|_0^2 + \|\boldsymbol{\tau}_h^t\|_0^2) \\
&\geq c\|\,\mathrm{dev_h}(\boldsymbol{\tau}_h^c + \boldsymbol{\tau}_h^t)\|_0^2 = c\|\,\mathrm{dev_h}\,\boldsymbol{\tau}_h^s\|_0^2.
\end{aligned}$$

To establish an upper bound for the norm of $\boldsymbol{\tau}_h^s$, we have to consider $\mathrm{sph_h}\,\boldsymbol{\tau}_h^s$ in more detail. Definitions of $\mathrm{sph_h}$, $\mathrm{dev_h}$ and S_h^s yield that $\mathrm{sph_h}\,\boldsymbol{\tau}_h^s = q_h^s\mathbf{1}$, $2q_h^s = \mathrm{tr}\,\mathrm{sph_h}\,\boldsymbol{\tau}_h^s \in \widetilde{M}_h^s$. Since $(V_h^s, \widetilde{M}_h^s)$ forms a stable Stokes pairing, we can find $\boldsymbol{v}_{q_h^s}^s \in V_h^s$ such that

$$(q_h^s, \mathrm{div}\,\boldsymbol{v}_{q_h^s}^s)_0 = 2\|q_h^s\|_0^2, \quad \|\boldsymbol{v}_{q_h^s}^s\|_1 \leq C\|q_h^s\|_0.$$

and the discrete spherical part of $\boldsymbol{\tau}_h^s$ can be bounded by

$$\begin{aligned}
\|\,\mathrm{sph_h}\,\boldsymbol{\tau}_h^s\|_0^2 = 2\|q_h^s\|_0^2 &= (q_h^s, \mathrm{div}\,\boldsymbol{v}_{q_h^s}^s)_0 = (q_h^s\mathbf{1}, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 \\
&= (\mathrm{sph_h}\,\boldsymbol{\tau}_h^s, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 \\
&= (\boldsymbol{\tau}_h^s, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 - (\mathrm{dev_h}\,\boldsymbol{\tau}_h^s, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 \\
&\leq (\boldsymbol{\tau}_h^s, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 + C\|\,\mathrm{dev_h}\,\boldsymbol{\tau}_h^s\|_0\|\,\mathrm{sph_h}\,\boldsymbol{\tau}_h^s\|_0.
\end{aligned}$$

Applying Young's inequality, we obtain

$$\|\boldsymbol{\tau}_h^s\|_0^2 = \|\,\mathrm{sph_h}\,\boldsymbol{\tau}_h^s\|_0^2 + \|\,\mathrm{dev_h}\,\boldsymbol{\tau}_h^s\|_0^2 \leq C\left((\boldsymbol{\tau}_h^s, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 + \hat{a}(\boldsymbol{\tau}_h^s, \boldsymbol{\tau}_h^s)\right). \quad (26)$$

Replacing $\boldsymbol{\tau}_h^s$ by $\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s$ and using the second equation in (25), the first term in the upper bound of (26) can be estimated by

$$\begin{aligned}
(\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s - p_h^s}^s))_0 &= (\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s - p_h^s}^s))_0 + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^s, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s - p_h^s}^s))_0 \\
&\leq \|\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}\|_0\|\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s\|_0.
\end{aligned}$$

To bound $\hat{a}(\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s, \boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)$, we have to use the first equation in (25). This gives

$$\begin{aligned}
\hat{a}(\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s, \boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s) &= (\mathcal{C}_h^{-1}\boldsymbol{\tau}_h^s, \boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)_0 - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)_0 \\
&= (\mathcal{C}_h^{-1}\boldsymbol{\tau}_h^s - \mathcal{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)_0 + (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)_0 \\
&\leq \|\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s\|_0(\|\mathcal{C}_h^{-1}\boldsymbol{\tau}_h^s - \mathcal{C}^{-1}\boldsymbol{\sigma}\|_0 + \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_0).
\end{aligned}$$

Combining the last two estimates with (26), we find that

$$\|\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s\|_0 \leq C(\|\mathcal{C}^{-1}\boldsymbol{\sigma} - \mathcal{C}_h^{-1}\boldsymbol{\tau}_h^s\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h^s\|_0 + \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_0). \quad (27)$$

Since $S^c_{2h} \subset S_{2h} \subset S^s_h$ Assumption 5.4 guarantees that the best approximation error in the subspace $S^c_h \cap S^s_h$ satisfies $\inf_{\boldsymbol{\tau}^s_h \in S^c_h \cap S^s_h} \|\boldsymbol{\tau}^s_h - \boldsymbol{\sigma}\|_0 \le Ch\|\boldsymbol{f}\|_0$. In terms of Theorem 5.2, the triangle inequality in combination with (27) yields

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^s_h\|_0 \le C\left(\inf_{\boldsymbol{\tau}^s_h \in S^c_h \cap S^s_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}^s_h\|_0 + \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_0\right) \le Ch\|\boldsymbol{f}\|_0.$$

We recall that $\mathcal{C}_h$ restricted to S^c_h is identical to $\mathcal{C}$.

7 Numerical examples

In this section, we illustrate the performance of the formulation discussed in the preceding sections in some numerical tests. In particular, we show the locking free response in the incompressible limit of the proposed formulation by comparing the results with the analytical solution and the results obtained from the standard Q_1-displacement approach. All examples are in two dimensions, and based on four-noded quadrilateral elements with standard bilinear interpolation of the displacement field. Furthermore, we assume isotropy and plane strain. The implementation is based on the finite element toolbox UG, [3]. In our two examples, we use the modified Hu–Washizu formulation (13) and apply static condensation.

Example 1 (Cook's membrane problem) In this popular benchmark problem [15, 17,20], we set $\Omega := \mathrm{conv}\{(0,0), (48,44), (48,60), (0,44)\}$, where $\mathrm{conv}\xi$ is the convex hull of the set ξ. The left boundary of the tapered panel Ω is clamped, and the right one is subjected to an in-plane shearing load of 100N along the y-direction, as shown in Fig. 2a. The material properties are taken to be $E = 250$ and $\nu = 0.4999$, so that a nearly incompressible response is obtained. The vertical tip displacement at the point T is computed for the different cases in Table 1, for different levels of uniform refinement, starting with the initial triangulation shown

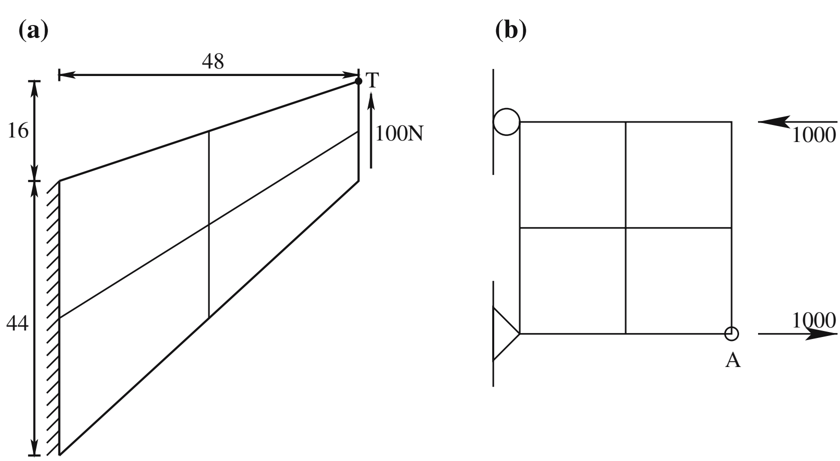

Fig. 2 **a** Cook's membrane problem with initial triangulation; **b** the square beam problem with a mesh of four squares

Table 2 Vertical tip displacement at point T, Example 1

	$\alpha = \frac{\lambda}{\mu}$		α Independent			$\alpha = \frac{1}{4}$		$\alpha = -\frac{1}{4}$		Q_1–P_0
lev	Q_1	I	II	III	V	I	IV	I	IV	
0	2.00	2.00	4.00	4.58	3.15	2.93	2.70	3.18	2.82	3.01
1	2.07	2.08	5.40	5.64	4.42	4.18	3.76	4.53	3.97	4.31
2	2.10	2.12	6.73	7.02	6.24	5.93	5.60	6.20	5.83	6.28
3	2.15	2.22	7.59	7.52	7.17	7.00	6.85	7.13	6.97	7.21
4	2.32	2.54	7.59	7.68	7.53	7.45	7.40	7.50	7.45	7.55
5	2.84	3.39	7.69	7.73	7.67	7.63	7.62	7.66	7.64	7.68
6	4.03	4.94	7.74	7.75	7.73	7.71	7.70	7.72	7.71	7.73

Table 3 Vertical tip displacement at point A, Example 2

	$\alpha = \frac{\lambda}{\mu}$		α Independent			$\alpha = \frac{1}{4}$		$\alpha = -\frac{1}{4}$		Q_1–P_0
ν	Q_1	I	II	III	V	I	IV	I	IV	
ν_1	2.64	2.80	3.36	4.07	3.13	3.04	2.85	3.13	2.92	3.45
ν_2	0.75	0.76	3.04	3.74	2.86	2.39	2.27	2.60	2.47	3.23
ν_3	0.09	0.09	3.00	3.70	2.83	2.30	2.19	2.53	2.41	3.20
ν_4	0.01	0.01	3.00	3.69	2.82	2.29	2.18	2.53	2.40	3.20

in Fig. 2a. As can be seen from Table 2, the standard displacement approach and standard Hu-Washizu formulation ($\alpha = \lambda/\mu$) with stress and strain spaces given in Cases I and IV exhibit locking whereas all other cases show rapid convergence. We also observe the coarse mesh accuracy of Cases II and III.

Example 2 (Square beam) In the second example, we consider the domain $\Omega := (0, 2) \times (0, 2)$, which is fixed in the x-direction at the point $(0, 2)$ and fixed in both directions at the origin. A linearly varying horizontal force is applied in the x-direction along the boundary $x = 2$, with resultant point forces $p = 1,000$ at $(2, 0)$ and $p = -1,000$ at $(2, 2)$ (Fig. 2b). This problem has also been considered in [23]. In Table 3, we present the vertical tip displacement at the point $A := (2, 0)$ for $E = 1,500$ and for different values of Poisson's ratio ν, where ν_1, ν_2, ν_3 and ν_4 are given by 0.4, 0.49, 0.499 and 0.4999, respectively. The exact vertical displacement at A is $4(1 - \nu^2)$.

As in the previous example, the standard displacement approach and the standard Hu–Washizu formulation with Cases I and IV show the locking effect whereas all other cases exhibit stable behavior. In particular, Cases II, V and Q_1–P_0 give better numerical results. Furthermore, the numerical results from Case II are almost exact.

To illustrate the dependence of the numerical solution $\boldsymbol{u}_h$ on α, we show in Fig. 3 the absolute error of the vertical tip displacement at A versus $\alpha/2$. The left picture shows the Case IV and the right pictures shows the Case I. We set $E = 1,500$ and $\nu = 0.4999$. As can be seen from Fig. 3, the locking effect increases with α. We note that $\mathcal{B}_\alpha$ is singular for $\alpha = -1$ but the displacement based formulation (18) with Q_h and $\mathcal{C}_h$ given as in Lemma 5.2 is well defined. Therefore, we use formulation (18) to compute the displacement here. For $\alpha = -2$, we find that $\mathcal{C}_h|_{S_h^t} = \mathbf{0}$ and $(S_h^4)^c = S_h^3 = (S_h^3)^c$, and $(S_h^1)^c = \mathcal{I}$. Thus Case IV reduces to Case

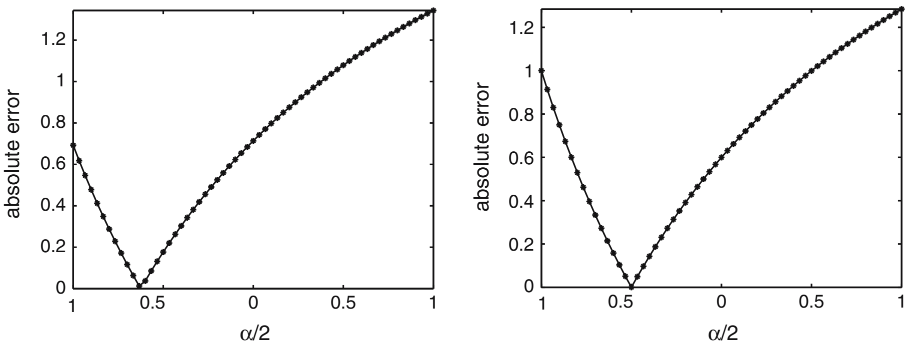

Fig. 3 Error of the vertical tip displacement at A versus $\alpha/2 \in [-1, 1]$, Case IV (*left*) and Case I (*right*)

III, whereas for the Case I Assumption 5.1(i) does not hold locally for this special choice of α.

8 Concluding remarks

In this work we have carried out a detailed analysis of a generalized Hu–Washizu formulation, of which the classical formulation is a special case. The generalization takes the form of a one-parameter family of formulations. Attention is paid to the question of well-posedness and convergence in the incompressible limit, and it is clear from a preliminary analysis that the classical Hu-Washizu formulation is not the appropriate setting in which to carry out such an analysis.

The first key outcome of this work is a result on well-posedness of the generalized formulation, subject to conditions on the spaces S_h and D_h. Furthermore, the discretization error is shown to satisfy a best approximation bound.

For the choices of bases considered here the pair $(V_h, \widetilde{M}_h)$ does not satisfy a uniform inf–sup condition, in that $\widetilde{M}_h$ contains checkerboard modes. Nevertheless it is shown that subject to conditions on a discrete elasticity operator and the choice of the parameter α, the displacement does satisfy a λ-independent optimal error estimate. Thus the lack of stability resulting from the checkerboard mode is confined to the stress. A λ-independent optimal error bound on the post-processed stress is derived.

Two numerical examples are presented. These illustrate the locking behavior that occurs in the incompressible limit when the conditions on the bases and on α necessary for stability are not met. The results also illustrate the good behavior that is achieved otherwise. A more extensive set of numerical examples is presented in [10], in a context the equivalence between various mixed and enhanced formulations is clarified. In that work the presence of the checkerboard modes in the stress is also illustrated.

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