

# Simply Extension to Quadrilateral Element of Three Field Hu-Washizu 2D Elasticity Formulation Based on Biorthogonal Systems

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## Abstract

New quadrilateral mixed finite element based on modified Hu-Washizu formulation are presented. Hu-Washizu is a three field formulation where the unknowns are: displacement, stress and strain. The stability and consistency of the element are obtained by adding different types of bubble functions at the displacement field. Different types of bubble functions are successfully investigated and analyzed. In order to obtain an efficient discretization scheme, we use a pair of finite element bases forming a biorthogonal system for the strain and stress. The biorthogonality relation allows us to statically condense out the strain and stress from the saddle-point system leading to a symmetric and positive-definite system. The strain and stress can be recovered in a post-processing step simply by inverting a diagonal matrix. Numerical examples prove the efficiency and the stability of the elements in the case of incompressible limit and distorted meshes. The extension at the 3D case is straightforward.

*Keywords:* mixed finite elements, quadrilateral element, Hu-Washizu, biorthogonal systems, elasticity.

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## 1. Introduction

In the linear and non-linear elasticity when the Lamé constant  $\lambda$  tends to infinity the standard finite element exhibit some problem of convergence may be locking phenomena. In this context mixed formulations are used to solve this inconvenient. One of the most popular mixed formulation adopted is the Hellinger-Raisner formulation, in that we describe the problem using with variables the stresses and the displacements. This type of formulation requires the fulfillment of the inf-sup condition as described in D.Boffi et al. (2013). For the Hellinger-Reissner formulation many elements are created such as D.N.Arnold and R.Winther (2002) for triangle elements and D.N.Arnold and G.Awanou (2005) for quadrilateral. Another type of formulation is the Hu-Washizu formulation (see H.Hu (1955); K.Washizu (1982)) that it was a three-field formulation. This was first introduced by B.M.Fraeijs de Veubeke (1951). One examples of Hu-Washizu formulation applied at linear and non-linear elasticity using quadrilaterals and hexahedra are reported in E.P. Kasper and R.L. Taylor (2000,?). In B.P.Lamichhane et al. (2006) are showed that the modify Hu-Washizu formulation are able to obtaining uniform convergence of the finite nelement approximation in the nearly incompressible regime in the case of quadrilateral elements. The goal of this work is to present an extension using the quadrilateral elements of the modified three-field formulation Hu-Washizu presented in B.P.Lamichhane (2009). In accordance to B.P.Lamichhane et al. (2013) we adopt the idea to create a biorthogonal system between the stresses and strains. This assumption is essentially for the static condensation of the stresses and strains for obtain a linear system in the only unknown displacement field. In order to ensure the stability of the element we enrich the space of displacement field with different type of bubble functions in accordance to W.Bai (1997); B.P.Lamichhane (2015) for the Stokes problem. In W.Bai (1997) we adopt two bubble functions to stabilize the element, while in B.P.Lamichhane (2015) use one single bubble function.

The structure of the papers are the following: first of all in Section 2 we recall the basic equations of linear elastic problem, in Section 3 we briefly recall the modified Hu-Washizu formulations, Section 4 we develop the finite element spaces. Finally in Section 5 we report some examples using the studied element and in Section 6 the conclusions of the work are detailed.

## 36 2. Linear elastic continuum problem

37 In this section we briefly recovery the equations governing the linear elas-  
38 tic problem. The equilibrium equation is:

$$-\operatorname{div}(\boldsymbol{\sigma}) = \mathbf{f} , \quad (1)$$

39 while in small deformation is:

$$\mathbf{d} = \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) . \quad (2)$$

40 In the case of linear elasticity we have:

$$\boldsymbol{\sigma} = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon} \quad (3)$$

41 where  $\mu$  and  $\lambda$  are the Lamé constant. By some algebra one obtains:

$$\boldsymbol{\sigma} = \begin{pmatrix} \lambda(\varepsilon_{11} + \varepsilon_{22}) & 0 \\ 0 & \lambda(\varepsilon_{11} + \varepsilon_{22}) \end{pmatrix} + 2\mu \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix} , \quad (4)$$

42 and rearranging the equation (4):

$$\boldsymbol{\sigma} = \begin{pmatrix} (\lambda + 2\mu)\varepsilon_{11} + \lambda \varepsilon_{22} & 2\mu \varepsilon_{12} \\ 2\mu \varepsilon_{12} & (\lambda + 2\mu) \varepsilon_{22} + \lambda \varepsilon_{11} \end{pmatrix} . \quad (5)$$

## 43 3. Briefly introduction to modify Hu-Washizu

44 We define the trial variables:  $\boldsymbol{\varepsilon}(\mathbf{u})$ ,  $\mathbf{d}$  and  $\boldsymbol{\sigma}$ , while the test variables are:  
45  $\boldsymbol{\varepsilon}(\mathbf{v})$ ,  $\mathbf{e}$  and  $\boldsymbol{\tau}$ .

$$-\int_{\Omega} \operatorname{div}(\mathbf{C} : \mathbf{d})) \cdot \mathbf{v} = \mathbf{f} \quad (6)$$

$$a((\mathbf{u}, \mathbf{d}), (\mathbf{v}, \mathbf{e})) + b((\mathbf{v}, \mathbf{e}), \boldsymbol{\sigma}) = l(\mathbf{v}) \quad (7)$$

$$b((\mathbf{u}, \mathbf{d}), \boldsymbol{\tau}) = 0 \quad (8)$$

46 where:

$$a((\mathbf{u}, \mathbf{d}), (\mathbf{v}, \mathbf{e})) = \int_{\Omega} \mathbf{d} : (\mathbf{C} : \mathbf{e}) dx + \alpha \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{d}) : (\boldsymbol{\varepsilon}(\mathbf{v}) - \mathbf{e}) dx \quad (9)$$

$$b((\mathbf{u}, \mathbf{d}), \boldsymbol{\tau}) = \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{d}) : \boldsymbol{\tau} dx . \quad (10)$$

47 The modify weak formulation of the problem is:

$$\left\{ \begin{array}{l} \alpha \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{d}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\sigma} \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx \\ \int_{\Omega} \mathbf{d} : \mathbf{C} \mathbf{e} \, dx - \alpha \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{d}) : \mathbf{e} \, dx - \int_{\Omega} \mathbf{e} : \boldsymbol{\sigma} \, dx = 0 \\ \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{d}) : \boldsymbol{\tau} \, dx = 0 \end{array} \right. \quad (11)$$

48 by rearranging:

$$\left\{ \begin{array}{ll} \alpha \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx - \alpha \int_{\Omega} \mathbf{d} : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\sigma} \, dx & = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx \\ -\alpha \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbf{e} \, dx + \int_{\Omega} \mathbf{d} : \mathbf{C} \mathbf{e} \, dx + \alpha \int_{\Omega} \mathbf{d} : \mathbf{e} \, dx - \int_{\Omega} \mathbf{e} : \boldsymbol{\sigma} \, dx & = 0 \\ \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\tau} \, dx - \int_{\Omega} \mathbf{d} : \boldsymbol{\tau} \, dx & = 0 \end{array} \right. \quad (12)$$

49 It is possible to rewrite the system in equation (12) in matrix form in the  
50 following way:

$$\begin{bmatrix} \alpha \mathbf{A} & -\alpha \mathbf{B} & \mathbf{W} \\ -\alpha \mathbf{B}^T & \mathbf{K} + \alpha \mathbf{M} & -\mathbf{D} \\ \mathbf{W}^T & -\mathbf{D}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_d \\ \mathbf{x}_\sigma \end{bmatrix} = \begin{bmatrix} \mathbf{b}_f \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad (13)$$

51 where  $\mathbf{A} = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})$ ,  $\mathbf{B} = \int_{\Omega} \mathbf{d} : \boldsymbol{\varepsilon}(\mathbf{v})$ ,  $\mathbf{W} = \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v})$ ,  $\mathbf{K} = \int_{\Omega} \mathbf{C} \mathbf{e} :$   
52  $\mathbf{d}$ ,  $\mathbf{M} = \int_{\Omega} \mathbf{e} : \mathbf{d}$ ,  $\mathbf{D} = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{e}$ .  $\mathbf{D}$  is a diagonal matrix. Using this property  
53 it is possible condense statically  $\mathbf{x}_d$  and  $\mathbf{x}_\sigma$ , and we obtain the following  
54 system in the only unknown  $\mathbf{x}_u$ :

$$[\alpha \mathbf{A} - \alpha (\mathbf{B} \mathbf{D}^{-1} \mathbf{W}^{-T} + \mathbf{W} \mathbf{D}^{-1} \mathbf{B}^T) + \mathbf{W} \mathbf{D}^{-1} (\mathbf{K} + \alpha \mathbf{M}) \mathbf{D}^{-1} \mathbf{W}^T] \mathbf{x}_u = \mathbf{b}_f \quad (14)$$

#### 55 4. Finite element discretization

56 We consider a quasi-uniform triangulation  $\mathcal{T}_h$  of the polygonal domain  
57  $\Omega$  consists of simply, either quadrilateral or hexahedral. We take into ac-  
58 count of standard bilinear finite element space  $K_h \subset H^1(\Omega)$  defined on the  
59 triangulation  $\mathcal{T}_h$ , where:

$$K_h := \{v \in C^0(\Omega) : v|_T \in \mathcal{Q}_1(T), T \in \mathcal{T}_h\}, \quad K_h^0 = K_h \cap H_0^1(\Omega), \quad (15)$$

60 and the space of bubble functions

$$B_h := \left\{ b_T \in H^1(T) : b_T|_{\partial T} = 0 \text{ and } \int_T b_T dx > 0, T \in \mathcal{T}_h \right\}, \quad (16)$$

61 and we define the spaces for strain and displacement as  $\mathbf{S}_h := [K_h]^{2 \times 2}$  and  
 62  $\mathbf{V}_h := [K_h^0 \oplus B_h]^2$ . In the next section we discuss the different choosing of  
 63 bubble functions. For the discrete stress space we use:

$$\mathbf{M}_h := \left\{ \boldsymbol{\tau}_h \in [M_h]^{2 \times 2} : \int_{\Omega} \boldsymbol{\tau}_h : \mathbf{1} dx = 0 \right\} \subset \mathbf{S}_0, \quad (17)$$

64 and let  $\{\phi_1, \dots, \phi_n\}$  and  $\{\mu_1, \dots, \mu_n\}$  the  $n$  the basis functions for the space  
 65  $V_h$  and  $M_h$  respectively, we construct the functions  $\mu_i$  using the following  
 66 biorthogonality property between the space  $V_h$  and  $M_h$ :

$$\int_{\Omega} \mu_i \phi_j dx = c_j \delta_{ij}, \quad c_j \neq 0, \quad 1 \leq i, j \leq n, \quad (18)$$

67 where  $\delta_{ij}$  is Kronecker symbol, and  $c_j$  is a scaling factor which can be  
 68 chosen to be proportion al to the area of support of  $\phi_j$ . The local ba-  
 69 sis function of  $K_h$  and  $M_h$  for the reference square element (see figure 1)  
 70  $\hat{T} := \{(\xi, \eta) : -1 \leq \xi \leq 1, -1 \leq \eta \leq 1\}$  are:

$$\begin{aligned} \phi_1 &= \frac{1}{4}(1 - \xi)(1 - \eta), & \phi_2 &= \frac{1}{4}(1 + \xi)(1 - \eta), \\ \phi_3 &= \frac{1}{4}(1 + \xi)(1 + \eta), & \phi_4 &= \frac{1}{4}(1 - \xi)(1 + \eta). \end{aligned} \quad (19)$$

71 and

$$\begin{aligned} \mu_1 &= 1 - 3\xi - 3\eta + 9\xi\eta, & \mu_2 &= 1 + 3\xi - 3\eta - 9\xi\eta, \\ \mu_3 &= 1 + 3\xi + 3\eta + 9\xi\eta, & \mu_4 &= 1 - 3\xi + 3\eta + 9\xi\eta. \end{aligned} \quad (20)$$

72 It is important to observe that the global basis functions of the space  $M_h$  are  
 73 not continuous.

#### 74 4.1. Bubble functions

75 In this section we detail the different choosing of the bubble functions.  
 76 Addition of the bubble functions is essential to create a stable space. we  
 77 have four types of bubbles. In the first two cases we use a modification of  
 78 the standard bubble function ,that is for the reference element:

$$b_T(\xi, \eta) = (1 - \xi^2)(1 - \eta^2), \quad (21)$$

79 while in the next two, we add to the standard bubble function another one.

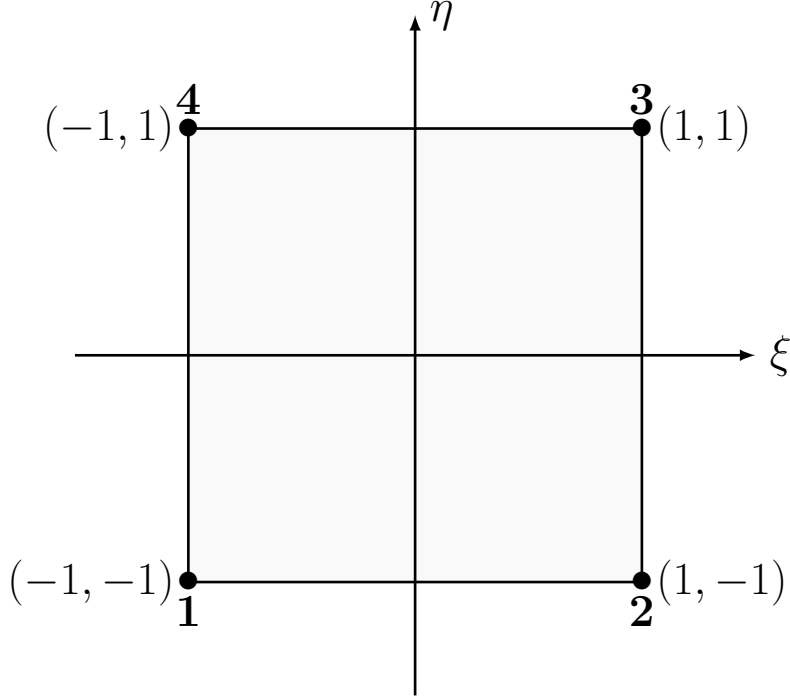


Figure 1: Reference Element

80 *4.1.1. One Bubble function (type 1)*

81 As a first choice of bubble function we use:

$$\hat{b}_T(\xi, \eta) = c_T \cdot \phi_T(\xi, \eta) \cdot b_T(\xi, \eta), \quad (22)$$

82 where  $c_T$  is a coefficient in order to obtain  $\hat{b}_T(\xi_g, \eta_g) = 1$  (where  $\mathbf{g}$  is the  
83 centroid of the elements),  $\phi_K$  is the standard bilinear basis function corre-  
84 sponding to the lower-left corner of the square  $T$ . In the case of reference  
85 square element we obtain:

$$\hat{b}_T(\xi, \eta) = (1 - \xi)(1 - \eta)(1 - \xi^2)(1 - \eta^2). \quad (23)$$

86 *4.1.2. One Bubble function (type 2)*

87 The second choice of bubble function we take:

$$\hat{b}_T(\xi, \eta) = c_T \cdot (a + b\xi + c\eta) \cdot b_T(\xi, \eta), \quad (24)$$

88 where  $a, b, c \in \mathbb{R}$  and  $a, b, c \neq 0$ . For simplicity we set  $a = b = c = 1$  and we  
 89 obtain for the reference square:

$$\hat{b}_T(\xi, \eta) = (1 + \xi + \eta)(1 - \xi^2)(1 - \eta^2) . \quad (25)$$

#### 90 4.1.3. Two Bubble functions

91 Using two bubble functions, where the first is the standard bubble func-  
 92 tion and the second bubble is a modification of the standard bubble:

$$\begin{aligned} \hat{b}_{T1}(\xi, \eta) &= b_T , \\ \hat{b}_{T2}(\xi, \eta) &= c_T \cdot (a\xi + b\eta) \cdot b_T , \end{aligned} \quad (26)$$

93 where  $a, b \in \mathbb{R}$  and  $a^2 + b^2 \neq 0$ . For the sake of simplicity we adopt  $a = b = 1$ .  
 94 One obtains:

$$\begin{aligned} \hat{b}_{T1}(\xi, \eta) &= (1 - \xi^2)(1 - \eta^2) , \\ \hat{b}_{T2}(\xi, \eta) &= (\xi + \eta)(1 - \xi^2)(1 - \eta^2) . \end{aligned} \quad (27)$$

#### 95 4.1.4. Two Bubble functions, which one mixed

96 As a finally choice of bubbles we use a standard bubble function plus one  
 97 mixed bubble function for the two components of displacement.

$$\begin{aligned} \hat{b}_{T1}(\xi, \eta) &= b_T , \\ \hat{b}_{T2,\xi}(\xi, \eta) &= (\nabla \phi_1)_\xi \cdot b_T , \\ \hat{b}_{T2,\eta}(\xi, \eta) &= (\nabla \phi_1)_\eta \cdot b_T , \end{aligned} \quad (28)$$

98 where  $(\nabla \phi_1)_i$  is  $i$ -th component of the gradient of the first shape function  $\phi$ .  
 99 In this way we have as shape function for the displacement using the mixed  
 100 bubble function the vector  $\left[ \hat{b}_{T2,x}(x, y), \hat{b}_{T2,y}(x, y) \right]$ .

## 101 5. Numerical example

102 In this section we report some examples using the presented formulation  
 103 to proven the good behaviour.

104 *5.1. Square problem*

105 First example is a unit square domain with homogeneous Dirichlet bound-  
 106 ary conditions. The Lamé constant are fix to  $\lambda = 123$  and  $\mu = 79.3$ . By  
 107 imposition of the previously exact solution one obtain for the body force  $f$

$$\begin{aligned} f_1 &= -\pi^2 \cos(\pi x) \sin(\pi y) (\lambda + \mu + 2\lambda \cos(\pi y) + 12\mu \cos(\pi y)), \\ f_2 &= -\pi^2 \sin(\pi x) (\lambda \cos(\pi y) + 3\mu \cos(\pi y) + 2\lambda (2 \cos(\pi y)^2 - 1) \\ &\quad + 2\mu (2 \cos(\pi y)^2 - 1)) \end{aligned} \quad (29)$$

108 The exact solution is

$$u_1 = \cos(\pi x) \sin(2\pi y), \quad u_2 = \sin(\pi x) \cos(\pi y). \quad (30)$$

109 The problem is study using two type of mesh: first of all using a square mesh  
 110 and before using a trapezoidal mesh. The two types of mesh are shown in  
 111 figures 2(a) and 2(b). Figures 3(a), 3(b), 4(a) and 4(b) shown the error in  
 112 norm  $L^2$  in the case of regular mesh for the different types of bubble functions  
 113 used and types of coefficient  $\alpha$ . All types of element converge in a good way.  
 114 In Figures 5(a), 5(b), 6(a) and 6(b) we report the previously results in the  
 case of trapezoidal meshes.

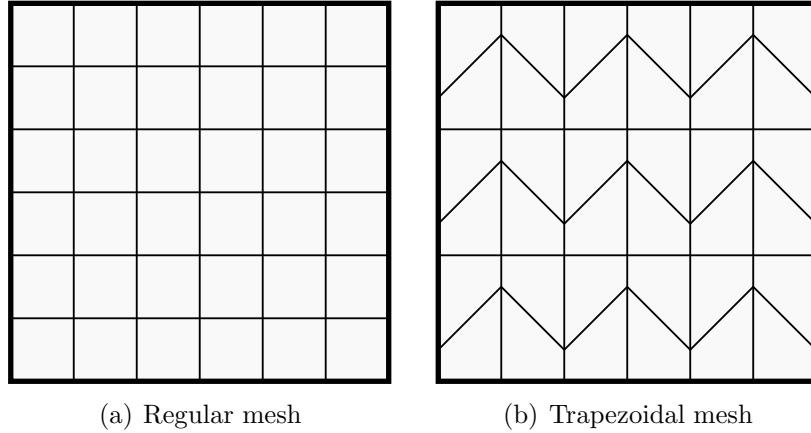
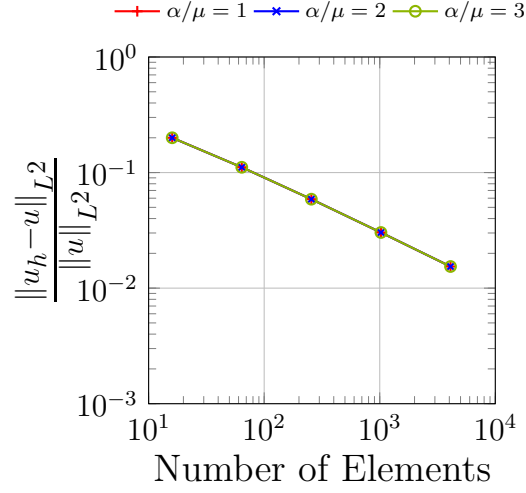
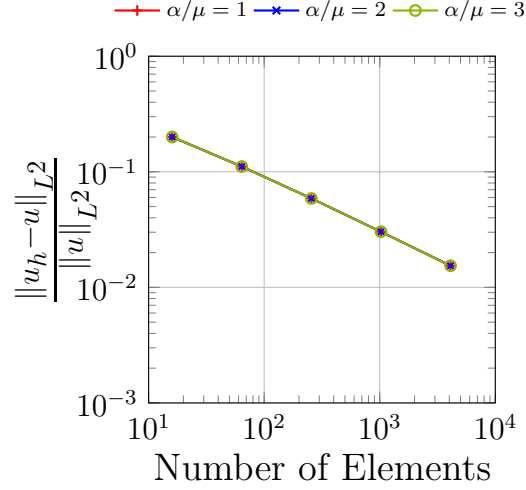


Figure 2: Square Problem





(a) type 1

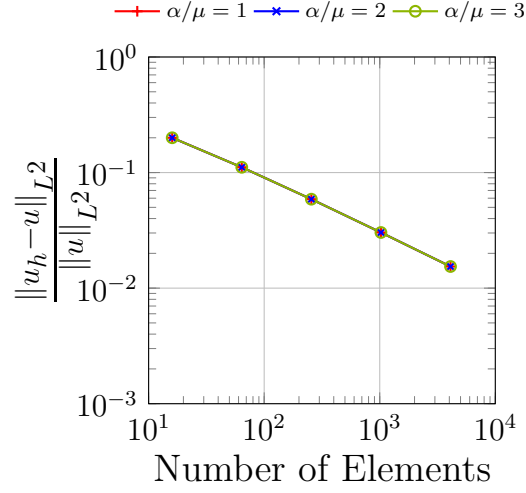


(b) type 2

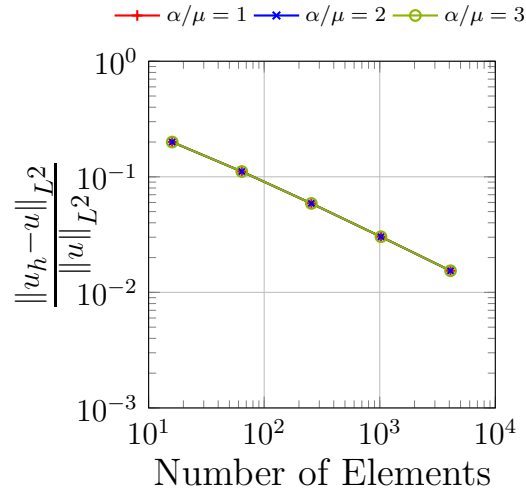
Figure 3: The relative error vs. the number of elements measured relative to the  $L^2$  norm (Case one bubble function and regular mesh)

## 116 5.2. Cantilever beam problem

117 Now we consider the beam with length  $L = 10$  and height  $l = 2$  as we  
 118 shown in figure 7. The Young modulus is set equal to  $E = 1500$  and the  
 119 Poisson  $\nu = 0.4999$  and subjected to a distributed load as in figure 7 with



(a) Case two bubble function

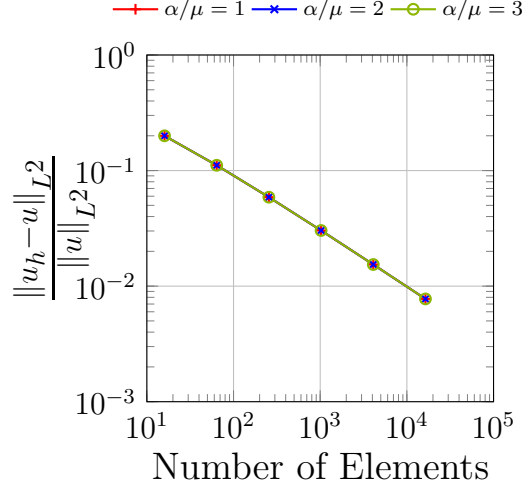


(b) Case two bubble function of which one mixed

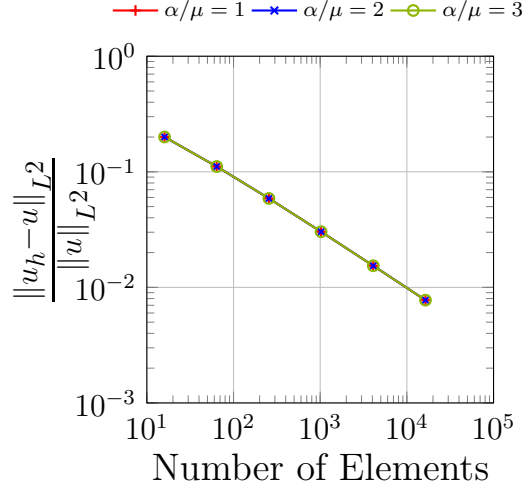
Figure 4: The relative error versus the number of elements measured relative to the  $L^2$  norm (regular mesh)

120  $f = 300$ . The exact solution is:

$$\begin{aligned} u(x, y) &= \frac{2f}{El}(1 - \nu^2)x \left( \frac{l}{2} - y \right), \\ v(x, y) &= \frac{f}{El} \left[ x^2 + \frac{\nu}{10^1 - \nu} (y^2 - ly) \right]. \end{aligned} \tag{31}$$



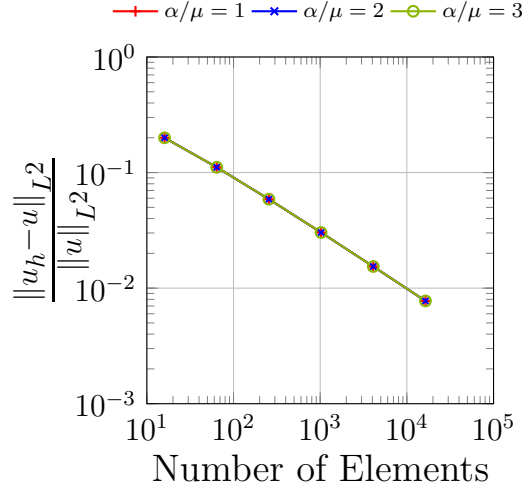
(a) type 1



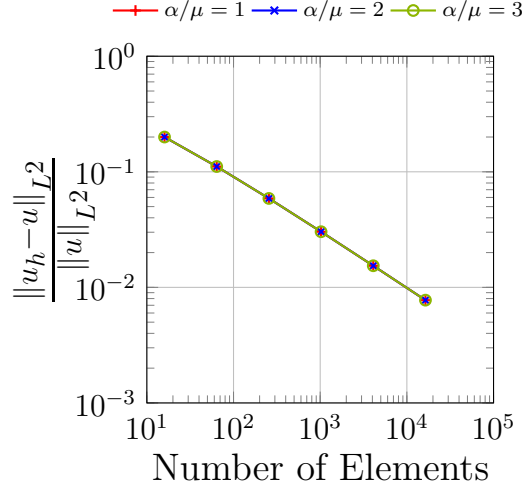
(b) type 2

Figure 5: The relative error vs. the number of elements measured relative to the  $L^2$  norm (Case one bubble function and Trapezoidal mesh)

121 We use to model the beam two types of mesh: regular and trapezoidal as  
 122 in the previously example (see figures 2(a) and 2(b)). we shown in figures  
 123 10(a), 10(b), 11(a) and 11(b) the  $L^2$ -norm error for different types of bubble  
 124 functions used in the case of  $\alpha/\mu := 1, 2, 3$ , while in figures 10(a), 10(b),



(a) Case two bubble function



(b) Case two bubble function of which one mixed

Figure 6: The relative error vs. the number of elements measured relative to the  $L^2$  norm (Case one bubble function and Trapezoidal mesh)

125 11(a) and 11(b) the same plots using trapezoidal meshes. In the all cases  
 126 the elements distorted have a good behaviour respect to the regular mesh.

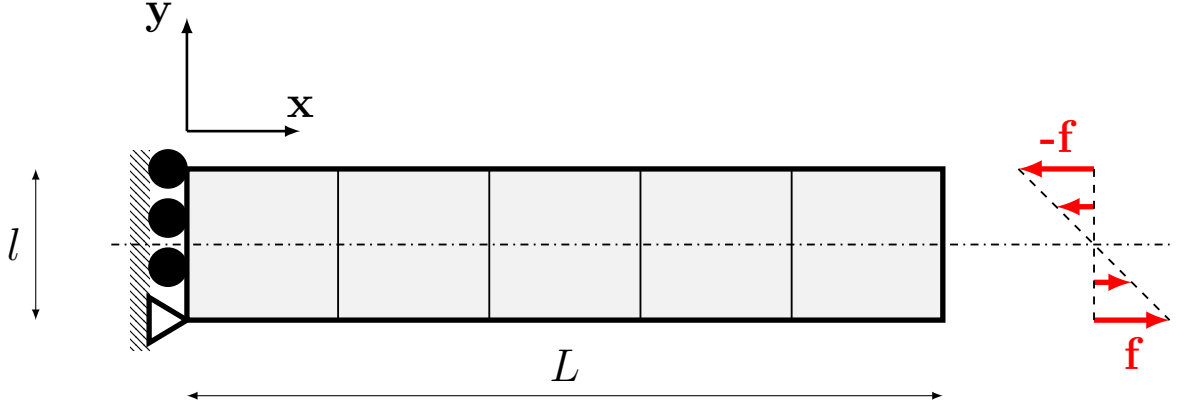


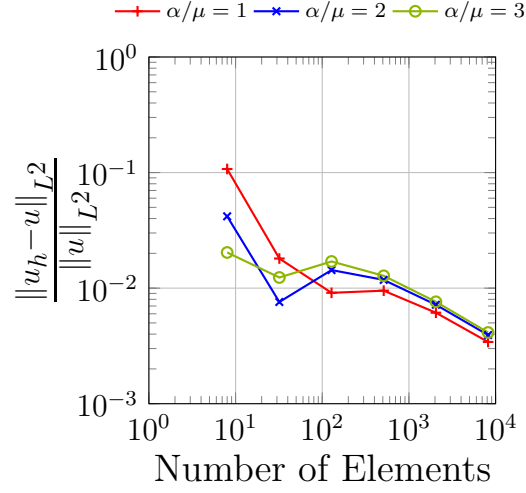
Figure 7: Beam cantilever geometry

### 127 5.3. Cook's membrane

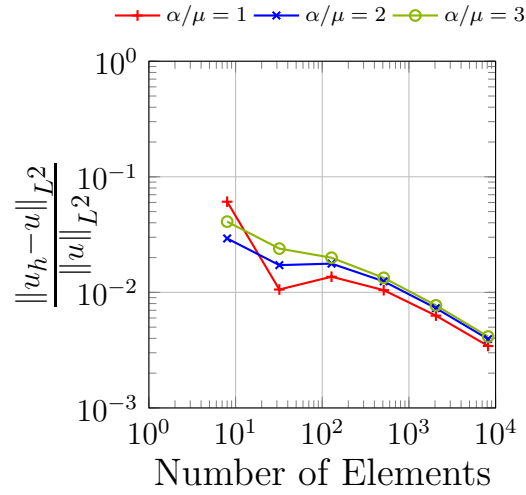
128 The final example is the Cook's membrane. That is a typical benchmark  
 129 and consist of a beam with vertex:  $(0, 0)$ ,  $(48, 44)$ ,  $(48, 60)$  and  $(0, 44)$ . The  
 130 left vertical edge is clamped and the right vertical edge subjected to the  
 131 vertical distributed forces with resultant  $F = 100$  as it shown in figure 12.  
 132 The material properties are taken to be  $E = 250$  and  $\nu = 0.4999$ , so that a  
 133 nearly incompressible response is obtained. We report in figures 13(a), 13(b),  
 134 13(c) and 13(d) the vertical displacement of the point  $A$  versus the number  
 135 of element per side for different choosing of the parameter  $\alpha = \{1, \mu, 2\mu, 3\mu\}$ .  
 136 All elements return different behaviour using different coefficients  $\alpha$ . In the  
 137 case of  $\alpha = 1$ , figure 13(a), the obtained results completely not converge to  
 138 the reference solution.

## 139 6. Conclusions

140 We present a new family of quadrilaterals mixed finite elements based on a  
 141 modified Hu-Washizu formulation. We test the different types of enrichment  
 142 and confirm the robustness of all elements created. Uniform convergence  
 143 are archived in the incompressible regime. We study the robustness of the  
 144 elements in the case of trapezoidal meshes. The extension to the 3D case  
 145 using hexahedra is straightforward.



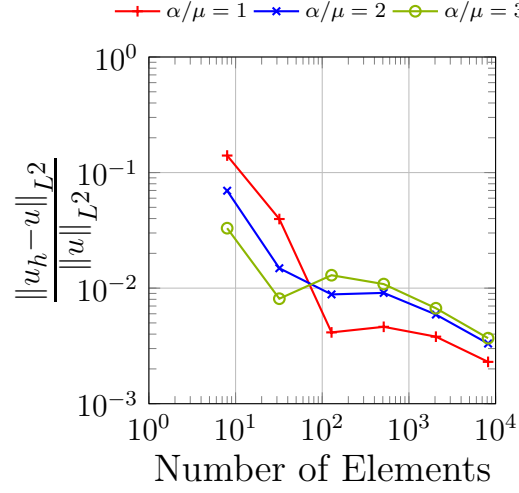
(a) Type 1



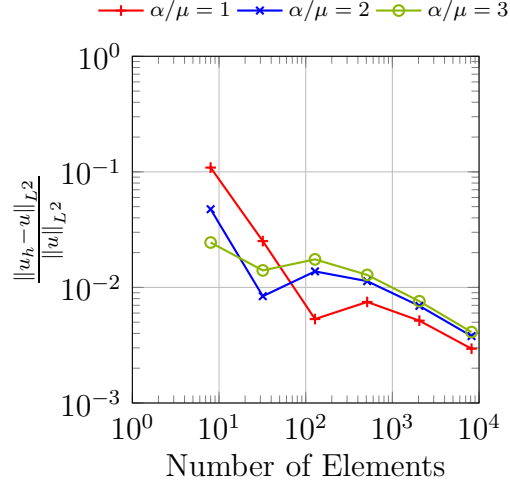
(b) Type 2

Figure 8: Beam Cantilever: the relative error vs. the number of elements measured relative to the  $L^2$  norm (regular mesh)

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 147 *Methods Appl. Mech. Engrg.*, 143: 41-47, 1997.
- 148 D.Boffi. "On the finite element method on quadrilateral meshes", *Appl. Num.*



(a) Case two bubble function

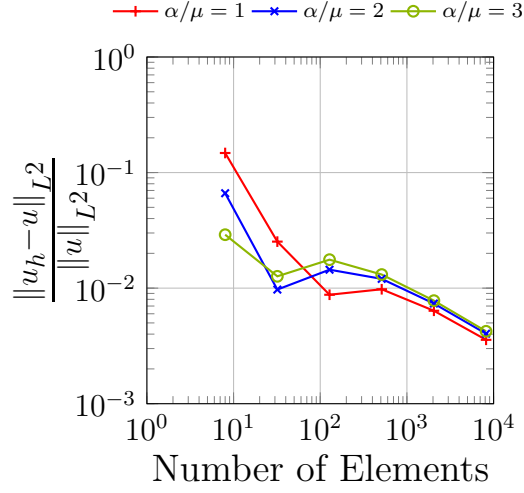


(b) Case two bubble function of which one mixed

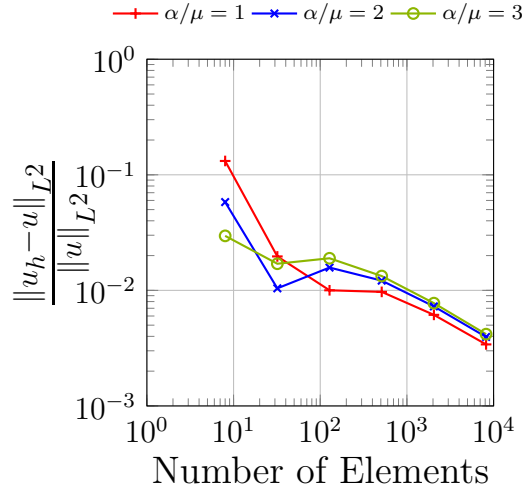
Figure 9: Beam Cantilever: the relative error vs. the number of elements measured relative to the  $L^2$  norm (regular mesh)

149 *Mathematics*, 56: 1271-1282, 2006.

150 B.P.Lamichhane. "A quadrilateral 'mini' finite element for the Stokes prob-



(a) Type 1



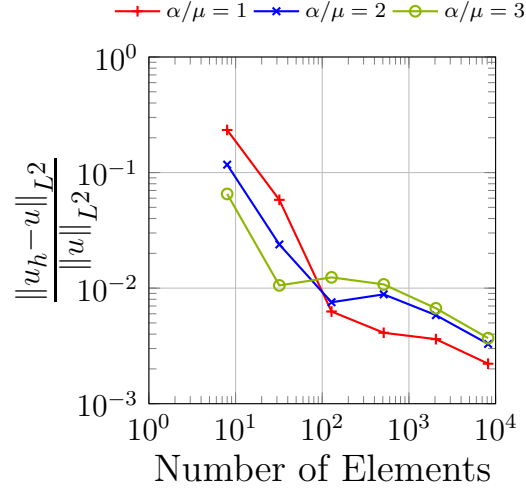
(b) Type 2

Figure 10: Beam Cantilever: the relative error vs. the number of elements measured relative to the  $L^2$  norm (trapezoidal mesh)

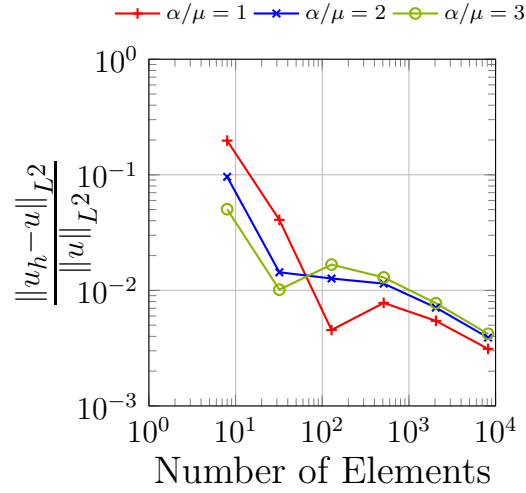
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(a) Case two bubble function



(b) Case two bubble function of which one mixed

Figure 11: Beam Cantilever: the relative error vs. the number of elements measured relative to the  $L^2$  norm (trapezoidal mesh)

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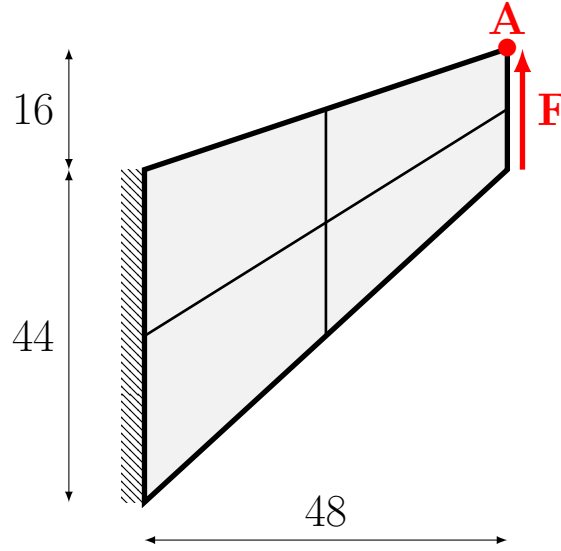
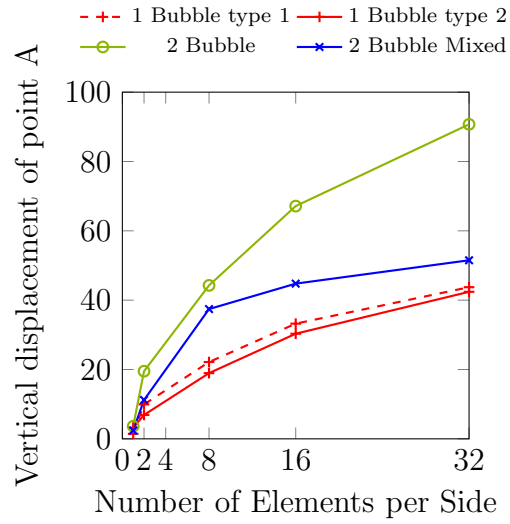


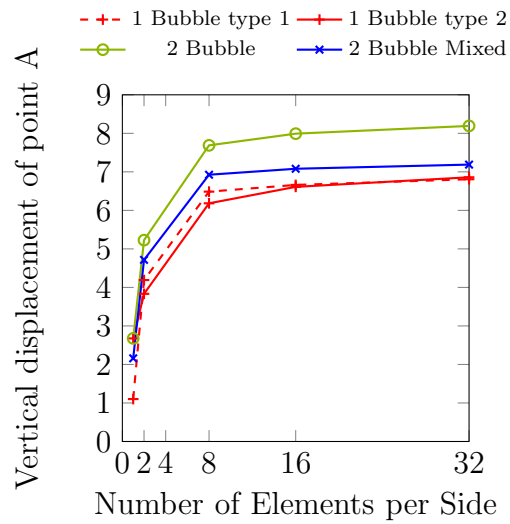
Figure 12: Cook's Membrane geometry

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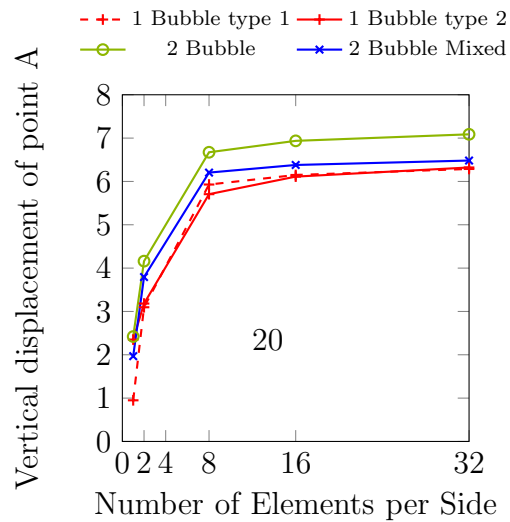
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(a)  $\alpha = 1$



(b)  $\alpha/\mu = 1$



(c)  $\alpha/\mu = 2$