A Mixed-FEM Formulation for Nonlinear Incompressible Elasticity in the Plane

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This article deals with an expanded mixed finite element formulation, based on the Hu-Washizu principle, for a nonlinear incompressible material in the plane. We follow our related previous works and introduce both the stress and the strain tensors as further unknowns, which yields a two-fold saddle point operator equation as the corresponding variational formulation. A slight generalization of the classical Babuška-Brezzi's theory is applied to prove unique solvability of the continuous and discrete formulations, and to derive the corresponding a priori error analysis. An extension of the well-known PEERS space is used to define an stable associated Galerkin scheme. Finally, we provide an a posteriori error analysis based on the classical Bank-Weiser approach. Numer Methods Partial Differential Eq 18: 105–128, 2002

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I. INTRODUCTION

It is well known that the displacement-based variational formulation yields inaccurate results for the finite element analysis of incompressible materials. This difficulty has been overcome through the use of mixed formulations with a pressure-like unknown as a secondary variable, which has

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shown to be very suitable to deal with the divergence free constraint (see, e.g. [1], [2] and the references therein). Moreover, the main advantage of using mixed finite element methods is that it allows to compute stresses more accurately than displacements. However, it must be noticed that most of the mixed variational formulations used for the incompressible and nearly incompressible cases, are of Stokes-like type, which are usually named *primal mixed* formulations. Only a few references consider the so called *dual-mixed* approach, which is characterized by the utilization of the spaces $[L^2(\Omega)]^2$ and $H(\mathbf{div};\Omega)$ to look for the displacement and stress tensor unknowns, respectively (see, e.g. [3], [4], and [2]). Moreover, the absence of dual-mixed formulations is also observed in the works concerning a posteriori error analyses, and in the case of incompressible media with nonlinear constitutive equations (see, e.g. [5]–[8]).

The purpose of this article is to analyze a dual-mixed formulation for a nonlinear incompressible material in the plane. More precisely, we consider the same non-Newtonian flow as in [7] and [8] and apply the dual-mixed approach from [9] (see also [10] and [11]), to study the solvability and finite element approximations of the corresponding variational formulation. This approach, also called expanded mixed finite element method in the literature, is based on the introduction of the strain tensor as an additional unknown, which yields a two fold saddle point operator equation as the associated weak formulation. Because of this structure, we have alternatively called them *dual-dual* variational formulations (see [12]–[14] for further details and applications).

On the other hand, we also develop in this work an a posteriori error analysis for the nonlinear problem under consideration. We proved in [11] that one can combine the classical Bank-Weiser method from [15] with the ideas from [5] and [7], to derive an a posteriori error estimate for the twofold saddle point formulation of an hyperelastic material in the plane. This analysis was recently extended in [16] and [10] to linear and nonlinear transmission problems in plane elastostatics. We prove in this article that it can also be successfully applied to nonlinear incompressible materials.

The rest of the article is organized as follows. The nonlinear boundary value problem is described in Section II. In Section III, we show that the associated variational formulation can be written as a twofold saddle point operator equation and prove its unique solvability. Section IV deals with the Galerkin approximation of the nonlinear problem. We extend the well known PEERS space to our present situation and demonstrate that it leads to a stable and convergent finite element scheme. Then, an a posteriori error analysis based on the Bank-Weiser method is carried out in Section V. We first give some preliminary results in Section A. In Section B, we introduce and analyze the Ritz projection of the finite element error. A reliable a posteriori error estimate, which depends on the exact solutions of certain local problems, is provided in Section C. In Section D, we propose a fully explicit a posteriori error estimate, which does not require either the exact or approximate solutions of the local problem. Finally, the main results concerning an extension of the Babuška-Brezzi theory to nonlinear twofold saddle point problems are recalled in the Appendix.

II. THE BOUNDARY VALUE PROBLEM

We first provide some notations. In what follows, C denotes a generic positive constant. In addition, given any Hilbert space U, U^2, and $U^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. In particular, $\mathbb{R}^{2\times 2}$ is the space of square matrices of order 2 with real entries, $\mathbf{I} := (\delta_{ij})$ is the identity matrix of $\mathbb{R}^{2\times 2}$ and given $\boldsymbol{\tau} := (\tau_{ij}), \boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we use the notations

$$\mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \qquad \boldsymbol{\zeta} : \boldsymbol{\tau} := \sum_{i,j=1}^{2} \zeta_{ij}\tau_{ij}, \qquad \text{and } \boldsymbol{\tau}^{\mathbf{T}} := (\tau_{ji}).$$

Now, let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with Lipschitz continuous boundary Γ. Our goal is to determine the displacement $\mathbf{u} := (u_1, u_2)^{\mathbf{T}}$ and the pressure-like unknown p of a nonlinear incompressible material occupying the region Ω, under the action of some external forces. More precisely, if $\boldsymbol{\sigma}(\mathbf{u}, p) \in \mathbb{R}^{2\times 2}$ and $\mathbf{e}(\mathbf{u}) \in \mathbb{R}^{2\times 2}$ denote the Cauchy stress tensor and the strain tensor of small deformations, respectively, the constitutive equation in Ω is given by

$$\boldsymbol{\sigma}(\mathbf{u}, p) = \mathcal{N}(\mathbf{e}(\mathbf{u})) + p\mathbf{I},$$

where $\mathcal{N} : [L^2(\Omega)]^{2\times 2} \to [L^2(\Omega)]^{2\times 2}$ is a nonlinear operator such that $\mathcal{N}(\mathbf{s}) = \mathcal{N}(\mathbf{s})^{\mathbf{T}}$ for each symmetric tensor $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$.

Throughout this article we assume that $\mathcal{N}$ induces a strongly monotone and Lipschitz continuous operator from $[L^2(\Omega)]^{2\times 2}$ into its dual. In other words, there exist $\alpha_1, \alpha_2 > 0$ such that for all $\mathbf{r}, \mathbf{s} \in [L^2(\Omega)]^{2\times 2}$

$$\int_\Omega [\mathcal{N}(\mathbf{r}) - \mathcal{N}(\mathbf{s})] : [\mathbf{r} - \mathbf{s}]\, dx \geq \alpha_1 \|\mathbf{r} - \mathbf{s}\|^2_{[L^2(\Omega)]^{2\times 2}}, \tag{2.1}$$

and

$$\|\mathcal{N}(\mathbf{r}) - \mathcal{N}(\mathbf{s})\|_{[L^2(\Omega)]^{2\times 2}} \leq \alpha_2 \|\mathbf{r} - \mathbf{s}\|_{[L^2(\Omega)]^{2\times 2}}. \tag{2.2}$$

An explicit nonlinear operator $\mathcal{N}$ verifying (2.1) and (2.2) will be given in Section V. It will correspond to a hyperelastic material satisfying the Hencky-von Mises stress-strain relation.

Then, given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, our nonlinear boundary value problem reads as follows: *Find* $(\boldsymbol{\sigma}, \mathbf{u}, p)$ *in appropriate spaces such that*

$$\boldsymbol{\sigma} = \mathcal{N}(\mathbf{e}(\mathbf{u})) + p\mathbf{I} \text{ in } \Omega,$$

$$\mathbf{div}\, \boldsymbol{\sigma} = -\mathbf{f} \text{ in } \Omega, \qquad \mathrm{div}\, \mathbf{u} = 0 \text{ in } \Omega,$$

$$\text{and} \qquad \mathbf{u} = \mathbf{g} \text{ on } \Gamma, \tag{2.3}$$

where $\boldsymbol{\nu}$ is the unit outward normal to Γ and **div** denotes the usual divergence operator div acting along each row of the corresponding tensor.

We remark that the Dirichlet data $\mathbf{g}$ must satisfy the compatibility condition:

$$\int_\Gamma \mathbf{g} \cdot \boldsymbol{\nu}\, ds = 0. \tag{2.4}$$

III. THE VARIATIONAL FORMULATION

A. A Twofold Saddle Point Operator Equation

Here we follow the analysis from [9] (see also [10], [11], and [26]) and derive a new mixed variational formulation for problem (2.3). Let us first define the tensor space

$$H(\mathbf{div}; \Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \mathbf{div}\, \boldsymbol{\tau} \in [L^2(\Omega)]^2\},$$

with inner product

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{H(\mathbf{div};\Omega)} := \langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{[L^2(\Omega)]^{2\times 2}} + \langle \mathbf{div}\,\boldsymbol{\zeta}, \mathbf{div}\,\boldsymbol{\tau} \rangle_{[L^2(\Omega)]^2},$$

where

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{[L^2(\Omega)]^{2\times 2}} := \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau}\, dx \qquad \forall \boldsymbol{\zeta}, \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$$

and

$$\langle \mathbf{z}, \mathbf{v} \rangle_{[L^2(\Omega)]^2} := \int_\Omega \mathbf{z} \cdot \mathbf{v}\, dx \qquad \forall \mathbf{z}, \mathbf{v} \in [L^2(\Omega)]^2.$$

The corresponding induced norms are $\|\cdot\|_{H(\mathbf{div};\Omega)}$, $\|\cdot\|_{[L^2(\Omega)]^{2\times 2}}$, and $\|\cdot\|_{[L^2(\Omega)]^2}$.

Then, we introduce the further unknown $\mathbf{t} := \mathbf{e}(\mathbf{u})$ in Ω, whence the nonlinear contitutive equation and the incompressibility condition become, respectively,

$$\boldsymbol{\sigma} = \mathcal{N}(\mathbf{t}) + p\mathbf{I} \qquad \text{in } \Omega, \tag{3.1}$$

and

$$\text{tr}\,(\mathbf{t}) = 0 \qquad \text{in } \Omega. \tag{3.2}$$

We remark that the idea of using the strain tensor as an additional unknown yields a variational approach called Hu-Washizu principle in the literature (see, e.g. [18]). Similarly, in some linear and nonlinear steady heat conduction problems, where the temperature gradient is also considered as an unknown, the approach has been named expanded mixed finite element method (see, e.g. [13], [19], [20]). However, the idea of writing the resulting variational formulation as a twofold saddle point operator equation, so that an extension of the classical Babuška-Brezzi theory can be easily applied, has only been utilized by the present authors and some coworkers (see e.g. [9]–[11], [13], and [14]).

Now, according to the definition of $\mathbf{e}(\mathbf{u})$ we can write

$$\mathbf{t} = \nabla \mathbf{u} - \boldsymbol{\gamma}, \tag{3.3}$$

where

$$\boldsymbol{\gamma} := \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^T) \tag{3.4}$$

represents rotations and lives in the space $\mathcal{R} := \{\boldsymbol{\delta} \in [L^2(\Omega)]^{2\times 2}\colon \boldsymbol{\delta} + \boldsymbol{\delta}^{\mathbf{T}} = 0\}$, which is equipped with the norm $\|\cdot\|_{\mathcal{R}} := \|\cdot\|_{[L^2(\Omega)]^{2\times 2}}$.

Next, we multiply (3.3) by a test function $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$, integrate by parts, and obtain

$$-\int_\Omega \mathbf{t} : \boldsymbol{\tau}\, dx - \int_\Omega \mathbf{u} \cdot \mathbf{div}\,\boldsymbol{\tau}\, dx - \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau}\, dx = -\int_\Gamma \mathbf{g} \cdot \boldsymbol{\tau}\boldsymbol{\nu}\, ds \qquad \forall \boldsymbol{\tau} \in H(\mathbf{div};\Omega) \tag{3.5}$$

Also, (3.1) and (3.2) yield, respectively,

$$\int_\Omega \mathcal{N}(\mathbf{t}) : \mathbf{s}\, dx - \int_\Omega \boldsymbol{\sigma} : \mathbf{s}\, dx + \int_\Omega p\, \text{tr}\,(\mathbf{s})\, dx = 0 \qquad \forall \mathbf{s} \in [L^2(\Omega)]^{2\times 2}, \tag{3.6}$$

and

$$\int_\Omega q\, \text{tr}\,(\mathbf{t})\, dx = 0 \qquad \forall q \in L^2(\Omega), \tag{3.7}$$

and the equilibrium equation becomes

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx \qquad \forall\mathbf{v}\in[L^2(\Omega)]^2. \tag{3.8}$$

Finally, the symmetry of $\boldsymbol{\sigma}$ is weakly required through the relation

$$\int_\Omega \boldsymbol{\delta}:\boldsymbol{\sigma}\,dx = 0 \qquad \forall\boldsymbol{\delta}\in\mathcal{R}. \tag{3.9}$$

Consequently, collecting (3.5), (3.6), (3.7), (3.8), and (3.9), we obtain the following variational formulation of (2.3): *Find* $(\mathbf{t},\boldsymbol{\sigma},p,\mathbf{u},\boldsymbol{\gamma})\in[L^2(\Omega)]^{2\times2}\times H(\mathbf{div};\Omega)\times L^2(\Omega)\times[L^2(\Omega)]^2\times\mathcal{R}$ *such that*

$$\int_\Omega \mathcal{N}(\mathbf{t}):\mathbf{s}\,dx-\int_\Omega\boldsymbol{\sigma}:\mathbf{s}\,dx+\int_\Omega p\,\mathrm{tr}\,(\mathbf{s})\,dx=0,$$

$$-\int_\Omega\mathbf{t}:\boldsymbol{\tau}\,dx+\int_\Omega q\,\mathrm{tr}\,(\mathbf{t})\,dx-\int_\Omega\mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}\,dx-\int_\Omega\boldsymbol{\gamma}:\boldsymbol{\tau}\,dx=-\int_\Gamma\mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds,$$

$$-\int_\Omega\mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}\,dx-\int_\Omega\boldsymbol{\delta}:\boldsymbol{\sigma}\,dx=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,dx,$$

for all $(\mathbf{s},\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\delta})\in[L^2(\Omega)]^{2\times2}\times H(\mathbf{div};\Omega)\times L^2(\Omega)\times[L^2(\Omega)]^2\times\mathcal{R}$.

We now observe that when adding $(c\mathbf{I},c)$ to $(\boldsymbol{\sigma},p)$, for any $c\in\mathbb{R}$, we obtain further solutions of our variational formulation.

Thus, in order to insure uniqueness of solution, and following a similar analysis to the one employed for linear incompressible materials (see, e.g. [2]–[4]), we need to impose the side condition

$$\eta\int_\Omega\mathrm{tr}\,\boldsymbol{\sigma}\,dx=0 \qquad \forall\eta\in\mathbb{R}.$$

Furthermore, in order to obtain a symmetric formulation, we consider an artificial unknown $\xi\in\mathbb{R}$ by adding the expression

$$\xi\int_\Omega\mathrm{tr}\,\boldsymbol{\tau}\,dx$$

to the second equation of the above formulation.

In this way, the variational formulation for (2.3) becomes: *Find* $(\mathbf{t},\boldsymbol{\sigma},p,\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\xi})\in[L^2(\Omega)]^{2\times2}\times H(\mathbf{div};\Omega)\times L^2(\Omega)\times[L^2(\Omega)]^2\times\mathcal{R}\times\mathbb{R}$ *such that*

$$\int_\Omega \mathcal{N}(\mathbf{t}):\mathbf{s}\,dx-\int_\Omega\boldsymbol{\sigma}:\mathbf{s}\,dx+\int_\Omega p\,\mathrm{tr}\,(\mathbf{s})\,dx=0,$$

$$-\int_\Omega\mathbf{t}:\boldsymbol{\tau}\,dx+\int_\Omega q\,\mathrm{tr}\,(\mathbf{t})\,dx$$

$$-\int_\Omega\mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}\,dx-\int_\Omega\boldsymbol{\gamma}:\boldsymbol{\tau}\,dx+\xi\int_\Omega\mathrm{tr}\,\boldsymbol{\tau}\,dx=-\int_\Gamma\mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds,$$

$$-\int_\Omega\mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}\,dx-\int_\Omega\boldsymbol{\delta}:\boldsymbol{\sigma}\,dx+\eta\int_\Omega\mathrm{tr}\,\boldsymbol{\sigma}\,dx=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,dx, \tag{3.10}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\delta}, \eta) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div}; \Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathcal{R} \times \mathbb{R}$.

The artificiality of the Lagrange multiplier ξ is seen by the fact that one knows in advance that $\xi = 0$. Indeed, taking $q = 1$ and $\boldsymbol{\tau} = \mathbf{I}$ in the second equation of (3.10), and using (2.4), we deduce that $\xi = 0$.

We now show that (3.10) can be written in the form of a twofold saddle point operator equation (see, e.g. [9], [21]). To this end we define the spaces

$$X_1 := [L^2(\Omega)]^{2\times 2}, \quad M_1 := H(\mathbf{div}; \Omega) \times L^2(\Omega), \quad M := [L^2(\Omega)]^2 \times \mathcal{R} \times \mathbb{R},$$

and define the operators $\mathcal{A}_1 : X_1 \to X_1', \mathcal{B}_1 : X_1 \to M_1', \mathcal{B}_1' : M_1 \to X_1', \mathcal{B} : M_1 \to M', \mathcal{B}' : M \to M_1'$, and the functionals $(\mathcal{F}_1, \mathcal{G}_1, \mathcal{G}) \in X_1' \times M_1' \times M'$ as follows

$$[\mathcal{A}_1(\mathbf{r}), \mathbf{s}] := \int_\Omega \mathcal{N}(\mathbf{r}) : \mathbf{s}\, dx \qquad \forall \mathbf{r}, \mathbf{s} \in X_1,$$

$$[\mathcal{B}_1(\mathbf{r}), (\boldsymbol{\tau}, q)] := -\int_\Omega \mathbf{r} : \boldsymbol{\tau}\, dx + \int_\Omega q \,\mathrm{tr}\,(\mathbf{r})\, dx \qquad \forall \mathbf{r} \in X_1, \forall (\boldsymbol{\tau}, q) \in M_1,$$

$$[\mathcal{B}_1'((\boldsymbol{\tau}, q), \mathbf{r}] := [\mathcal{B}_1(\mathbf{r}), (\boldsymbol{\tau}, q)] \qquad \forall (\boldsymbol{\tau}, q) \in M_1, \forall \mathbf{r} \in X_1,$$

$$[\mathcal{B}(\boldsymbol{\zeta}, \rho), (\mathbf{v}, \boldsymbol{\delta}, \eta)] := -\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\zeta}\, dx - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\zeta}\, dx + \eta \int_\Omega \mathrm{tr}\, \boldsymbol{\zeta}\, dx$$

$$\forall (\boldsymbol{\zeta}, \rho) \in M_1, \forall (\mathbf{v}, \boldsymbol{\delta}, \eta) \in M,$$

$$[\mathcal{B}'(\mathbf{v}, \boldsymbol{\delta}, \eta), (\boldsymbol{\zeta}, \rho)] := [\mathcal{B}(\boldsymbol{\zeta}, \rho), (\mathbf{v}, \boldsymbol{\delta}, \eta)] \quad \forall (\mathbf{v}, \boldsymbol{\delta}, \eta) \in M, \forall (\boldsymbol{\zeta}, \rho) \in M_1,$$

$$[\mathcal{F}_1, \mathbf{s}] := 0, [\mathcal{G}_1, (\boldsymbol{\tau}, q)] := -\int_\Gamma \mathbf{g} \cdot \boldsymbol{\tau\nu}, \qquad \text{and} \qquad [\mathcal{G}, (\mathbf{v}, \boldsymbol{\delta}, \eta)] := \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx,$$

for all $(\mathbf{s}, (\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \eta)) \in X_1 \times M_1 \times M$. Hereafter, $[\cdot, \cdot]$ stands for the duality pairing indicated by the corresponding operators and functionals. Also, from now on $\mathcal{O}$ denotes a generic null operator/functional.

According to the above, the variational formulation (3.10) can be set as the following twofold saddle point operator equation: *Find* $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) \in X_1 \times M_1 \times M$ *such that*

$$\begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1' & \mathcal{O} \\ \mathcal{B}_1 & \mathcal{O} & \mathcal{B}' \\ \mathcal{O} & \mathcal{B} & \mathcal{O} \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ (\boldsymbol{\sigma}, p) \\ (\mathbf{u}, \boldsymbol{\gamma}, \xi) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{G}_1 \\ \mathcal{G} \end{pmatrix}. \tag{3.11}$$

The purpose of the next section is to prove that (3.11) has a unique solution. To this end, we apply a general theory from [22] and [23], which constitutes an extension to nonlinear twofold saddle point problems of the classical Babuška-Brezzi analysis for constrained variational problems. These abstract results are recalled in the Appendix of this article and will also be used in Section IV to study the solvability and stability of a finite element scheme for (3.11).

B. Unique Solvability

We first state the following result.

Lemma 3.1. *The nonlinear operator $\mathcal{A}_1 : X_1 \to X_1'$ is strongly monotone and Lipschitz continuous; that is, there exist $\alpha_1, \alpha_2 > 0$ such that*

$$[\mathcal{A}_1(\mathbf{r}) - \mathcal{A}_1(\mathbf{s}), \mathbf{r} - \mathbf{s}] \geq \alpha_1 \|\mathbf{r} - \mathbf{s}\|_{X_1}^2,$$

and

$$\|\mathcal{A}_1(\mathbf{r}) - \mathcal{A}_1(\mathbf{s})\|_{X_1'} \leq \alpha_2 \|\mathbf{r} - \mathbf{s}\|_{X_1}$$

for all $\mathbf{r}, \mathbf{s} \in X_1 := [L^2(\Omega)]^{2\times 2}$.

Proof. It follows straightforward from the assumptions (2.1) and (2.2). ∎

Next, we prove that $\mathcal{B}$ satisfies a continuous inf-sup condition.

Lemma 3.2. *There exists $\beta > 0$ such that for all $(\mathbf{v}, \boldsymbol{\delta}, \eta) \in M$ there holds*

$$\sup_{\substack{(\boldsymbol{\tau},q)\in M_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \eta)]}{\|(\boldsymbol{\tau}, q)\|_{M_1}} \geq \beta \|(\mathbf{v}, \boldsymbol{\delta}, \eta)\|_M.$$

Proof. Let $(\mathbf{v}, \boldsymbol{\delta}, \eta) \in M$. Then, we clearly have

$$\sup_{\substack{(\boldsymbol{\tau},q)\in M_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \eta)]}{\|(\boldsymbol{\tau}, q)\|_{M_1}} \geq \frac{[\mathcal{B}(\eta\mathbf{I}, 0), (\mathbf{v}, \boldsymbol{\delta}, \eta)]}{\|\eta\mathbf{I}\|_{H(\mathbf{div};\Omega)}} = C|\eta|, \tag{3.12}$$

with $C = (2|\Omega|)^{1/2}$. Now, we observe that $H(\mathbf{div};\Omega) = \tilde{H}(\mathbf{div};\Omega) + \mathbb{R}\mathbf{I}$, where

$$\tilde{H}(\mathbf{div};\Omega) := \left\{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau})\,dx = 0\right\}.$$

In fact, given $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$ we have the unique decomposition $\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}} + c\mathbf{I}$ with $\tilde{\boldsymbol{\tau}} \in \tilde{H}(\mathbf{div};\Omega)$ and $c = \frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}\,(\boldsymbol{\tau})\,dx \in \mathbb{R}$. Note that $\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}^2 = \|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)}^2 + 2c^2|\Omega|$. Then it is easy to see that for all $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$ it holds

$$\begin{aligned}[\mathcal{B}(\boldsymbol{\tau}, 0), (\mathbf{v}, \boldsymbol{\delta}, 0)] &:= -\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau}\,dx - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\tau}\,dx\\ &= [\mathcal{B}(\tilde{\boldsymbol{\tau}}, 0), (\mathbf{v}, \boldsymbol{\delta}, 0)] = [\mathcal{B}(\tilde{\boldsymbol{\tau}}, 0), (\mathbf{v}, \boldsymbol{\delta}, \eta)].\end{aligned}$$

It follows that

$$\begin{aligned}\sup_{\substack{(\boldsymbol{\tau},q)\in M_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \eta)]}{\|(\boldsymbol{\tau}, q)\|_{M_1}} &\geq \sup_{\substack{\tilde{\boldsymbol{\tau}}\in\tilde{H}(\mathbf{div};\Omega)\\\tilde{\boldsymbol{\tau}}\neq 0}} \frac{[\mathcal{B}(\tilde{\boldsymbol{\tau}}, 0), (\mathbf{v}, \boldsymbol{\delta}, \eta)]}{\|\tilde{\boldsymbol{\tau}}\|_{H(\mathbf{div};\Omega)}}\\ &= \sup_{\substack{\boldsymbol{\tau}\in H(\mathbf{div};\Omega)\\\boldsymbol{\tau}\neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, 0), (\mathbf{v}, \boldsymbol{\delta}, 0)]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}}\\ &= \sup_{\substack{\boldsymbol{\tau}\in H(\mathbf{div};\Omega)\\\boldsymbol{\tau}\neq 0}} \frac{-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau}\,dx - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\tau}\,dx}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}}.\end{aligned} \tag{3.13}$$

Next, according to the analysis shown in Theorem 4.3 of [9] (see also Lemma 4.5 in [26] or Lemma 4.3 in [11]), we know that there exists $\bar{\beta} > 0$, independent of $(\mathbf{v}, \boldsymbol{\delta})$, such that

$$\sup_{\substack{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, 0), (\mathbf{v}, \boldsymbol{\delta}, 0)]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} \geq \bar{\beta} \|(\mathbf{v}, \boldsymbol{\delta})\|_{[L^2(\Omega)]^2 \times \mathcal{R}}. \tag{3.14}$$

In this way, (3.12), (3.13), and (3.14) imply the inf-sup condition for $\mathcal{B}$ and complete the proof. ∎

In order to continue our analysis, we need to characterize the null space of the operator $\mathcal{B}$. In fact, let

$$\tilde{M}_1 := \{(\boldsymbol{\tau}, q) \in M_1 : [\mathcal{B}(\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \eta)] = 0 \quad \forall (\mathbf{v}, \boldsymbol{\delta}, \eta) \in M\}.$$

Then, it is not difficult to see that

$$\tilde{M}_1 = \tilde{M}_1^{\boldsymbol{\sigma}} \times L^2(\Omega),$$

where

$$\tilde{M}_1^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \mathbf{div}\, \boldsymbol{\tau} = 0 \text{ in } \Omega, \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathbf{T}} \text{ in } \Omega, \int_\Omega \operatorname{tr}(\boldsymbol{\tau})\, dx = 0 \right\}.$$

Lemma 3.3. *There exists $\beta_1 > 0$ such that for all $(\boldsymbol{\tau}, q) \in \tilde{M}_1$ there holds*

$$\sup_{\substack{\mathbf{s} \in X_1 \\ \mathbf{s} \neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), (\boldsymbol{\tau}, q)]}{\|\mathbf{s}\|_{X_1}} \geq \beta_1 \|(\boldsymbol{\tau}, q)\|_{M_1}.$$

Proof. Let $(\boldsymbol{\tau}, q) \in \tilde{M}_1$. We separate the proof into two possible cases.

First, assume that $\|q\|_{L^2(\Omega)} \leq \|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}$. Then, taking $\mathbf{s} = -[\boldsymbol{\tau} - \frac{1}{2} \operatorname{tr}(\boldsymbol{\tau})\mathbf{I}]$ (which verifies $\operatorname{tr}(\mathbf{s}) = 0$), we obtain

$$\sup_{\substack{\mathbf{s} \in X_1 \\ \mathbf{s} \neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), (\boldsymbol{\tau}, q)]}{\|\mathbf{s}\|_{X_1}} \geq \frac{\int_\Omega \boldsymbol{\tau} : [\boldsymbol{\tau} - \frac{1}{2} \operatorname{tr}(\boldsymbol{\tau})\mathbf{I}]\, dx}{\|\boldsymbol{\tau} - \frac{1}{2} \operatorname{tr}(\boldsymbol{\tau})\mathbf{I}\|_{[L^2(\Omega)]^{2\times 2}}} = \left\| \boldsymbol{\tau} - \frac{1}{2} \operatorname{tr}(\boldsymbol{\tau})\mathbf{I} \right\|_{[L^2(\Omega)]^{2\times 2}}.$$

Next, using that $\int_\Omega \operatorname{tr}(\boldsymbol{\tau})\, dx = 0$, that $\mathbf{div}\, \boldsymbol{\tau} = 0$ in Ω, and applying Lemma 3.1 from [4], we deduce that there exists $C > 0$, depending only on Ω, such that

$$\|\boldsymbol{\tau} - \frac{1}{2} \operatorname{tr}(\boldsymbol{\tau})\mathbf{I}|_{[L^2(\Omega)]^{2\times 2}} \geq C \|\boldsymbol{\tau}\|_{[L^2(\Omega)]^{2\times 2}} = C \|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)},$$

and hence, with $\tilde{\beta}_1 = \frac{C}{2}$, we find that

$$\sup_{\substack{\mathbf{s} \in X_1 \\ \mathbf{s} \neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), (\boldsymbol{\tau}, q)]}{\|\mathbf{s}\|_{X_1}} \geq \tilde{\beta}_1 \|(\boldsymbol{\tau}, q)\|_{M_1}.$$

Now, assume that $\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} \leq \|q\|_{L^2(\Omega)}$. Then, taking $\mathbf{s} = q\mathbf{I} + \boldsymbol{\tau}$ we get

$$\sup_{\substack{\mathbf{s} \in X_1 \\ \mathbf{s} \neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), (\boldsymbol{\tau}, q)]}{\|\mathbf{s}\|_{X_1}} \geq \frac{2\|q\|^2_{L^2(\Omega)} - \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)}}{\|q\mathbf{I} + \boldsymbol{\tau}\|_{[L^2(\Omega)]^{2\times 2}}} \geq \frac{1}{1 + \sqrt{2}} \|q\|_{L^2(\Omega)},$$

and therefore, with $\hat{\beta}_1 := \frac{1}{2(1+\sqrt{2})}$, we conclude that

$$\sup_{\substack{\mathbf{s} \in X_1 \\ \mathbf{s} \neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), (\boldsymbol{\tau}, q)]}{\|\mathbf{s}\|_{X_1}} \geq \hat{\beta}_1 \|(\boldsymbol{\tau}, q)\|_{M_1}.$$

Thus, choosing $\beta_1 := \min\{\tilde{\beta}_1, \hat{\beta}_1\}$, the proof is completed. ∎

We are now in position to state our main result concerning the solvability of (3.11).

Theorem 3.4. *There exists a unique* $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \xi)) \in X_1 \times M_1 \times M$ *solution of the twofold saddle point operator equation* (3.11). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \xi))\|_{X_1\times M_1\times M} \le C\{\|\mathbf{f}\|_{[L^2(\Omega)]^2} + \|\mathbf{g}\|_{[H^{1/2}(\Gamma)]^2} + \|\mathcal{N}(0)\|_{[L^2(\Omega)]^{2\times 2}}\}.$$

Proof. In virtue of Lemmata 3.1, 3.2, and 3.3, the proof follows straightforward from the abstract Theorem 6.1 in the Appendix. ∎

IV. A FINITE ELEMENT SOLUTION

We assume that Γ is a polygonal curve, and let $\{\mathcal{T}_h\}_{h\in\mathbf{I}}$ be a regular family of triangulations of Ω, made up of triangles T of diameter h_T, such that $h := \sup_{T\in\mathcal{T}_h} h_T$ and $\bar{\Omega} = \cup\{T: T \in \mathcal{T}_h\}$. Here $\mathbf{I}$ is a set of indexes, say $\mathbf{I} := \{h_j\}_{j\in\mathbb{N}}$, with $h_j \ge h_{j+1} \forall j \in \mathbb{N}$.

Hereafter, given integers $l \ge 0, k \in \{1, 2\}$ and a subset S of $\mathbb{R}^k, \mathbb{P}_l(S)$ stands for the space of polynomials in k variables defined in S of total degree $\le l$. Also we use the abbreviation RT_0 to mean the Raviart-Thomas space of order 0. For a detailed description of this space we refer to [2] or [24].

Then, we extend the PEERS space from [3] to the present situation and define the following finite element subspaces:

$$X_{1,h} := \{\text{piecewise } \mathbb{P}_0 + \text{rotations of cubic bubble functions} \\ + \text{gradients of cubic bubble functions}\}, \quad (4.1)$$

$$M^{\boldsymbol{\sigma}}_{1,h} := \{RT_0 + \text{rotations of cubic bubble functions}\},$$

$$M^{p}_{1,h} := \{q_h \in L^2(\Omega): q_h|_T \in \mathbb{P}_0(T) \quad \forall T \in \mathcal{T}_h\},$$

$$M_{1,h} := M^{\boldsymbol{\sigma}}_{1,h} \times M^{p}_{1,h},$$

$$M^{\mathbf{u}}_{h} := \{\mathbf{v}_h \in [L^2(\Omega)]^2: \mathbf{v}_h|_T \in [\mathbb{P}_0(T)]^2 \quad \forall T \in \mathcal{T}_h\},$$

$$\mathcal{R}_h := \left\{ \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} \in [H^1(\Omega)]^{2\times 2}: \delta|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h \right\}, \quad (4.2)$$

and

$$M_h := M^{\mathbf{u}}_h \times \mathcal{R}_h \times \mathbb{R}. \quad (4.3)$$

We recall that $M^{\boldsymbol{\sigma}}_{1,h} \times M^{\mathbf{u}}_h \times \mathcal{R}_h$ corresponds precisely to the PEERS space approximating $\boldsymbol{\sigma}, \mathbf{u}$, and $\boldsymbol{\gamma}$.

Then, the Galerkin scheme associated with (3.11) reads as follows: *Find* $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \xi_h)) \in X_{1,h} \times M_{1,h} \times M_h$ *such that*

$$[\mathcal{A}_1(\mathbf{t}_h), \mathbf{s}_h] + [\mathcal{B}_1(\mathbf{s}_h), (\boldsymbol{\sigma}_h, p_h)] = [\mathcal{F}_1, \mathbf{s}_h],$$

$$[\mathcal{B}_1(\mathbf{t}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathcal{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \xi_h)] = [\mathcal{G}_1, (\boldsymbol{\tau}_h, q_h)],$$

$$[\mathcal{B}(\boldsymbol{\sigma}_h, p_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h)] = [\mathcal{G}, (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h)], \tag{4.4}$$

for all $(\mathbf{s}_h, (\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h)) \in X_{1,h} \times M_{1,h} \times M_h$.

The following theorem provides the unique solvability of (4.4) and the corresponding quasi-optimal error estimate.

Theorem 4.1. *There exists a unique* $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \xi_h)) \in X_{1,h} \times M_{1,h} \times M_h$ *solution of the finite element scheme* (4.4). *In addition, there exists* $C > 0$, *independent of* h, *such that the following Cea estimate holds*

$$\begin{aligned}&\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \xi)) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \xi_h))\|_{X_1 \times M_1 \times M}\\ &\le C \inf_{\substack{(\mathbf{s}_h, (\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h))\\ \in X_{1,h} \times M_{1,h} \times M_h^{\mathbf{u}} \times \mathcal{R}_h}} \|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma})) - (\mathbf{s}_h, (\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h))\|_{X_1 \times M_1 \times [L^2(\Omega)]^{2\times 2} \times \mathcal{R}}.\end{aligned}$$

Proof. According to Lemma 3.1 and the abstract Theorems 6.2 and 6.3 in the Appendix, we only need to show that $\mathcal{B}$ and $\mathcal{B}_1$ satisfy the corresponding discrete inf-sup conditions with constants independent of $h \in \mathbf{I}$. To this end we proceed similarly as in the proofs of Lemmata 3.2 and 3.3.

First, given $(\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h) \in M_h$ we observe that $\eta_h \mathbf{I} \in M_{1,h}^{\boldsymbol{\sigma}}$ and hence

$$\sup_{\substack{(\boldsymbol{\tau}_h, q_h) \in M_{1,h}\\ (\boldsymbol{\tau}_h, q_h) \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{M_1}} \ge \frac{[\mathcal{B}(\eta_h \mathbf{I}, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h)]}{\|\eta_h \mathbf{I}\|_{H(\mathbf{div};\Omega)}} = C|\eta_h|, \tag{4.5}$$

with $C = (2|\Omega|)^{1/2}$.

Also, using the decomposition $M_{1,h}^{\boldsymbol{\sigma}} = \tilde{M}_{1,h}^{\boldsymbol{\sigma}} + \mathbb{R}\mathbf{I}$, with $\tilde{M}_{1,h}^{\boldsymbol{\sigma}} := \{\boldsymbol{\tau}_h \in M_{1,h}^{\boldsymbol{\sigma}} : \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h)\, dx = 0\}$, we easily deduce that

$$\begin{aligned}\sup_{\substack{(\boldsymbol{\tau}_h, q_h) \in M_{1,h}\\ (\boldsymbol{\tau}_h, q_h) \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{M_1}} &\ge \sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h}^{\boldsymbol{\sigma}}\\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}_h, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, 0)]}{\|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}}\\ &= \sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h}^{\boldsymbol{\sigma}}\\ \boldsymbol{\tau}_h \neq 0}} \frac{-\int_\Omega \mathbf{v}_h \cdot \mathbf{div}\, \boldsymbol{\tau}_h\, dx - \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau}_h\, dx}{\|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}}.\end{aligned}$$

Then, from Lemma 4.4 in [3] we know that, given $(\mathbf{v}_h, \boldsymbol{\delta}_h) \in M_h^{\mathbf{u}} \times \mathcal{R}_h$, there exists $\boldsymbol{\tau}_h \in M_{1,h}^{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h \neq 0$, such that

$$[\mathcal{B}(\boldsymbol{\tau}_h, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, 0)] \ge \bar{\beta} \|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} \|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^{2\times 2} \times \mathcal{R}}, \tag{4.6}$$

with a constant $\bar{\beta} > 0$, independent of h.

In this way, (4.5) and (4.6) yield the existence of $\beta^* > 0$, independent of h, such that for all $(\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h) \in M_h$,

$$\sup_{\substack{(\boldsymbol{\tau}_h, q_h) \in M_{1,h}\\ (\boldsymbol{\tau}_h, q_h) \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{M_1}} \ge \beta^* \|(\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h)\|_M,$$

which is the discrete inf-sup condition for $\mathcal{B}$.

Now, the discrete kernel of $\mathcal{B}$ is given by

$$\tilde{M}_{1,h} := \{(\boldsymbol{\tau}_h, q_h) \in M_{1,h}: [\mathcal{B}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h)] = 0 \qquad \forall (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h) \in M_h\}.$$

It follows that

$$\tilde{M}_{1,h} := \tilde{M}^{\boldsymbol{\sigma}}_{1,h} \times M^{p}_{1,h},$$

where

$$\tilde{M}^{\boldsymbol{\sigma}}_{1,h} := \left\{ \boldsymbol{\tau}_h \in M^{\boldsymbol{\sigma}}_{1,h}: -\int_\Omega \mathbf{v}_h \cdot \mathbf{div}\, \boldsymbol{\tau}_h \, dx - \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau}_h \, dx + \eta_h \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) \, dx = 0 \qquad \forall (\mathbf{v}_h, \boldsymbol{\delta}_h, \eta_h) \in M_h \right\}.$$

Since $(\mathbf{div}\, \boldsymbol{\tau}_h)|_T$ and $\mathbf{v}_h|_T$ are constant vectors $\forall \boldsymbol{\tau}_h \in M^{\boldsymbol{\sigma}}_{1,h}, \forall \mathbf{v}_h \in M^{\mathbf{u}}_h, \forall T \in \mathcal{T}_h$, we deduce that $\mathbf{div}\, \boldsymbol{\tau}_h = 0$ in Ω $\forall \boldsymbol{\tau}_h \in \tilde{M}^{\boldsymbol{\sigma}}_{1,h}$. Further, it follows easily that $\int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h)\, dx = 0$ $\forall \boldsymbol{\tau}_h \in \tilde{M}^{\boldsymbol{\sigma}}_{1,h}$. Hence, we can write

$$\tilde{M}^{\boldsymbol{\sigma}}_{1,h} := \left\{ \boldsymbol{\tau}_h \in M^{\boldsymbol{\sigma}}_{1,h}: \mathbf{div}\, \boldsymbol{\tau}_h = 0 \quad \text{in } \Omega, \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h)\, dx = 0, \quad \text{and} \quad \int_\Omega \delta_h : \boldsymbol{\tau}_h \, dx = 0 \qquad \forall \boldsymbol{\delta}_h \in \mathcal{R}_h \right\}.$$

Thus, one can prove the discrete inf-sup condition for $\mathcal{B}_1$ following the same lines of the proof of Lemma 3.3. In fact, given $(\boldsymbol{\tau}_h, q_h) \in \tilde{M}_{1,h}$, we have that $\mathbf{s}_h := \boldsymbol{\tau}_h - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau}_h)\mathbf{I}$ belongs to $X_{1,h}$ and verifies $\operatorname{tr}(\boldsymbol{\tau}_h) = 0$. In addition, since $q_h|_T \in \mathbb{P}_0(T)$ $\forall T \in \mathcal{T}_h$, it is clear that $q_h\mathbf{I} \in X_{1,h}$, and then $\mathbf{s}_h := q_h\mathbf{I} + \boldsymbol{\tau}_h$ also belongs to $X_{1,h}$. Therefore we conclude that there exists $\beta_1^* > 0$, independent of h, such that

$$\sup_{\mathbf{s}_h \in X_{1,h}} \frac{[\mathcal{B}_1(\mathbf{s}_h), (\boldsymbol{\tau}_h, q_h)]}{\|\mathbf{s}_h\|_{X_1}} \geq \beta_1^* \|(\boldsymbol{\tau}_h, q_h)\|_{M_1} \qquad \forall (\boldsymbol{\tau}_h, q_h) \in \tilde{M}_{1,h}.$$

Consequently, a straightforward application of the abstract Theorems 6.2 and 6.3 in the Appendix, yields the usual Cea estimate. Finally, the fact that the artificial unknown ξ and its finite element approximation ξ_h belong to $\mathbb{R}$ imply the estimate of the present theorem and complete the proof. ∎

We now recall the following approximation properties of the subspaces $X_{1,h}, M^{\boldsymbol{\sigma}}_{1,h}, M^{p}_{1,h}, M^{\mathbf{u}}_h$, and $\mathcal{R}_h$, respectively, which follow from classical error estimates for projection and equilibrium interpolation operators (see, e.g. [3] and [24])

($\mathrm{AP}_{1,h}$) For all $\mathbf{s} \in X_1 := [L^2(\Omega)]^{2\times 2}$ with $\mathbf{s}|_T \in [H^1(T)]^{2\times 2}$ $\forall T \in \mathcal{T}_h$, there exists $\mathbf{s}_h \in X_{1,h}$ such that

$$\|\mathbf{s} - \mathbf{s}_h\|_{[L^2(\Omega)]^{2\times 2}} \leq Ch \sum_{T \in \mathcal{T}_h} \|\mathbf{s}\|_{[H^1(T)]^{2\times 2}}.$$

($\mathrm{AP}^{\boldsymbol{\sigma}}_{1,h}$) For all $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}$ with $\mathbf{div}\, \boldsymbol{\tau} \in [H^1(\Omega)]^2$, there exists $\boldsymbol{\tau}_h \in M^{\boldsymbol{\sigma}}_{1,h}$ such that

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} \leq Ch\{\|\boldsymbol{\tau}\|_{[H^1(\Omega)]^{2\times 2}} + \|\mathbf{div}\, \boldsymbol{\tau}\|_{[H^1(\Omega)]^2}\}.$$

($\mathrm{AP}^{p}_{1,h}$) For all $q \in H^1(\Omega)$ there exists $q_h \in M^{p}_{1,h}$ such that

$$\|q - q_h\|_{L^2(\Omega)} \leq Ch\|q\|_{H^1(\Omega)}.$$

($\mathrm{AP}_h^{\mathbf{u}}$) For all $\mathbf{v} \in [H^1(\Omega)]^2$ there exists $\mathbf{v}_h \in M_h^{\mathbf{u}}$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{[L^2(\Omega)]^2} \leq Ch\|\mathbf{v}\|_{[H^1(\Omega)]^2}.$$

(AP_h) For all $\boldsymbol{\delta} \in [H^1(\Omega)]^{2\times 2}$ there exists $\boldsymbol{\delta}_h \in \mathcal{R}_h$ such that

$$\|\boldsymbol{\delta} - \boldsymbol{\delta}_h\|_{[L^2(\Omega)]^{2\times 2}} \leq Ch\|\boldsymbol{\delta}\|_{[H^1(\Omega)]^{2\times 2}}.$$

Then we can establish the following result on the rate of convergence of the finite element solution.

Theorem 4.2. *Let $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \xi))$ and $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \xi_h))$ be the unique solutions of the continuous and discrete formulations* (3.11) *and* (4.4), *respectively. Assume that $\mathbf{t}|_T \in [H^1(T)]^{2\times 2}\ \forall T \in \mathcal{T}_h, \boldsymbol{\sigma} \in [H^1(\Omega)]^{2\times 2}, \mathbf{div}\ \boldsymbol{\sigma} \in [H^1(\Omega)]^2, p \in H^1(\Omega), \mathbf{u} \in [H^1(\Omega)]^2$, and $\boldsymbol{\gamma} \in [H^1(\Omega)]^{2\times 2}$. Then there exists $C > 0$, independent of h, such that*

$$\begin{aligned}
&\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \xi)) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \xi_h))\|_{X_1 \times M_1 \times M} \\
&\leq\ Ch\left\{\sum_{T\in\mathcal{T}_h} \|\mathbf{t}\|_{[H^1(T)]^{2\times 2}} + \|\boldsymbol{\sigma}\|_{[H^1(\Omega)]^{2\times 2}} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{[H^1(\Omega)]^2}\right. \\
&\qquad \left. + \|p\|_{H^1(\Omega)} + \|\mathbf{u}\|_{[H^1(\Omega)]^2} + \|\boldsymbol{\gamma}\|_{[H^1(\Omega)]^{2\times 2}}\right\}.
\end{aligned}$$

Proof. It follows straightforward from the Cea estimate in Theorem 4.1 and the above approximation properties. ∎

V. A POSTERIORI ERROR ANALYSIS

In this section we follow the approach from [5] (see also [6], [11], and [16]) to derive a Bank-Weiser's type a posteriori error estimate for the finite element solution introduced in Section IV. The original method due to Bank and Weiser was proposed in [15]. We begin with the following preliminary subsection.

A. Preliminaries

The a posteriori error analysis to be developed here can be carried out for a general nonlinear operator $\mathcal{N}$ satisfying (2.1)–(2.2) and other continuity assumptions. Nevertheless, for clarity of exposition we consider from now on an hyperelastic material whose constitutive equation is given by the Hencky-von Mises stress-strain relation. In other words, we define

$$\mathcal{N}(\mathbf{r}) := [\kappa - \mu(\text{dev } \mathbf{r})] \text{ tr } (\mathbf{r})\mathbf{I} + 2\mu(\text{dev } \mathbf{r})\mathbf{r} \qquad \forall \mathbf{r} \in [L^2(\Omega)]^{2\times 2}, \tag{5.1}$$

where κ is a positive constant, called the bulk modulus, $\mu : \mathbb{R}^+ \to \mathbb{R}$ is a nonlinear Lamé function, and $\text{dev } \mathbf{r} := (\mathbf{r} - \frac{1}{2}\text{tr } (\mathbf{r})\mathbf{I}) : (\mathbf{r} - \frac{1}{2} \text{ tr } (\mathbf{r})\mathbf{I})$ for all $\mathbf{r} \in \mathbb{R}^{2\times 2}$.

We assume that $\mu \in C^1(\mathbb{R}^+)$ and that there exists constants μ_1, μ_2 such that

$$0 < \mu_1 \leq \mu(t) < \kappa \qquad \text{and}$$

$$0 < \mu_1 \leq \mu(t) + 2t\mu'(t) \leq \mu_2 \qquad \forall t \in \mathbb{R}. \tag{5.2}$$

For specific examples of nonlinear functions μ verifying (5.2) we refer to [25].

We now introduce the mappings $\hat{\lambda}, \hat{\mu} : \mathbb{R}^{2\times 2} \to \mathbb{R}$, defined by

$$\hat{\lambda}(\mathbf{r}) := [\kappa - \mu(\text{dev}\,\mathbf{r})] \quad \text{and} \quad \hat{\mu}(\mathbf{r}) := \mu(\text{dev}\,\mathbf{r}) \quad \forall \mathbf{r} \in \mathbb{R}^{2\times 2}.$$

Then, the corresponding nonlinear operator $\mathcal{A}_1 : X_1 \to X_1'$ is given by

$$[\mathcal{A}_1(\mathbf{r}), \mathbf{s}] := \int_\Omega [\hat{\lambda}(\mathbf{r})\,\text{tr}\,(\mathbf{r})\,\text{tr}\,(\mathbf{s}) + 2\hat{\mu}(\mathbf{r})\mathbf{r} : \mathbf{s}]\,dx \qquad \forall \mathbf{r}, \mathbf{s} \in X_1, \tag{5.3}$$

or equivalently, with $\mathbf{r} := (r_{ij})$ and $\mathbf{s} := (s_{ij})$,

$$[\mathcal{A}_1(\mathbf{r}), \mathbf{s}] := \sum_{i,j=1}^{2} \int_\Omega a_{ij}(\mathbf{r}) s_{ij}\,dx,$$

where $a_{ij} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is the nonlinear mapping defined by

$$a_{ij}(\mathbf{r}) := \hat{\lambda}(\mathbf{r})\,\text{tr}\,(\mathbf{r})\delta_{ij} + 2\hat{\mu}(\mathbf{r}) r_{ij}.$$

It can be proved that, under the previous assumptions on μ, $\mathcal{A}_1$ becomes strongly monotone and Lipschitz continuous. More precisely, we have the following lemma (see also Lemmata 2.3 and 3.2 in [17] or Lemma 1 in [25] for similar results).

Lemma 5.1. *The operator $\mathcal{A}_1$ has a continuous first order Gâteaux derivative $D\mathcal{A}_1$, and there exist $\tilde{C}_1, \tilde{C}_2 > 0$, such that*

$$|D\mathcal{A}_1(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s})| \le \tilde{C}_1 \|\mathbf{r}\|_{X_1} \|\mathbf{s}\|_{X_1}$$

and

$$D\mathcal{A}_1(\tilde{\mathbf{r}})(\mathbf{s}, \mathbf{s}) \ge \tilde{C}_2 \|\mathbf{s}\|_{X_1}^2,$$

for all $\tilde{\mathbf{r}}, \mathbf{r}, \mathbf{s} \in X_1 := [L^2(\Omega)]^{2\times 2}$.

Proof. Because of (5.2), one finds that a_{ij} is of class C^1 and that there exists $C_1, C_2 > 0$ such that

$$\left| \frac{\partial}{\partial \tilde{r}_{kl}} a_{ij}(\tilde{\mathbf{r}}) \right| \le C_1 \qquad \text{and} \qquad \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} a_{ij}(\tilde{\mathbf{r}}) s_{kl} s_{ij} \ge C_2 \sum_{i,j=1}^{2} s_{ij}^2$$

for all $\tilde{\mathbf{r}} := (\tilde{r}_{ij}), \mathbf{s} := (s_{ij}) \in \mathbb{R}^{2\times 2}$ (see [2] for details). Then the proof is completed by observing that

$$D\mathcal{A}_1(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) = \int_\Omega \left\{ \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} a_{ij}(\tilde{\mathbf{r}}) r_{kl} s_{ij} \right\} dx \tag{5.4}$$

for all $\tilde{\mathbf{r}}, \mathbf{r}, \mathbf{s} \in X_1$. ∎

It is important to remark that for all $\tilde{\mathbf{r}} \in X_1$, $D\mathcal{A}_1(\tilde{\mathbf{r}})$ can be identified with a bilinear form on $X_1 \times X_1$ (defined by (5.4)), which, according to the previous lemma, is uniformly bounded and uniformly X_1-elliptic. Moreover, these conditions easily yield the strong monotonicity and Lipschitz continuity of the operator $\mathcal{A}_1$. In addition, if $X_{1,T}$ denotes the local space $[L^2(T)]^{2\times 2}, \forall T \in \mathcal{T}_h$, and $\mathcal{A}_{1,T} : X_{1,T} \to X_{1,T}'$ is the restriction of $\mathcal{A}_1$ to the triangle T, then the same arguments show that $\mathcal{A}_{1,T}$ is also strongly monotone and Lipschitz continuous. This fact will be used on the next page to prove what we call the local quasi-efficiency of our main a posteriori error estimate.

B. Ritz Projection of the Error

Let $X := X_1 \times M_1$ and introduce the nonlinear saddle point operator $\mathcal{A} := \left(\begin{smallmatrix}\mathcal{A}_1 & \mathcal{B}_1' \\ \mathcal{B}_1 & \mathcal{O}\end{smallmatrix}\right) : X \to X'$, that is

$$[\mathcal{A}(\mathbf{r},\boldsymbol{\zeta},\rho),(\mathbf{s},\boldsymbol{\tau},q)] := [\mathcal{A}_1(\mathbf{r}),\mathbf{s}] + [\mathcal{B}_1(\mathbf{s}),(\boldsymbol{\zeta},\rho)] + [\mathcal{B}_1(\mathbf{r}),(\boldsymbol{\tau},q)], \tag{5.5}$$

for all $(\mathbf{r},\boldsymbol{\zeta},\rho),(\mathbf{s},\boldsymbol{\tau},q) \in X$. In addition, let $\hat{\mathcal{A}} : X \to X'$ be the linear and bounded operator induced by the inner product of X, that is

$$[\hat{\mathcal{A}}(\mathbf{r},\boldsymbol{\zeta},\rho),(\mathbf{s},\boldsymbol{\tau},q)] := \langle \mathbf{r},\mathbf{s}\rangle_{[L^2(\Omega)]^{2\times 2}} + \langle \boldsymbol{\zeta},\boldsymbol{\tau}\rangle_{H(\mathbf{div};\Omega)} + \langle \rho,q\rangle_{L^2(\Omega)}, \tag{5.6}$$

for all $(\mathbf{r},\boldsymbol{\zeta},\rho),(\mathbf{s},\boldsymbol{\tau},q) \in X$.

Then we define the X-Ritz projection of the finite element error, with respect to $\hat{\mathcal{A}}$, as the unique $(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}) \in X$ such that

$$\begin{aligned}[\hat{\mathcal{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)] = [\mathcal{A}(\mathbf{t},\boldsymbol{\sigma},p),(\mathbf{s},\boldsymbol{\tau},q)] - [\mathcal{A}(\mathbf{t}_h,\boldsymbol{\sigma}_h,p_h),(\mathbf{s},\boldsymbol{\tau},q)] \\ +[\mathcal{B}(\boldsymbol{\tau},q),(\mathbf{u},\boldsymbol{\gamma},\xi) - (\mathbf{u}_h,\boldsymbol{\gamma}_h,\xi_h)]\end{aligned} \tag{5.7}$$

for all $(\mathbf{s},\boldsymbol{\tau},q) \in X$. The existence of $(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})$ is guaranteed by the fact that the right hand side of (5.7) constitutes a linear and bounded functional in X (as a mapping acting on $(\mathbf{s},\boldsymbol{\tau},q)$).

Now, given $T \in \mathcal{T}_h$ we denote by $\langle\cdot,\cdot\rangle_{[L^2(T)]^{2\times 2}}$, $\langle\cdot,\cdot\rangle_{H(\mathbf{div};T)}$, and $\langle\cdot,\cdot\rangle_{L^2(T)}$ the inner products of $[L^2(T)]^{2\times 2}, H(\mathbf{div};T)$, and $L^2(T)$, respectively, and let $(\mathbf{t}_{h,T},\boldsymbol{\sigma}_{h,T},p_{h,T},\mathbf{u}_{h,T},\boldsymbol{\gamma}_{h,T})$ be the restriction of $(\mathbf{t}_h,\boldsymbol{\sigma}_h,p_h,\mathbf{u}_h,\boldsymbol{\gamma}_h)$ to T. Then we can prove the following theorem, which provides an important upper bound for the X-Ritz projection of the error.

Theorem 5.2. *Let $\tilde{\boldsymbol{\varphi}}_h$ be a function in $[H^1(\Omega)]^2$. For each $T \in \mathcal{T}_h$ define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\mathbf{I} - 2\hat{\mu}(\mathbf{t}_{h,T})\mathbf{t}_{h,T} - p_{h,T}\mathbf{I}, \tag{5.8}$$

and let $\hat{\boldsymbol{\sigma}}_T \in H(\mathbf{div};T)$ be the unique solution of the local problem

$$\langle \hat{\boldsymbol{\sigma}}_T,\boldsymbol{\tau}\rangle_{H(\mathbf{div};T)} = F_{h,T}(\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in H(\mathbf{div};T), \tag{5.9}$$

where $F_{h,T} \in H(\mathbf{div};T)'$ is defined by

$$\begin{aligned}F_{h,T}(\boldsymbol{\tau}) := \int_T (\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T}) : \boldsymbol{\tau}\,dx + \int_T \mathbf{u}_{h,T}\cdot \mathbf{div}\,\boldsymbol{\tau}\,dx - \xi_h \int_T \mathrm{tr}\,(\boldsymbol{\tau})\,dx \\ - \int_{\partial T} \tilde{\boldsymbol{\varphi}}_h \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\,ds - \int_{\partial T\cap\Gamma} (\mathbf{g} - \tilde{\boldsymbol{\varphi}}_h)\cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\,ds,\end{aligned} \tag{5.10}$$

with $\boldsymbol{\nu}_T$ being the unit outward normal to ∂T. Then there holds

$$\|(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})\|_X^2 \le \sum_{T\in\mathcal{T}_h} \left\{\|\hat{\mathbf{t}}_T\|^2_{[L^2(T)]^{2\times 2}} + \|\mathrm{tr}\,(\mathbf{t}_{h,T})\|^2_{L^2(T)} + \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)}\right\}. \tag{5.11}$$

Proof. From the variational formulation (3.11) we deduce that

$$[\mathcal{A}(\mathbf{t},\boldsymbol{\sigma},p),(\mathbf{s},\boldsymbol{\tau},q)] + [\mathcal{B}(\boldsymbol{\tau},q),(\mathbf{u},\boldsymbol{\gamma},\xi)] = -\int_\Gamma \mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds,$$

whence (5.7) yields

$$\begin{aligned}[\hat{\mathcal{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)] = -\int_\Gamma \mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds - [\mathcal{A}(\mathbf{t}_h,\boldsymbol{\sigma}_h,p_h),(\mathbf{s},\boldsymbol{\tau},q)] \\ - [\mathcal{B}(\boldsymbol{\tau},q),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\xi_h)] \qquad \forall(\mathbf{s},\boldsymbol{\tau},q) \in X.\end{aligned} \tag{5.12}$$

Next, we clearly have that

$$-\frac{1}{2}\|(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})\|_X^2 = \min_{(\mathbf{s},\boldsymbol{\tau},q)\in X} \left\{\frac{1}{2}\|(\mathbf{s},\boldsymbol{\tau},q)\|_X^2 - [\hat{\mathcal{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)]\right\},$$

which, using (5.12), becomes

$$-\frac{1}{2}\|(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})\|_X^2 = \min_{(\mathbf{s},\boldsymbol{\tau},q)\in X} \mathbf{J}(\mathbf{s},\boldsymbol{\tau},q), \tag{5.13}$$

where

$$\mathbf{J}(\mathbf{s},\boldsymbol{\tau},q) := \frac{1}{2}\|(\mathbf{s},\boldsymbol{\tau},q)\|_X^2 + \int_\Gamma \mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds + [\mathcal{A}(\mathbf{t}_h,\boldsymbol{\sigma}_h,p_h),(\mathbf{s},\boldsymbol{\tau},q)] + [\mathcal{B}(\boldsymbol{\tau},q),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\xi_h)].$$

Since the function $\tilde{\boldsymbol{\varphi}}_h$ belongs to $[H^1(\Omega)]^2$, we can write

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\tilde{\boldsymbol{\varphi}}_h\cdot\boldsymbol{\tau}\boldsymbol{\nu}_T\,ds - \int_\Gamma \tilde{\boldsymbol{\varphi}}_h\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds = 0,$$

which is then added to the quadratic functional $\mathbf{J}$.

In this way, and using the definitions of $\mathcal{A}$ and $\mathcal{B}$, we obtain that

$$\mathbf{J}(\mathbf{s},\boldsymbol{\tau},q) = \sum_{T\in\mathcal{T}_h}\left\{\mathbf{J}_{1,T}(\mathbf{s}_T) + \mathbf{J}_{2,T}(\boldsymbol{\tau}_T) + \mathbf{J}_{3,T}(q_T)\right\}, \tag{5.15}$$

where $(\mathbf{s}_T,\boldsymbol{\tau}_T,q_T)$ is the restriction of $(\mathbf{s},\boldsymbol{\tau},q)$ to T,

$$\mathbf{J}_{1,T}(\mathbf{s}_T) := \frac{1}{2}\|\mathbf{s}_T\|^2_{[L^2(T)]^{2\times 2}} - \langle\hat{\mathbf{t}}_T,\mathbf{s}_T\rangle_{[L^2(T)]^{2\times 2}}, \tag{5.16}$$

$$\mathbf{J}_{2,T}(\boldsymbol{\tau}_T) := \frac{1}{2}\|\boldsymbol{\tau}_T\|^2_{H(\mathbf{div};T)} - F_{h,T}(\boldsymbol{\tau}_T), \tag{5.17}$$

and

$$\mathbf{J}_{3,T}(q_T) := \frac{1}{2}\|q_T\|^2_{L^2(T)} + \langle \mathrm{tr}\,(\mathbf{t}_{h,T}),q_T\rangle_{L^2(T)}. \tag{5.18}$$

We note that $\hat{\mathbf{t}}_T$ (defined by (5.8)) and tr $(\mathbf{t}_{h,T})$ belong to $[L^2(T)]^{2\times 2}$ and $L^2(T)$, respectively, and hence it easily follows that

$$\min_{\mathbf{s}_T\in[L^2(T)]^{2\times 2}} \mathbf{J}_{1,T}(\mathbf{s}_T) = -\frac{1}{2}\|\hat{\mathbf{t}}_T\|^2_{[L^2(T)]^{2\times 2}}, \tag{5.19}$$

$$\min_{\boldsymbol{\tau}_T\in H(\mathbf{div};T)} \mathbf{J}_{2,T}(\boldsymbol{\tau}_T) = -\frac{1}{2}\|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)}, \tag{5.20}$$

and

$$\min_{q_T\in L^2(T)} \mathbf{J}_{3,T}(q_T) = -\frac{1}{2}\|\mathrm{tr}\,(\mathbf{t}_{h,T})\|^2_{L^2(T)}. \tag{5.21}$$

Therefore, replacing (5.15) up to (5.18) back into (5.13), noting that $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$ (resp. $q \in L^2(\Omega)$) if and only if $\mathbf{s}_T \in [L^2(T)]^{2\times 2}$ (resp. $q_T \in L^2(T)$) for all $T \in \mathcal{T}_h$, observing that $H(\mathbf{div};\Omega)$ is contained in the *broken* space

$$H(\mathbf{div};\Omega)^{br} := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}\colon\ \boldsymbol{\tau}_T \in H(\mathbf{div};T) \qquad \forall T \in \mathcal{T}_h\},$$

and using (5.19)–(5.21), we conclude that

$$-\frac{1}{2}\,\|(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})\|_X^2 \geq -\frac{1}{2}\sum_{T\in\mathcal{T}_h}\left\{\|\hat{\mathbf{t}}_T\|^2_{[L^2(T)]^{2\times 2}} + \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)} + \|\mathrm{tr}\,(\mathbf{t}_{h,T})\|^2_{L^2(T)}\right\},$$

which completes the proof. ∎

At this point we remark that there is no side restriction on $\tilde{\boldsymbol{\varphi}}_h$, and hence this function can be chosen arbitrarily in $[H^1(\Omega)]^2$. Nevertheless, we will show in the next section that, under the assumption that $\mathbf{u} \in [H^1(\Omega)]^2$, the proposed a posteriori error estimate is efficient up to a term depending on $(\mathbf{u} - \tilde{\boldsymbol{\varphi}}_h)$. This property is called, from now on, quasi-efficiency. Consequently, one should try to choose $\tilde{\boldsymbol{\varphi}}_h$ as close as possible, at least empirically, to the exact solution $\mathbf{u}$.

C. The Main a Posteriori Error Estimate

The following theorem establishes a reliable a posteriori error estimate for our finite element solution. It makes use of the X-Ritz projection of the error and of the corresponding upper bound given in Theorem 5.2.

Theorem 5.3. *Let $\tilde{\boldsymbol{\varphi}}_h$ be an arbitrary function in $[H^1(\Omega)]^2$. For each $T \in \mathcal{T}_h$ define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\mathbf{I} - 2\hat{\mu}(\mathbf{t}_{h,T})\mathbf{t}_{h,T} - p_{h,T}\mathbf{I},$$

and let $\hat{\boldsymbol{\sigma}}_T \in H(\mathbf{div};T)$ be the unique solution of the local problem (5.9). *Then there exists $C > 0$, independent of h, such that*

$$\|(\mathbf{t},\boldsymbol{\sigma},p,\mathbf{u},\boldsymbol{\gamma},\xi) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,p_h,\mathbf{u}_h,\boldsymbol{\gamma}_h,\xi_h)\|_{X\times M} \leq C\left\{\sum_{T\in\mathcal{T}_h}\theta_T^2\right\}^{1/2},$$

where

$$\theta_T^2 := \|\hat{\mathbf{t}}_T\|^2_{[L^2(T)]^{2\times 2}} + \|\mathrm{tr}\,(\mathbf{t}_{h,T})\|^2_{L^2(T)} + \|\hat{\boldsymbol{\sigma}}_T\|^2_{H(\mathbf{div};T)} + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{h,T}\|^2_{[L^2(T)]^2} + \|\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}^{\mathbf{T}}_{h,T}\|^2_{[L^2(T)]^{2\times 2}}.$$

Proof. We first recall from Lemma 5.1 that $D\mathcal{A}_1(\tilde{\mathbf{r}})$ becomes a uniformly bounded and elliptic bilinear form on $X_1 \times X_1$, for all $\tilde{\mathbf{r}} \in X_1$. In addition, we know from Lemmata 3.2 and 3.3 that $\mathcal{B}$ and $\mathcal{B}_1$ satisfy the continuous inf-sup conditions. Then, by the continuous dependence result for the linear version of Theorem 6.1 in the Appendix (see also Theorem 2 in [23] or Theorem 2.3 in [21]), we conclude that there exists $\tilde{C} > 0$, independent of $\tilde{\mathbf{r}} \in X_1$, such that

$$\|(\tilde{\mathbf{t}},\tilde{\boldsymbol{\sigma}},\tilde{p},\tilde{\mathbf{u}},\tilde{\boldsymbol{\gamma}},\tilde{\boldsymbol{\xi}})\|_{X\times M} \leq \tilde{C}\sup_{\substack{(\mathbf{s},\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\delta},\eta)\in X\times M\\ \|(\mathbf{s},\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\delta},\eta)\|\leq 1}}\{D\mathcal{A}_1(\tilde{\mathbf{r}})(\tilde{\mathbf{t}},s) + [\mathcal{B}_1(\tilde{\mathbf{t}}),(\boldsymbol{\tau},q)] + [\mathcal{B}_1(\mathbf{s}),(\tilde{\sigma},\tilde{p})] + [\mathcal{B}(\tilde{\sigma},\tilde{p}),(\mathbf{v},\boldsymbol{\delta},\eta)] + [\mathcal{B}(\boldsymbol{\tau},q),(\tilde{\mathbf{u}},\tilde{\gamma},\tilde{\xi})]\}, \quad (5.22)$$

for all $(\tilde{\mathbf{t}},\tilde{\boldsymbol{\sigma}},\tilde{p},\tilde{\mathbf{u}},\tilde{\boldsymbol{\gamma}},\tilde{\boldsymbol{\xi}}) \in X \times M$, and for all $\tilde{\mathbf{r}} \in X_1$.

Because of the continuity of $D\mathcal{A}_1$, there exists $\tilde{\mathbf{r}}_h \in X_1$ such that

$$D\mathcal{A}_1(\tilde{\mathbf{r}}_h)(\mathbf{t}-\mathbf{t}_h,\mathbf{s}) = [\mathcal{A}_1(\mathbf{t}),\mathbf{s}] - [\mathcal{A}_1(\mathbf{t}_h),\mathbf{s}] \qquad \forall \mathbf{s} \in X_1. \tag{5.23}$$

Therefore, we apply (5.22) (with $\tilde{\mathbf{r}} = \tilde{\mathbf{r}}_h$) to

$$(\tilde{\mathbf{t}},\tilde{\boldsymbol{\sigma}},\tilde{p},\tilde{\mathbf{u}},\tilde{\boldsymbol{\gamma}},\tilde{\xi}) := (\mathbf{t}-\mathbf{t}_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,p-p_h,\mathbf{u}-\mathbf{u}_h,\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\xi-\xi_h),$$

and use (5.23), the X-Ritz projection $(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})$, and the statements of the continuous and Galerkin formulations (3.11) and (4.4), to obtain

$$\begin{aligned}
&\frac{1}{\tilde{C}}\,\|(\mathbf{t},\boldsymbol{\sigma},p,\mathbf{u},\boldsymbol{\gamma},\xi)-(\mathbf{t}_h,\boldsymbol{\sigma}_h,p_h,\mathbf{u}_h,\boldsymbol{\gamma}_h,\xi_h)\|_{X\times M}\\
&\le \sup_{\substack{(\mathbf{s},\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\delta},\eta)\in X\times M\\ \|(\mathbf{s},\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\delta},\eta)\|\le 1}} \left\{[\hat{\mathcal{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)] + \int_\Omega (\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h)\cdot\mathbf{v}\,dx + \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\delta}\,dx\right\}
\end{aligned} \tag{5.24}$$

Now, it is easily seen that

$$\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\delta}\,dx = \frac{1}{2}\int_\Omega (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathbf{T}}) : \boldsymbol{\delta}\,dx \qquad \forall \boldsymbol{\delta} \in \mathcal{R}.$$

Consequently, by applying Cauchy-Schwarz's inequality in (5.24), we deduce that

$$\begin{aligned}
&\frac{1}{\tilde{C}}\,\|(\mathbf{t},\boldsymbol{\sigma},p,\mathbf{u},\boldsymbol{\gamma},\xi)-(\mathbf{t}_h,\boldsymbol{\sigma}_h,p_h,\mathbf{u}_h,\boldsymbol{\gamma}_h,\xi_h)\|_{X\times M}\\
&\le \sup_{\substack{(\mathbf{s},\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\delta},\eta)\in X\times M\\ \|(\mathbf{s},\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\delta},\eta)\|\le 1}} \Big\{\|(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})\|_X\|(\mathbf{s},\boldsymbol{\tau},q)\|_X + \|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^2}\|\mathbf{v}\|_{[L^2(\Omega)]^2}\\
&\qquad\qquad + \frac{1}{2}\,\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathbf{T}}\|_{[L^2(\Omega)]^{2\times 2}}\|\boldsymbol{\delta}\|_{\mathcal{R}}\Big\}\\
&\le C\left\{\|(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p})\|_X^2 + \sum_{T\in\mathcal{T}_h}\left(\|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_{h,T}\|^2_{[L^2(T)]^2} + \|\boldsymbol{\sigma}_{h,T}-\boldsymbol{\sigma}_{h,T}^{\mathbf{T}}\|^2_{[L^2(T)]^{2\times 2}}\right)\right\}^{1/2},
\end{aligned}$$

which, together with the upper bound (5.11) provided by Theorem 5.2, completes the proof. ∎

The following lemma provides a priori estimates for the solution of the local problem (5.9). They will be used to prove the quasi-efficiency of the estimate given in the previous theorem and to derive a fully explicit reliable a posteriori error estimate for the finite element solution.

Lemma 5.4. *Let $\tilde{\varphi}_h$ be a function in $[H^1(\Omega)]^2$ and let $\hat{\boldsymbol{\sigma}}_T \in H(\mathbf{div};T)$ be the unique solution of the local problem*

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau}\rangle_{H(\mathbf{div};T)} = F_{h,T}(\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in H(\mathbf{div};T),$$

where $F_{h,T} \in H(\mathbf{div};T)'$ is defined by

$$\begin{aligned}
F_{h,T}(\boldsymbol{\tau}) := \int_T (\mathbf{t}_{h,T}+\boldsymbol{\gamma}_{h,T}) : \boldsymbol{\tau}\,dx + \int_T \mathbf{u}_{h,T}\cdot\mathbf{div}\,\boldsymbol{\tau}\,dx - \xi_h\int_T \mathrm{tr}\,(\boldsymbol{\tau})\,dx\\
- \int_{\partial T}\tilde{\varphi}_h\cdot\boldsymbol{\tau}\boldsymbol{\nu}_T\,ds - \int_{\partial T\cap\Gamma}(\mathbf{g}-\tilde{\varphi}_h)\cdot\boldsymbol{\tau}\boldsymbol{\nu}_T\,ds.
\end{aligned}$$

Then there exists $C > 0$, independent of h and T, such that

$$\|\hat{\boldsymbol{\sigma}}_T\|_{H(\mathbf{div};T)} \leq C\Big\{\|\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\tilde{\boldsymbol{\varphi}}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_{h,T} - \tilde{\boldsymbol{\varphi}}_h\|^2_{[L^2(T)]^2} + |\xi_h|^2|T| + \|\mathbf{g} - \tilde{\boldsymbol{\varphi}}_h\|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2}\Big\}^{1/2}. \quad (5.25)$$

In addition, for any $\mathbf{z} \in [H^1(\Omega)]^2$ such that $\mathbf{z} = \mathbf{g}$ on Γ, we get

$$\|\hat{\boldsymbol{\sigma}}_T\|_{H(\mathbf{div};T)} \leq C\Big\{\|\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\mathbf{z}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_{h,T} - \mathbf{z}\|^2_{[L^2(T)]^2} + |\xi_h|^2|T| + \|\mathbf{z} - \tilde{\boldsymbol{\varphi}}_h\|^2_{[H^{1/2}(\tilde{\partial T})]^2}\Big\}^{1/2}, \quad (5.26)$$

where $\tilde{\partial T}$ is the part of ∂T not contained in Γ.

Proof. Since $\tilde{\boldsymbol{\varphi}}_h \in [H^1(\Omega)]^2$, a simple integration by parts gives

$$\int_{\partial T} \tilde{\boldsymbol{\varphi}}_h \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\, ds = \int_T \nabla\tilde{\boldsymbol{\varphi}}_h : \boldsymbol{\tau}\, dx + \int_T \tilde{\boldsymbol{\varphi}}_h \cdot \mathbf{div}\,\boldsymbol{\tau}\, dx.$$

Then, replacing the above expression back into the definition of $F_{h,T}$, applying Cauchy Schwarz's inequality, and noting that $\|\hat{\boldsymbol{\sigma}}_T\|_{H(\mathbf{div};T)} = \|F_{h,T}\|_{H(\mathbf{div};T)'}$, we obtain (5.25).

Now, given $\mathbf{z} \in [H^1(\Omega)]^2$ such that $\mathbf{z} = \mathbf{g}$ on Γ, we can write

$$\begin{aligned} &- \int_{\partial T} \tilde{\boldsymbol{\varphi}}_h \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\, ds - \int_{\partial T\cap\Gamma} (\mathbf{g} - \tilde{\boldsymbol{\varphi}}_h) \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\, ds \\ &= - \int_{\partial T} \mathbf{z} \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\, ds + \int_{\partial T} (\mathbf{z} - \tilde{\boldsymbol{\varphi}}_h) \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\, ds - \int_{\partial T\cap\Gamma} (\mathbf{z} - \tilde{\boldsymbol{\varphi}}_h) \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\, ds \\ &= - \int_T \nabla\mathbf{z} : \boldsymbol{\tau}\, dx - \int_T \mathbf{z} \cdot \mathbf{div}\,\boldsymbol{\tau}\, dx + \int_{\tilde{\partial T}} (\mathbf{z} - \tilde{\boldsymbol{\varphi}}_h) \cdot \boldsymbol{\tau}\boldsymbol{\nu}_T\, ds, \end{aligned}$$

which, replaced back into $F_{h,T}$, leads (5.26) and concludes the proof. ∎

The above lemma motivates the forthcoming discussion in which we remark on the importance of choosing $\tilde{\boldsymbol{\varphi}}_h|_{\tilde{\partial T}}$ as close as possible, at least empirically, to $\mathbf{u}|_{\tilde{\partial T}}, \forall T \in \mathcal{T}_h$. Indeed, the following result shows that the reliable a posteriori error estimate given by Theorem 5.3 is locally (and hence globally) quasi-efficient in the sense that it is efficient up to a term depending on the traces of $(\mathbf{u} - \tilde{\boldsymbol{\varphi}}_h)$ on the element boundaries not contained in Γ.

Lemma 5.5. *Assume that $\mathbf{u}$ and $\tilde{\boldsymbol{\varphi}}_h \in [H^1(\Omega)]^2$. Then there exists $C > 0$, independent of h, such that for all $T \in \mathcal{T}_h$*

$$\theta_T^2 \leq C\Big\{\|\mathbf{t}_{h,T} - \mathbf{t}\|^2_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}\|^2_{H(\mathbf{div};T)} + \|p_{h,T} - p\|^2_{L^2(T)} + \|\mathbf{u}_{h,T} - \mathbf{u}\|^2_{[L^2(T)]^2} + \|\boldsymbol{\gamma}_{h,T} - \boldsymbol{\gamma}\|^2_{[L^2(T)]^{2\times 2}} + |\xi_h - \xi|^2|T| + \|\mathbf{u} - \tilde{\boldsymbol{\varphi}}_h\|^2_{[H^{1/2}(\tilde{\partial T})]^2}\Big\},$$

and hence

$$\sum_{T\in\mathcal{T}_h} \theta_T^2 \leq C\left\{\|(\mathbf{t}, \boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}, \xi) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \xi_h)\|^2_{X\times M} + \sum_{S\in\mathcal{E}_h(\Omega)} \|\mathbf{u} - \tilde{\boldsymbol{\varphi}}_h\|^2_{[H^{1/2}(S)]^2}\right\},$$

where $\mathcal{E}_h(\Omega)$ is the set of edges of $\mathcal{T}_h$ not contained in Γ.

Proof. Since $\mathbf{u} \in [H^1(\Omega)]^2$, we can integrate by parts in the second equation of (3.10) and deduce that

$$\nabla \mathbf{u} = \mathbf{t} + \boldsymbol{\gamma} \text{ in } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \Gamma, \quad \xi = 0, \quad \text{and tr}\,(\mathbf{t}) = 0 \text{ in } \Omega.$$

Then, a straightforward application of (5.26) yields

$$\begin{aligned}\|\hat{\boldsymbol{\sigma}}_T\|_{H(\mathbf{div};T)} &\leq C\Big\{\|\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\mathbf{u}\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_{h,T} - \mathbf{u}\|^2_{[L^2(T)]^2} \\ &\quad + |\xi_h - \xi|^2|T| + \|\mathbf{u} - \tilde{\varphi}_h\|^2_{[H^{1/2}(\partial\tilde{T})]^2}\Big\}^{1/2} \\ &\leq C\Big\{\|\mathbf{t}_{h,T} - \mathbf{t}\|^2_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{\gamma}_{h,T} - \boldsymbol{\gamma}\|^2_{[L^2(T)]^{2\times 2}} \\ &\quad + \|\mathbf{u}_{h,T} - \mathbf{u}\|^2_{[L^2(T)]^2} + |\xi_h - \xi|^2|T| + \|\mathbf{u} - \tilde{\varphi}_h\|^2_{[H^{1/2}(\partial\tilde{T})]^2}\Big\}^{1/2}. \end{aligned} \tag{5.27}$$

Also, we have that

$$\|\text{tr}\,(\mathbf{t}_{h,T})\|_{L^2(T)} = \|\text{tr}\,(\mathbf{t}_{h,T} - \mathbf{t})\|_{L^2(T)} \leq C\|\mathbf{t}_{h,T} - \mathbf{t}\|_{[L^2(T)]^{2\times 2}}. \tag{5.28}$$

Now, from the first equation of (3.10) we obtain

$$\boldsymbol{\sigma} = \mathcal{N}(\mathbf{t}) + p\mathbf{I} \qquad \text{in } \Omega,$$

and hence

$$\begin{aligned}\|\hat{\mathbf{t}}_T\|_{[L^2(T)]^{2\times 2}} &:= \|\boldsymbol{\sigma}_{h,T} - \mathcal{N}(\mathbf{t}_{h,T}) - p_{h,T}\mathbf{I}\|_{[L^2(T)]^{2\times 2}} \\ &\leq \|\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}\|_{[L^2(T)]^{2\times 2}} + \|\mathcal{N}(\mathbf{t}_{h,T}) - \mathcal{N}(\mathbf{t})\|_{[L^2(T)]^{2\times 2}} + \|(p_h - p)\mathbf{I}\|_{[L^2(T)]^{2\times 2}} \\ &\leq C\{\|\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}\|_{[L^2(T)]^{2\times 2}} + \|\mathbf{t}_{h,T} - \mathbf{t}\|_{[L^2(T)]^{2\times 2}} + \|p_h - p\|_{L^2(T)}\}. \end{aligned} \tag{5.29}$$

Here we used the Lipschitz continuity of $\mathcal{A}_{1,T}$ (restriction of $\mathcal{A}_1$ to T), which is equivalent to the local Lipschitz continuity of $\mathcal{N}$.

Next, the third equation of (3.10) gives

$$-\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \text{ in } \Omega \qquad \text{and} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \text{ in } \Omega,$$

whence we get

$$\|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{h,T}\|_{[L^2(T)]^2} = \|\mathbf{div}\,(\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma})\|_{[L^2(T)]^2}, \tag{5.30}$$

and

$$\|\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}^{\mathbf{T}}_{h,T}\|_{[L^2(T)]^{2\times 2}} \leq 2\|\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}\|_{[L^2(T)]^{2\times 2}}. \tag{5.31}$$

Therefore, the estimates (5.27), (5.28), (5.29), (5.30), and (5.31) imply the local quasi-efficiency of the error estimator θ_T provided in Theorem 5.3. Finally, the proof is completed by summing up over all the elements $T \in \mathcal{T}_h$. ■

The results given in the previous lemma are in full agreement with those obtained by the classical Bank-Weiser theory for the a posteriori error analysis of elliptic boundary value problems

([15]). In fact, it is well known that this kind of estimate does not yield *efficiency* but only *quasi-efficiency* (as shown here), and that it is possible to provide an explicit lower bound of the error only through the use of another estimator, usually of residual type.

At this point we remark that the local problem defining $\hat{\boldsymbol{\sigma}}_T \in H(\mathbf{div};T)$ is set in a space of infinite dimension, and hence it can only be solved approximately by replacing $H(\mathbf{div};T)$ by a finite dimensional subspace. Since it is a linear problem, we suggest to apply the p or the $h-p$ version, as indicated in [27]. Alternatively, we propose in the next subsection a fully explicit reliable a posteriori error estimate, which does not require the approximate solution of the local problems, and which is based on an appropriate choice of the function $\tilde{\boldsymbol{\varphi}}_h$ (see also [10]). The bound given by (5.25) plays here an essential role.

D. A Fully Explicit Reliable A-posteriori Error Estimate

We proceed similarly as in [10] and define $\tilde{\mathbf{u}}_h \in [C(\bar{\Omega})]^2$ as the unique function satisfying the following conditions:

- $\tilde{\mathbf{u}}_{h,T} := \tilde{\mathbf{u}}_h|_T \in [\mathbb{P}_1(T)]^2$ for all $T \in \mathcal{T}_h$.
- $\tilde{\mathbf{u}}_h(\bar{\mathbf{x}}) = \mathbf{g}(\bar{\mathbf{x}})$ for each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ lying on Γ.
- $\tilde{\mathbf{u}}_h(\bar{\mathbf{x}})$ is the weighted average of the constant values of $\mathbf{u}_h$ on all the triangles $T \in \mathcal{T}_h$ to which $\bar{\mathbf{x}}$ belongs, for any interior vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$. Here, the weighting is according to the relative area of each triangle.

It follows easily that $\tilde{\mathbf{u}}_h \in [H^1(\Omega)]^2$. Hence, we choose $\tilde{\boldsymbol{\varphi}}_h = \tilde{\mathbf{u}}_h$ and obtain the following result.

Theorem 5.6. *Let $\tilde{\mathbf{u}}_h$ be as indicated above. For each $T \in \mathcal{T}_h$ we let $\{\bar{\mathbf{x}}_{k,T}\}_{k=1}^3$ be its vertices, and define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T})\ \mathrm{tr}\ (\mathbf{t}_{h,T})\mathbf{I} - 2\hat{\mu}(\mathbf{t}_{h,T})\mathbf{t}_{h,T} - p_{h,T}\mathbf{I}.$$

Then there exists $C > 0$, independent of h, such that

$$\|(\mathbf{t},\boldsymbol{\sigma},p,\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\xi}) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,p_h,\mathbf{u}_h,\boldsymbol{\gamma}_h,\xi_h)\|_{X\times M} \le C\left\{\sum_{T\in\mathcal{T}_h}\tilde{\theta}_T^2\right\}^{1/2}, \tag{5.32}$$

where

$$\begin{aligned}\tilde{\theta}_T^2 := \Big\{&\|\hat{\mathbf{t}}_T\|^2_{[L^2(T)]^{2\times2}} + \|\mathrm{tr}\ (\mathbf{t}_{h,T})\|^2_{L^2(T)} + \|\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\tilde{\mathbf{u}}_{h,T}\|^2_{[L^2(T)]^{2\times2}}\\ &+ h_T^2\sum_{k=1}^{3}\|\mathbf{u}_{h,T} - \tilde{\mathbf{u}}_{h,T}(\bar{\mathbf{x}}_{k,T})\|^2_{\mathbb{R}^2} + h_T^2|\xi_h|^2 + \|\mathbf{g} - \tilde{\mathbf{u}}_{h,T}\|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2}\\ &+ \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{h,T}\|^2_{[L^2(T)]^2} + \|\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}^{\mathbf{T}}_{h,T}\|^2_{[L^2(T)]^{2\times2}}\Big\}.\end{aligned} \tag{5.33}$$

Proof. Let $\hat{\boldsymbol{\sigma}}_T \in H(\mathbf{div};T)$ be the solution of the local problem (5.9) with $\tilde{\boldsymbol{\varphi}}_h = \tilde{\mathbf{u}}_h$. Then, according to Lemma 5.4 (cf. (5.25)), and using that $|T| \le h_T^2$, we obtain that

$$\begin{aligned}\|\hat{\boldsymbol{\sigma}}_T\|_{H(\mathbf{div};T)} \le C\Big\{&\|\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\tilde{\mathbf{u}}_{h,T}\|^2_{[L^2(T)]^{2\times2}} + \|\mathbf{u}_{h,T} - \tilde{\mathbf{u}}_{h,T}\|^2_{[L^2(T)]^2}\\ &+ h_T^2|\xi_h|^2 + \|\mathbf{g} - \tilde{\mathbf{u}}_{h,T}\|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2}\Big\}^{1/2}.\end{aligned} \tag{5.34}$$

Next, since $\mathbf{u}_{h,T} \in [\mathbb{P}_0(T)]^2$ and $\tilde{\mathbf{u}}_{h,T} \in [\mathbb{P}_1(T)]^2$, we find that

$$\|\mathbf{u}_{h,T} - \tilde{\mathbf{u}}_{h,T}\|^2_{[L^2(T)]^2} \le h_T^2 \sum_{k=1}^{3} \|\mathbf{u}_{h,T} - \tilde{\mathbf{u}}_{h,T}(\bar{\mathbf{x}}_{k,T})\|^2_{\mathbb{R}^2}. \tag{5.35}$$

In this way, (5.34), (5.35), and the estimate from Theorem 5.3 yield (5.32) and complete the proof. ∎

It is important to remark, assuming that $\mathbf{u} \in [H^1(\Omega)]^2$ and according to Lemma 5.5, that the fully explicit reliable a posteriori error estimate provided by Theorem 5.6 is certainly quasi-efficient since it is just a particular case of the general one analyzed before.

Also, we observe that the third term on the right hand side of (5.33) can be bounded by a simpler expression. In fact, since $\boldsymbol{\gamma}_{h,T} \in [\mathbb{P}_1(T)]^{2\times 2}$ and $\nabla\tilde{\mathbf{u}}_{h,T} \in [\mathbb{P}_0(T)]^{2\times 2}$, we deduce that

$$\begin{aligned}
\|\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\tilde{\mathbf{u}}_{h,T}\|^2_{[L^2(T)]^{2\times 2}} &\le 2\left\{\|\mathbf{t}_{h,T} - \frac{1}{2}[\nabla\tilde{\mathbf{u}}_{h,T} + (\nabla\tilde{\mathbf{u}}_{h,T})^{\mathbf{T}}]\|^2_{[L^2(T)]^{2\times 2}}\right. \\
&\quad \left. + \|\boldsymbol{\gamma}_{h,T} - \frac{1}{2}[\nabla\tilde{\mathbf{u}}_{h,T} - (\nabla\tilde{\mathbf{u}}_{h,T})^{\mathbf{T}}]\|^2_{[L^2(T)]^{2\times 2}}\right\} \\
&\le 2\left\{\|\mathbf{t}_{h,T} - \frac{1}{2}\,[\nabla\tilde{\mathbf{u}}_{h,T} + (\nabla\tilde{\mathbf{u}}_{h,T})^{\mathbf{T}}]\|^2_{[L^2(T)]^{2\times 2}}\right. \\
&\quad \left. + h_T^2 \sum_{k=1}^{3} \|\boldsymbol{\gamma}_{h,T}(\bar{\mathbf{x}}_{k,T}) - \frac{1}{2}\,[\nabla\tilde{\mathbf{u}}_{h,T} - (\nabla\tilde{\mathbf{u}}_{h,T})^{\mathbf{T}}]\|^2_{\mathbb{R}^{2\times 2}}\right\}.
\end{aligned}$$

On the other hand, for implementation purposes, the $[H^{1/2}(\partial T \cap \Gamma)]^2$-norms can be bounded by using the interpolation theorem. More precisely, assuming that $\mathbf{g} \in [H^1(\partial T \cap \Gamma)]^2$, we can write

$$\|\mathbf{g} - \tilde{\mathbf{u}}_{h,T}\|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} \le \|\mathbf{g} - \tilde{\mathbf{u}}_{h,T}\|_{[L^2(\partial T\cap\Gamma)]^2}\ \|\mathbf{g} - \tilde{\mathbf{u}}_{h,T}\|_{[H^1(\partial T\cap\Gamma)]^2}.$$

We end this section by emphasizing that, up to the authors' knowledge, there are no other dual-mixed variational formulations available for nonlinear incompressible materials. This fact naturally complicates the eventual comparison of the present method with approaches of the same kind. Nevertheless, in a forthcoming article we will provide several numerical examples, including the computation of the efficiency indices, and compare our method with the performance of the Stokes-type mixed formulation studied in [8] and its corresponding a posteriori error estimates.

VI. APPENDIX

Here we recall from [22] and [23] the main results concerning the solvability and Galerkin approximations of the class of twofold saddle point problems defined by (3.11).

In order to set the abstract problem of interest, we let X_1, M_1, M be Hilbert spaces and consider a nonlinear operator $\mathcal{A}_1 : X_1 \to X_1'$, and linear bounded operators $\mathcal{B}_1 : X_1 \to M_1'$ and $\mathcal{B} : M_1 \to M'$, with transposes $\mathcal{B}_1' : M_1 \to X_1'$ and $\mathcal{B}' : M \to M_1'$, respectively.

Then, we are interested in the following nonlinear variational problem:

Given $(\mathcal{F}_1, \mathcal{G}_1, \mathcal{G}) \in X_1' \times M_1' \times M'$, *find* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *such that*

$$\begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1' & \mathcal{O} \\ \mathcal{B}_1 & \mathcal{O} & \mathcal{B}' \\ \mathcal{O} & \mathcal{B} & \mathcal{O} \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \boldsymbol{\sigma} \\ u \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{G}_1 \\ \mathcal{G} \end{pmatrix}. \tag{6.1}$$

We have the following theorem.

Theorem 6.1. *Let* $\tilde{M}_1 := ker(\mathcal{B})$, *define* $V_1 := \{\mathbf{s} \in X_1 : [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\tau}] = 0 \;\forall \boldsymbol{\tau} \in \tilde{M}_1\}$, *and let* $\Pi_1 : X_1' \to V_1'$ *be the canonical imbedding defined by* $\Pi_1(\mathcal{F}_1) = \mathcal{F}_1|_{V_1}$ *for all* $\mathcal{F}_1 \in X_1'$. *Assume that*

i) *there exists* $\beta > 0$ *such that for all* $v \in M$

$$\sup_{\substack{\boldsymbol{\tau} \in M_1 \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}), v]}{\|\boldsymbol{\tau}\|_{M_1}} \geq \beta \|v\|_M; \tag{6.2}$$

ii) *there exists* $\beta_1 > 0$ *such that for all* $\boldsymbol{\tau} \in \tilde{M}_1$

$$\sup_{\substack{\mathbf{s} \in X_1 \\ \mathbf{s} \neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{\|\mathbf{s}\|_{X_1}} \geq \beta_1 \|\boldsymbol{\tau}\|_{M_1}; \tag{6.3}$$

iii) *the nonlinear operator* $\mathcal{A}_1 : X_1 \to X_1'$ *is Lipschitz continuous with a Lipschitz constant* $\gamma > 0$, *and for any* $\tilde{\mathbf{t}} \in X_1$, *the nonlinear operator* $\Pi_1 \mathcal{A}_1(\cdot + \tilde{\mathbf{t}}) : V_1 \to V_1'$ *is strongly monotone*.

Then, for each $(\mathcal{F}_1, \mathcal{G}_1, \mathcal{G}) \in X_1' \times M_1' \times M'$ *there exists a unique* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *solution of* (6.1). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u)\|_{X_1 \times M_1 \times M} \leq C\{\|\mathcal{F}_1\| + \|\mathcal{G}_1\| + \|\mathcal{G}\| + \|\mathcal{A}_1(0)\|\}.$$

Proof. See Theorem 1 in [23] (see also Theorem 2.4 in [22] or Theorem 4.1 in [14]. ∎

Now, let $\mathbf{I}$ be a set of indexes, say $\mathbf{I} := \{h_j\}_{j \in \mathbb{N}}$, with $h_j \geq h_{j+1} \forall j \in \mathbb{N}$, and let $X_{1,h}$, $M_{1,h}$ and M_h be finite dimensional subspaces of X_1, M_1 and M, respectively. Then the Galerkin scheme associated with (6.1) reads as follows: *Given* $(\mathcal{F}_1, \mathcal{G}_1, \mathcal{G}) \in X_1' \times M_1' \times M'$, *find* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *such that*

$$[\mathcal{A}_1(\mathbf{t}_h), \mathbf{s}_h] + [\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] = [\mathcal{F}_1, \mathbf{s}_h],$$

$$[\mathcal{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathcal{B}(\boldsymbol{\tau}_h), u_h] = [\mathcal{G}_1, \boldsymbol{\tau}_h],$$

$$[\mathcal{B}(\boldsymbol{\sigma}_h), v_h] = [\mathcal{G}, v_h], \tag{6.4}$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \in X_{1,h} \times M_{1,h} \times M_h$.

The discrete analogue of Theorem 6.1 is as follows.

Theorem 6.2. *Let* $\tilde{M}_{1,h} := \{\boldsymbol{\tau}_h \in M_{1,h}\text{: } [\mathcal{B}(\boldsymbol{\tau}_h), v_h] = 0 \;\forall v_h \in M_h\}$, *define* $V_{1,h} := \{\mathbf{s}_h \in X_{1,h}\text{: } [\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h] = 0 \;\forall \boldsymbol{\tau}_h \in \tilde{M}_{1,h}\}$ *and let* $\Pi_{1,h} : X_{1,h}' \to V_{1,h}'$ *be the canonical imbedding*. *Further, let* $\mathcal{A}_{1,h} := p_h' \mathcal{A}_1 : X_1 \to X_{1,h}'$ *where* $p_h : X_{1,h} \to X_1$ *is the canonical injection with adjoint* $p_h' : X_1' \to X_{1,h}'$. *Assume that* *i*) *there exists* $\beta^* > 0$, *independent of the subspaces involved, such that for all* $v_h \in M_h$

$$\sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{v}_h]}{\|\boldsymbol{\tau}_h\|_{M_1}} \geq \beta^* \|v_h\|_M; \tag{6.5}$$

ii) *there exists* $\beta_1^* > 0$, *independent of the subspaces involved, such that for all* $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$

$$\sup_{\substack{\mathbf{s}_h \in X_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{\|\mathbf{s}_h\|_{X_1}} \geq \beta_1^* \|\boldsymbol{\tau}_h\|_{M_1}; \tag{6.6}$$

iii) *the nonlinear operator* $\mathcal{A}_{1,h} : X_1 \to X'_{1,h}$ *is Lipschitz-continuous, and for any* $\tilde{\mathbf{t}} \in X_{1,h}$, *the nonlinear operator* $\Pi_{1,h}\mathcal{A}_{1,h}(\cdot + \tilde{\mathbf{t}}) : V_{1,h} \to V'_{1,h}$ *is strongly monotone with a monotonicity constant* $\alpha_h > 0$ *independent of* $\tilde{\mathbf{t}}$.

Then, for each $(\mathcal{F}_1, \mathcal{G}_1, \mathcal{G}) \in X'_1 \times M'_1 \times M'$ *there exists a unique* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *solution of* (6.4).

Proof. See Theorem 3 in [23] (see also Theorem 3.2 in [22] or Theorem 4.2 in [14]). ∎

Finally, concerning the error analysis, we have the following result.

Theorem 6.3. *Assume that all the hypotheses of both Theorem* 6.1 *and Theorem* 6.2 *are satisfied, and let* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *and* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *be the unique solutions of* (6.1) *and* (6.4), *respectively. In addition, suppose that the family of nonlinear operators* $\{\Pi_{1,h}\mathcal{A}_{1,h}(\cdot + \tilde{\mathbf{t}})\colon \tilde{\mathbf{t}} \in X_{1,h}, h \in \mathbf{I}\}$ *is uniformly strongly monotone, i.e., there exists* $\alpha > 0$ *such that* $\alpha_h \geq \alpha$ *for all* $h \in \mathbf{I}$. *Then, there exists* $C > 0$, *depending only on* $\alpha, \gamma, \|\mathcal{B}_1\|, \beta_1^*, \|\mathcal{B}\|$ *and* β^*, *such that the following Cea error estimate holds for all* $h \in \mathbf{I}$:

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\| \leq C \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \\ \in X_{1,h} \times M_{1,h} \times M_h}} \|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{s}_h, \boldsymbol{\tau}_h, v_h)\|. \tag{6.7}$$

Proof. See Theorem 5 in [23] (see also Section 4 in [22]). ∎

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