Comput. Methods Appl. Mech. Engrg. 258 (2013) 109–117

Contents lists available at SciVerse ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

A finite element method for a three-field formulation of linear elasticity based on biorthogonal systems

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ARTICLE INFO

Article history:
Received 28 October 2012
Received in revised form 9 February 2013
Accepted 16 February 2013
Available online 27 February 2013

Keywords:
Elasticity
Three-field formulation
Biorthogonal system
Mixed finite elements
A priori estimate

ABSTRACT

We consider a mixed finite element method based on simplicial triangulations for a three-field formulation of linear elasticity. The three-field formulation is based on three unknowns: displacement, stress and strain. In order to obtain an efficient discretization scheme, we use a pair of finite element bases forming a biorthogonal system for the strain and stress. The biorthogonality relation allows us to statically condense out the strain and stress from the saddle-point system leading to a symmetric and positive-definite system. The strain and stress can be recovered in a post-processing step simply by inverting a diagonal matrix. Moreover, we show a uniform convergence of the finite element approximation in the incompressible limit. Numerical experiments are presented to support the theoretical results.

1. Introduction

It is well known that low-order finite elements with quadrilaterals, hexahedra or simplices constructed from a standard displacement-based formulation of a nearly incompressible elasticity problem exhibit poor performance, in the form of poor coarse-mesh approximations and the locking effect, in which they do not converge uniformly with respect to the Lamé parameter λ. Some relevant works in a substantial literature on the subject include [1,9,11,39].

An important approach for eliminating the locking effect is to use a mixed method. Mixed formulations are generally obtained by formulating a saddle-point problem with additional unknown variables. The linear elasticity problem can be formulated in mixed form in many different ways, see [15,10]. The resulting formulation not only provides an approach to alleviating the locking effect, but may also be used to compute accurately other variables of interest – sometimes called dual variables. These include the stress or pressure in elasticity, while for the Poisson equation the gradient of the solution may be of interest. In standard formulations these additional variables have to be obtained a posteriori by differentiation, potentially resulting in a loss of accuracy.

One of the most popular mixed formulations in elasticity is the Hellinger–Reissner formulation, which is based on the stress and displacement as unknown variables. A stable discretization of the stress-displacement formulation of the elasticity problem requires the construction of a compatible pair of finite element spaces: one for the space of stresses, which are symmetric tensor fields, and the other for the displacements, which are vector fields. The pair of finite element spaces should also be compatible in the sense that they satisfy a suitable inf–sup condition [15,11]. Such a compatible stable pair of finite element spaces using polynomial shape functions was first presented in [4] for triangles and in [2] for rectangles in the case of plane elasticity. It is interesting to note that 24 degrees of freedom are needed for the triangular case and 45 for the rectangular case. These elements are therefore expensive in the two-dimensional case, and their use in three-dimensional elasticity would be prohibitive.

Another popular mixed formulation of elasticity used to overcome the locking effect is the three-field formulation commonly known as the Hu–Washizu formulation [21,43], which was first introduced by Fraeijs de Veubeke [18]. In this formulation, the unknown variables are displacement, stress and strain. The formulation incorporates weak statements of the equation of equilibrium, the strain-displacement equation, and the elasticity relation. The Hu–Washizu formulation has been used frequently to obtain locking-free methods in linear and non-linear elasticity based on bilinear or trilinear finite elements on quadrilaterals or

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http://dx.doi.org/10.1016/j.cma.2013.02.008

hexahedra [40,39,20,22,23,37]. The mathematical analysis for well-posedness of the Hu–Washizu formulation for nearly-incompressible elasticity has been investigated in [31], where it has been shown that a modified version of of the Hu–Washizu formulation is more amenable to obtaining uniform convergence of the finite element approximation in the nearly incompressible regime. However, the analysis is restricted to a class of quadrilateral meshes. Many existing methods such as the assumed strain method, assumed stress method, mixed enhanced method, strain gap method, B-bar method, etc. have been shown in [16] to be special cases of the modified Hu–Washizu formulation.

The objective of this work is to present an extension of the finite element analysis of the three-field formulation to simplicial triangulations. We start with the stabilized Hu–Washizu formulation presented in [29], where the formulation is used to analyze the stability and convergence of the average nodal strain formulation. In this work we introduce a new discretization based on the use of a pair of finite element bases for the stress and strain that form a biorthogonal system. That is, each component of the strain is discretized by the standard linear finite element space, whereas the discrete space of stresses is spanned by basis functions which form a biorthogonal system with the standard finite element space. The biorthogonality relation is an important component of the formulation, inasmuch as it allows the strain and stress to be statically condensed out of the system. The static condensation leads to a reduced system, which is symmetric and positive-definite. The uniform convergence of the finite element solution is shown by using an analysis similar to that in [29]. However, in contrast to [29], uniform convergence is shown without assuming the full H^2-regularity of the solution. Moreover, we prove a priori error estimate for the stress and present a set of numerical results that illustrate the performance and properties of the proposed formulation.

A finite element method using a biorthogonal system for a displacement–pressure formulation of linear elasticity has been presented in [27,28,32]. Although the finite element approximation converges uniformly in the nearly incompressible case, stress cannot be directly computed in these formulations.

There are a few publications devoted to the analysis of enhanced strain techniques for simplicial elements. For example, the mixed enhanced formulation is extended to simplicial meshes in [42]. However, the formulation in [42] is derived using the mini element, requiring that the pressure variable be continuous. Similar enhanced strain methods are discussed in [34,5]. Recovering a displacement-based formulation is not so straightforward.

The structure of the rest of the paper is as follows. In the next section, we fix some notation and briefly recall the standard and the mixed formulations of linear elasticity. We introduce our finite element discretization in Section 3. Section 4 is devoted to the mathematical analysis of the discrete problem. In this section, we show that the finite element approximation converges optimally to the true solution without assuming the full H^2-regularity of the solution, and that convergence does not depend on the Lamé parameter λ. This proves that the method does not exhibit locking in the nearly incompressible regime. Section 5 is devoted to a number of numerical examples which illustrate the performance of the method. Finally, some conclusions are presented in Section 6.

2. Governing equations and weak formulation

Vector- and tensor- or matrix-valued functions will be written in boldface form. The scalar product of two tensors or matrices $\boldsymbol{d}$ and $\boldsymbol{e}$ will be denoted by $\boldsymbol{d} : \boldsymbol{e}$, and is given by $\boldsymbol{a} : \boldsymbol{b} = a_{ij}b_{ij}$, the summation convention on repeated indices being invoked.

We start with the boundary value problem of homogeneous and isotropic linear elastic body occupying a bounded domain Ω in $\mathbb{R}^d, d \in \{2,3\}$ with Lipschitz boundary Γ. Let $L^2(\Omega)$ be the set of all square-integrable functions in Ω, and $\boldsymbol{S} := \{\boldsymbol{d} \in L^2(\Omega)^{d\times d} : \boldsymbol{d}$ is symmetric$\}$ is the set of symmetric tensors in Ω with each component being square-integrable. For a prescribed body force $\boldsymbol{f} \in L^2(\Omega)^d$, the governing equilibrium equation in Ω is

$$-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f} \tag{1}$$

with $\boldsymbol{\sigma}$ being the symmetric Cauchy stress tensor.

The strain $\boldsymbol{d}$ is related to the displacement through the relation

$$\boldsymbol{d} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + [\nabla\boldsymbol{u}]^t), \tag{2}$$

in which $\boldsymbol{\varepsilon}$ is the infinitesimal strain.

Assuming isotropic linear elastic behavior the constitutive relation is given by

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{d} := \lambda(\mathrm{tr}\boldsymbol{d})\mathbf{1} + 2\mu\boldsymbol{d}, \tag{3}$$

where $\mathcal{C}$ denotes the fourth-order elasticity tensor, $\mathbf{1}$ is the identity tensor, and λ and μ are the Lamé parameters. We assume that the body occupying the domain Ω is homogeneous, and λ and μ are positive constants[1]. We focus on the problem of uniform approximation of finite element approximations in the incompressible limit, which corresponds to $\lambda \to \infty$. The inverse of (3) is given by

$$\boldsymbol{d} = \mathcal{C}^{-1}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\mu + d\lambda}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{1}\right).$$

We assume that the displacement satisfies the homogeneous Dirichlet boundary condition

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma. \tag{4}$$

Introducing the Sobolev space $\boldsymbol{V} := [H^1_0(\Omega)]^d$ of displacements with standard inner product $(\cdot,\cdot)_{1,\Omega}$, semi-norm $|\cdot|_{1,\Omega}$, and norm $\|\cdot\|_{1,\Omega}$, see, e.g., [13], we define the bilinear form $A(\cdot,\cdot)$ and the linear functional $\ell(\cdot)$ by

$$A : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}, \quad A(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})dx,$$

$$\ell : \boldsymbol{V} \to \mathbb{R}, \quad \ell(\boldsymbol{v}) := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx.$$

Then the standard weak form of the linear elasticity problem is as follows: given $\ell \in \boldsymbol{V}'$, find $\boldsymbol{u} \in \boldsymbol{V}$ that satisfies

$$A(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{5}$$

Here $A(\cdot,\cdot)$ is symmetric, continuous, and $\boldsymbol{V}$-elliptic due to Korn's inequality. Hence standard arguments can be used to show that (5) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}$. Furthermore, if the domain Ω is convex polygonal or polyhedral we have $\boldsymbol{u} \in [H^2(\Omega)]^d \cap \boldsymbol{V}$, and there exists a constant C independent of λ such that

$$\|\boldsymbol{u}\|_{2,\Omega} + \lambda\|\operatorname{div}\boldsymbol{u}\|_{1,\Omega} \leqslant C\|\ell\|_{0,\Omega}. \tag{6}$$

We refer to [14] for a proof of the a priori estimate (6) for two-dimensional linear elasticity and [25] for three-dimensional linear elasticity. In order to derive a suitable mixed formulation for the strain–displacement formulation of linear elasticity, we start with the following minimization problem. The variational formulation of the linear elastic problem with homogeneous Dirichlet boundary condition can be written as the following problem:

$$\min_{\substack{(\boldsymbol{u},\boldsymbol{d}) \in \boldsymbol{V}\times\boldsymbol{S} \\ \boldsymbol{d} = \boldsymbol{\epsilon}(\boldsymbol{u})}} \frac{1}{2}\int_\Omega \boldsymbol{d} : \mathcal{C}\boldsymbol{d}dx - \ell(\boldsymbol{u}). \tag{7}$$

[1] This assumption is somewhat stronger than required for the standard elasticity problem where one assumes $\mathcal{C}$ to be pointwise stable and hence $\mu > 0$ and $\lambda > -2/3\mu$. Our interest here, however, is the problem of quasi-incompressible ealasticity where $\lambda \to \infty$.

We write a weak variational equation for the relation between the strain and the displacement in terms of the Lagrange multiplier space $\boldsymbol{S}$ to obtain the saddle-point problem of the minimization problem (7), which is the standard Hu–Washizu formulation of linear elasticity. The Lagrange multiplier space $\boldsymbol{S}$ is precisely the space of stress in the standard Hu–Washizu formulation. As in [31], we restrict the space of stress to $\boldsymbol{S}_0$, where

$$\boldsymbol{S}_0 := \left\{\boldsymbol{\tau} \in \boldsymbol{S} | \int_\Omega \boldsymbol{\tau} : \mathbf{1} dx = 0\right\}. \tag{8}$$

The standard Hu–Washizu formulation is to find $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in \boldsymbol{V} \times \boldsymbol{S} \times \boldsymbol{S}_0$ such that

$$\begin{aligned} &\tilde{a}((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + b((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) = \ell(\boldsymbol{v}), \quad (\boldsymbol{v}, \boldsymbol{e}) \in \boldsymbol{V} \times \boldsymbol{S}, \\ &b((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in \boldsymbol{S}_0, \end{aligned} \tag{9}$$

where

$$\tilde{a}((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) = \int_\Omega \boldsymbol{d} : \mathcal{C}\boldsymbol{e} dx, \text{ and } b((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) = \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{d}) : \boldsymbol{\tau} dx.$$

The main difficulty here is that the bilinear form $\tilde{a}(\cdot, \cdot)$ is not elliptic on the whole space $\boldsymbol{V} \times \boldsymbol{S}$. This difficulty prevents us from using some simple finite element spaces for the displacement, stress and strain as we have to satisfy an inf–sup condition among the discrete spaces for the stress, strain and displacement, and coercivity of the bilinear from $\tilde{a}(\cdot, \cdot)$ on a suitable kernel space. This problem is well-known in the context of Mindlin–Reissner plate theory and the Darcy equation [3,35,7].

This provides a motivation for modifying the bilinear form $\tilde{a}(\cdot, \cdot)$ consistently by adding a stabilization term so that we obtain the ellipticity on the whole space $\boldsymbol{V} \times \boldsymbol{S}$. Such a modification for the Hu–Washizu formulation was first proposed in [29] to obtain an average nodal strain formulation. The modification of the bilinear form $\tilde{a}(\cdot, \cdot)$ is obtained by adding the term

$$\alpha \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{d}) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{e})\, dx$$

to the bilinear form $\tilde{a}(\cdot, \cdot)$ so that our modified saddle-point problem is to find $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in \boldsymbol{V} \times \boldsymbol{S} \times \boldsymbol{S}_0$ such that

$$\begin{aligned} &a((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + b((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) = \ell(\boldsymbol{v}), \quad (\boldsymbol{v}, \boldsymbol{e}) \in \boldsymbol{V} \times \boldsymbol{S}, \\ &b((u, \boldsymbol{d}), \boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in \boldsymbol{S}_0, \end{aligned} \tag{10}$$

where

$$a((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) = \int_\Omega \boldsymbol{d} : \mathcal{C}\boldsymbol{e}\, dx + \alpha \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{d}) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{e})\, dx,$$

$b(\cdot, \cdot)$ is defined as previously, and $\alpha > 0$ is a parameter. Since the stabilization term is consistent, the parameter $\alpha > 0$ can be arbitrary, can be utilized for accelerating the solver as in an augmented Lagrangian formulation [8].

3. Finite element discretization

We consider a quasi-uniform triangulation $\mathcal{T}_h$ of the polygonal or polyhedral domain Ω, where $\mathcal{T}_h$ consists of simplices, either triangles or tetrahedra. Making use of the standard linear finite element space $K_h \subset H^1(\Omega)$ defined on the triangulation $\mathcal{T}_h$, where

$$K_h := \{v \in C^0(\Omega) : v_{|_T} \in \mathcal{P}_1(T), T \in \mathcal{T}_h\}, \quad K_h^0 = K_h \cap H_0^1(\Omega)$$

and the space of bubble functions

$$B_h := \left\{b_T \in H^1(T) : b_{T|\partial T} = 0 \text{ and } \int_T b_T dx > 0, T \in \mathcal{T}_h\right\},$$

we introduce the finite element spaces for the strain and displacement as $\boldsymbol{S}_h := [K_h]^{d \times d}$, and $\boldsymbol{V}_h := [K_h^0 \oplus B_h]^d$. The bubble function on an element T can be defined by $b_T(x) = c_b \Pi_{i=1}^{d+1} \lambda_{T^i}(x)$, where $\lambda_{T^i}(x)$ are the barycentric coordinates of the element T associated with vertices x_{T^i} of $T, i = 1, \ldots, d+1$, and the constant c_b is computed in such a way that the value of the bubble function b_T is 1 at the barycenter of T.

Let $M_h \subset L^2(\Omega)$ be a piecewise polynomial space to be specified later which satisfies the following assumptions:

Assumption 1.

(i) $\dim M_h = \dim K_h$;
(ii) There is a constant $\beta > 0$ independent of the triangulation $\mathcal{T}_h$ such that

$$\|\phi_h\|_{L^2(\Omega)} \leqslant \beta \sup_{\mu_h \in M_h \setminus \{0\}} \frac{\int_\Omega \mu_h \phi_h dx}{\|\mu_h\|_{L^2(\Omega)}}, \quad \phi_h \in K_h; \tag{11}$$

These assumptions guarantee that the resulting system matrix is square and invertible. The discrete space for stresses is

$$\boldsymbol{M}_h = \left\{\boldsymbol{\tau}_h \in [M_h]^{d \times d} : \int_\Omega \boldsymbol{\tau}_h : \mathbf{1} dx = 0\right\} \subset \boldsymbol{S}_0.$$

Then the discrete saddle-point formulation is written as: find $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h) \in \boldsymbol{V}_h \times \boldsymbol{S}_h \times \boldsymbol{M}_h$ such that

$$\begin{aligned} &a((\boldsymbol{u}_h, \boldsymbol{d}_h), (\boldsymbol{v}_h, \boldsymbol{e}_h)) + b((\boldsymbol{v}_h, \boldsymbol{e}_h), \boldsymbol{\sigma}_h) = \ell(\boldsymbol{v}_h), \quad (\boldsymbol{v}_h, \boldsymbol{e}_h) \in \boldsymbol{V}_h \times \boldsymbol{S}_h, \\ &b((u_h, \boldsymbol{d}_h), \boldsymbol{\tau}_h) = 0, \quad \boldsymbol{\tau}_h \in \boldsymbol{M}_h. \end{aligned} \tag{12}$$

We are now going to specify the discrete space $\boldsymbol{M}_h$. Let $\{\phi_1, \cdots, \phi_N\}$ be the finite element basis of K_h where N is the of number of basis functions of K_h. Starting with the basis of K_h, we construct the space M_h spanned by the basis $\{\mu_1, \cdots, \mu_N\}$ so that the basis functions of K_h and M_h satisfy the biorthogonality relation

$$\int_\Omega \mu_i \phi_j dx = c_j \delta_{ij}, \quad c_j \neq 0, \ 1 \leqslant i, j \leqslant N, \tag{13}$$

where δ_{ij} is the Kronecker symbol, and c_j is a scaling factor which can be chosen to be proportional to the area of support of ϕ_j. Hence the sets of basis functions of K_h and M_h form a biorthogonal system. The basis functions of M_h can be constructed locally on the reference element $\hat{T}$ by inverting a local mass matrix so that basis functions of K_h and M_h have the same support.

The local basis functions of M_h for the reference triangle $\hat{T} := \{(x, y) : 0 \leqslant x, 0 \leqslant y, x + y \leqslant 1\}$ are

$$\hat{\mu}_1 := 3 - 4x - 4y, \quad \hat{\mu}_2 := 4x - 1, \text{ and } \hat{\mu}_3 := 4y - 1,$$

where the basis functions $\hat{\mu}_1, \hat{\mu}_2$ and $\hat{\mu}_3$ are associated with the three vertices $(0,0), (1,0)$ and $(0,1)$ of the reference triangle. Similarly, the local basis functions of M_h for the reference tetrahedron $\hat{T} := \{(x, y, z) : 0 \leqslant x, 0 \leqslant y, 0 \leqslant z, x + y + z \leqslant 1\}$ are

$$\hat{\mu}_1 := 4 - 5x - 5y - 5z, \quad \hat{\mu}_2 := 5x - 1, \quad \hat{\mu}_3 := 5y - 1, \text{ and } \hat{\mu}_4 := 5z - 1,$$

where the basis functions $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3$ and $\hat{\mu}_4$ are associated with the four vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,1)$ of the reference tetrahedron. The global basis functions of M_h are constructed by affinely transforming the local basis functions on the reference element to the physical element and 'glueing' them together exactly as in the construction of global basis functions of K_h from the local ones. However, the global basis functions of M_h are not continuous, see [24,26,30].

In order to present the algebraic formulation of the discrete saddle-point problem (12), we use the same notation for the vector representation of the solution and the solution as elements in $\boldsymbol{V}_h, \boldsymbol{S}_h$ and $\boldsymbol{M}_h$. Let $\mathtt{A}, \mathtt{B}, \mathtt{W}, \mathtt{K}, \mathtt{M}$ and $\mathtt{D}$ be the matrices associated with the bilinear forms $\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}_h):\boldsymbol{\varepsilon}(\boldsymbol{v}_h)dx$, $\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}_h):\boldsymbol{e}_h dx$, $\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}_h):\boldsymbol{\tau}_h dx$, $\int_\Omega \mathcal{C}\boldsymbol{e}_h:\boldsymbol{d}_h dx$, $\int_\Omega \boldsymbol{e}_h:\boldsymbol{d}_h dx$ and $\int_\Omega \boldsymbol{\tau}_h:\boldsymbol{d}_h dx$, respectively. The matrix $\mathtt{D}$ associated with the bilinear form $\int_\Omega \boldsymbol{\tau}_h:\boldsymbol{d}_h dx$ is also referred to as a Gram matrix. In the case of the saddle-point formulation, $\boldsymbol{u}_h, \boldsymbol{d}_h$ and $\boldsymbol{\sigma}_h$ are three independent unknowns. Then the algebraic formulation of the saddle-point problem (12) can be written as

$$\begin{bmatrix} \alpha\mathtt{A} & -\alpha\mathtt{B}^T & \mathtt{W}^T \\ -\alpha\mathtt{B} & \mathtt{K}+\alpha\mathtt{M} & -\mathtt{D}^T \\ \mathtt{W} & -\mathtt{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_h \\ \boldsymbol{d}_h \\ \boldsymbol{\sigma}_h \end{bmatrix} = \begin{bmatrix} \ell_h \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{14}$$

where ℓ_h is the vector form of the discretization of the linear form $\ell(\cdot)$. Since our goal is to obtain an efficient numerical scheme, we want to statically condense out the degrees of freedom associated with $\boldsymbol{\sigma}_h$ and $\boldsymbol{d}_h$. Here the matrix $\mathtt{D}$ is diagonal, so we can easily eliminate the degrees of freedom corresponding to $\boldsymbol{\sigma}_h$ and $\boldsymbol{d}_h$ arriving at the formulation involving only one unknown $\boldsymbol{u}_h$. After statically condensing out variables $\boldsymbol{d}_h$ and $\boldsymbol{\sigma}_h$, we arrive at a reduced system

$$\left(\alpha\mathtt{A} - \alpha(\mathtt{B}^T\mathtt{D}^{-1}\mathtt{W} + \mathtt{W}^T\mathtt{D}^{-1}\mathtt{B}) + \mathtt{W}^T\mathtt{D}^{-1}(\mathtt{K}+\alpha\mathtt{M})\mathtt{D}^{-1}\mathtt{W}\right)\boldsymbol{u}_h = \ell_h. \tag{15}$$

As the matrix $\mathtt{D}^{-1}$ is also diagonal, the condensed system matrix is sparse. Interestingly, we get a similar system matrix when we discretize a biharmonic problem using a biorthogonal system [30].

4. A priori error estimates

In order to show that the finite element approximation converges uniformly to the continuous solution, we introduce two quasi-projection operators: $Q_h : L^2(\Omega) \to K_h$, and $Q_h^* : L^2(\Omega) \to M_h$, which for $v \in L^2(\Omega)$ are defined by

$$\int_\Omega Q_h v \mu_h dx = \int_\Omega v\mu_h dx,\ v \in L^2(\Omega), \mu_h \in M_h \text{ and}$$
$$\int_\Omega Q_h^* v \varphi_h dx = \int_\Omega v\varphi_h dx,\ v \in L^2(\Omega), \varphi_h \in K_h. \tag{16}$$

These operators are introduced in [33] in the context of mortar finite elements, see also [38,6,26]. The definition of Q_h allows us to write the weak strain as

$$\boldsymbol{d}_h = Q_h(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)),$$

where the operator Q_h is applied to the tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}_h)$ componentwise. We see that Q_h and Q_h^* are well-defined due to Assumptions 1(i) and 1ii). Furthermore, Q_h is the identity operator if restricted to K_h, and Q_h^* is identity operator if restricted to M_h. Hence they are oblique projectors onto K_h and M_h, respetively, see [19,41] for more details on oblique projections. The biorthogonality relation between the basis functions of K_h and M_h allows to write the action of operator Q_h on a function $v \in L^2(\Omega)$ as

$$Q_h v = \sum_{i=1}^N \frac{\int_\Omega \mu_i v dx}{c_i}\varphi_i. \tag{17}$$

We find that Q_h^* and Q_h satisfy for $v, w \in L^2(\Omega)$

$$\int_\Omega Q_h v Q_h w dx = \int_\Omega Q_h^* Q_h v w dx. \tag{18}$$

In the following, we will use a generic constant C, which will take different values at different places but will be always independent of the mesh-size h and Lamé parameter λ. The stability of Q_h and Q_h^* in L^2-norm can be shown as in [24,26].

Lemma 1. *Under Assumption 1–(ii)*

$$\|Q_h v\|_{0,\Omega} \leqslant C\|v\|_{0,\Omega} \text{ and } \|Q_h^* v\|_{0,\Omega} \leqslant C\|v\|_{0,\Omega}, \quad v \in L^2(\Omega). \tag{19}$$

The following lemma establishes the approximation property of the operator Q_h and Q_h^* for a function $v \in H^{1+s}(\Omega), 0 < s \leqslant 1$, and $v \in H^s(\Omega), 0 < s \leqslant 1$, which can be proved as in [24,33,26].

Lemma 2. *For $0 < s \leqslant 1$, there exists a constant C independent of the mesh-size h such that*

$$\begin{aligned} &\|v - Q_h v\|_{0,\Omega} \leqslant Ch^{1+s}|v|_{s+1,\Omega}, \quad v \in H^{s+1}(\Omega), \text{ and} \\ &\|v - Q_h^* v\|_{0,\Omega} \leqslant Ch^s|v|_{s,\Omega}, \quad v \in H^s(\Omega). \end{aligned} \tag{20}$$

The biorthogonality relation allows us to eliminate the degrees of freedom corresponding to $\boldsymbol{d}_h$ and $\boldsymbol{\sigma}_h$. These variables can be obtained a posteriori by inverting a diagonal matirx.

Lemma 3. *Using the operator Q_h, we can eliminate the degrees of freedom corresponding to $\boldsymbol{d}_h$ and $\boldsymbol{\sigma}_h$ so that our problem is to find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that*

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{21}$$

where

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{d}_h : \mathcal{C}\boldsymbol{e}_h dx + \alpha\int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{d}_h) : (\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \boldsymbol{e}_h)dx$$

with $\boldsymbol{d}_h = Q_h(\boldsymbol{\varepsilon}(\boldsymbol{u}_h))$ and $\boldsymbol{e}_h = Q_h(\boldsymbol{\varepsilon}(\boldsymbol{v}_h))$.

Thus the expression for $A_h(\cdot,\cdot)$ is

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = &\int_\Omega Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h) \\ &: \mathcal{C}Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h)dx + \alpha\int_\Omega(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h)) \\ &: (\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h))dx. \end{aligned} \tag{22}$$

The following lemma proves the coercivity of the bilinear form $A_h(\cdot,\cdot)$ on $\boldsymbol{V}_h \times \boldsymbol{V}_h$. Its proof is similar to the proof of [29, Lemma 4.2].

Lemma 4. *The bilinear from $A_h(\cdot,\cdot)$ is coercive on $\boldsymbol{V}_h \times \boldsymbol{V}_h$ uniformly with respect to λ, i.e., there exists a constant C independent of λ such that*

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geqslant C\|\boldsymbol{u}_h\|_{1,\Omega}^2 \tag{23}$$

with

$$C = \frac{1}{\max\left(\frac{C_K}{\alpha}, \frac{C_K}{2\mu}\right)}.$$

Note that the bilinear form $A_h(\cdot,\cdot)$ is also coercive on $[K_h^0]^d \times [K_h^0]^d$. However, we need to use the space $\boldsymbol{V}_h = [K_h^0 \oplus B_h]^d$ for the displacement to obtain the uniform convergence of the finite element approximation in the incompressible limit.

Remark 2. We can see that the coercivity constant C depends on μ and α but not on λ. A smaller value of α decreases the coercivity constant in the previous lemma, and a larger value of α can pollute the approximation. Therefore, an appropriate choice of the parameter $\alpha > 0$ is necessary. As suggested by the previous lemma, one natural choice of the parameter α is 2μ [36]. For convenience α is set equal to 1 in the analysis that follows. The influence of α will be investigated numerically in Section 5.

Setting $\alpha = 1$ and $P_h = Q_h^* Q_h$, where P_h is an operator from $L^2(\Omega)$ to M_h, we have

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = &\int_\Omega P_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h) : \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)dx + \int_\Omega(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h)) \\ &: (\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h))dx. \end{aligned} \tag{24}$$

In the following, we show the uniform convergence of the finite element approximation in the incompressible limit. Although our approach here is similar to the one presented in [11,31,29], we focus on getting the approximation when the solution does not have full H^2-regularity. Introducing the pressure variable $p = \lambda \operatorname{div} \boldsymbol{u}$, the solution $\boldsymbol{u}$ of (5) is also a solution of the following saddle-point problem

$$\begin{aligned} &2\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{u}) dx + \int_\Omega \operatorname{div} \boldsymbol{v}\, p\, dx = \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}, \\ &\int_\Omega \operatorname{div} \boldsymbol{u} q dx - \frac{1}{\lambda} \int_\Omega p\, q\, dx = 0, \quad q \in L_0^2(\Omega), \end{aligned} \tag{25}$$

where $L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}$. We also need a subspace of S_h and a subspace of M_h having zero average on Ω defined as

$$S_h^0 := \left\{ \nu_h \in S_h : \int_\Omega \nu_h dx = 0 \right\}, \quad M_h^0 := \left\{ \eta_h \in M_h : \int_\Omega \eta_h dx = 0 \right\}.$$

A finite element discretization of the saddle-point problem (25) is to find $(\tilde{\boldsymbol{u}}_h, p_h) \in \boldsymbol{V}_h \times S_h^0$ such that

$$\begin{aligned} &2\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}_h) : \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_h) dx + \int_\Omega \operatorname{div} \boldsymbol{v}_h p_h dx = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h, \\ &\int_\Omega \operatorname{div} \tilde{\boldsymbol{u}}_h q_h dx - \frac{1}{\lambda} \int_\Omega p_h q_h dx = 0, \quad q_h \in M_h^0. \end{aligned} \tag{26}$$

Since the pair $(\boldsymbol{V}_h, M_h^0)$ satisfies a uniform inf–sup condition [27], i.e.,

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}_h \psi_h dx}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geqslant \beta \|\psi_h\|_{0,\Omega}, \quad \psi_h \in M_h^0, \tag{27}$$

we have the existence, uniqueness and stability of the discrete saddle-point problem (26), see [15,27].

Lemma 5. *The saddle-point problem (26) has a unique solution $(\tilde{\boldsymbol{u}}_h, p_h) \in \boldsymbol{V}_h \times S_h^0$, and*

$$\|\tilde{\boldsymbol{u}}_h\|_{1,\Omega} + \|p_h\|_{0,\Omega} \leqslant C \|\ell\|_{0,\Omega}.$$

Furthermore, if $(\boldsymbol{u}, p) \in \boldsymbol{V} \times L_0^2(\Omega)$ is the solution of (25) with $\boldsymbol{u} \in [H^{r+1}(\Omega)]^d$ and $p \in H^r(\Omega)$ for some $r > 0$, we have

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leqslant C h^r \left(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega} \right), \quad 0 < r \leqslant 1. \tag{28}$$

Since $p = \lambda \operatorname{div} \boldsymbol{u}$ and $p_h = \lambda Q_h \operatorname{div} \tilde{\boldsymbol{u}}_h$, we can write the finite element estimate (28) as

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_{1,\Omega} + \lambda \|\operatorname{div} \boldsymbol{u} - Q_h \operatorname{div} \tilde{\boldsymbol{u}}_h\|_{0,\Omega} \leqslant h^r \left(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega} \right). \tag{29}$$

For the uniform convergence of the finite element solutions of the statically condensed formulation (21) in the incompressible limit, we need the following lemma. Although the proof of this lemma is similar to [[29, Lemma 4.7], we have extended the proof to the case where we do not have full H^2-regularity of the solution.

Lemma 6. *Assume that $\boldsymbol{u}$ and $\boldsymbol{u}_h$ are the solutions of (5) and (21), respectively, with $\boldsymbol{u} \in [H^{1+r}(\Omega) \cap H_0^1(\Omega)]^d, 0 < r \leqslant 1$. Then there exists a $\boldsymbol{v}_h \in \boldsymbol{V}_h$ such that*

$$|A(\boldsymbol{w}_h, \boldsymbol{u}) - A_h(\boldsymbol{w}_h, \boldsymbol{v}_h)| \leqslant C |\boldsymbol{w}_h|_{1,\Omega} h^r (\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega}), \quad \boldsymbol{w}_h \in \boldsymbol{V}_h.$$

Proof. Using the expression of the bilinear form $A_h(\cdot,\cdot)$ from (24), a combination of the Cauchy–Schwarz and triangle inequalities yields

$$\begin{aligned} |A(\boldsymbol{w}_h, \boldsymbol{u}) - A_h(\boldsymbol{w}_h, \boldsymbol{v}_h)| &= \left| \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{w}_h) : (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{C}P_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)) dx \right. \\ &\quad \left. + \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)) : (\boldsymbol{\varepsilon}(\boldsymbol{w}_h) - Q_h \boldsymbol{\varepsilon}(\boldsymbol{w}_h)) dx \right| \\ &\leqslant C |\boldsymbol{w}_h|_{1,\Omega} \Big(\|\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_h \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} \\ &\quad + \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} \Big). \end{aligned} \tag{30}$$

Now we analyze the second term of the right-hand side of the last line of (30), *viz.*

$$\|\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_h \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} + \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}. \tag{31}$$

For the first term of (31), we have

$$\begin{aligned} \|\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_h \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} \leqslant 2\mu \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} + \lambda \|\operatorname{div} \boldsymbol{u} \\ - P_h \operatorname{div} \boldsymbol{v}_h\|_{0,\Omega} \end{aligned} \tag{32}$$

and for the second term of (31), we have

$$\begin{aligned} \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} \leqslant \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}) \\ - Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}. \end{aligned} \tag{33}$$

We use the triangle inequality, the approximation properties of Q_h and Q_h^* from Lemma 2 and the stability of Q_h and Q_h^* in L^2-norm from Lemma 1 to obtain

$$\begin{aligned} \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} &\leqslant \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_h \boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega} + \|P_h \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ &\quad - P_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} \\ &\leqslant C \left(h^r \|\boldsymbol{u}\|_{1+r,\Omega} + |\boldsymbol{u} - \boldsymbol{v}_h|_{1,\Omega} \right). \end{aligned} \tag{34}$$

Combining the estimates (30), (33) and (34), we can bound (31) according to

$$\begin{aligned} &\|\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_h \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} + \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} \\ &\quad \leqslant C \left(h^r \|\boldsymbol{u}\|_{1+r,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,\Omega} + \lambda \|\operatorname{div} \boldsymbol{u} - P_h \operatorname{div} \boldsymbol{v}_h\|_{0,\Omega} \right). \end{aligned} \tag{35}$$

The result now follows by choosing $\boldsymbol{v}_h$ to be the displacement solution of problem (26) and noting that

$$\begin{aligned} \|p - P_h \lambda \nabla \cdot \boldsymbol{v}_h\|_{0,\Omega} &\leqslant \|p - Q_h^* p\|_{0,\Omega} + \|Q_h^* p - P_h \lambda \nabla \cdot \boldsymbol{v}_h\|_{0,\Omega} \\ &\leqslant \|p - Q_h^* p\|_{0,\Omega} + \|p - \lambda Q_h \nabla \cdot \boldsymbol{v}_h\|_{0,\Omega}, \end{aligned}$$

where $p = \lambda \nabla \cdot \boldsymbol{u}$. $\square$

Now we formulate the main result of this section.

Theorem 3. *Assume that $\boldsymbol{u}$ and $\boldsymbol{u}_h$ are the solutions of (5) and (21), respectively, and $\boldsymbol{u} \in [H^{1+s}(\Omega) \cap H_0^1(\Omega)]^d$ with $s > 0$. Then the error in the displacement satisfies*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \leqslant C h^r \left(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega} \right), \quad r = \min\{1, s\}. \tag{36}$$

where $C < \infty$ is independent of λ and h, where $p = \lambda \operatorname{div} \boldsymbol{u}$.

Proof. The central point in the proof is to adapt the first lemma of Strang, see [10]. Let $\boldsymbol{v}_h$ be the displacement solution of (26). Using the coercivity of $A_h(\cdot,\cdot)$, Lemma 6 and (5), we find that

$$\begin{aligned} \|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{1,\Omega}^2 &\leqslant C |A_h(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{v}_h)| \\ &= C |A_h(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{u}_h) - A_h(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{v}_h)| \\ &= C |A(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{u}) - A_h(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{v}_h)| \\ &\leqslant C \|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{1,\Omega} h^r \left(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega} \right). \end{aligned}$$

The result follows from

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega} \leqslant \|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,\Omega} + \|\boldsymbol{v}_h-\boldsymbol{u}_h\|_{1,\Omega} \leqslant Ch^r\left(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega}\right). \qquad \square$$

Corollary 1. *Let Ω be a convex polygonal or polyhedral domain, and $\ell \in L^2(\Omega)$. We have the a priori error estimate for the discretization error in the displacement*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega} \leqslant Ch\|\ell\|_{0,\Omega}.$$

where $C < \infty$ is independent of λ and h.

Proof. Since $\boldsymbol{u} \in H^2(\Omega)$, we use $r = 1$ in (36) and use (6). $\square$

4.1. A priori error estimate for stress

Note that the second equation of (12) yields $\boldsymbol{d}_h = Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h)$. Putting $\boldsymbol{v}_h = \boldsymbol{0}$ in the first equation of (12) and utilizing $\boldsymbol{d}_h = Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h)$, we have the following formula for the stress

$$\boldsymbol{\sigma}_h = \alpha Q_h^*(Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h)) + \mathcal{C}Q_h^*Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h). \tag{37}$$

Using the triangle inequality we have the following bound for the first term of (37)

$$\alpha\|Q_h^*(Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h))\|_{0,\Omega} \leqslant \alpha\Big[\|Q_h^*Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}\Big],$$

which has the correct order of convergence to zero. Now we focus on estimating the term

$$\|\boldsymbol{\sigma}_h^1 - \boldsymbol{\sigma}\|_{0,\Omega},$$

where

$$\boldsymbol{\sigma}_h^1 = \mathcal{C}Q_h^*Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h) = \mathcal{C}P_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h) \text{ and } \boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}).$$

Using the definition of $\mathcal{C}$, we can write

$$\boldsymbol{\sigma}_h^1 = 2\mu P_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \lambda P_h(\nabla\cdot\boldsymbol{u}_h)\mathbf{1}.$$

Let $p_h = \lambda P_h(\nabla\cdot\boldsymbol{u}_h)$ and $p = \lambda\nabla\cdot\boldsymbol{u}$. Now using the triangle inequality

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^1\|_{0,\Omega} \leqslant 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega} + \|p\mathbf{1} - p_h\mathbf{1}\|_{0,\Omega}.$$

Since we have an optimal estimate for $\|\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega}$ independent of λ, the a priori estimate for the stress will be independent of λ, if we can show that the estimate for $\|p - p_h\|_{0,\Omega}$ does not depend on λ.

Since the displacement solution $\boldsymbol{u}_h$ satisfies

$$\int_\Omega P_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h) : \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)dx + \int_\Omega(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h)) : (\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h))dx = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

the pressure $p_h = \lambda P_h(\nabla\cdot\boldsymbol{u}_h)$ and the displacement $\boldsymbol{u}_h$ satisfy the variational equation

$$\tilde{a}(\boldsymbol{u}_h, \boldsymbol{v}_h) + \tilde{b}(\boldsymbol{v}_h, p_h) = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h, \tag{38}$$

where $\tilde{b}(\boldsymbol{v}_h, p_h) = \int_\Omega \nabla\cdot\boldsymbol{v}_h p_h dx$, and

$$\tilde{a}(\boldsymbol{u}_h, \boldsymbol{v}_h) = 2\mu\int_\Omega P_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h) : \boldsymbol{\varepsilon}(\boldsymbol{u}_h)dx + \int_\Omega(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - Q_h\boldsymbol{\varepsilon}(\boldsymbol{u}_h)) : (\boldsymbol{\varepsilon}(\boldsymbol{v}_h) - Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h))dx.$$

We know that the pair $(\boldsymbol{V}_h, M_h)$ satisfies a uniform inf–sup condition [27], and hence for $p_h - q_h \in M_h$

$$\|p_h - q_h\|_{0,\Omega} \leqslant C\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{\int_\Omega \operatorname{div}\boldsymbol{v}_h(p_h - q_h)dx}{\|\boldsymbol{v}_h\|_{1,\Omega}} = C\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{\int_\Omega\tilde{b}(\boldsymbol{v}_h, p_h - q_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}}.$$

Moreover, we have

$$\tilde{b}(\boldsymbol{v}_h, p_h - q_h) = \tilde{b}(\boldsymbol{v}_h, p_h - p + p - q_h) = \tilde{b}(\boldsymbol{v}_h, p_h - p) + \tilde{b}(\boldsymbol{v}_h, p - q_h).$$

Note that the exact displacement $\boldsymbol{u}$ and the pressure $p = \lambda\nabla\cdot\boldsymbol{u}$ satisfy the variational equation

$$\hat{a}(\boldsymbol{u}, \boldsymbol{v}_h) + \tilde{b}(\boldsymbol{v}_h, p) = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h, \tag{39}$$

where

$$\hat{a}(\boldsymbol{u}, \boldsymbol{v}) = 2\mu\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{u})dx.$$

Subtracting (39) from (38) and rearranging, we obtain

$$\tilde{b}(\boldsymbol{v}_h, p_h - p) = \hat{a}(\boldsymbol{u}, \boldsymbol{v}_h) - \tilde{a}(\boldsymbol{u}_h, \boldsymbol{v}_h),$$

so that

$$\tilde{b}(\boldsymbol{v}_h, p_h - p) + \tilde{b}(\boldsymbol{v}_h, p - q_h) = \hat{a}(\boldsymbol{u}, \boldsymbol{v}_h) - \tilde{a}(\boldsymbol{u}_h, \boldsymbol{v}_h) + \tilde{b}(\boldsymbol{v}_h, p - q_h).$$

Thus we have

$$\|p_h - q_h\|_{0,\Omega} \leqslant C\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{\int_\Omega\tilde{b}(\boldsymbol{v}_h, p_h - q_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} = C\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{\hat{a}(\boldsymbol{u}, \boldsymbol{v}_h) - \tilde{a}(\boldsymbol{u}_h, \boldsymbol{v}_h) + \tilde{b}(\boldsymbol{v}_h, p - q_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}}.$$

We can use the same analysis as in Lemma 6 to arrive at

$$\|p_h - q_h\|_{0,\Omega} \leqslant \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|p - q_h\|_{0,\Omega}.$$

Thus the final result follows from the triangle inequality

$$\|p - p_h\|_{0,\Omega} \leqslant \|p - q_h\|_{0,\Omega} + \|q_h - p_h\|_{0,\Omega}.$$

Note that the estimate does not depend on λ.

5. Numerical results

In this section we illustrate the performance and features of the proposed formulation via a series of numerical examples. The assumption of plane strain is made throughout.

5.1. Cook's membrane problem

Consider the tapered cantilever, fully fixed on the left edge and subjected to a shearing load $f = 100$ on the right edge, shown in Fig. 1. The linear elastic version of this problem, frequently referred to as the "Cook's membrane problem", is used to ascertain the performance of a formulation under bending-dominated deformation, see, e.g., [40,22]. The Cook's membrane problem is also a widely used benchmark for nearly incompressible elasticity. The domain of the problem Ω is the convex hull of the set $\{(0,0),(48,44),(48,60),(0,44)\}$. The material properties, chosen so as to reproduce a nearly-incompressible response, are $E = 250$ for the Young's modulus and $\nu = 0.4999$ for the Poisson ratio. The value of the parameter α is set at $\alpha = 2\mu$.

The vertical displacement of the upper-right vertex of the cantilever (point A) is used to assess the performance of the formulation in the quasi-incompressible regime. Fig. 1 shows the relationship between the vertical deflection of point A and the number of elements per side (a measure of the mesh size). The poor performance of the conventional linear P_1 element (denoted "standard formulation") in bending-dominated problems is

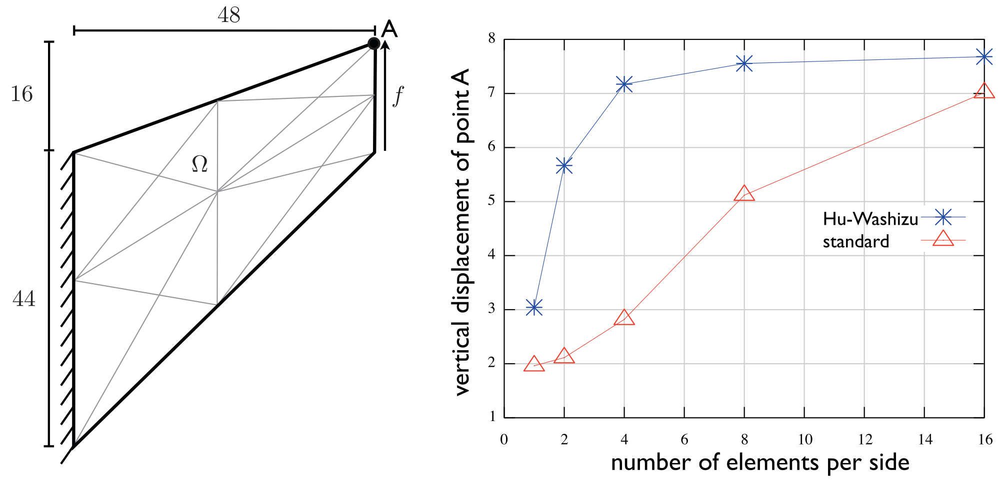

Fig. 1. Geometry of the Cook's membrane problem (left) and a comparison using the vertical displacement of point A, for various levels of spatial refinement, for both the Hu–Washizu-based and standard formulations (right).

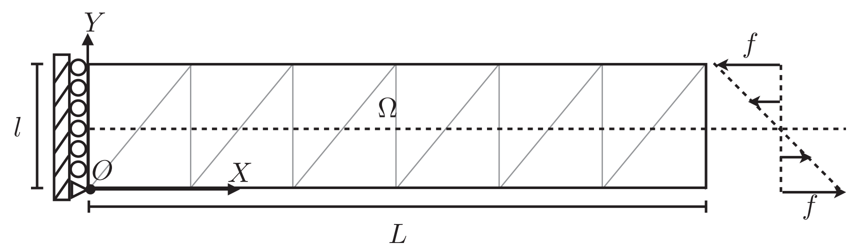

Fig. 2. Geometry of the rectangular beam problem.

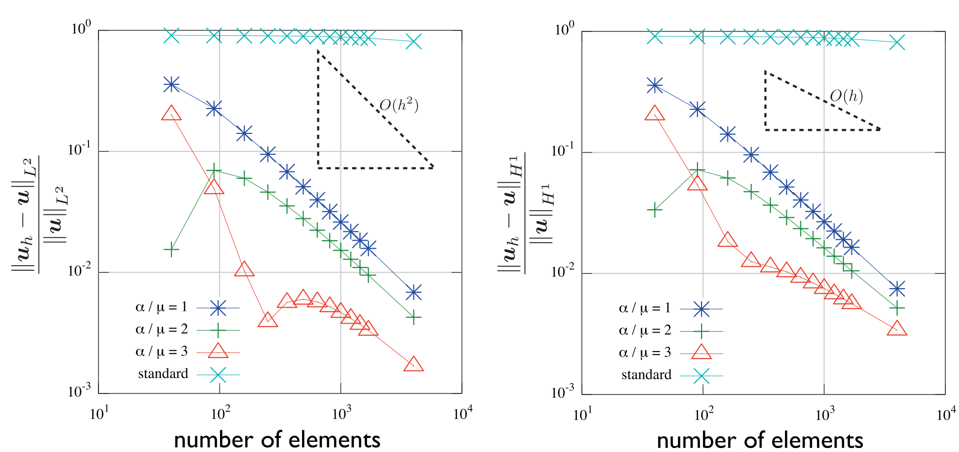

Fig. 3. The relative error versus the number of elements measured relative to the L^2 (left) and H^1 norms (right) for the rectangular beam problem.

responsible for the extremely slow convergence rate of the standard problem.

The Hu–Washizu-based formulation exhibits rapid convergence to the value reported in the literature, even for relatively coarse discretizations.

5.2. Rectangular beam

Consider the rectangular beam (length $L = 10$ and height $l = 2$) subjected to a couple ($f = 300$) as shown in Fig. 2. The left edge of the domain Ω is constrained to move only in the vertical direction Y. Furthermore, the material point located at the origin O is fully fixed. This problem is also a widely-used benchmark due to the availability of an exact solution, see, e.g., [17,28]. The material properties, chosen so as to reproduce a nearly-incompressible response, are $E = 1500$ for the Young's modulus and $\nu = 0.4999$ for the Poisson ratio. The value of the parameter α is varied between μ and 3μ to assess its influence.

The relative errors in the solution measured in the L^2 and H^1 norms are shown in Fig. 3. We can see that the standard formulation locks. By contrast, the Hu–Washizu-based formulation demonstrates near-optimal convergence for all values of α investigated. The parameter α influences the magnitude of the relative error but not the rate of convergence. The fluctuations in the relative error produced by the Hu–Washizu-based formulation for relatively coarse discretisations disappear beyond a certain refine-

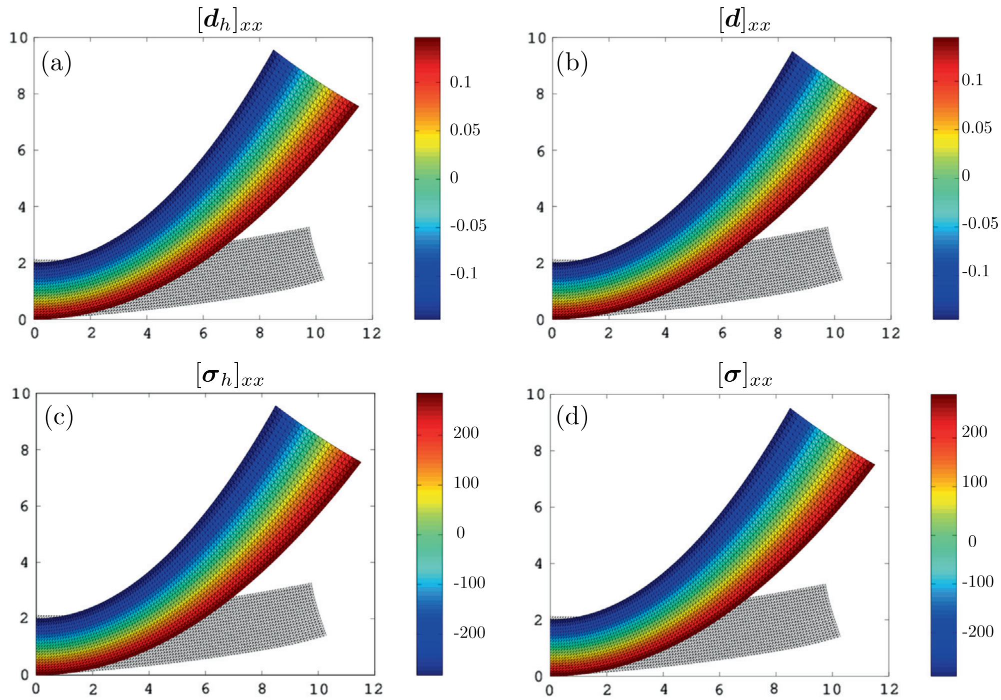

Fig. 4. Comparison of the Hu–Washizu-based approximation of the *xx*-component of the strain and stress, (a) and (c), with the exact solution, (b) and (d). The deformed configuration obtained using the standard formulation is also shown (the stress distribution is not shown for this case). The mesh contains 4000 elements and $\alpha = \mu$.

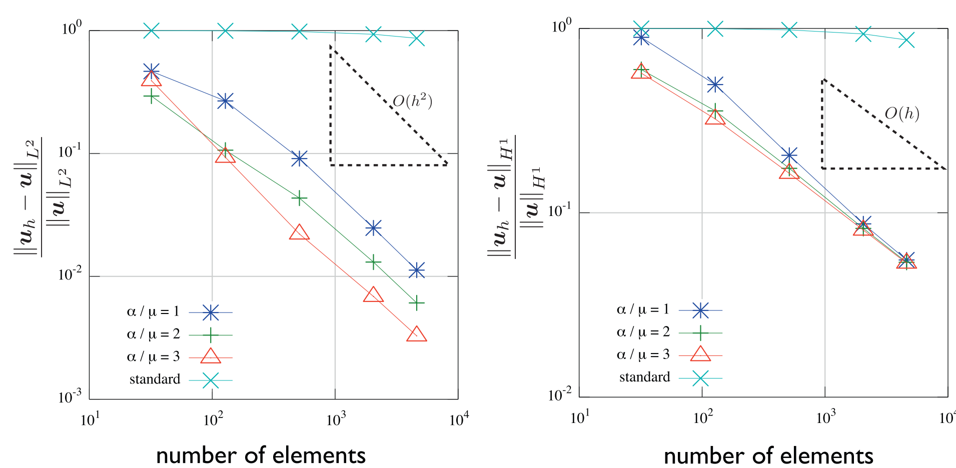

Fig. 5. The relative error versus the number of elements measured relative to the L^2 (left) and H^1-norms (right) for the square plate problem.

ment level. These initial fluctuations arise due to the interplay between the parameter α and the increasing number of degrees of freedom possessed by the discrete solution upon refinement.

The Hu–Washizu-based approximation of the stress and strain fields are compared to the exact solution in Fig. 4. It is clear that both approximations are in good agreement. The deformed configuration obtained using the standard formulation is also shown. It is clear that the standard formulation locks.

5.3. Square plate

Consider the benchmark problem analyzed in [12] of a unit square plate subject to a body force $\boldsymbol{f}$ given by

$$[\boldsymbol{f}]_1 = \beta(\pi^2(4\sin(2\pi y)(-1 + 2\cos(2\pi x)) - \cos(\pi(x+y)) + A \times \sin(\pi x)\sin(\pi y))), \tag{40}$$

$$[\boldsymbol{f}]_2 = \beta(\pi^2(4\sin(2\pi x)(1 - 2\cos(2\pi y)) - \cos(\pi(x+y)) + A \times \sin(\pi x)\sin(\pi y))), \tag{41}$$

where

$$A = \frac{2}{1+\lambda} \text{ and } \beta = \frac{1}{25}.$$

The domain is subject to homogeneous Dirichlet boundary conditions. The material properties are $\mu = 1$ and $\nu = 0.49995$. The exact solution $\boldsymbol{u}$ is given by

$$[\boldsymbol{u}]_1 = \beta(\sin(2\pi y)(-1 + \cos(2\pi x)) + B), \tag{42}$$
$$[\boldsymbol{u}]_2 = \beta(\sin(2\pi x)(1 - \cos(2\pi y)) + B), \tag{43}$$

where

$$B = 0.5A\sin(\pi x)\sin(\pi y).$$

The domain is discretized using an equal number of elements in the horizontal and vertical directions. The relative errors in the solution measured in the L^2- and H^1- norms are shown in Fig. 5. As in the previous examples, the standard formulation does not converge at all – it locks. By contrast, the Hu–Washizu-based formulation demonstrates optimal convergence for all values of α investigated.

6. Concluding remarks

We have presented a novel finite element method for a stabilized Hu–Washizu formulation for simplicial meshes. We have carefully chosen the finite element bases for the stress and strain so that they form a biorthogonal system. This choice allows us to statically condense out the stress and strain variables from the linear system leading to an efficient finite element method. The space of displacement is enriched with element-wise bubble functions in order to achieve the uniform convergence of the finite element approximation in the incompressible regime. An optimal a priori error estimate is proved, and numerical results are presented to demonstrate the efficiency and optimality of the approach.

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