ELSEVIER

Comput. Methods Appl. Mech. Engrg. 143 (1997) 41–47

Computer methods in applied mechanics and engineering

The quadrilateral 'Mini' finite element for the Stokes problem ☆

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Received 20 March 1995; revised 24 February 1996

Abstract

We propose two mixed finite element methods for the Stokes equations in two space dimensions. Starting from a Q_1–Q_1 combination for the velocity and pressure, it is studied how to introduce internal degrees of freedom into the velocity space in order to obtain a LBB-stable combination. The LBB stability conditions of the two proposed methods are checked by using the macroelement technique.

1. Introduction

The recent researches [1–3] reveal that the stabilized finite element methods (or Galerkin-least-squares methods) are equivalent to the standard Galerkin methods with some proper 'bubble functions'. For the triangular case, the 'Mini' element of Arnold–Brezzi–Fortin [4] plays the role. Also, it is observed that the 'Mini' element might be the simplest one among Taylor–Hood elements for the Stokes problem. A systematic numerical experiment had been carried out on various continuous pressure elements for studying incompressible flows in two dimensions of space, and the 'Mini' element appeared sufficiently accurate and efficient and the most susceptible to be generalized in the three-dimensional case [5]. But, we have not found its valuable quadrilateral counterpart in the literature except that a somewhat cumbersome bilinear continuous pressure element had been mentioned in the paper [6] which essentially added *three internal nodes* to the standard bilinear velocity finite element space in order to achieve a LBB-stable combination.

It is well known that the equal-order bilinear velocity-bilinear continuous pressure element—the Q_1–Q_1 element—exhibits a certain spurious pressure mode, see [7, Chapter 4] or [8, Chapter 6] for more details. In the paper we propose a new stabilized Q_1–Q_1 combination for the velocity and pressure with *three* internal degrees of freedom added to the velocity space, that is, one degree of freedom for each component of the velocity and one degree of freedom shared by both components of the velocity. Since it is very easily seen that by simply adding one internal node (two internal degrees of freedom) to the bilinear velocity space is not sufficient to obtain a LBB-stable combination even for a rectangular mesh, three degrees of freedom might be the least degrees of freedom needed to stabilize the Q_1–Q_1 combination. The element is referred to as the quadrilateral 'Mini' finite element. In addition, a new bilinear velocity-bilinear continuous pressure combination with *two* stabilizing internal nodes is proposed as a flexible alternative.

☆ Supported by Chinese Aeronautical Science Foundation.

PII 0045-7825(96)01146-2

In Section 2, we briefly review the macroelement technique to check the LBB stability condition. The two new methods are proposed and analyzed in Sections 3 and 4, respectively.

2. The macroelement technique

Let $V = H_0^1(\Omega)^2$, $P = L_0^2(\Omega)$ be the usual Sobolev spaces, where Ω is a bounded polygonal domain in R^2 with boundary Γ. The variational formulation of the Stokes problem reads:

Find a pair $(\boldsymbol{u}, p) \in V \times P$ such that

$$\begin{aligned} &\nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}) - (\operatorname{div} \boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle && \forall \boldsymbol{v} \in V , \\ &(\operatorname{div} \boldsymbol{u}, q) = 0 && \forall q \in P , \end{aligned} \tag{1}$$

where $\boldsymbol{u} = (u, v)$ is the velocity of the fluid, p is the pressure, $\nu > 0$ is the viscosity, and $\boldsymbol{f} = (f_1, f_2) \in H^{-1}(\Omega)^2$ is the given body force.

Let $V_h \subset V$ and $P_h \subset P$ be two finite element subspaces, the mixed finite element formulation of problem (1) reads:

Find a pair $(\boldsymbol{u}_h, p_h) \in V_h \times P_h$ such that

$$\begin{aligned} &\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}_h) - (\operatorname{div} \boldsymbol{v}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle && \forall \boldsymbol{v}_h \in V_h , \\ &(\operatorname{div} \boldsymbol{u}_h, q_h) = 0 && \forall q_h \in P_h . \end{aligned} \tag{2}$$

It is well known that in order to obtain a working method, the spaces V_h and P_h cannot be chosen arbitrarily. The method can be expected to behave well only if the following LBB stability condition is satisfied [8,9]:

$$\inf_{0 \neq p \in P_h} \sup_{0 \neq \boldsymbol{v} \in V_h} \frac{(\operatorname{div} \boldsymbol{v}, p)}{\|\boldsymbol{v}\|_{1,\Omega} \|p\|_{0,\Omega}} \geq C > 0 . \tag{3}$$

A practical criterion to check the LBB stability condition is the so-called macroelement technique [10,11].

Let $\mathcal{T}_h$ be a regular triangulation of $\bar{\Omega}$, let $\hat{\kappa}$ be the reference element, and for $\kappa \in \mathcal{T}_h$ denote by F_κ the affine, bilinear, or trilinear mapping from $\hat{\kappa}$ onto κ. Further, let $\hat{V}$ and $\hat{P}$ be two polynomial spaces defined on $\hat{\kappa}$. We now assume that V_h and P_h are defined as

$$\begin{aligned} V_h &= \{\boldsymbol{v} \in H_0^1(\Omega)^2 | \boldsymbol{v}(\boldsymbol{x}) = \hat{\boldsymbol{v}}(F_\kappa^{-1}(\boldsymbol{x})),\ \hat{\boldsymbol{v}} \in \hat{V},\ \kappa \in \mathcal{T}_h\} , \\ P_h &= \{p \in C(\Omega) \cap L_0^2(\Omega) | p(\boldsymbol{x}) = \hat{p}(F_\kappa^{-1}(\boldsymbol{x})),\ \hat{p} \in \hat{P},\ \kappa \in \mathcal{T}_h\} . \end{aligned} \tag{4}$$

By a macroelement M we define a connected set of elements of which the intersection of any two is either empty, a vertex, or an edge. Two macroelements M and $\tilde{M}$ are said to be equivalent if they can be mapped continuously onto each other [11]. For a macroelement M we define the spaces

$$\begin{aligned} V_{0,M} &= \{\boldsymbol{v} \in H_0^1(M)^2 | \boldsymbol{v}(\boldsymbol{x}) = \hat{\boldsymbol{v}}(F_\kappa^{-1}(\boldsymbol{x})),\ \hat{\boldsymbol{v}} \in \hat{V},\ \boldsymbol{x} \in \kappa,\ \kappa \in M\} , \\ P_M &= \{p \in C(M) | p(\boldsymbol{x}) = \hat{p}(F_\kappa^{-1}(\boldsymbol{x})),\ \hat{p} \in \hat{P},\ \boldsymbol{x} \in \kappa,\ \kappa \in M\} , \\ N_M &= \{p \in P_M | (\operatorname{div} \boldsymbol{v}, p)_M = 0,\ \boldsymbol{v} \in V_{0,M}\} . \end{aligned} \tag{5}$$

The macroelement technique is given by the following [11].

THEOREM 2.1. Suppose that there is a fixed set of equivalent classes $\mathscr{E}_i$, $i = 1, 2, \ldots, l$, of macroelements, a positive integer L, and a macroelement partitioning $\mathcal{M}_h$ such that

- *(M1) For each $M \in \mathscr{E}_i$, $i = 1, 2, \ldots, l$, the space N_M is one-dimensional, consisting of functions that are constant on M.*
- *(M2) Each $M \in \mathcal{M}_h$ belongs to one of the classes $\mathscr{E}_i$, $i = 1, 2, \ldots, l$.*
- *(M3) Each $\kappa \in \mathcal{T}_h$ is contained in at least one and not more than L macroelements of $\mathcal{M}_h$.*

(M4) Each $T \in \Gamma_h$ is contained in the interior of at least one and not more than L macroelements of $\mathcal{M}_h$, where Γ_h denotes the collection of edges, of the elements of $\mathcal{T}_h$, in the interior of Ω.
Then the LBB stability inequality (3) is valid.

REMARK 2.1. The macroelement technique is essentially a localized criterion. For other references relevant to the localized criteria, see [12,13] in which the identical approach was named the patch test of rank non-deficiency.

3. The quadrilateral 'Mini' finite element

Let $\mathcal{T}_h$ be a regular partitioning of $\bar{\Omega}$ into convex quadrilaterals with diameters bounded by h. For an arbitrary quadrilateral κ with vertices a_i, $1 \leq i \leq 4$, let $\hat{\kappa}$ be the reference square $[-1,1] \times [-1,1]$ in the (ξ, η) reference space, with vertices $\hat{a}_i$, $1 \leq i \leq 4$, as in Fig. 1.

Let F_κ be the bilinear mapping from $\hat{\kappa}$ onto κ, that is, $(x,y) = F_\kappa(\xi,\eta) = (F_\kappa^1, F_\kappa^2) = (\Sigma_{i=1}^4 x_i\hat{\phi}_i, \Sigma_{i=1}^4 y_i\hat{\phi}_i)$, where $\hat{\phi}_1 = (1-\xi)(1-\eta)/4$, $\hat{\phi}_2 = (1+\xi)(1-\eta)/4$, $\hat{\phi}_3 = (1+\xi)(1+\eta)/4$, $\hat{\phi}_4 = (1-\xi)(1+\eta)/4$.

With the above notations, we construct the following velocity–pressure finite element spaces:

$$\begin{aligned} V_h &= \{ \boldsymbol{v} \in H_0^1(\Omega)^2 | \boldsymbol{v}(\boldsymbol{x}) = \hat{\boldsymbol{v}}(F_\kappa^{-1}(\boldsymbol{x})),\ \hat{\boldsymbol{v}} \in Q_1^+(\hat{\kappa}),\ \kappa \in \mathcal{T}_h \}\,, \\ P_h &= \{ p \in C(\Omega) \cap L_0^2(\Omega) | p(\boldsymbol{x}) = \hat{p}(F_\kappa^{-1}(\boldsymbol{x})),\ \hat{p} \in Q_1(\hat{\kappa}),\ \kappa \in \mathcal{T}_h \}\,. \end{aligned} \tag{6}$$

where

$$\begin{aligned} Q_1^+(\hat{\kappa}) &= \left\{ \hat{\boldsymbol{v}} | \hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_* + \left[\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + w_0 \nabla \hat{\phi}_1 \right] \hat{b}_0,\ \hat{\boldsymbol{v}}_* \in Q_1(\hat{\kappa}),\ u_0, v_0, w_0 \in R \right\}, \\ Q_1(\hat{\kappa}) &= \operatorname{span}\{ \hat{\phi}_i,\ i = 1, \ldots, 4 \}\,, \end{aligned} \tag{7}$$

where $\hat{b}_0$ is the bubble function, $\hat{b}_0 = (1-\xi^2)(1-\eta^2)$.

A lemma is proposed and proved to support our LBB stability analysis.

LEMMA 3.1. For any $p \in \{ p \in C(\kappa) | p(\boldsymbol{x}) = \hat{p}(F_\kappa^{-1}(\boldsymbol{x})), \hat{p} \in Q_1(\hat{\kappa}), \boldsymbol{x} \in \kappa \}$, we have

$$\nabla p = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + c_0 \nabla \hat{\phi}_1\,, \tag{8}$$

where $\hat{\phi}_1 = (1-\xi)(1-\eta)/4$, c_1, c_2 and c_0 are constants dependent on $p_i = p(a_i)$ and a_i, $i = 1, \ldots, 4$.

PROOF. For any $p \in \{ p \in C(\kappa) | p(\boldsymbol{x}) = \hat{p}(F_\kappa^{-1}(\boldsymbol{x})), \hat{p} \in Q_1(\hat{\kappa}), \boldsymbol{x} \in \kappa \}$, we have

$$p(x,y) = \hat{p}(F_\kappa^{-1}(x,y)) = \hat{p}(\xi,\eta) = \sum_{i=1}^4 p_i \hat{\phi}_i\,. \tag{9}$$

Let us introduce the matrix–vector notations:

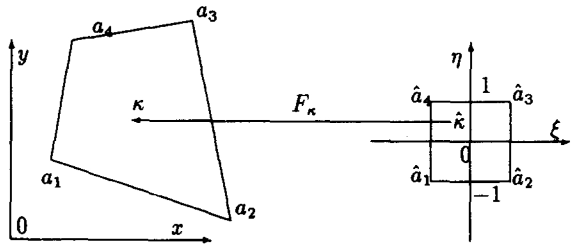

Fig. 1. Illustration of quadrilateral κ and its reference square $\hat{\kappa}$.

$$G_x = \begin{bmatrix} x_1 & \cdots & x_4 \\ y_1 & \cdots & y_4 \end{bmatrix}, \qquad G_f = \nabla_{\xi,\eta}[\hat{\phi}_i]_i \equiv \begin{bmatrix} \dfrac{\partial \hat{\phi}_1}{\partial \xi} & \cdots & \dfrac{\partial \hat{\phi}_4}{\partial \xi} \\ \dfrac{\partial \hat{\phi}_1}{\partial \eta} & \cdots & \dfrac{\partial \hat{\phi}_4}{\partial \eta} \end{bmatrix}, \tag{10}$$

$$p_v^{\mathrm{t}} = \{p_1, \ldots, p_4\}, \qquad A_{2,4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} := \begin{bmatrix} E_{1,4} \\ I_{1,4} \end{bmatrix}. \tag{11}$$

By virtue of

$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F_\kappa^1}{\partial \xi} & \dfrac{\partial F_\kappa^2}{\partial \xi} \\ \dfrac{\partial F_\kappa^1}{\partial \eta} & \dfrac{\partial F_\kappa^2}{\partial \eta} \end{bmatrix}^{-1} = (G_f \cdot G_x^{\mathrm{t}})^{-1},$$

we get

$$\nabla p = \begin{bmatrix} \dfrac{\partial \hat{p}}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \hat{p}}{\partial \eta}\dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \hat{p}}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial \hat{p}}{\partial \eta}\dfrac{\partial \eta}{\partial y} \end{bmatrix} = (G_f \cdot G_x^{\mathrm{t}})^{-1} \nabla_{\xi,\eta} \hat{p}. \tag{12}$$

Since the 4×4 matrix $[G_x^{\mathrm{t}}, A_{2,4}^{\mathrm{t}}]$ is rank-full due to the partition $\mathcal{T}_h$ being regular, there exists a constant vector $\boldsymbol{c} = (c_1, c_2, c_3, c_0)^{\mathrm{t}}$ such that

$$p_v = G_x^{\mathrm{t}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + c_3 E_{1,4}^{\mathrm{t}} + c_0 I_{1,4}^{\mathrm{t}}. \tag{13}$$

By virtue of Eq. (12), we further get

$$\nabla p = (G_f \cdot G_x^{\mathrm{t}})^{-1} \nabla_{\xi,\eta}(\hat{p} - c_3), \tag{14}$$

where (by virtue of Eq. (13))

$$\nabla_{\xi,\eta}(\hat{p} - c_3) = \nabla_{\xi,\eta} \sum_{i=1}^{4} (p_i - c_3)\hat{\phi}_i = G_f(p_v - c_3 E_{1,4}^{\mathrm{t}}) = G_f \left[G_x^{\mathrm{t}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + c_0 I_{1,4}^{\mathrm{t}} \right], \tag{15}$$

which implies (by virtue of Eq. (12)) that

$$\nabla p = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + c_0 (G_f \cdot G_x^{\mathrm{t}})^{-1} G_f I_{1,4}^{\mathrm{t}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + c_0 \nabla \hat{\phi}_1. \tag{16}$$

The proof is complete. □

We now show that the proposed method is LBB-stable by using the macroelement technique (Theorem 2.1).

THEOREM 3.1. For the Stokes problem (1), the velocity–pressure finite element formulation (6), namely the quadrilateral 'Mini' element, satisfies the LBB stability condition (3).

PROOF. Consider a one-element macroelement $M = \kappa$, as in Fig. 1. By virtue of the macroelement technique, it is sufficient to show that

$$N_M = \{p \in P_\kappa | (\operatorname{div} \boldsymbol{v}, p)_\kappa = 0, \ \boldsymbol{v} \in V_{0,\kappa}\} = \{\text{constants}\}, \tag{17}$$

where

$$\begin{aligned} V_{0,\kappa} &= \{\boldsymbol{v} \in H_0^1(\kappa)^2 | \boldsymbol{v}(\boldsymbol{x}) = \hat{\boldsymbol{v}}(F_\kappa^{-1}(\boldsymbol{x})), \hat{\boldsymbol{v}} \in Q_1^+(\hat{\kappa}), \boldsymbol{x} \in \kappa\}, \\ P_\kappa &= \{p \in C(\kappa) | p(\boldsymbol{x}) = \hat{p}(F_\kappa^{-1}(\boldsymbol{x})), \hat{p} \in Q_1(\hat{\kappa}), \boldsymbol{x} \in \kappa\}. \end{aligned} \tag{18}$$

For any $p \in P_\kappa$, by virtue of Lemma 3.1, we have

$$\nabla p = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + c_0 \nabla \hat{\phi}_1 . \tag{19}$$

If $(\operatorname{div} \boldsymbol{v}, p)_\kappa = 0$, $\forall\, \boldsymbol{v} \in V_{0,\kappa}$, let

$$\boldsymbol{v} = \left[\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + c_0 \nabla \hat{\phi}_1 \right] \hat{b}_0 \in V_{0,\kappa},$$

we have

$$0 = -(\operatorname{div} \boldsymbol{v}, p)_\kappa = (\boldsymbol{v}, \nabla p)_\kappa = \int_\kappa \boldsymbol{v} \cdot \nabla p \, \mathrm{d}x = \int_\kappa (\nabla p)^2 \cdot \hat{b}_0 \, \mathrm{d}x \tag{20}$$

i.e. $\nabla p = 0$, $p = \text{constant}$. The theorem follows directly from Theorem 2.1. □

4. An alternative approach

In this section a new bilinear velocity-bilinear continuous pressure combination with *two* stabilizing internal nodes (four internal degrees of freedom) is proposed as a flexible alternative. Again, the LBB stability of the proposed method is checked by using the macroelement technique.

Let the velocity–pressure finite element spaces be respectively:

$$\begin{aligned} V_h &= \{\boldsymbol{v} \in H_0^1(\Omega)^2 | \boldsymbol{v}(\boldsymbol{x}) = \hat{\boldsymbol{v}}(F_\kappa^{-1}(\boldsymbol{x})), \hat{\boldsymbol{v}} \in Q_1^{++}(\hat{\kappa}), \kappa \in \mathcal{T}_h\}, \\ P_h &= \{p \in C(\Omega) \cap L_0^2(\Omega) | p(\boldsymbol{x}) = \hat{p}(F_\kappa^{-1}(\boldsymbol{x})), \hat{p} \in Q_1(\hat{\kappa}), \kappa \in \mathcal{T}_h\}, \end{aligned} \tag{21}$$

where

$$Q_1^{++}(\hat{\kappa}) = \{\hat{\boldsymbol{v}} | \hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_* + \boldsymbol{v}_0 \hat{b}_0 + \boldsymbol{v}_{01}(c_1\xi + c_2\eta)\hat{b}_0, \hat{\boldsymbol{v}}_* \in Q_1(\hat{\kappa}), \boldsymbol{v}_0, \boldsymbol{v}_{01} \in R^2\}, \tag{22}$$

where $\boldsymbol{v}_0 = (u_0, v_0)$ and $\boldsymbol{v}_{01} = (u_{01}, v_{01})$ are the bubble unknowns, c_1 and c_2 are two arbitrary fixed constants satisfying $c_1^2 + c_2^2 \neq 0$, $\hat{\boldsymbol{v}}_* = \Sigma_{i=1}^4 \boldsymbol{v}_i \hat{\phi}_i$, $\boldsymbol{v}_i \in R^2$. In general, we should take $c_1 = c_2 = 1$, that is,

$$Q_1^{++}(\hat{\kappa}) = \{\hat{\boldsymbol{v}} | \hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}_* + \boldsymbol{v}_0 \hat{b}_0 + \boldsymbol{v}_{01}(\xi + \eta)\hat{b}_0, \hat{\boldsymbol{v}}_* \in Q_1(\hat{\kappa}), \boldsymbol{v}_0, \boldsymbol{v}_{01} \in R^2\}. \tag{23}$$

THEOREM 4.1. The velocity–pressure finite element formulation (21) for the Stokes problem (1), satisfies the LBB stability condition (3).

PROOF. Consider a one-element macroelement $M = \kappa$, as in Fig. 1. By virtue of the macroelement technique, it is sufficient to show that

$$N_M = \{p \in P_\kappa | (\operatorname{div} \boldsymbol{v}, p)_\kappa = 0, \boldsymbol{v} \in V_{0,\kappa}\} = \{\text{constants}\}, \tag{24}$$

where

$$\begin{aligned} V_{0,\kappa} &= \{\boldsymbol{v} \in H_0^1(\kappa)^2 | \boldsymbol{v}(\boldsymbol{x}) = \hat{\boldsymbol{v}}(F_\kappa^{-1}(\boldsymbol{x})), \hat{\boldsymbol{v}} \in Q_1^{++}(\hat{\kappa}), \boldsymbol{x} \in \kappa\}, \\ P_\kappa &= \{p \in C(\kappa) | p(\boldsymbol{x}) = \hat{p}(F_\kappa^{-1}(\boldsymbol{x})), \hat{p} \in Q_1(\hat{\kappa}), \boldsymbol{x} \in \kappa\}. \end{aligned} \tag{25}$$

Clearly, we have

$$\int_{\kappa} \boldsymbol{v} \cdot \nabla p \, \mathrm{d}x \, \mathrm{d}y = \int_{\hat{\kappa}} \boldsymbol{v} \cdot J^{-1} \cdot \nabla_{\xi,\eta} p |J| \, \mathrm{d}\xi \, \mathrm{d}\eta$$
$$= \int_{\hat{\kappa}} u \left(\frac{\partial y}{\partial \eta} \frac{\partial p}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial p}{\partial \eta} \right) \mathrm{d}\xi \, \mathrm{d}\eta + \int_{\hat{\kappa}} v \left(\frac{-\partial x}{\partial \eta} \frac{\partial p}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial p}{\partial \eta} \right) \mathrm{d}\xi \, \mathrm{d}\eta$$
$$= \sum_{i,j=1}^{4} p_i y_j \int_{\hat{\kappa}} u \left(\frac{\partial \hat{\phi}_j}{\partial \eta} \frac{\partial \hat{\phi}_i}{\partial \xi} - \frac{\partial \hat{\phi}_j}{\partial \xi} \frac{\partial \hat{\phi}_i}{\partial \eta} \right) \mathrm{d}\xi \, \mathrm{d}\eta$$
$$+ \sum_{i,j=1}^{4} p_i x_j \int_{\hat{\kappa}} v \left(- \frac{\partial \hat{\phi}_j}{\partial \eta} \frac{\partial \hat{\phi}_i}{\partial \xi} + \frac{\partial \hat{\phi}_j}{\partial \xi} \frac{\partial \hat{\phi}_i}{\partial \eta} \right) \mathrm{d}\xi \, \mathrm{d}\eta \,, \tag{26}$$

where

$$\nabla_{\xi,\eta} \equiv \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix}, \quad J = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}, \quad J^{-1} = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}. \tag{27}$$

For any $p \in P_{\kappa}$, let $\boldsymbol{v}_i = (u_i, v_i) = (0, 0)$, $1 \leq i \leq 4$, $\boldsymbol{v}_0 = (u_0, v_0) = (1, 0)$, $\boldsymbol{v}_{01} = (u_{01}, v_{01}) = (0, 0)$, i.e. $\boldsymbol{v} = \hat{\boldsymbol{v}} = (\hat{b}_0, 0) \in \boldsymbol{V}_{0,\kappa}$, by virtue of Eq. (26), after a trivial calculation we get

$$\int_{\kappa} \boldsymbol{v} \cdot \nabla p \, \mathrm{d}x \, \mathrm{d}y = \frac{2}{9} [(p_1 - p_3)(y_2 - y_4) + (p_2 - p_4)(y_3 - y_1)] \,. \tag{28}$$

Similarly, let $\boldsymbol{v}_i = (u_i, v_i) = (0, 0)$, $1 \leq i \leq 4$, $\boldsymbol{v}_0 = (u_0, v_0) = (0, 1)$, $\boldsymbol{v}_{01} = (u_{01}, v_{01}) = (0, 0)$, i.e. $\boldsymbol{v} = \hat{\boldsymbol{v}} = (0, \hat{b}_0) \in \boldsymbol{V}_{0,\kappa}$, we get

$$\int_{\kappa} \boldsymbol{v} \cdot \nabla p \, \mathrm{d}x \, \mathrm{d}y = -\frac{2}{9} [(p_1 - p_3)(x_2 - x_4) + (p_2 - p_4)(x_3 - x_1)] \,. \tag{29}$$

If $0 = -(\operatorname{div} \boldsymbol{v}, p)_{\kappa} = (\boldsymbol{v}, \nabla p)_{\kappa}$, $\forall \boldsymbol{v} \in \boldsymbol{V}_{0,\kappa}$, then

$$\begin{aligned} &(p_1 - p_3)(y_2 - y_4) + (p_2 - p_4)(y_3 - y_1) = 0 \,, \\ &(p_1 - p_3)(x_2 - x_4) + (p_2 - p_4)(x_3 - x_1) = 0 \,. \end{aligned} \tag{30}$$

The Jacobian determinant

$$|J| = \begin{vmatrix} y_2 - y_4 & y_3 - y_1 \\ x_2 - x_4 & x_3 - x_1 \end{vmatrix} \neq 0$$

($|J| = 0$ would imply that the two diagonals $a_2 - a_4$ and $a_3 - a_1$ are parallel) yields $p_1 = p_3 = a$, $p_2 = p_4 = b$.

Let $\boldsymbol{v}_i = (u_i, v_i) = (0, 0)$, $1 \leq i \leq 4$, $\boldsymbol{v}_0 = (u_0, v_0) = (0, 0)$, $\boldsymbol{v}_{01} = (u_{01}, v_{01}) = (1, 0)$, i.e. $\boldsymbol{v} = \hat{\boldsymbol{v}} = ((c_1 \xi + c_2 \eta)\hat{b}_0, 0) \in \boldsymbol{V}_{0,\kappa}$, by virtue of Eq. (26), after a trivial calculation we get

$$\begin{aligned} 0 &= -(\operatorname{div} \boldsymbol{v}, p)_{\kappa} = (\boldsymbol{v}, \nabla p)_{\kappa} \\ &= \frac{2}{45} \{c_1[p_1(y_4 - y_3) + p_2(y_3 - y_4) + p_3(y_1 - y_2) + p_4(y_2 - y_1)] \\ &\quad + c_2[p_1(y_3 - y_2) + p_2(y_1 - y_4) + p_3(y_4 - y_1) + p_4(y_2 - y_3)]\} \,, \end{aligned} \tag{31}$$

and owing to $p_1 = p_3 = a$, $p_2 = p_4 = b$, we further get

$$(a - b)(c_1 - c_2)(y_1 - y_3) + (a - b)(c_1 + c_2)(y_4 - y_2) = 0 \,. \tag{32}$$

Similarly, let $\boldsymbol{v}_i = (u_i, v_i) = (0, 0)$, $1 \leq i \leq 4$, $\boldsymbol{v}_0 = (u_0, v_0) = (0, 0)$, $\boldsymbol{v}_{01} = (u_{01}, v_{01}) = (0, 1)$, i.e. $\boldsymbol{v} = \hat{\boldsymbol{v}} = (0, (c_1 \xi + c_2 \eta)\hat{b}_0) \in \boldsymbol{V}_{0,\kappa}$, we get

$$(a - b)(c_1 - c_2)(x_1 - x_3) + (a - b)(c_1 + c_2)(x_4 - x_2) = 0 \,. \tag{33}$$

Since the Jacobian determinant

$$|J| = \begin{vmatrix} y_1 - y_3 & y_4 - y_2 \\ x_1 - x_3 & x_4 - x_2 \end{vmatrix} \neq 0$$

($|J| = 0$ would imply that the two diagonals $a_1 - a_3$ and $a_4 - a_2$ are parallel), we get

$$(a - b)(c_1 - c_2) = (a - b)(c_1 + c_2) = 0 \tag{34}$$

which implies that $a = b$ due to $c_1^2 + c_2^2 \neq 0$, i.e. $p_1 = p_2 = p_3 = p_4$, i.e. p is a constant on κ. The theorem follows directly from Theorem 2.1. □

REMARK 4.1. The degrees of freedom associated with the bubble functions can be eliminated by the static condensation. It is demonstrated that eliminating the additional degrees of freedom does not lead to a loss of accuracy, and it avoids the storage problem they generate [5]. This makes stabilization by bubbles much more affordable.

Acknowledgment

The author is most grateful to Professor Tian-Xiao Zhou for suggesting this study and for numerous fruitful discussions during the course of the work.

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