On the extension of mixed finite element PEERS for linear elasticity with a weakly imposed symmetry condition

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Abstract

We present an analysis of the extension of the lower order $PEERS$ element for the quadrilateral element. The approach is based on a modification of the Hellinger–Reissner functional in which the symmetry of the stress field is enforced weakly through the introduction of a Lagrange multiplier. It uses the lowest order RT_0 space for the stress, and piecewise constants for the displacement and bilinear for the rotation. We focus the attention of the different enrichment of stress field using different choise of bubble functions. In the case of no optimal convergence we correct the spaces of approximations as in such a way of ABF. More examples are developed to study the capacity of the different element in the case of the regular meshes such as in the distorted meshes.

Keywords: mixed finite elements, linear elasticity, Hellinger–Reissner, weak symmetry, quadrilateral element, $PEERS$

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1. Introduction

2. Reduced symmetry formulation

First of all, we recall the weak formulation of the elasticity system using the weak imposition of the symmetry of the stress tensor. We define $\mathbb{M}$ and $\mathbb{S}$ for the spaces of 2×2 matrices and symmetric matrices, respectively.

$$A\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\mu + 2\lambda}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right), \quad \boldsymbol{\sigma} \in \mathbb{M}, \tag{1}$$

First of all, we recall the equations governing the linear elastic problems. The classical theory of linear elasticity then requires that

$$\begin{aligned} \mathrm{div}\, \boldsymbol{\sigma} = \boldsymbol{f} \quad & \text{on } \Omega, \\ \boldsymbol{\varepsilon} = \nabla^{\mathrm{s}}\boldsymbol{u} \quad & \text{on } \Omega, \end{aligned} \tag{2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the strain tensor equal to the symmetric part of the displacement field, i.e.:

$$\nabla^{\mathrm{s}}\boldsymbol{u} = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) \\ \frac{1}{2}(u_{1,2} + u_{2,1}) & u_{2,2} \end{bmatrix}, \tag{3}$$

where $u_{i,j}$ indicate the derivative of the i component respect to the j direction. On the boundary we have

$$\begin{aligned} \boldsymbol{u} = \bar{\boldsymbol{u}} \quad & \text{on } \Gamma_D, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t} \quad & \text{on } \Gamma_N, \end{aligned} \tag{4}$$

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}, \tag{5}$$

where $\mathbb{C}$ is a fourth order tensor. The equation (5) can be recall in terms of the Lamé constant in the following:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}, \tag{6}$$

where μ and λ are the Lamé constants and $\mathrm{tr}(\boldsymbol{\varepsilon})$ is the trace of the strain tensor and $\boldsymbol{I}$ is the identity matrix.

Let $\Omega \in R^2$ and $\boldsymbol{f} \in L^2(\Omega)$

$$\begin{cases} \displaystyle\int_\Omega \boldsymbol{\varepsilon} : \boldsymbol{\tau} \,\mathrm{d}\Omega - \int_\Omega \nabla^{\mathrm{s}} \boldsymbol{u} : \boldsymbol{\tau} \,\mathrm{d}\Omega = 0 \,, \\ \displaystyle\int_\Omega \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v} \,\mathrm{d}\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\Omega \,, \end{cases} \tag{7}$$

$$\begin{cases} \displaystyle\int_\Omega \mathbb{C}^{-1} \boldsymbol{\sigma} : \boldsymbol{\tau} \,\mathrm{d}\Omega - \int_\Omega \nabla \boldsymbol{u} : \boldsymbol{\tau} \,\mathrm{d}\Omega + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau}^{\mathrm{e}} \,\mathrm{d}\Omega & = 0 \,, \\ \displaystyle\int_\Omega \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v} \,\mathrm{d}\Omega & = \displaystyle\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\Omega \,, \\ \displaystyle\int_\Omega \boldsymbol{\sigma}^{\mathrm{e}} : \boldsymbol{\delta} \,\mathrm{d}\Omega & = 0 \,, \end{cases} \tag{8}$$

$$\begin{cases} \displaystyle\int_\Omega \mathbb{C}^{-1} \boldsymbol{\sigma} : \boldsymbol{\tau} \,\mathrm{d}\Omega + \int_\Omega \boldsymbol{u} \cdot \mathrm{div}\ \boldsymbol{\tau} \,\mathrm{d}\Omega + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau}^{\mathrm{e}} \,\mathrm{d}\Omega & = \displaystyle\int_{\Gamma_D} [\boldsymbol{\tau} \cdot \boldsymbol{n}] \cdot \boldsymbol{u}_d \,\mathrm{d}\Gamma \,, \\ \displaystyle\int_\Omega \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v} \,\mathrm{d}\Omega & = \displaystyle\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\Omega \,, \\ \displaystyle\int_\Omega \boldsymbol{\sigma}^{\mathrm{e}} : \boldsymbol{\delta} \,\mathrm{d}\Omega & = 0 \,, \end{cases} \tag{9}$$

3. PEERSQ element

In this section we define the element derived by the PEERS with different enrichment of bubble functions.

3.1. PEERSQ: one bubble function

In the following we refer as **PEERS1B** for this element.

3.2. PEERSQ: two bubble functions

We define two types of element with two bubble functions: one using two standard bubbles (**PEERSQ2B**) and the second with a standard bubble plus a mixed bubble function (**PEERSQ2BM**).

3.3. PEERSQ: three bubbles

This element are defined in the **?** and its the first enrichments used.

4. Numerical example

In this section we report some examples using the presented formulation to proven the good behaviour. All examples are studied in nearly incompressible limit.

4.1. Square problem

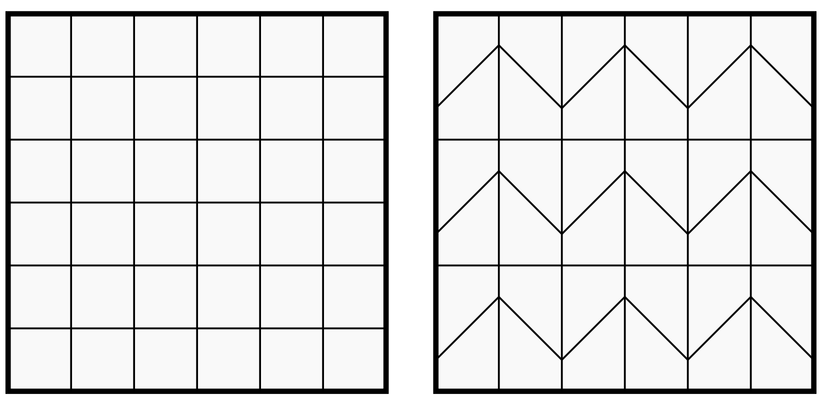
First example is a unit square domain with homogeneous Dirichlet boundary conditions and we the exact solution is

$$u_1 = \cos(\pi x)\sin(2\pi y), \quad u_2 = \sin(\pi x)\cos(\pi y)\,. \tag{10}$$

The Lamé constant are fix to $\lambda = 123$ and $\mu = 79.3$. By imposition of the previously exact solution one obtain for the body force f

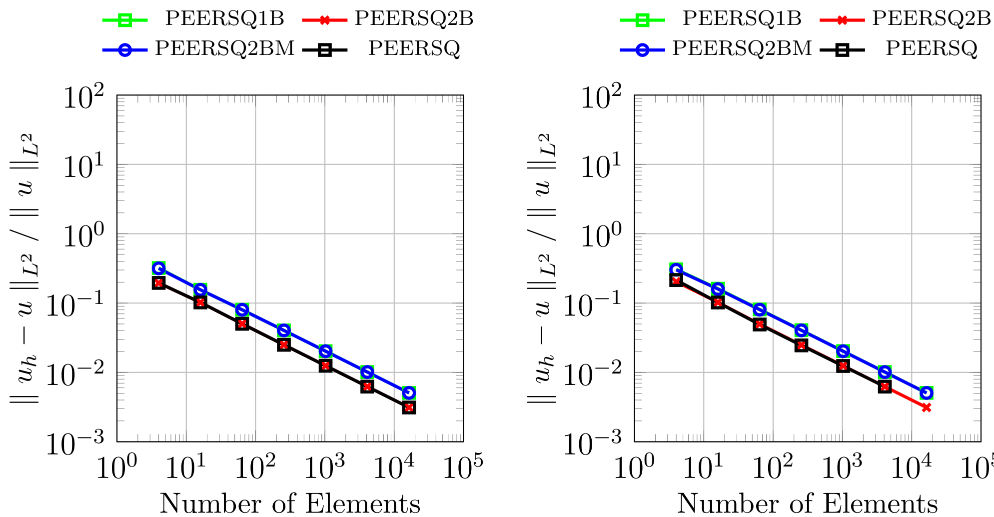
$$\begin{aligned} f_1 &= -\pi^2\cos(\pi x)\sin(\pi y)\left(\lambda+\mu+2\lambda\cos(\pi y)+12\mu\cos(\pi y)\right),\\ f_2 &= -\pi^2\sin(\pi x)\left(\lambda\cos(\pi y)+3\mu\cos(\pi y)+2\lambda\left(2\cos(\pi y)^2-1\right)+2\mu\left(2\cos(\pi y)^2-1\right)\right) \end{aligned} \tag{11}$$

The problem is study using two type of mesh, first of all using a square mesh and before using a trapezoidal mesh. The two different types of meshes are shown in Figures 2(a) and 2(b).



(a) Regular mesh (b) Trapezoidal mesh

Figura 1: Square Problem



(a) Regular mesh

(b) Trapezoidal mesh

Figura 2: Error in L^2–norm of square problem

4.2. Cantilever beam problem

Now we consider the beam with length $L = 10$ and height $l = 2$ as we shown in Figure 3. The Young modulus is set equal to $E = 1500$ and the Poisson $\nu = 0.4999$. The beam are fixed in the bottom left corner and subjected to a distributed load with $f = 300$ on the right edge as shown in Figure 3. We use to model the beam with two types of mesh: regular and trapezoidal as in the previous example (see Figures 2(a) and 2(b)).

4.3. Cook's membrane

The final example is the Cook's membrane. That is a typical benchmark and consist of a beam with vertex: $(0, 0)$, $(48, 44)$, $(48, 60)$ and $(0, 44)$. The left vertical edge is clamped and the right vertical edge is subjected to the vertical distributed forces with resultant $F = 100$ as shown in Figure 4. The material properties are taken to be $E = 250$ and $\nu = 0.4999$, so that a nearly incompressible response is obtained. We take into account the case of uniform meshes and the case of random distorted meshes (see Figure). We report in Figures 5(a) and 6(a), the vertical displacement of the point A versus the number of element per side for different PEERS and ABF elements in the case of regular mesh. In the case of random distorted mesh the same results

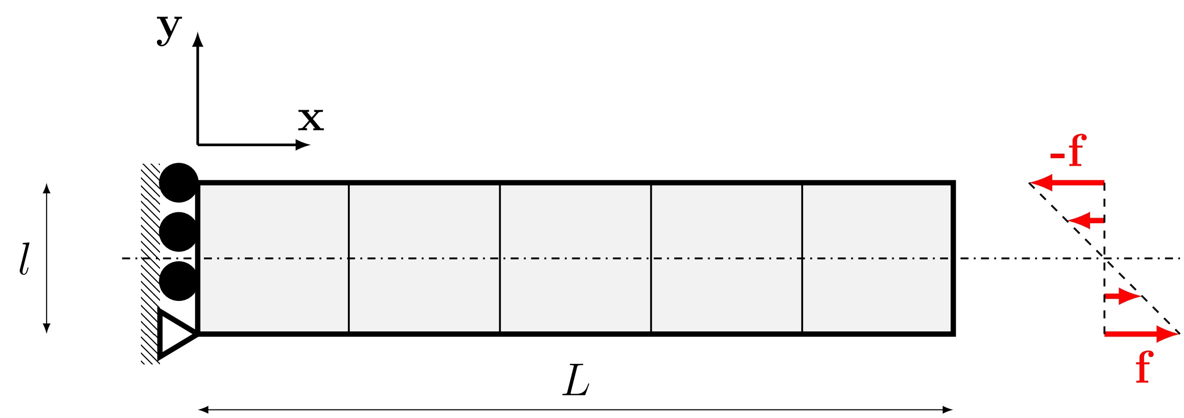


Figura 3: Cantilever Beam: Geometry problems

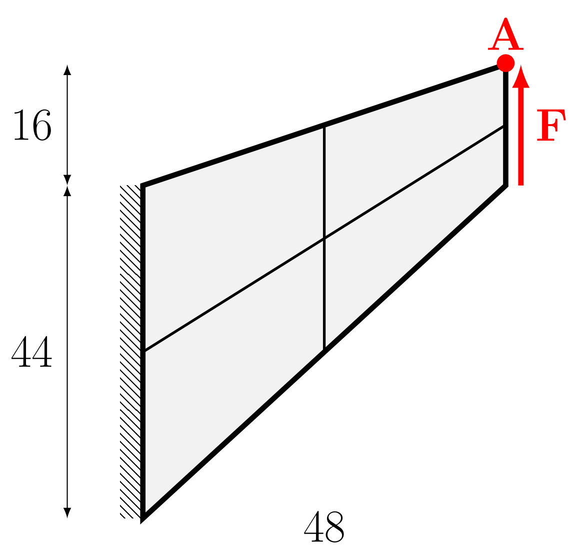
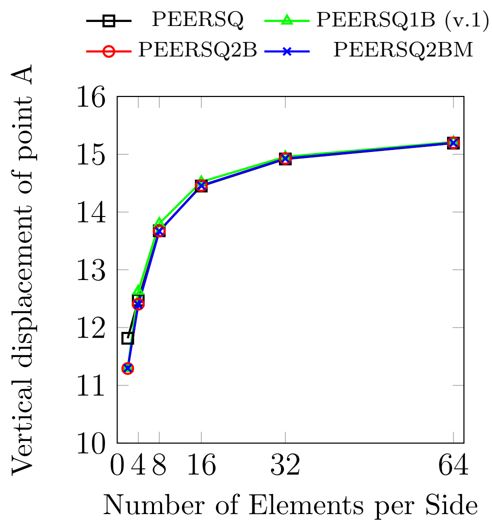


Figura 4: Cook's Membrane Geometry

are shown in Figures **??** and **??**. It can be observe that the standard solution obtained



(a) Regular mesh

Figura 5: Cook's Membrane (PEERSQ): Vertical Displacement of Point A vs. Element per Side

5. Conclusions

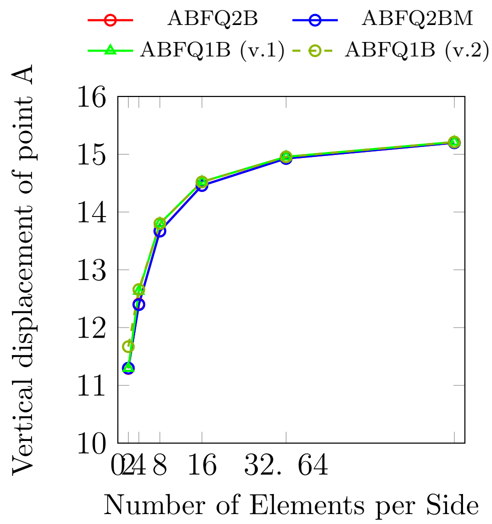
$$\| \operatorname{div}(\sigma_h - \sigma) \|_{L^2} = \left(\int_\Omega (\operatorname{div}(\sigma_h) - \operatorname{div}(\sigma))^2 \, d\Omega \right)^{1/2} \tag{12}$$

Ricordando che la divergenza:

$$\operatorname{div}(\sigma_h) = \frac{1}{det(J)} \operatorname{div}(\hat{\sigma}_h) \tag{13}$$

$$\| \operatorname{div}(\sigma_h - \sigma) \|_{L^2} = \left(\int_{\hat{\Omega}} \left(\frac{1}{det(J)} \operatorname{div}(\hat{\sigma}_h) - \operatorname{div}(\sigma) \right)^2 det(J) \, d\hat{\Omega} \right)^{1/2} \tag{14}$$

$$\| \operatorname{div}(\sigma_h - \sigma) \|_{L^2} = \left(\sum_{i=1}^{\#el} \sum_{j=1}^{nqp} \left(\frac{1}{det(J(\xi_j, \eta_j))} \operatorname{div}(\hat{\sigma}_h(\xi_j, \eta_j)) - \operatorname{div}(\sigma(x_j, y_j)) \right)^2 det(J(\xi_j, \eta_j)) \, w_j \right)^{1/2} \tag{15}$$



(a) Regular mesh

Figura 6: Cook's Membrane (ABF): Vertical Displacement of Point A vs. Element per Side

dove ξ_j, η_j sono i gauss points nell'elemento di referenza $[-1, 1] \times [-1, 1]$ e x_j, y_j sono i gauss points mappati nell'elemento fisico.

In modo analogo per gli spostamenti e per le pressioni:

$$\| u_h - u \|_{L^2} = \left(\sum_{i=1}^{\#el} \sum_{j=1}^{nqp} \left(u_h(\xi_j, \eta_j) - u(x_j, y_j) \right)^2 \; det(J(\xi_j, \eta_j)) \, w_j \right)^{1/2} \tag{16}$$

$$\| p_h - p \|_{L^2} = \left(\sum_{i=1}^{\#el} \sum_{j=1}^{nqp} \left(p_h(\xi_j, \eta_j) - p(x_j, y_j) \right)^2 \; det(J(\xi_j, \eta_j)) \, w_j \right)^{1/2} \tag{17}$$

infine per gli sforzi

$$\| \sigma_h - \sigma \|_{L^2} = \left(\sum_{i=1}^{\#el} \sum_{j=1}^{nqp} \left[(J(\xi_j, \eta_j)) \hat{\sigma}_h(\xi_j, \eta_j) / det(J(\xi_j, \eta_j)) - \sigma(x_j, y_j) \right]^2 \; det(J(\xi_j, \eta_j)) \, w_j \right)^{1/2} \tag{18}$$