Bead on a tilted wire

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Consider a bead of mass m (see figure below) sliding along a straight wire inclined at an angle θ with respect to the horizontal. The mass is attached to a spring of stiffness k and relaxed length L_0, and is also acted on by gravity. For simplicity, choose coordinates along the wire so that $x = 0$ occurs at the point closest to the support point of the spring; let a be the distance between this support point and the wire.

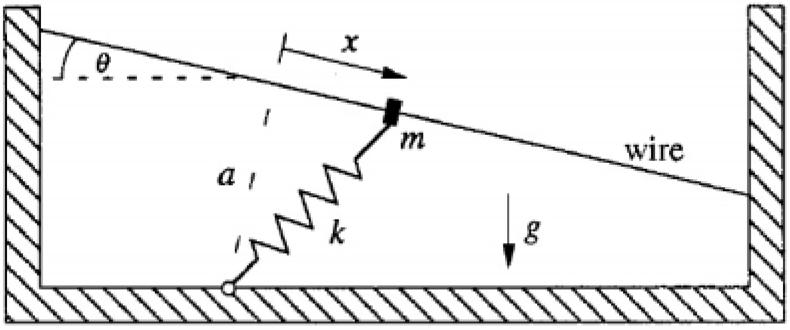

Figure 1: Illustration of the mechanical system under study.

In this mechanical situation, the actual distention of the spring is

$$\Delta L = \sqrt{x^2 + a^2} - L_0.$$

Denoting by ϕ the angle that between the spring and the wire (in the clockwise sense), we can decompose the forces that act on the bead as shown in the next figure.

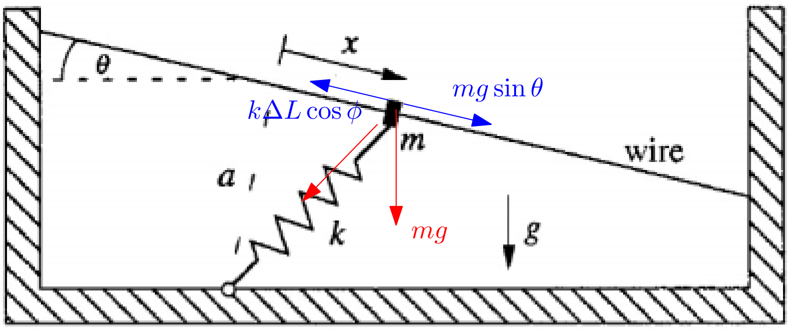

Figure 2: Diagram of the forces acting on the bead.

Since $\cos\phi = \dfrac{x}{\sqrt{x^2+a^2}}$, a straightforward application of Newton's second law yields:

$$m\ddot{x} = mg\sin\theta - kx\left(1 - \frac{L_0}{\sqrt{a^2+x^2}}\right). \tag{1}$$

Setting $\ddot{x} = 0$, the equilibrium points of the bead are found to respect the equation

$$mg\sin\theta = kx\left(1 - \frac{L_0}{\sqrt{a^2+x^2}}\right). \tag{2}$$

Next, notice that we can cast this equation in a dimensionless form as follows:

$$\frac{mg\sin\theta}{kx} = 1 - \frac{L_0}{a\sqrt{1+(x/a)^2}} \xrightarrow{u=x/a} \frac{mg\sin\theta}{kua} = 1 - \frac{L_0}{a\sqrt{1+u^2}}$$

Then, defining the dimensionless parameters $R = \dfrac{L_0}{a}$ and $h = \dfrac{mg\sin\theta}{ka}$, we get

$$1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}. \tag{3}$$

The number of fixed points can be guessed by a graphical analysis of this equation. Consider the functions $y_1(u) = 1 - \dfrac{h}{u}$ and $y_2(u) = \dfrac{R}{\sqrt{1+u^2}}$. First, we claim that, irrespective of the value of h and R, for $u > 0$ these functions always intercept once and just once.

Indeed: for $u > 0$, the graph of y_1 is a branch of a hyperbole that monotonically increases from $-\infty$ to the horizontal asymptote $y = 1$, whereas the function y_2 attains its maximum value, R, at $u = 0$ and monotonically decreases to zero as $u \to \infty$. Equivalently, if we define the function

$$g(u) = 1 - \frac{h}{u} - \frac{R}{\sqrt{1+u^2}},$$

it is easy to see that one can always choose a sufficiently small value of $u > 0$ for which $g(u) < 0$, and a sufficiently large one for which $g(u) > 0$. This guarantees that we have a solution of (3) for $u > 0$, and the monotonic behavior of the functions y_1 and y_2 assures that this solution is unique.

As for the branch $u < 0$, we need to separate the analysis into two cases: $R < 1$ and $R > 1$. In the former, there is no solution, since $y_1(u) > 1\ \forall u < 0$ and $y_2(u) \leq R < 1\ \forall u$. On the other hand, if $R > 1$ we can have none, one or two solutions, what will depend on the value of h. The figure below illustrate the representative cases.

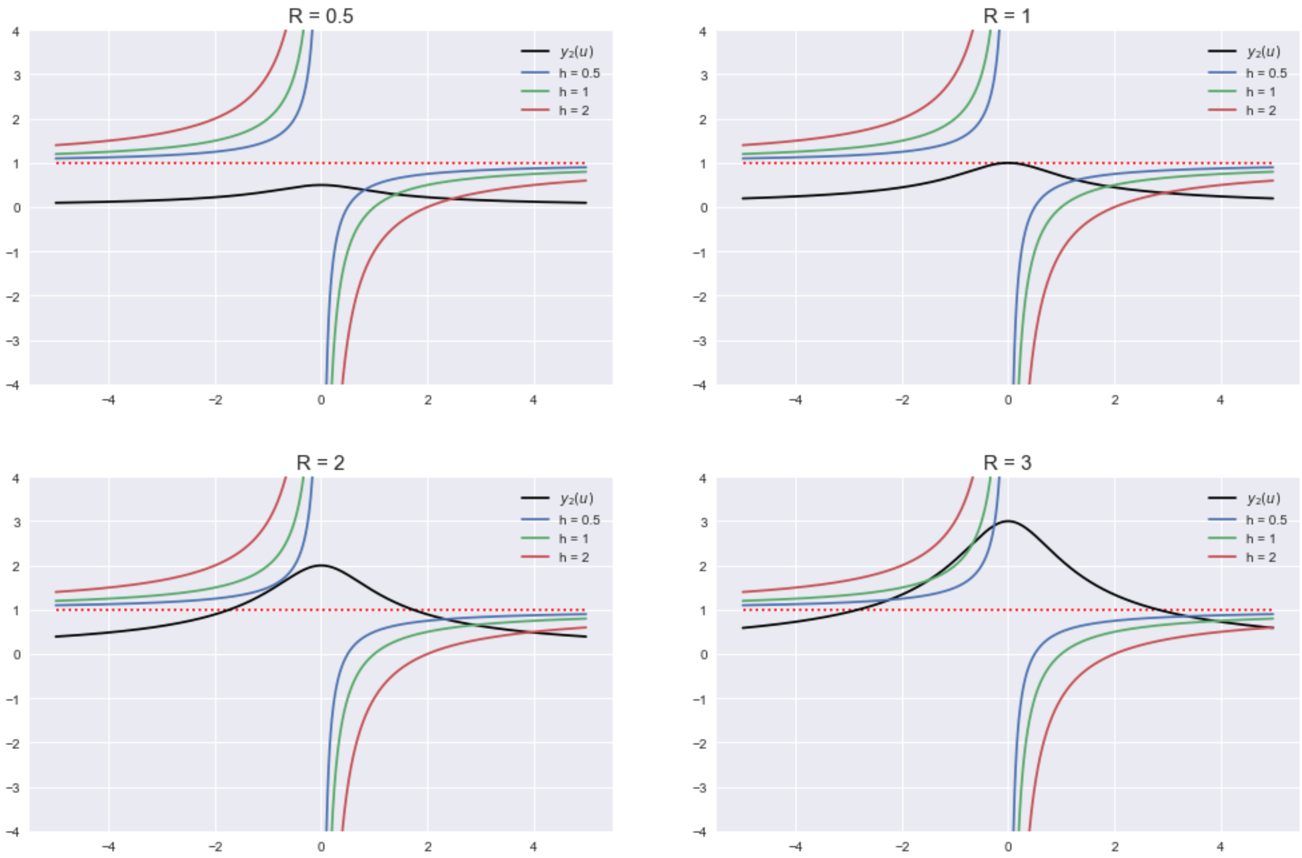

Figure 3: Representative cases for the determination of the number of fixed points.

In order to further analyze equation (3), we consider the case where $|u| \ll 1$, $h \ll 1$, and $r = R - 1 \ll 1$. Under these assumptions, we can write

$$1 - \frac{u}{h} = \frac{R}{\sqrt{1+u^2}} \implies u - h = Ru\left(1 - \frac{u}{2} + \mathcal{O}(u^4)\right)$$

$$\implies u - h = u + ru - \frac{ru^3}{2} - \frac{u^3}{2} + \mathcal{O}(u^5)$$

$$\implies h + ru - \frac{u^3}{2} + \mathcal{O}(4) = 0.$$

So, within this context equation (3) is well approximated by (up to order four in the outlined quantities):

$$h + ru - \frac{u^3}{2} = 0. \tag{4}$$

Notice that this equation is essentially the prototype of a catastrophe, by means of the addition of the *imperfection parameter* h to the normal form of a *supercritical pitchfork bifurcation*. Recall that the condition for the occurrence of saddle-node bifurcations is that the functions $y(u) = h$ and $y(u) = ru - \dfrac{u^3}{2}$ intercept tangentially. In this case:

$$\frac{d(ru - u^3/2)}{du} = 0 \implies r - \frac{3u^2}{2} = 0 \implies u = \pm\sqrt{\frac{2r}{3}}.$$

Returning to (4), we get the *bifurcation curves* in the (r, h) plane:

$$h_c(r) = \pm\frac{2r}{3}\sqrt{\frac{2r}{3}}. \tag{5}$$

Now, let us interpret what is going on here in physical terms. Since h was identified as the imperfection parameter, it is instructive to begin by considering the case $h = 0$, where the bifurcation diagram is that of a supercritical pitchfork bifurcation. Recalling the definition of such parameter, we see that this case corresponds to $\theta = 0$, i.e., a horizontal wire. In this scenario, $x = 0$ is always a equilibrium point; if $R = \dfrac{L_0}{a} < 1$, it means that the spring is strained and, thus, the equilibrium is stable. Notice that even with $\theta \neq 0$ the condition $R < 1$ just allows for the existence of one fixed (stable) point, as we can verify by looking at the plot with $R = 0,5$ in figure 3. However, if $R > 1$ the spring is compressed, so that $x = 0$ becomes a unstable equilibrium point. If the wire is horizontal, we expect two more stable equilibrium positions to appear in a symmetric disposition with respect to $x = 0$. Moreover, if the wire is slightly tilted ($h \ll 1$), we expect that this scenario will not change. Notice that this is indeed the case for $h = 0,01$ and $R = 3$, as shown in figure 1.

But what if we begin to tilt the wire more and more? By our physical intuition, we expect the upper equilibrium position (the one that appeared to the left of $x = 0$) to suddenly disappears, causing the bead to jump to a stable equilibrium position that is now located downhill. Such behavior is illustrated in the plots with $R = 2$ and $R = 3$. Notice that the values of $R > 1$ and $h > 0$ for which the predefined curves $y_1(u)$ and $y_2(u)$ intercept tangentially (the situation for $R = 2$ and $h = 0,5$ seems to be very close of it) correspond to the imminence of the occurrence of such effect ("catastrophe").

Bifurcation curves

As mentioned before, a saddle-node bifurcation is observed when the curves $y_1(u)$ and $y_2(u)$ intercept tangentially. Let us derive the exact expression of the bifurcation curves in the parameter space (R, h).

Taking the derivative with respect to u of equation (3), we get

$$\frac{h}{u^2} = -Ru^3(1 + u^2)^{-3/2}.$$

So, in the parametric form, the bifurcation curves are given by the expressions

$$h(u) = -u^3 \tag{6}$$

and

$$R(u) = (1 + u^2)^{3/2}. \tag{7}$$

Notice that for $u \ll 1$, equation (7) can be approximated by

$$R(u) \approx 1 + \frac{3u^2}{2} \implies r(u) = \frac{3u^2}{2}. \tag{8}$$

One can easily verify that (6) and (8) give the parametric form of the bifurcation curves expressed in equation (5).

Below, we show a numerically accurate plot of the exact bifurcation diagram of our system, comparing with the approximation in (5).

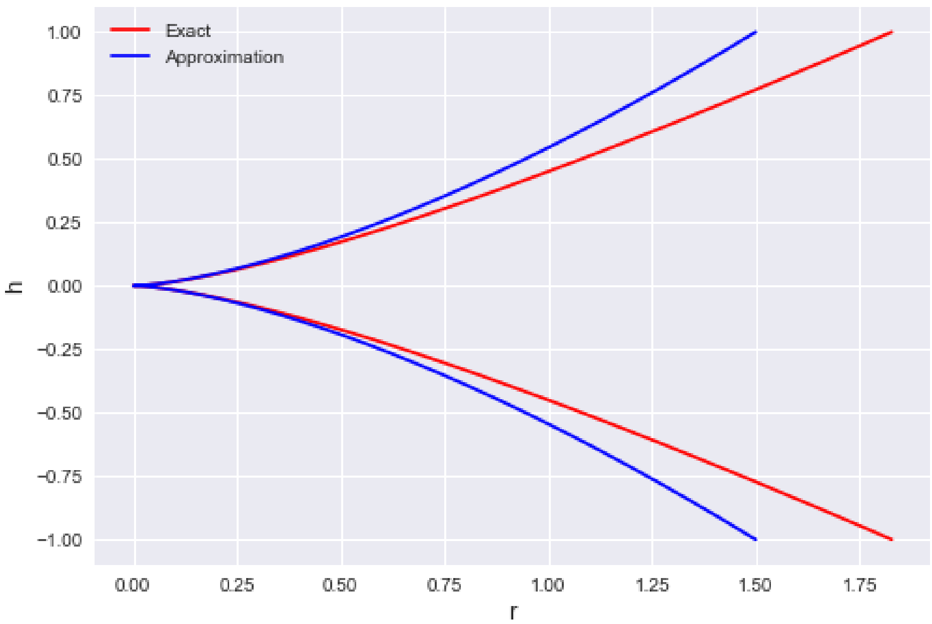

Figure 4: Bifurcation diagram in the (r, h) plane. Once again, we recall that $r = R - 1$. In red, we have the exact bifurcation curves, while the blue color is reserved for approximated curves of equation (5).

Just as expected, the approximation is very accurate in the region where $r \ll 1$ and $h \ll 1$.