Demostraciones

1. $I = \int_a^b f(x)\,dx$

$\Omega = \{(a, f(a)), (b, f(b))\}$

$$L_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

$$L_0(x) = \frac{x-b}{a-b}$$

$$L_1(x) = \frac{x-a}{b-a}$$

$$P(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}$$

$$I = \int_a^b p(x)\,dx = \int_a^b f(a)\frac{x-b}{a-b} + \int_a^b f(b)\frac{x-a}{b-a}\,dx$$

$$= \frac{f(a)}{a-b}\int_a^b x-b\,dx + \frac{f(b)}{b-a}\int_a^b x-a\,dx$$

$$= \frac{f(a)}{2(a-b)} \cdot (x-b)^2\Big|_a^b + \frac{f(b)}{2(b-a)} \cdot (x-a)^2\Big|_a^b$$

$$= -\frac{f(a)}{2(a-b)}(a-b)^2 + \frac{f(b)}{2(b-a)}(b-a)^2$$

$$= \frac{f(a)(b-a)}{2} + \frac{f(b)(b-a)}{2} = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

2. $\varepsilon = f(x) - p(x) = \dfrac{f^{n+1}\,\varepsilon_x}{(n+1)!}\,(x-x_0)(x-x_1)\dots(x-x_n)$ $\quad n=1$

$$\varepsilon = \frac{f''\,\varepsilon(x)}{2!}(x-a)(x-b) \qquad a \leq \varepsilon_x \leq b$$

$$\varepsilon = \frac{f''(\varepsilon_x)}{2}\left(x^2 - x(a+b) + ab\right)$$

$$\int_a^b \varepsilon\,dx = \frac{f''(\varepsilon_x)}{2}\int_a^b x^2 - x(a+b) + ab\,dx = \frac{f''\varepsilon_x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2}(a+b) + abx\Big|_a^b\right)$$

$$= \left(\frac{b^3}{3} - \frac{b^2}{2}(a+b) + ab^2 - \frac{a^3}{3} + \frac{a^2}{2}(a+b) - a^2b\right)\frac{f''\varepsilon_x}{2}$$

$$= \frac{2b^3 - 3b^2a - 3b^3 + 6ab^2 - 2a^3 + 3a^3 + 3a^2b - 6a^2b}{6} \cdot \frac{f''\varepsilon_x}{2}$$

$$= \frac{-b^3 + 3ab^2 - 3a^2b + a^3}{6} \cdot \frac{f''\varepsilon_x}{2}$$

$$= -\frac{(b-a)^3}{6} \cdot \frac{f''\varepsilon_x}{2} = -\frac{f''\varepsilon_x \cdot (b-a)^3}{12}$$

3. $I = \int_a^b f(x)\,dx$

$$V = \{(a, f(a)), (x_m, f(x_m)), (b, f(b))\} \qquad x_m = \frac{a+b}{2}$$

$$f(x) \approx p_2(x) = \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a) + \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) + \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)$$

$$\int_a^b p_2(x) = \frac{f(a)}{(a-b)(a-x_m)} \int_a^b (x-b)(x-x_m)\,dx + \frac{f(x_m)}{(x_m-a)(x_m-b)} \int_a^b (x-a)(x-b)\,dx + \frac{f(b)}{(b-a)(b-x_m)} \int_a^b (x-a)(x-x_m)\,dx$$

$$= \frac{f(a)}{(a-b)(a-x_m)} \int_a^b x^2 - x(b+x_m) + x_m b + \int_a^b \frac{f(x_m) \cdot (x^2 - x(b+a) + ab)}{(x_m-a)(x_m-b)}\,dx + \frac{f(b)}{(b-a)(b-x_m)} \int_a^b x^2 - x(a+x_m) + a x_m\,dx$$

$$= \frac{f(a)}{(a-b)(a-x_m)} \left(\frac{b^3}{3} - \frac{b^2}{2}(b+x_m) + x_m b^2 - \frac{a^3}{3} + \frac{a^2}{2}(b+x_m) - a x_m b \right) + \frac{f(x_m)}{(x_m-a)(x_m-b)} \cdot \frac{(b-a)^3}{(-6)}$$

$$+ \frac{f(b)}{(b-a)(b-x_m)} \cdot \left(\frac{b^3}{3} - \frac{b^2}{2}(a+x_m) + a x_m b - \frac{a^3}{3} + \frac{a^2}{2}(a+x_m) - x_m a^2 \right)$$

$$\to \frac{f(a)}{(a-b)(a-x_m)} \cdot \frac{-b^3+3x_m b^2-2a^3+3a^2b+3a^2x_m-6ax_mb}{6}$$

$$= \left('' \quad '' \right) \cdot \frac{-b^3+\frac{3(b+a)}{2}b^2-2a^3+3a^2b+3a^2\left(\frac{a+b}{2}\right)-6ab\left(\frac{a+b}{2}\right)}{6}$$

$$= \left('' \quad '' \right) \frac{-2b^3+3ab^2+3b^2-4a^3+6a^2b+3a^3+3a^2b-6a^2b-6ab^2}{2\cdot 6}$$

$$= \left('' \quad '' \right) \frac{(b-a)^3}{12}$$

$$\to \frac{f(b)}{(b-a)(b-x_m)} \cdot \frac{2b^3-3b^2a-3b^2x_m+a^3+3a^2x_m-6a^2x_m+6abx_m}{6}$$

$$\left('' \quad '' \right) \cdot \frac{4b^3-6b^2a-3b^2(a+b)+2a^3+3a^2(a+b)-6a^2(a+b)+6ab(a+b)}{12}$$

$$= \left('' \quad '' \right) \frac{4b^3-\cancel{6b^2a}-3b^2a-3b^3+2a^3+3a^3+3a^2b-6a^3-\cancel{6a^2b}+\cancel{6a^2b}+\cancel{6ab^2}}{12}$$

$$= \frac{b^3-3b^2a-a^3+3a^2b}{12} = \frac{(b-a)^3}{12}$$

$$\int p_x = \frac{f(a)}{(a-b)(a-x_m)}\frac{(b-a)^3}{12} - \frac{f(x_m)}{(x_m-a)(x_m-b)}\frac{(b-a)^3}{6} + \frac{f(b)}{(b-a)(b-x_m)}\frac{(b-a)^3}{12}$$

$$= \frac{f(a)}{(a-b)\left(\frac{a-b}{2}\right)}\frac{(b-a)^3}{12} - \frac{(b-a)^3}{6}\cdot\frac{f(x_m)}{\frac{(b-a)}{2}\frac{(a-b)}{2}} + \frac{f(b)}{(b-a)\left(\frac{b-a}{2}\right)}\left(\frac{b-a}{12}\right)^3$$

$$f(a)\frac{(b-a)^3}{6(a-b)^2} + \frac{4}{6}\frac{f(x_m)}{(b-a)^2}(b-a)^3 + \frac{f(b)\,(b-a)^3}{(b-a)^2\,6}$$

$$= f(a)\frac{b-a}{6} + \frac{4f(x_m)(b-a)}{6} + \frac{f(b)\,(b-a)}{6} \qquad h = \frac{b-a}{2}$$

$$= \frac{f(a)\,h}{3} + \frac{4f(x_m)\,h}{3} + \frac{f(b)\,h}{3} = \frac{h}{3}\left(f(a)+4f(x_m)+f(b)\right)$$