Matrix equilibrium states

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July 23, 2019

Equilibrium/Gibbs states

Let $\Sigma = \{0, \dots, M-1\}$ and $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ be the shift map.

- "Scalar" equilibrium/Gibbs states. Let $\varphi : \Sigma^{\mathbb{Z}} \to \mathbb{R}$.
  - Equilibrium: Maximize
    $$h_\mu(\sigma) + \int \varphi d\mu$$
    over σ invariant measures.
  - Gibbs: There exist constants $C > 0$ and P such that
    $$C^{-1} e^{-nP + S_n\varphi(x)} \le \mu_\varphi[x_0 x_1 \cdots x_{n-1}] \le C e^{-nP + S_n\varphi(x)}$$
    for all $x \in \Sigma^{\mathbb{Z}}$.
- "Matrix" equilibrium/Gibbs states.
  - $\varphi : \Sigma^{\mathbb{Z}} \to \mathbb{R}$ is replaced by $\mathcal{A} : \Sigma^{\mathbb{Z}} \to M_d(\mathbb{R})$.
  - $S_n\varphi(x)$ is replaced by $\log \|\mathcal{A}(\sigma^{n-1}x) \cdots \mathcal{A}(\sigma x)\mathcal{A}(x)\|$.

Equilibrium/Gibbs states

Let $\mathcal{A} : \Sigma^{\mathbb{Z}} \to M_d(\mathbb{R})$ depend only on 1-coordinate,
$\mathcal{A} = (A_0, A_1, \ldots, A_{M-1})$ and $\beta \in \mathbb{R}$.

- Equilibrium: Maximize

$$h_\mu(\sigma) + \beta \underbrace{\lim_{n\to\infty} \frac{1}{n} \int \log \|A_{x_{n-1}} \cdots A_{x_1} A_{x_0}\| \, d\mu}_{\Lambda(\mathcal{A},\mu)}$$

  over σ invariant measures.

- Gibbs: There exist constants $C > 0$ and P such that

$$C^{-1} e^{-nP} \|A_{x_{n-1}} \cdots A_{x_0}\|^\beta \le \mu_{\mathcal{A},\beta}[x_0 \cdots x_{n-1}] \le C e^{-nP} \|A_{x_{n-1}} \cdots A_{x_0}\|^\beta$$

  for all $x \in \Sigma^{\mathbb{Z}}$.

Comment: P is the maximum of $h_\mu(\sigma) + \beta\Lambda(\mathcal{A}, \mu)$, $\beta \mapsto P(\beta)$ is called the (matrix) pressure function.

Some reasons to care about MES

- They appear "in the wild"
  - Hidden Markov measures.
  - The Kusuoka measure.
- You care about their "cousin": equilibrium states for the singular value potential.
  - Applications to self-affine fractals.
- You care about the function $P(\beta)$.
  - $$\frac{dP}{d\beta}\Big|_{\beta=0} = \lambda$$

    where λ is maximal Lyapunov exponent with respect to the uniform Bernoulli measure.
  - Provided you take some assumptions on $\mathcal{A}$ and $\lambda = 0$

$$\frac{\log \|A_{x_{n-1}} \cdots A_{x_1} A_{x_0}\|}{\sqrt{n}} \xrightarrow{\text{dist}} \mathcal{N}(0, \sigma^2) \text{ and } \frac{d^2 P}{d\beta^2}\Big|_{\beta=0} = \sigma^2.$$

Some questions about MES

- Existence and uniqueness:
  - Provided the collection $\mathcal{A}$ does not preserve a proper and non-trivial subspace and $\beta > 0$ matrix equilibrium states are unique and satisfy the Gibbs inequality. (Feng and Käenmäki '11)
- Ergodic and statistical properties:
  - Mixing? If yes can we find a mixing rate?
  - Isomorphic to a Bernoulli shift?
- Equilibrium states for (regular enough) scalar potentials are known to have very nice properties to what extent is the same true for matrix equilibrium states?

Constructing matrix Gibbs states

- **Sketch:** Assume $\mathcal{A}$ is irreducible and $\beta > 0$. Define

$$\nu_n[x_0 \cdots x_{n-1}] = \frac{\|A_{x_{n-1}} \cdots A_{x_1} A_{x_0}\|^\beta}{\sum_{y_0 \cdots y_{n-1}} \|A_{y_{n-1}} \cdots A_{x_1} A_{x_0}\|^\beta}$$

take a weak* limit point ν. Satisfies Gibbs inequality but might not be shift invariant so average it

$$\frac{1}{N} \sum_{k=0}^{N-1} \sigma_*^k \nu$$

Take a weak* limit point μ which is the matrix Gibbs state for $(\mathcal{A}, \beta)$.

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- If you use this construction you have to rely exclusively on the Gibbs inequality to prove ergodic properties.
- We need a new construction.

1-Gibbs states for non-negative matrices

Set $\mathcal{A} = (A_0, A_1)$ where $A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

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$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ notice } A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, A^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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The formula

$$\mu_{\mathcal{A},1}[x_0 x_1 \cdots x_{n-1}] = 3^{-n} \begin{bmatrix} 1 & 1 \end{bmatrix} A_{x_{n-1}} \cdots A_{x_1} A_{x_0} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

extends to a measure on $\{0,1\}^{\mathbb{Z}}$ by Kolmogorov consistency.

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extends to a measure on $\{0,1\}^{\mathbb{Z}}$ by Kolmogorov consistency. This is the 1-Gibbs state for $\mathcal{A}$ as

$$\begin{bmatrix} 1 & 1 \end{bmatrix} A_{x_{n-1}} \cdots A_{x_1} A_{x_0} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \approx \| A_{x_{n-1}} \cdots A_{x_1} A_{x_0} \|_1$$

moreover we can see that $P = \log 3$.

1-Gibbs states for non-negative matrices

We can deduce mixing properties from spectral properties of $A = A_0 + A_1$.

$$\mu_{\mathcal{A},1}([i_0 \cdots i_{m-1}] \cap \sigma^{-n}[j_0 \cdots j_{m'-1}])$$

$$= 3^{-(m+m'+n-m)} \sum_{k_0 \cdots k_{n-m-1}} \left\langle \underbrace{A_{j_{m'}} \cdots A_{j_0}}_{A_J} \underbrace{A_{k_{n-m-1}} \cdots A_{k_0}}_{A_K} \underbrace{A_{i_m} \cdots A_{i_0}}_{A_I} [1/2], [1] \right\rangle$$

$$= 3^{-(m+m')} \left\langle A_J 3^{-(n-m)} A^{n-m} A_I [1/2], [1] \right\rangle \text{ because } \sum_{|K|=n-m} A_K = A^{n-m}$$

$$\xrightarrow{n\to\infty} 3^{-m} \langle A_I[1/2], [1] \rangle \, 3^{-m'} \langle A_J[1/2], [1] \rangle \text{ because } A \text{ is positive}$$

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Question: How do we deal with $\beta \neq 1$? Matrices which are not non-negative?

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Trick: Consider the action of the matrices on norm one vectors weighted by $\|A_i u\|^\beta$.Define

$$L_i : C(\mathbb{RP}^{d-1}) \to C(\mathbb{RP}^{d-1}) \text{ by } L_i f(\overline{u}) = \left\| A_i \frac{u}{\|u\|} \right\|^\beta f(\overline{A_i u}).$$

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Transfer operator:

$$L_{\mathcal{A},\beta} = \sum_i L_i \text{ that is } L_{\mathcal{A},\beta} f(\overline{u}) = \sum_i \left\| A_i \frac{u}{\|u\|} \right\|^\beta f(\overline{A_i u}).$$

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Take (if they exist) $h > 0$ ν with $L_{\mathcal{A},\beta} h = \rho(L_{\mathcal{A},\beta}) h$ and $L^*_{\mathcal{A},\beta} \nu = \rho(L_{\mathcal{A},\beta}) \nu$.

$$\mu_{\mathcal{A},\beta}[x_0 x_1 \cdots x_{n-1}] = \rho(L_{\mathcal{A},\beta})^{-n} \int_{\mathbb{RP}^{d-1}} L_{x_0} L_{x_1} \cdots L_{x_{n-1}} h d\nu.$$

$$\mu_{\mathcal{A},\beta}[x_0x_1\cdots x_{n-1}] = \rho(L_{\mathcal{A},\beta})^{-n}\int_{\mathbb{RP}^{d-1}} L_{x_0}L_{x_1}\cdots L_{x_{n-1}}hd\nu.$$

Provided the collection $\mathcal{A}$ is proximal (read contracting) and strongly irreducible (read topologically mixing) then $L_{\mathcal{A},\beta}$ has a spectral gap (Guivarc'h and Le Page '04). This implies h and ν exist and

$$\rho(L_{\mathcal{A},\beta})^{-n}L^n_{\mathcal{A},\beta}f \xrightarrow{n\to\infty} \langle f,\nu\rangle\, h.$$

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This makes the analog of the computation

$$\begin{aligned}\mu_{\mathcal{A},1}([I]\cap\sigma^{-n}[J]) &= 3^{-|I|-|J|}\left\langle A_J 3^{-n+|J|}A^{n-|J|}A_I[1/2],[1]\right\rangle\\ &\xrightarrow{n\to\infty} 3^{-|I|}\langle A_I[1/2],[1]\rangle\, 3^{-|J|}\langle A_J[1/2],[1]\rangle\\ &= \mu_{\mathcal{A},1}([I])\mu_{\mathcal{A},1}([J]).\end{aligned}$$

work which yields the following result...

Result

Theorem (P.)

Suppose that $\mathcal{A} = (A_0, \dots A_{M-1})$ *is a collection of real invertible* $d \times d$ *matrices which is proximal and strongly irreducible. Then for any* $\beta \geq 0$ *there exists a unique Gibbs state for* $(\mathcal{A}, \beta)$, $\mu_{\mathcal{A},\beta}$. *Moreover*

1. $\mu_{\mathcal{A},\beta}$ *is weak Bernoulli.*
2. $\mu_{\mathcal{A},\beta}$ *has exponential decay of correlations for Hölder continuous functions. That is, for a fixed* $\theta \in (0,1)$ *there are constants* D *and* $\gamma \in (0,1)$ *such that*

$$\left| \int f \cdot g \circ \sigma^n d\mu_{\mathcal{A},\beta} - \int f d\mu_{\mathcal{A},\beta} \int g d\mu_{\mathcal{A},\beta} \right| \leq D \left\| f \right\|_\theta \left\| g \right\|_\theta \gamma^n$$

for all $f, g \in \mathcal{H}_\theta$, $n \geq 0$.

Where do we go from here?

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- What is the connection between $L_{\mathcal{A},\beta} : C(\mathbb{RP}^{d-1}) \to C(\mathbb{RP}^{d-1})$ and the transfer operator $\mathcal{L} : L^1(\mu_{\mathcal{A},\beta}) \to L^1(\mu_{\mathcal{A},\beta})$?

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- Lots other questions:
  - $\beta < 0$?
  - Singular value potential?
  - $\mathcal{A} : \Sigma^{\mathbb{Z}} \to M_d(\mathbb{R})$ Hölder?
  - etc...