1-Block factors of g measures

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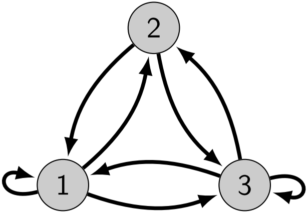
University of Victoria

Topics in Mathematical Physics

Hidden Markov measures



$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \quad u = \begin{bmatrix} 3/8 \\ 1/4 \\ 3/8 \end{bmatrix}$$

- $\Sigma_A = \left\{ (x_i)_{i=0}^{\infty} \in \{1,2,3\}^{\mathbb{N}} : P_{x_i x_{i+1}} > 0 \right\}$
- P gives rise to a Markov measure μ_P on Σ_A.

$$\mu_P[x_0 x_1 \cdots x_n] = u_{x_0} P_{x_0 x_1} P_{x_1 x_2} \cdots P_{x_{n-1} x_n}$$

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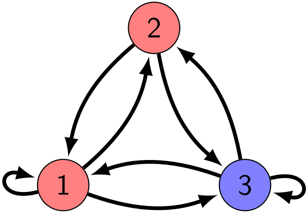

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$$\pi(1) = r \ , \ \pi(2) = r \text{ and } \pi(3) = b$$

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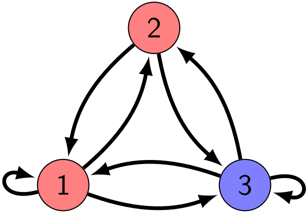

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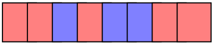
  $\downarrow \pi$

 $\cdots$

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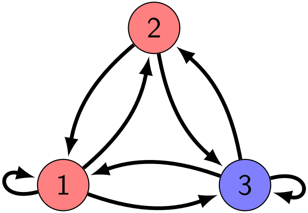

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  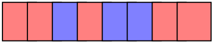 $\cdots$

- We would like to understand $\pi_* \mu_P$.

Hidden Markov measures

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$$\pi_* \mu_P [rrbr] = 1^T \mathcal{P}_{rb} \mathcal{P}_{br} \mathcal{P}_{rr} u_r$$

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- Is $\operatorname{var}_n g = O(\theta^n)$ for $0 < \theta < 1$?

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Theorem (Chazottes-Ugalde '03, Yoo '10)

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- This property that all products are positive is "mixing" in fibers.
- Is the class of g measures with $\mathrm{var}_n \log g = O(\theta^n)$ for some $0 < \theta < 1$ closed under factor maps which are mixing in fibers?

Some Results on g measures

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Suppose that Σ_A is a topologically mixing shift of finite type, μ a g measure and π a factor map which is mixing in fibers. If $\log g$ is Bowen (respectively Walters, Hölder) then the logarithm of the g function for $\pi_*\mu$ is Bowen (respectively Walters, Hölder).

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All of the "classical" uniqueness regimes for g measures are closed under closed under factor maps which are mixing in fibers.

(Not a) proof

The replacement for P is the Ruelle operator $L_g : C(\Sigma_A) \to C(\Sigma_A)$

$$L_g f(x) = \sum_{A_{ix_0}=1} g(ix) f(ix).$$

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$$L_g = \begin{bmatrix} L_{11} & L_{21} & L_{31} \\ L_{12} & 0 & L_{32} \\ L_{13} & L_{23} & L_{33} \end{bmatrix}$$

$$C(\Sigma_A) = \bigoplus_{i=1,2,3} C([i])$$

$$L_{ij} : C([i]) \to C([j])$$

$$L_{ij} f = \chi_{[j]} L_g (f \chi_{[i]})$$

$$L_{ij} f(x) = g(ix) f(ix) \chi_{[j]}(x)$$

$$\mu_g[x_0 x_1 \cdots x_n] = \langle L_{x_{n-1}x_n} \cdots L_{x_1 x_2} L_{x_0 x_1} 1, \mu_g \rangle$$

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$$\pi_* \mu_g[rrbr] = \langle \mathcal{L}_{rb} \mathcal{L}_{br} \mathcal{L}_{rr} 1, \mu_g \rangle$$

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$$\Lambda_r \subseteq \mathcal{X}_r \text{ and } \Lambda_b \subseteq \mathcal{X}_b$$

$$\Lambda_r = \{f \in \mathcal{X}_r : f \geq 0 f(x) \leq e^{\alpha_k} f(y) x \sim_k y\} \text{ where } \alpha_k \xrightarrow{k\to\infty} 0.$$

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Regulairty of g + “Mixing in fibers”

$\downarrow$

Finite Projective diameter

$\downarrow$

Regularty of g function for $\pi_* \mu_g$