On the numerical solution of a transmission problem for the Klein-Gordon equation

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In this work we considered the integral equation approach for finding a solution of a transmission problem for the Klein-Gordon equation.

Let $D \subset R^2$ a domain with the boundary $\Gamma_1 \in C^2$, $D_2 \subset D$ a domain with the boundary $\Gamma_2 \in C^2$. Denote $D_1 := D \backslash \overline{D_2}$. We need to find a solution to the next boundary value problem:

$$\begin{cases} \Delta u_1 - k_1^2 u_1 = 0 \quad in\ D_1, \\ \Delta u_2 - k_2^2 u_2 = 0 \quad in\ D_2, \\ u_1 = u_2 \quad on\ \Gamma_2, \\ k_1 \frac{\partial u_1}{\partial \vartheta} = k_2 \frac{\partial u_2}{\partial \vartheta}\ on\ \Gamma_2, \\ u_1 = f \quad on\ \Gamma_1\ . \end{cases} \qquad (1)$$

Here f is a known function and k_1, k_2 are known positive constants, ϑ is the unit outward normal to the boundary. Using features of potentials, we transformed problem (1) of finding solutions u_1, u_2 in D_1, D_2 to the system of integral equations with unknown densities $\varphi_1, \varphi_2, \varphi_3$. We present the solution of (1) in the form of single layer potentials with densities $\varphi_1, \varphi_2, \varphi_3$.

$$\begin{cases} \frac{1}{2\pi} \sum_{i=1}^{2} \int_{\Gamma_i} K_0(k_1, x, y) \varphi_i(y) ds(y) = f(x), \qquad x \in \Gamma_1\ , \\ \frac{1}{2\pi} \sum_{i=1}^{2} \int_{\Gamma_i} K_0(k_1, x, y) \varphi_i(y) ds(y) - \frac{1}{2\pi} \int_{\Gamma_2} K_0(k_2, x, y) \varphi_3(y) ds(y) = 0, x \in \Gamma_2\ , \\ k_1 \left(\frac{1}{2} \varphi_2(x) + \frac{1}{2\pi} \sum_{i=1}^{2} \int_{\Gamma_i} \frac{\partial K_0(k_1, x, y)}{\partial \vartheta(x)} \varphi_i(y) ds(y) \right) - \\ \quad -k_2 \left(\frac{1}{2} \varphi_3(x) + \frac{1}{2\pi} \int_{\Gamma_2} \frac{\partial K_0(k_2, x, y)}{\partial \vartheta(x)} \varphi_3(y) ds(y) \right) = 0, \quad x \in \Gamma_2\ . \end{cases}$$

Where $K_0(k_i, x, y)$ is a modified Bessel function.

Assuming that $\Gamma_i = \{x_i(s) = (x_{i1}(s), x_{i2}(s)): 0 \le s \le 2\pi\}$, $i = 1, 2$ are parametrized boundaries, we rewrite the system as:

$$\begin{cases} \dfrac{1}{2\pi}\sum_{i=1}^{2}\int_0^{2\pi} H_{1i}(k_1, s, \sigma)\mu_i(\sigma)d\sigma = \tilde{f}(s), \\ \dfrac{1}{2\pi}\sum_{i=1}^{2}\int_0^{2\pi} H_{2i}(k_1, s, \sigma)\mu_i(\sigma)d\sigma - \dfrac{1}{2\pi}\int_0^{2\pi} H_{22}(k_2, s, \sigma)\mu_3(\sigma)d\sigma = 0, \\ k_1\left(\dfrac{1}{2}\mu_2(s) + \dfrac{1}{2\pi}\sum_{i=1}^{2}\int_0^{2\pi} L_{2i}(k_1, s, \sigma)\mu_i(\sigma)d\sigma\right) - \\ \quad -k_2\left(\dfrac{1}{2}\mu_3(s) + \dfrac{1}{2\pi}\int_0^{2\pi} L_{22}(k_2, s, \sigma)\mu_3(\sigma)d\sigma\right) = 0, \\ \qquad s\epsilon[0; 2\pi]. \end{cases}$$

$$H_{ij}(k_l, s, \sigma) = K_0\left(k_l, x_i(s), x_j(\sigma)\right)\left|x_j'(\sigma)\right|,$$

$$L_{ij}(k_l, s, \sigma) = -k_l K_1\left(k_l, x_i(s), x_j(\sigma)\right) h_{ij}\left(k_l, x_i(s), x_j(\sigma)\right)\left|x_j'(\sigma)\right|,$$

$$h_{ij}\left(k_l, x_i(s), x_j(\sigma)\right) = \frac{\partial\left(k_l\left|x_i(s) - x_j(\sigma)\right|\right)}{\partial\vartheta\left(x_i(s)\right)}, \quad \tilde{f}(s) = f\left(x_1(s)\right),$$

$$\mu_i(s) = \varphi_i\left(x_i(s)\right), \qquad i = 1,2,3, \qquad j, l = 1,2.$$

We discretized the system using trigonometrical quadrature method with help of special functions features. As a result, we obtained a system of linear equations. The numerical solution of the transmission problem has the form

$$u_1^M(x) = \sum_{i=1}^{2}\sum_{k=1}^{2M-1}\frac{H_{0i}(k_i, x, s_k)}{2M}\widetilde{\mu_i}(s_k), \quad x\epsilon D_1,$$

$$u_2^M(x) = \sum_{k=1}^{2M-1}\frac{H_{02}(k_2, x, s_k)}{2M}\widetilde{\mu_3}(s_k), \quad x\epsilon D_2,$$

where $H_{oi}(k, x, s_k) = \frac{1}{2\pi}K_o(k, |x - x_i(s_k)|)$.

The accuracy of the method was confirmed by numerical experiments.

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