Analysis of complex networks with algebra

· **Workshop 39** ·

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Agenda

Analysis of complex networks with algebra

1. Introduction (plotting multigraphs)
2. Elementary structures
   - ➟ Example 1: Dihedral group
3. Group structure in social networks
   - ➟ Example 2: Kariera kinship
4. Multiplex and signed networks
   - ➟ Example 3: Monastery novices
   - ➟ Example 4: Incubator network A
5. Affiliation and multilevel networks
   - ➟ Example 5: Group of Twenty (valued)

1. Introduction

Plotting multigraphs

‘multiplex’ *for computations of multiple networks in* **R**

Package ‘multiplex’

August 28, 2013

Type Package

Title Analysis of Multiple Social Networks with Algebra

Version 1.0

Depends R (>= 3.0.1)

Date 2013-08-28

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Description multiplex - Analysis of Multiple Social Networks with Algebra is a package for the study of social systems made of different types of relationships. It is possible to create and manipulate multivariate network data with different formats, and there are effective ways available to treat multiple networks with routines that combine algebraic systems like the partially ordered semigroup or the semiring structure together with the relational bundles occurring in different types of multivariate network data sets.

License GPL-3

Suggests Rgraphviz

Encoding latin1

Collate
'as.semigroup.R' 'as.strings.R' 'bundle.census.R' 'bundles.R''cngr.R' 'convert.R' 'cph.R'

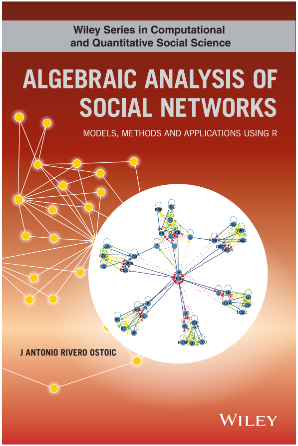
NeedsCompilation no

Repository CRAN

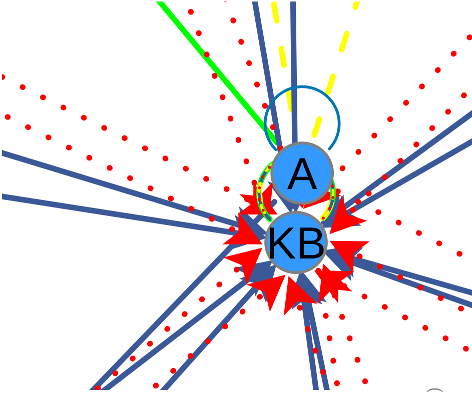
Date/Publication 2013-08-28 13:53:11

R topics documented:

‘multigraph’ *to plot multiplex networks in **R***



A JOHN WILEY & SONS, INC., PUBLICATION



(Cayley graph)

Download and install packages in **R** console (or IDE Rstudio):

```r
# from CRAN
install.packages("multiplex", "multigraph")
# or versions from GitHub
devtools::install_github("mplex/multiplex")
devtools::install_github("mplex/multigraph")
```

```r
# load packages
library("multigraph")
# Loading required package: multiplex
```

Representing network data

Different ways to represent network data

$(1, 2)$

```r
multiplex::transf("1, 2")
```

```
   1  2
1  0  1
2  0  0
```

```r
multigraph("1, 2", cex = 18, lwd = 20, rot = -90, pos = 0, vedist = -2)
```

```r
scp <- list(cex = 18, lwd = 20, rot = -90, pos = 0, vedist = -2)
multigraph("1, 2", scope = scp)
```

Undirected


```r
matrix(c(0,1,1,0), nrow = 2, ncol = 2)
```

$\{1, 2\}$

```
     [,1] [,2]
[1,]    0    1
[2,]    1    0
```

```r
multigraph("1, 2", directed = FALSE, scope = scp)
```

Multiplex

$(1,2); (2,1)$

```
, , 1

  1 2
1 0 1
2 0 0

, , 2

  1 2
1 0 0
2 1 0
```

```
multigraph(list("1, 2", "2, 1"), scope = scp, ecol = 1, bwd = .7)
```

Multiplex

$(1, 2); (2, 1)$

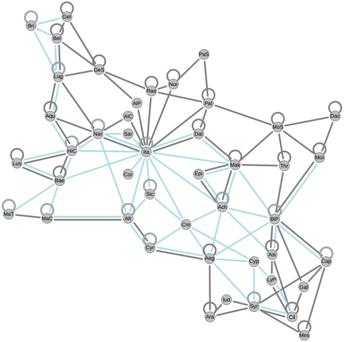
```
, , 1

  1 2
1 0 1
2 0 0

, , 2

  1 2
1 0 0
2 1 0
```

```r
net <- list("1, 2", "2, 1")
multigraph(net, scope = scp, ecol = 1, bwd = .7, swp = TRUE)
```

Roman Empire transport network: Multiplex and undirected

ca. AD 125

routes based on Rodrigue (2013)

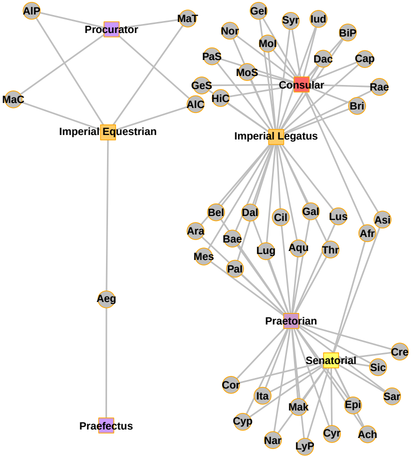



```r
sdam::plot.map(type = "rp")
```

(Forthcomming)

Roman provinces political affiliations: Two-mode

ca. AD 117



(Forthcomming)

2. Elementary structures

Example 1: Dihedral groups

Typology of multiple network structures

Simple networks:

- *(Simple) graphs*, *matrices*
  ➠ for relations between actors

Multiplex networks:

- *Multigraphs*, *arrays*
  ➠ for (types of) relations between actors
- *Cayley graphs*, *tables*
  ➠ for relationships between relations

☛ *Different types of algebraic structures are represented by tables*

Algebraic representation of multiplex networks

Typology

Type of structure	Algebraic object
Elementary	*Group*
Complex	*Semigroup*, *Semiring*, *Lattice*, etc.

Group: Elementary structure

A ***group*** is an algebraic structure with an *element set* and an endowed *operation*:

$$\langle\ G,\ \cdot\ \rangle$$

That for all a, b, c, and a neutral element $e \in G$ satisfies axioms:

$$\begin{aligned}
\text{Identity:}\quad & a \cdot e = e \cdot a = a \\
\textit{Inversion}\text{:}\quad & a \cdot a^{-1} = a^{-1} \cdot a = e \\
\text{Associativity:}\quad & (a \cdot b) \cdot c = a \cdot (b \cdot c) \\
\text{Closure:}\quad & a \cdot b \in G \qquad \text{(for all} \quad a, b\text{)}
\end{aligned}$$

Group structure by permutations

Theorem (Cayley)

All of group theory can be found in permutations.

➠ we focus on permutation symmetry

A *permutation* operator is represented by a *permutation matrix*

➠ having one entry in each row and in each column, and 0 elsewhere

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Group Structures

Definition (Permutation Group on X)

The *permutation group on* X is the set of all permutations S_X on X

Definition (Symmetric Group of order n, S_n)

The *symmetric group* on a n-element set $\{1, 2, \dots, n\}$ is the set of all permutations with $n!$ bijections σ, $S_n = \{\sigma_1, \sigma_2, \dots, \sigma_{n!}\}$.

- If $X = \{1, 2, \dots, n\}$ then $S_X = S_n$
  - ➟ the symmetric groups on n-elements are permutation groups

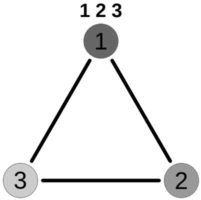
Definition (Dihedral Group of degree n, D_n, $n > 2$)

The set of all permutations which are symmetries on a regular n-sided polygon and the composition operation $\circ$ makes the *dihedral group* $(D_n, \circ)$.

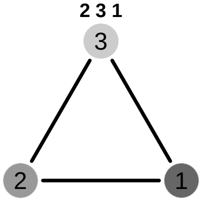
- the order of a dihedral group is twice its degree

Group of symmetries of the equilateral triangle (Dihedral group, D_3)

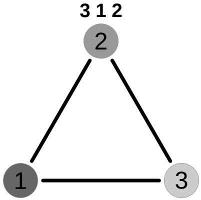
Rotations



R_0

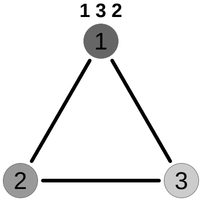


R_{120}

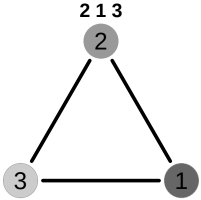


R_{240}

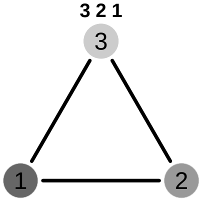
Reflections



S_V



S_D



S_L

Cayley table of D_3

$\mathsf{R}_\circ$ rotations; S_V: mirror; S_D and S_L are diagonals

$\circ$	R_0	R_{120}	R_{240}	S_V	S_D	S_L
R_0	R_0	R_{120}	R_{240}	S_V	S_D	S_L
R_{120}	R_{120}	R_{240}	R_0	S_D	S_L	S_V
R_{240}	R_{240}	R_0	R_{120}	S_L	S_V	S_D
S_V	S_V	S_L	S_D	R_0	R_{240}	R_{120}
S_D	S_D	S_V	S_L	R_{120}	R_0	R_{240}
S_L	S_L	S_D	S_V	R_{240}	R_{120}	R_0

Generators of D_3

Define dihedral group family generators as permutation matrices

```
# sort for a lexicographic order
D3 <- transf(list(F = c("1, 3","2, 1","3, 2"), G = c("1, 1","2, 3","3, 2")),
+              type = "toarray", sort = TRUE)
```

```
, , F

  1 2 3
1 0 0 1
2 1 0 0
3 0 1 0

, , G

  1 2 3
1 1 0 0
2 0 0 1
3 0 1 0
```

String relations in D_3

word tables

```r
multiplex::strings(D3)
```

```
$wt
, , F           , , FF          , , GF

  1 2 3           1 2 3           1 2 3
1 0 0 1         1 0 1 0         1 0 0 1
2 1 0 0         2 0 0 1         2 0 1 0
3 0 1 0         3 1 0 0         3 1 0 0

, , G           , , FG          , , GG

  1 2 3           1 2 3           1 2 3
1 1 0 0         1 0 1 0         1 1 0 0
2 0 0 1         2 1 0 0         2 0 1 0
3 0 1 0         3 0 0 1         3 0 0 1
```

Equations in group structure, D_3 $(k = 3)$

Argument `equat` to find group equations with the identity

```r
strings(D3, equat = TRUE, k = 3)
```

```
$equat
$equat$F
[1] "F"   "GGF" "FGG"

$equat$G
[1] "G"   "GGG" "FGF"

$equat$FF
[1] "FF"  "GFG"

$equat$FG
[1] "FG"  "GFF"

$equat$GF
[1] "GF"  "FFG"

$equat$GG
[1] "GG"  "FFF"


$equate
$equate$e
[1] "e"   "GG"  "FFF"
```

Group structure, D_3

symbolic format

Function semigroup() allows finding the group structure

➟ since "any group is a semigroup as well"

```
# semigroup structure with symbolic format
semigroup(D3, type = "symbolic")$S
```

```
    F  G FF FG GF GG
F  FF FG GG GF  G  F
G  GF GG FG FF  F  G
FF GG GF  F  G FG FF
FG  G  F GF GG FF FG
GF FG FF  G  F GG GF
GG  F  G FF FG GF GG
```

Group structure, D_3

```r
# record semigroup with numerical format
D3S <- multiplex::semigroup(D3)
```

```
...

$st
[1] "F"  "G"  "FF" "FG" "GF" "GG"

$S
  1 2 3 4 5 6
1 3 4 6 5 2 1
2 5 6 4 3 1 2
3 6 5 1 2 4 3
4 2 1 5 6 3 4
5 4 3 2 1 6 5
6 1 2 3 4 5 6

attr(,"class")
[1] "Semigroup" "numerical"
```

Permutation of the group structure, D_3

`perm()` for rearrangement of elements' group structure in `D3S`

```
D3S <- multiplex::perm(D3S$S, clu = c(2,4,3,5,6,1))
```

```
  6 1 3 2 4 5
6 6 1 3 2 4 5
1 1 3 6 4 5 2
3 3 6 1 5 2 4
2 2 5 4 6 3 1
4 4 2 5 1 6 3
5 5 4 2 3 1 6
```

This comes from the string labels where `GG` is the identity element

```
..
$st
[1] "F"  "G"  "FF" "FG" "GF" "GG"
...
```

Depiction of group structure: Cayley graph

Definition (Cayley graph)

The *Cayley graph* Γ of a group G with respect to a generating set $C \subseteq G$:

$$\Gamma = \Gamma(G, C).$$

- G is the node set in Γ
- A generator $c \in C$ connects two nodes $a, b \in G$ whenever $b = ca$

  ➟ i.e. all pairs of the form $(a, c \cdot b)$ make the edge set in Γ

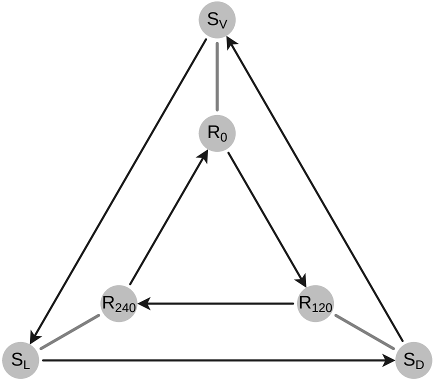
Cayley colour graph

Example (Cayley graph, integers under addition $\mathbb{Z}_2$)

```
                       e=ee  => solid loop
    e x                e=xx  => solid loop
  e e x
  x x e                x=ex  => dashed arc
                       x=xe  => dashed arc
```

Cayley graph of Dihedral group D_3

Group of symmetries of the equilateral triangle



Depiction of the group structure, D_3

Relabeling semigroup for Cayley table

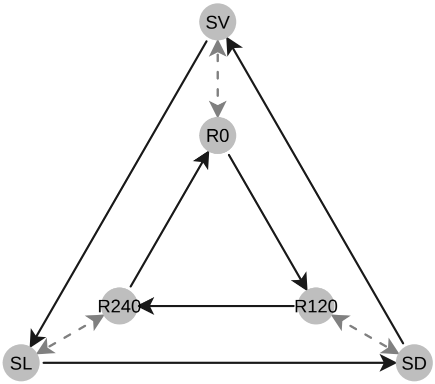
```r
# relabel strings in D3S specifying generators
D3S <- multiplex::as.semigroup(D3S, gens = c(2, 4),
+                          lbs = c("R0", "R120", "R240", "SV", "SD", "SL"))
```

```
...
$st
[1] "R0"   "R120" "R240" "SV"   "SD"   "SL"

$gens
[1] "R120" "SV"

$S
       R0 R120 R240   SV   SD   SL
R0     R0 R120 R240   SV   SD   SL
R120 R120 R240   R0   SD   SL   SV
R240 R240   R0 R120   SL   SV   SD
SV     SV   SL   SD   R0 R240 R120
SD     SD   SV   SL R120   R0 R240
SL     SL   SD   SV R240 R120   R0

attr(,"class")
[1] "Semigroup" "symbolic"
```

Depiction of group structure

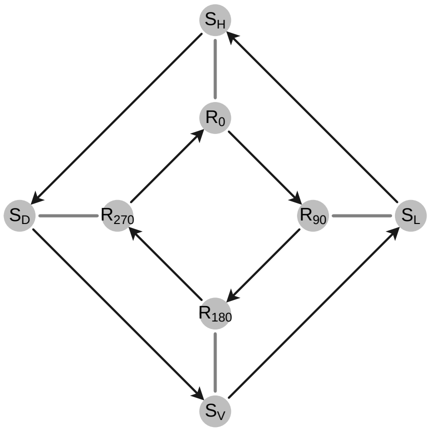
Cayley graph, D_3

```
# plot Cayley colour graph with a 2-radii concentric layout
scpD3 <- list(cex = 7, lwd = 3, pos = 0, col = 8, fsize = 16)
multigraph::ccgraph(D3S, conc = TRUE, nr = 2, scope = scpD3)
```



Group of symmetries of the square

Cayley graph of dihedral group D_4



$\mathit{3.}$ Group structure in social networks

Example 2: Kariera kinship

Kariera society kinship system and group structure

- Despite the symmetry, algebraic groups can model human societies

- Some primitive societies like the *Kariera* from Western Australia have kinship networks that follow the rules of a group structure

  ➟ where *primitive* means "first of its class"

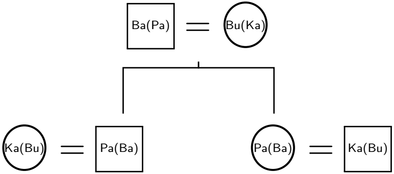
- The Karieras have four clans with specific rules of marriage & descent: *Banaka*, *Burung*, *Karimera*, and *Palyeri*.

  ➟ data collected by Radcliffe-Brown, analysed by White (1963)

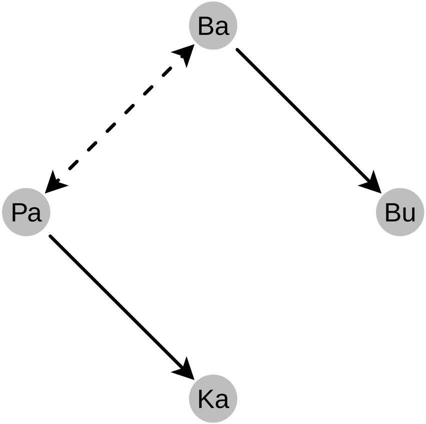
Kariera rules for marriage & descent (I)

Clans: Banaka (Ba), Burung (Bu), Karimera (Ka), Palyeri (Pa)

Two types of descent rules among Banaka and Palyeri (ego male)



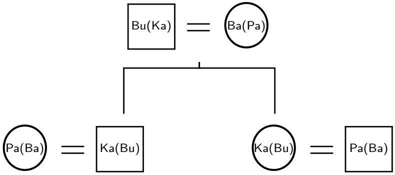
square: male, circle: female



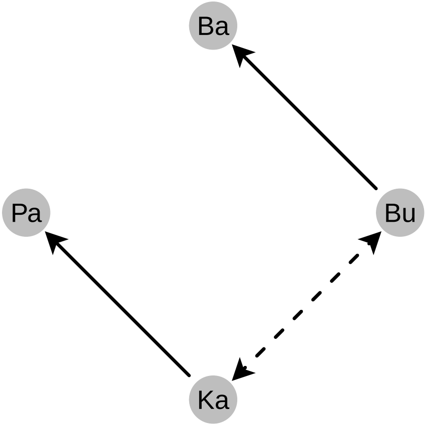
Kariera rules for marriage & descent (II)

Clans: Banaka (Ba), Burung (Bu), Karimera (Ka), Palyeri (Pa)

Two types of descent rules among Burung and Karimera (ego male)



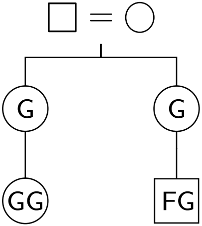
square: male, circle: female



Parallel-cousins marriages in kinship networks

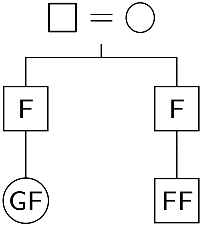
identifiers, F for male and G for female, are with right multiplication

$$FG = GG$$



(a) Matrilineal

$$GF = FF$$

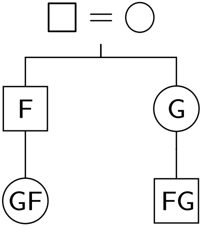


(b) Patrilineal

Cross-cousins marriages in kinship networks

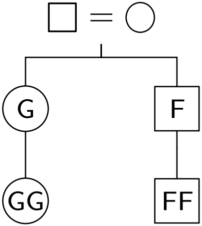
identifiers, F for male and G for female, are with right multiplication

$$\mathsf{FG} = \mathsf{GF}$$



(a) Matrilineal

$$\mathsf{FF} = \mathsf{GG}$$



(b) Patrilineal

Permutation matrices for marriage & descent

Kariera kinship system

```
# create permutation matrices for marriage & descent rules
kks <- transf(list(F = c("1, 2", "3, 4", "2, 1", "4, 3"),
+                  G = c("1, 4", "2, 3", "3, 2", "4, 1")))
```

```
, , F

  1 2 3 4
1 0 1 0 0
2 1 0 0 0
3 0 0 0 1
4 0 0 1 0

, , G

  1 2 3 4
1 0 0 0 1
2 0 0 1 0
3 0 1 0 0
4 1 0 0 0
```

Group structure as multiplication table

Kariera kinship system

The *multiplication table* reflects the group structure of the clan system

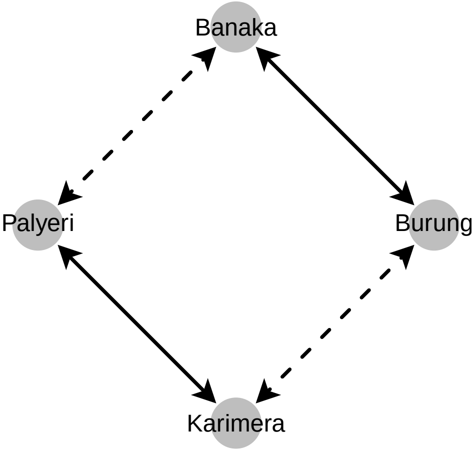
```
# Group structure with a symbolic format
semigroup(kks, type = "symbolic")
```

```
$dim
[1] 4
...

$ord
[1] 4

$st
[1] "F"  "G"  "FF" "FG"

$S
    F  G FF FG
F  FF FG  F  G
G  FG FF  G  F
FF  F  G FF FG
FG  G  F FG FF

attr(,"class")
[1] "Semigroup" "symbolic"
```

Rules of marriage & descent

Kariera kinship system

```
# visualize marriage & descent rules in the Kariera
multigraph(kks, scope = scpD3, ecol = 1, collRecip = TRUE,
+          lbs = c("Banaka", "Burung", "Karimera", "Palyeri"))
```



Set of equations

identify cross- and parallel-cousins marriages

The *set of equations* to detect allowed marriage types by commutation

```r
# equations allows finding marriage types in 'kks'
strings(kks, equat = TRUE)
```

```
...
$st
[1] "F"  "G"  "FF" "FG"

$equat
$equat$FF
[1] "FF" "GG"

$equat$FG
[1] "FG" "GF"


$equate
$equate$e
[1] "e"  "FF" "GG"
```

☛ *Both cross-cousins marriages are permitted in the Kariera*

Algebraic constraints in group structures

Two *algebraic constraints* for the analysis of the elementary structures:

- *Multiplication table* with relations between the different types of tie
- *Set of equations* among different types of tie

☛ *Complex structures have additional algebraic constraints*

$4a$. Multiplex networks

Example 3: Monastery novices

Monastery novices: Directed, multiplex, signed, valued, and longitudinal

```
# read Sampson Monastery dataset as Ucinet DL file
samp <- multiplex::read.dl("http://vlado.fmf.uni-lj.si/pub/networks/data/ucinet/sampson.dat")
```

```
# what types of tie the network has?
dimnames(samp)[[3]]
```

```
[1] "SAMPLK1" "SAMPLK2" "SAMPLK3" "SAMPDLK" "SAMPES"  "SAMPDES" "SAMPIN"  "SAMPNIN" "SAMPPR"  "SAMNPR"
```

➠ "like T1-T3", "dislike", "esteem", "disesteem", "influence" (pos/neg), "praise" (pos/neg)

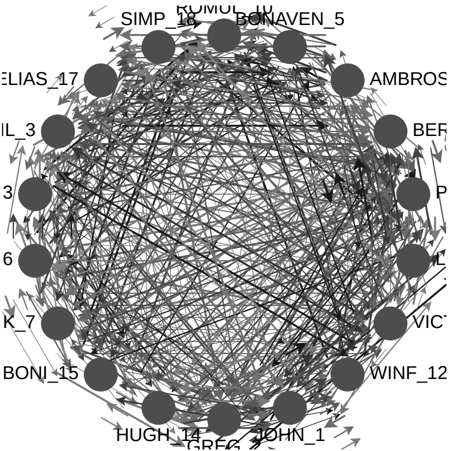
```
# plot Monastery novices network as valued multigraph (default)
multigraph(samp, valued = TRUE)
```

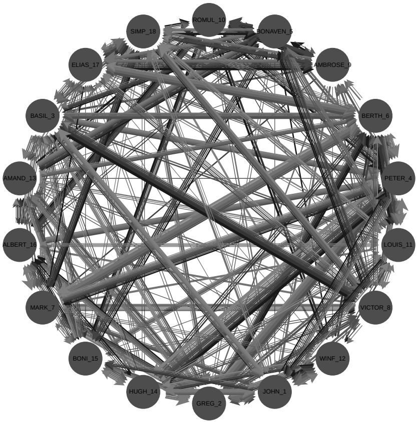
```
# plot valued network with customized values
multigraph(samp, valued = TRUE, bwd = .1, pos = 0, fsize = 6)
```

(Sampson, 1969)

Monastery novices network plot

multigraph circular





Monastery novices: Bundle patterns

```
# enumeration of bundle class types
multiplex::bundle.census(samp)
```

```
      BUNDLES NULL ASYMM RECIP T.ENTR T.EXCH MIXED FULL
TOTAL     134   19    20     1     37      8    68    0
```

```
# bundle patterns in the Monastery novices network
multiplex::summaryBundles(multiplex::bundles(samp))
```

```
                                                                  Bundles
Asym1                                           ->{SAMPLK1} (WINF_12, BONAVEN_5)
Asym2                                            ->{SAMPLK1} (BASIL_3, ROMUL_10)
...                                                                          ...
Asym20                                            ->{SAMNPR} (AMAND_13, SIMP_18)
Recp                                            <->{SAMPLK3} (BONI_15, VICTOR_8)
Tent1                       ->{SAMPDLK} ->{SAMPDES} ->{SAMNPR}  (ALBERT_16, ELIAS_17)
Tent2             ->{SAMPDLK} ->{SAMPDES} ->{SAMPNIN} ->{SAMNPR}  (ALBERT_16, PETER_4)
```

Monastery novices: Define a system & plot

Recall the types of tie in this network:

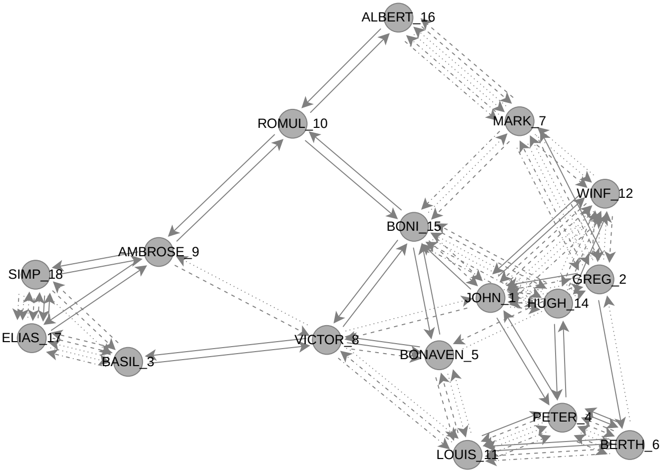
- "like T1-T3", "dislike",
- "esteem", "disesteem",
- "influence" (pos/neg),
- "praise" (pos/neg)

```
# extract system of strong bonds having positive ties
sampsb <- multiplex::rel.sys(samp[,,c(3,5,7,9)], type = "toarray", bonds = "strong")
```

```
# define a new scope
scps <- list(fsize = 8, pos = 0, vcol = "#AEAEAE", vcol0 = "#808080",
             ecol = "#808080", bwd = .4, rot = 145, mirrorY = TRUE)

# plot graph with a reproducible force-directed layout
multigraph(sampsb, layout = "force", seed = 12, scope = scps)
```

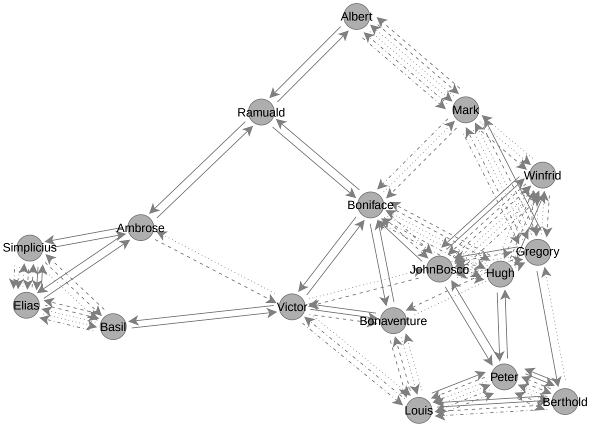
System of strong bonds

Monastery novices network



System of strong bonds: Customized node labels

```
monks <- c("Ramuald","Bonaventure","Ambrose","Berthold","Peter",
+          "Louis","Victor","Winfrid","JohnBosco","Gregory","Hugh",
+          "Boniface","Mark","Albert","Basil","Elias","Simplicius")

multigraph(sampsb, layout = "force", seed = 12, scope = scps, lbs = monks)
```



$4b.$ Signed networks

Example 4: Incubator network A

Structural Balance

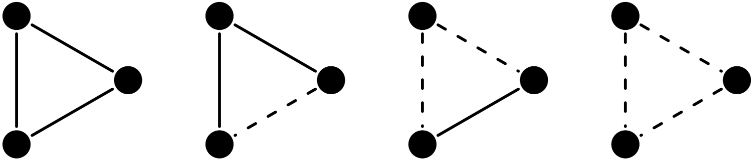
- Simmel (1950) studied "conflict as a mechanism for integration" in triadic relations
- Heider (1958) developed the *Structural Balance* theory as a special cases of transitivity
- Structural Balance theory applies to networks to see whether the system has an inherent equilibrium or not

  "all positive ties within groups; all negative ties between groups"

Structural Balance

- A balanced structure is represented by a *signed network*

  ➟ a special case of multiplex network



- Paths in signed graphs are positive when they have an even number of negative edges; otherwise negative

☛ *extension*: a path/semipath is ambivalent iff contains at least one ambivalent edge

Structures in Balance theory

balanced $\rightarrow$ **clusterable** $\rightarrow$ '*weak*' clusterable

(Cartwright & Harary, 1956) (Davis, 1967)

$\circ$	p	n
p	p	n
n	n	p

Classical

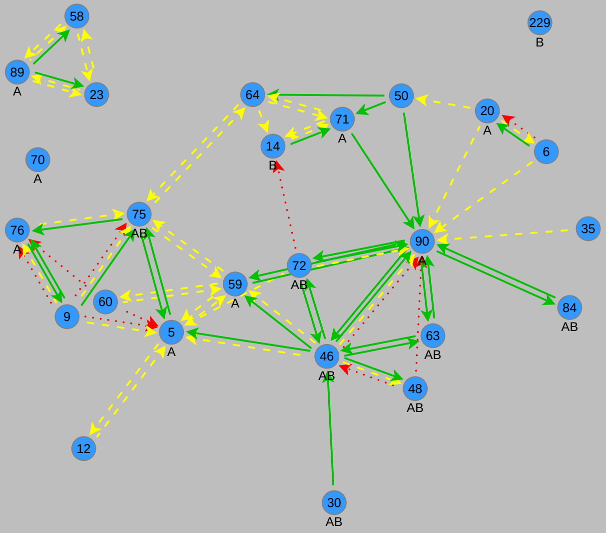
$\circ$	p	n	a
p	p	n	a
n	n	a	a
a	a	a	a

Extended

p $\rightarrow$ positive n $\rightarrow$ negative a $\rightarrow$ ambivalent

Incubator network “A”

Collaboration (green), Friendship (yellow), Competition (red)



Incubator network A

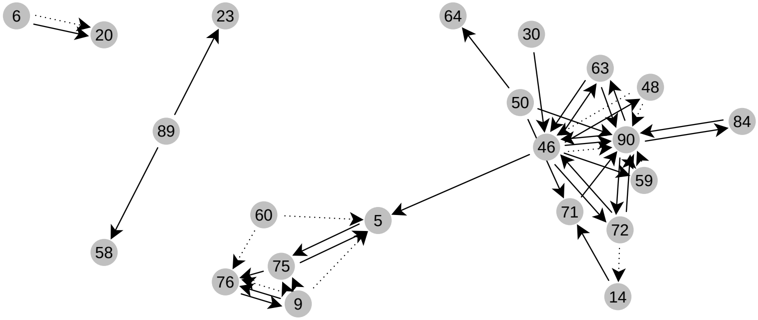
```r
# incubator network A dataset and structure
data("incA")
str(incA)
```

```
List of 5
 $ net  : num [1:26, 1:26, 1:5] 0 0 0 0 0 0 0 0 0 1 ...
  ..- attr(*, "dimnames")=List of 3
  .. ..$ : chr [1:26] "5" "6" "9" "12" ...
  .. ..$ : chr [1:26] "5" "6" "9" "12" ...
  .. ..$ : chr [1:5] "C" "F" "K" "A" ...
 $ atnet:List of 1
  ..$ : num [1:5] 0 0 0 1 1
 $ IM   : num [1:4, 1:4, 1:7] 1 1 1 0 0 1 0 0 1 0 ...
  ..- attr(*, "dimnames")=List of 3
  .. ..$ : NULL
  .. ..$ : NULL
  .. ..$ : chr [1:7] "C" "F" "K" "D" ...
 $ atIM : num [1:7] 0 0 0 0 0 0 1
 $ ...
```

```r
# cooperation and competition ties in 'incA' without isolated actors
netA <- rm.isol(incA$net[,,c(1,3)])
```

Signed structure in Incubator network A

```r
# plot signed multigraph
scpA <- list(ecol = 1, vcol = "#C0C0C0", cex = 3, fsize = 8, pos = 0, bwd = .5)
multigraph(netA, scope = scpA, signed = TRUE, layout = "force", seed = 9)
```



Signed Network *C and K in Incubator A*

```r
# create a "Signed" class object from matrices in netA
netAsg <- signed(netA)
```

```
$val
[1] p o n a

$s
   5 6 9 14 20 23 30 46 48 50 58 59 60 63 64 71 72 75 76 84 89 90
5  o o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  p  o  o  o  o
6  o o o o  a  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o
9  n o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  a  a  o  o  o
14 o o o o  o  o  o  o  o  o  o  o  o  o  o  p  o  o  o  o  o  o
20 o o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o
23 o o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o
30 o o o o  o  o  o  p  o  o  o  o  o  o  o  o  o  o  o  o  o  o
46 p o o o  o  o  o  o  p  o  o  p  o  p  o  o  p  o  o  o  o  a
48 o o o o  o  o  o  n  o  o  o  o  o  o  o  o  o  o  o  o  o  n
50 o o o o  o  o  o  o  o  o  o  o  o  o  p  p  o  o  o  o  o  p
58 o o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o
59 o o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  p
60 n o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  n  o  o  o
63 o o o o  o  o  o  p  o  o  o  o  o  o  o  o  o  o  o  o  o  p
64 o o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o
71 o o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  p
72 o o o n  o  o  o  p  o  o  o  o  o  o  o  o  o  o  o  o  o  p
75 p o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  p  o  o  o
76 o o p o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o
84 o o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  p
89 o o o o  o  p  o  o  o  o  p  o  o  o  o  o  o  o  o  o  o  o
90 o o o o  o  o  o  p  o  o  o  p  o  p  o  o  p  o  o  p  o  o
```

Semiring

Algebraic structure

A *semiring* is an object set endowed with a pair operations, multiplication and addition, together with two neutral elements:

$$\langle\, Q,\ +,\ \cdot\ ,\ 0,\ 1\,\rangle$$

properties:

- closed, associative, and commutative under addition
- multiplication distributes over addition, i.e. for all $\mathsf{p}, \mathsf{n}, \mathsf{a} \in Q$:

$$\mathsf{p}\ \cdot\ (\mathsf{n} + \mathsf{a}) = (\mathsf{p}\ \cdot\ \mathsf{n}) + (\mathsf{p}\ \cdot\ \mathsf{a}) \quad \text{and} \quad (\mathsf{p} + \mathsf{n})\ \cdot\ \mathsf{a} = (\mathsf{p}\ \cdot\ \mathsf{a}) + (\mathsf{n}\ \cdot\ \mathsf{a})$$

☛ *Semirings help us to evaluate the relational system in terms of balance theory by looking at paths and semipaths*

Semiring operations

$\cdot$	o	n	p	a
o	o	o	o	o
n	o	p	n	a
p	o	n	p	a
a	o	a	a	a

$+$	o	n	p	a
o	o	n	p	a
n	n	n	a	a
p	p	a	p	a
a	a	a	a	a

Balance

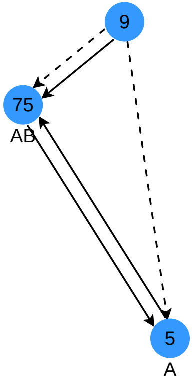
$\cdot$	o	n	p	a	q
o	o	o	o	o	o
n	o	q	n	n	q
p	o	n	p	a	q
a	o	n	a	a	q
q	o	q	q	q	q

$+$	o	n	p	a	q
o	o	n	p	a	q
n	n	n	a	a	n
p	p	a	p	a	p
a	a	a	a	a	a
q	q	n	p	a	q

Clustering

Balance semiring *(Signed triad)*

```
# 2-Paths  (9, 75)
"n, p" "o, a" "a, o"

# multiplication
"n" "o" "o"

# addition
n
```



	5	9	75
5	o	o	p
9	n	o	a
75	p	o	o

t^{α}

	5	9	75
5	p	o	o
9	a	o	n
75	o	o	p

t^{α} paths, $k > 1$

	5	9	75
5	p	a	a
9	a	a	n
75	a	n	a

t^{α} semipaths, $k = 2$

	5	9	75
5	a	a	a
9	a	a	a
75	a	a	a

t^{α} semipaths, $k > 2$

Semiring function

```r
# arguments in function semiring()
formals("semiring")
```

```
$x


$type
c("balance", "cluster")

$symclos
[1] TRUE

$transclos
[1] TRUE

$k
[1] 2

$lbs
```

Semiring structures

```
# balance semiring 2-paths (deafult)
semiring(netAsg, type = "balance")

# 3-paths
semiring(netAsg, type = "balance", k = 3)

# 2-semipaths
semiring(netAsg, type = "balance", symclos = FALSE)
# ...
```

```
# cluster semiring 2-paths (deafult)
semiring(netAsg, type = "cluster")

# 3-paths
semiring(netAsg, type = "cluster", k = 3)

# 2-semipaths
semiring(netAsg, type = "cluster", symclos = FALSE)
# ...
```

Checking for equilibrium in Balance semiring

```r
identical(semiring(netAsg, type = "balance", k = 3)$Q,
+           semiring(netAsg, type = "balance", k = 2)$Q )
```

```
[1] FALSE
```

```r
identical(semiring(netAsg, type = "balance", k = 3)$Q,
+           semiring(netAsg, type = "balance", k = 4)$Q )
```

```
[1] FALSE
```

```r
identical(semiring(netAsg, type = "balance", k = 4)$Q,
+           semiring(netAsg, type = "balance", k = 5)$Q )
```

```
[1] TRUE
```

Checking for equilibrium in Cluster semiring

```
identical(semiring(netAsg, type = "cluster", k = 3)$Q,
+          semiring(netAsg, type = "cluster", k = 2)$Q )
```

```
[1] FALSE
```

```
identical(semiring(netAsg, type = "cluster", k = 3)$Q,
+          semiring(netAsg, type = "cluster", k = 4)$Q )
```

```
[1] FALSE
```

```
identical(semiring(netAsg, type = "cluster", k = 4)$Q,
+          semiring(netAsg, type = "cluster", k = 5)$Q )
```

```
[1] TRUE
```

Weak balance structure with semipaths

```r
# length four and permutation with clustering
QnetA <- semiring(netAsg, type = "balance", k = 4)
multiplex::perm(QnetA$Q, clu = c(1,6,1,2,6,6,3,2,2,3,6,2,4,2,5,2,2,1,1,2,6,2))
```

```
   5 9 75 76 14 46 48 59 63 71 72 84 90 30 50 60 64 6 20 23 58 89
5  a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
9  a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
75 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
76 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
14 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
46 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
48 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
59 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
63 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
71 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
72 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
84 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
90 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
30 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
50 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
60 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
64 a a a  a  a  a  a  a  a  a  a  a  a  a  a  a  a  o o  o  o  o
6  o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  a o  o  o  o
20 o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o a  o  o  o
23 o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  p  p  o
58 o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  p  p  o
89 o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  p
```

Weak balance structure with paths

```r
# length four and permutation with clustering
QnetA <- semiring(netAsg, type = "balance", symclos = FALSE, k = 4)
perm(QnetA$Q, clu = c(1,6,1,2,6,6,3,2,2,3,6,2,4,2,5,2,2,1,1,2,6,2))
```

```
   5 9 75 76 14 46 48 59 63 71 72 84 90 30 50 60 64 6 20 23 58 89
5  a a a  a  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
9  a a a  a  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
75 a a a  a  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
76 a a a  a  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
14 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
46 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
48 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
59 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
63 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
71 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
72 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
84 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
90 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
30 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
50 a a a  a  a  a  a  a  a  a  a  a  a  o  o  o  o  o o  o  o  o
60 a a a  a  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
64 o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
6  o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
20 o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
23 o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
58 o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
89 o o o  o  o  o  o  o  o  o  o  o  o  o  o  o  o  o o  o  o  o
```

Main component of Incubator A

weak balance structure

```
# comps() finds components and isolates
multiplex::comps(netA)
```

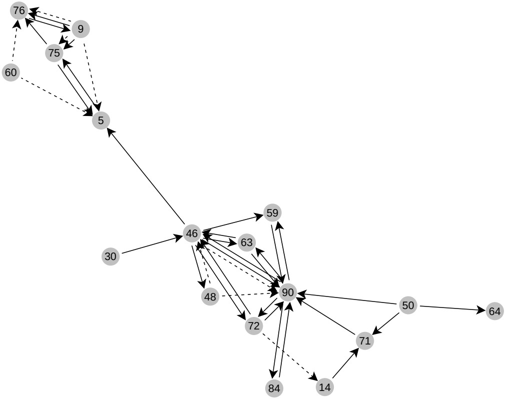
```
$com
$com[[1]]
 [1] "5"  "50" "59" "60" "63" "64" "71" "72" "75" "76" "84" "90" "9"  "14" "30" "46" "48"

$com[[2]]
[1] "58" "89" "23"

$com[[3]]
[1] "6"  "20"
```

```
# extract the 17 ties from main component in 'netA'
com <- comps(netA)$com[[1]]
# select in component two types of tie and actor attributes
nsA <- rel.sys(incA$net[,,c(1,3,4:5)], type = "toarray", sel = com)
```

Main component of Incubator A

weak balance structure

```
# plot network relations 'C' and 'K' in main component
multigraph(nsA, layout = "force", seed = 123, scope = scpA)
```



Factions with weak balance structure

paths main component of Incubator A

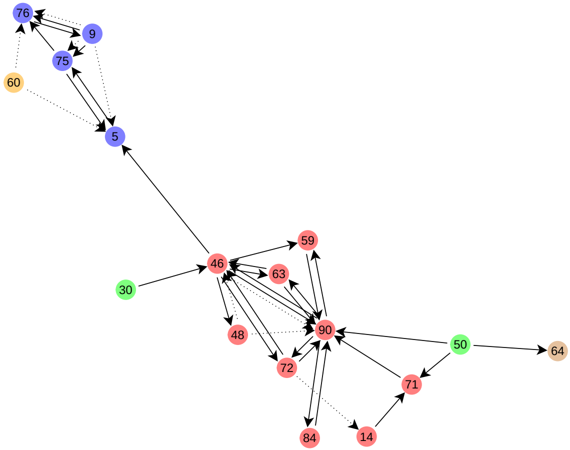
```
# scope with factions
scpAc <- list(lty = c(1,3), clu = c(1,1,2,3,2,2,3,2,4,2,5,2,2,1,1,2,2),
+              vcol = c("blue","red","green","orange","peru"), alpha = .5)
# list by concatenating scopes
c(scpAc, scpA)
```

```
$lty
[1] 1 3

$clu
 [1] 1 1 2 3 2 2 3 2 4 2 5 2 2 1 1 2 2

$vcol
[1] "blue"   "red"    "green"  "orange" "peru"

$alpha
[1] 0.5

$ecol
[1] 1

$vcol
[1] "#C0C0C0"
...
```

Factions with weak balance structure

paths main component of Incubator A

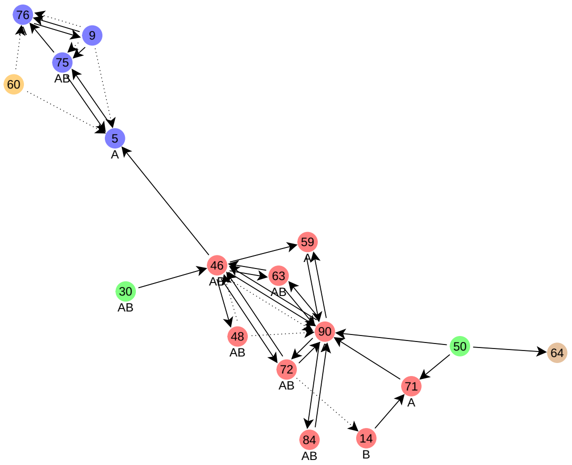
```
# plot combining scopes
multigraph(nsA, layout = "force", seed = 123, scope = c(scpA, scpAc))
```



Social influence through comparison

weak balance structure with paths

```
multigraph(nsA, att = nsA[,,3:4], layout = "force", seed = 123,
+     scope = c(scpA, scpAc))
```



$5.$ Affiliation networks

Example 5: Group of Twenty

Affiliation networks

- Ties between two sets of entities represent two-mode, bipartite, or *affiliations networks*

  ➠ like the duality between *"people and groups"*, *"person and events"*, *"actors and their attributes"*

- In a 2-mode matrix data the domain and the codomain are not equal

  ➠ serves to represent affiliations networks

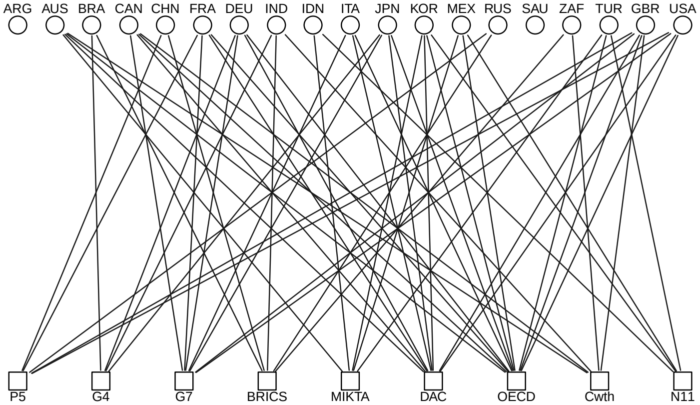
Group of Twenty (G20) affiliation network

```r
G20 <- data.frame( P5    = c(0,0,0,0,1,0,1,1,0,0,0,0,0,0,1,0,0,1,0),
                   G4    = c(0,0,1,0,0,1,0,0,0,1,0,1,0,0,0,0,0,0,0),
                   G7    = c(0,0,0,1,0,1,1,1,0,0,1,1,0,0,0,0,0,1,0),
                   BRICS = c(0,0,1,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,1),
                   MITKA = c(0,1,0,0,0,0,0,0,1,0,0,0,1,1,0,0,1,0,0),
                   DAC   = c(0,1,0,1,0,1,1,1,0,0,1,1,1,0,0,0,0,1,0),
                   OECD  = c(0,1,0,1,0,1,1,1,0,0,1,1,1,1,0,0,1,1,0),
                   Cwth  = c(0,1,0,1,0,0,0,1,0,1,0,0,0,0,0,0,0,0,1),
                   N11   = c(0,0,0,0,0,0,0,0,1,0,0,0,1,1,0,0,1,0,0) )
rownames(G20) <- c("ARG","AUS","BRA","CAN","CHN","DEU","FRA","GBR","IDN","IND",
                   "ITA","JPN","KOR","MEX","RUS","SAU","TUR","USA","ZAF")
```

```
    P5 G4 G7 BRICS MITKA DAC OECD Cwth N11
ARG  0  0  0     0     0   0    0    0   0
AUS  0  0  0     0     1   1    1    1   0
BRA  0  1  0     1     0   0    0    0   0
CAN  0  0  1     0     0   1    1    1   0
CHN  1  0  0     1     0   0    0    0   0
DEU  0  1  1     0     0   1    1    0   0
FRA  1  0  1     0     0   1    1    0   0
GBR  1  0  1     0     0   1    1    1   0
IDN  0  0  0     0     1   0    0    0   1
IND  0  1  0     1     0   0    0    1   0
ITA  0  0  1     0     0   1    1    0   0
JPN  0  1  1     0     0   1    1    0   0
KOR  0  0  0     0     1   1    1    0   1
MEX  0  0  0     0     1   0    1    0   1
RUS  1  0  0     1     0   0    0    0   0
SAU  0  0  0     0     0   0    0    0   0
TUR  0  0  0     0     1   0    1    0   1
USA  1  0  1     0     0   1    1    0   0
ZAF  0  0  0     1     0   0    0    1   0
```

Bipartite graph

```
# bipartite graph
bmgraph(G20, rot = 90, mirrorX = TRUE)
```



Clustering information in G20

```r
# actor clustering (IMF economic classification of countries)
ac <- c(1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1)
ac <- replace(ac, ac==0, "Emerging")
ac <- replace(ac, ac==1, "Advanced")
```

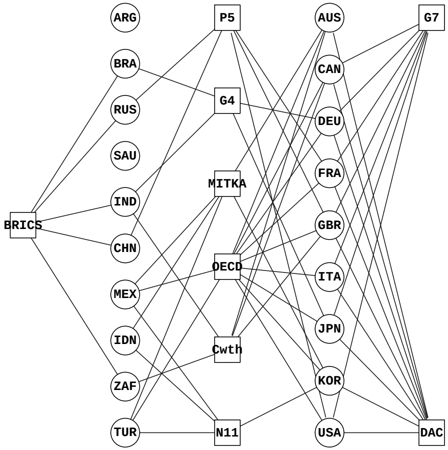
```
 [1] "Advanced" "Emerging" "Advanced" "Emerging" "Advanced" "Emerging" "Emerging"
 [8] "Emerging" "Advanced" "Advanced" "Emerging" "Emerging" "Emerging" "Advanced"
[15] "Advanced" "Advanced" "Advanced" "Emerging" "Advanced"
```

```r
# event clustering information
ec <- c(1, 1, 2, 0, 1, 2, 1, 1, 1)
ec <- replace(ec, ec==0, "Emerging")
ec <- replace(ec, ec==1, "Mixed")
ec <- replace(ec, ec==2, "Advanced")
```

```
[1] "Mixed"    "Mixed"    "Advanced" "Emerging" "Mixed"    "Advanced" "Mixed"
[8] "Mixed"    "Mixed"
```

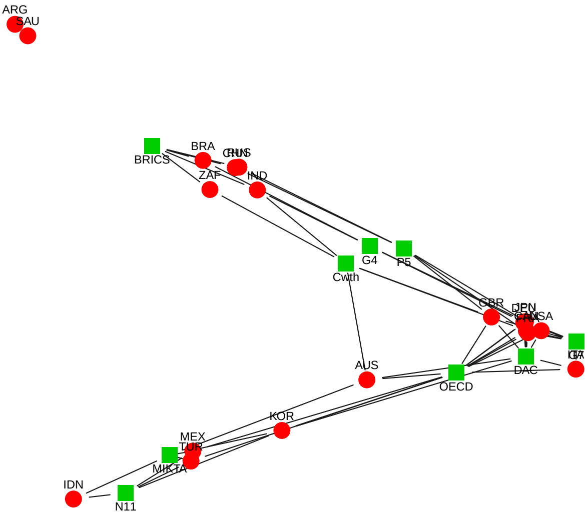
Clustered bipartite graph

```
# clustering info and permutation as lists
bmgraph(G20, layout = "bipc", clu = list(ac,ec), perm = list(c(...),c(...)) )
```



Correspondence analysis

```
# a binomial projection
bmgraph(G20, layout = "CA", rot = 99, vcol = 2:3, pch = c(19, 15), jitter = .1)
```



Formal Concept Analysis (Ganter & Wille, 1996)

algebraic approach

- *Formal concept analysis* is an analytical framework for the study of affiliation networks

- Elements in the domain and codomain are called *objects* and *attributes* resp.

- The set of objects G and the set of attributes M are associated with an incident relation $I \subseteq G \times M$ in a *formal context*

- The *formal concept* of a formal context is a pair of sets of maximally contained objects A and attributes B

  ➟ (i.e. maximal rectangles in the formal context)

  A and B are said to be the *extent* and *intent* of the formal concept

Galois Derivations

- A *Galois derivation* between G and M is defined for any subsets $A \subseteq G$ and $B \subseteq M$ by

$$
\begin{aligned}
A' &= m \in M \mid (g,m) \in I \quad \text{(for all } g \in A) \\
B' &= g \in G \mid (g,m) \in I \quad \text{(for all } m \in B)
\end{aligned}
$$

– A' is the set of attributes common to all the objects in the intent

– B' the set of objects possessing the attributes in the extent

```r
formals("galois")
```

```
$x


$labeling
c("full", "reduced")
```

Galois derivations in G20

```r
galois(G20)
```

```
$P5
[1] "CHN, FRA, GBR, RUS, USA"

$G4
[1] "BRA, DEU, IND, JPN"

$`DAC, G7, OECD`
[1] "CAN, DEU, FRA, GBR, ITA, JPN, USA"

$BRICS
[1] "BRA, CHN, IND, RUS, ZAF"

$MIKTA
[1] "AUS, IDN, KOR, MEX, TUR"

$`DAC, OECD`
[1] "AUS, CAN, DEU, FRA, GBR, ITA, JPN, KOR, USA"

$OECD
[1] "AUS, CAN, DEU, FRA, GBR, ITA, JPN, KOR, MEX, TUR, USA"

$Cwth
[1] "AUS, CAN, GBR, IND, ZAF"

$`MIKTA, N11`
[1] "IDN, KOR, MEX, TUR"

$`BRICS, Cwth, DAC, G4, G7, MIKTA, N11, OECD, P5`
character(0)
```

Galois derivations in G20 – Reduced labeling

```r
g20gc <- galois(G20, labeling = "reduced")
```

```
$reduc
$reduc$P5
character(0)

$reduc$G4
character(0)

$reduc$G7
[1] "ITA"

$reduc$BRICS
character(0)

$reduc$MIKTA
character(0)

$reduc$DAC
character(0)

$reduc$OECD
character(0)

$reduc$Cwth
character(0)
```

```
$reduc$N11
[1] "IDN"

$reduc[[10]]
character(0)

$reduc[[11]]
[1] "FRA, USA"

$reduc[[12]]
[1] "CHN, RUS"

$reduc[[13]]
[1] "GBR"

$reduc[[14]]
[1] "DEU, JPN"

$reduc[[15]]
[1] "BRA"

$reduc[[16]]
[1] "IND"

$reduc[[17]]
[1] "CAN"
```

```
$reduc[[18]]
[1] "ZAF"

$reduc[[19]]
[1] ""

$reduc[[20]]
character(0)

$reduc[[21]]
[1] "AUS"

$reduc[[22]]
character(0)

$reduc[[23]]
[1] "KOR"

$reduc[[24]]
[1] "MEX, TUR"

$reduc[[25]]
[1] "ARG, SAU"
```

Galois derivations and partial ordering

```r
# structure of g20gc object created with a reduced labeling
str(g20gc)
```

```
List of 2
 $ full :List of 25
  ..$ P5                                        : chr "CHN, FRA, GBR, RUS, USA"
  ..$ G4                                        : chr "BRA, DEU, IND, JPN"
  ..$ DAC, G7, OECD                             : chr "CAN, DEU, FRA, GBR, ITA, JPN, USA"
  ..$ BRICS                                     : chr "BRA, CHN, IND, RUS, ZAF"
  ..$ MIKTA                                     : chr "AUS, IDN, KOR, MEX, TUR"
  ..$ DAC, OECD                                 : chr "AUS, CAN, DEU, FRA, GBR, ITA, JPN,
  ..$ OECD                                      : chr "AUS, CAN, DEU, FRA, GBR, ITA, JPN,
  ..$ Cwth                                      : chr "AUS, CAN, GBR, IND, ZAF"
  ..$ MIKTA, N11                                : chr "IDN, KOR, MEX, TUR"
  ..$ BRICS, Cwth, DAC, G4, G7, MIKTA, N11, OECD, P5: chr(0)
...

..- attr(*, "class")= chr [1:2] "Galois" "full"
 $ reduc:List of 25
  ..$ P5    : chr(0)
  ..$ G4    : chr(0)
  ..$ G7    : chr "ITA"
  ..$ BRICS: chr(0)
  ..$ MIKTA: chr(0)
  ..$ DAC   : chr(0)
  ..$ OECD : chr(0)
  ..$ Cwth : chr(0)
  ..$ N11   : chr "IDN"
  ..$       : chr(0)
...
```

Partial ordering of the Concepts

A *hierarchy* of concepts is given by the sub–superconcept relation

$$(A, B) \leq (A_2, B_2) \quad \Leftrightarrow \quad A_1 \subseteq A_2 \qquad (\Leftrightarrow \quad B_1 \subseteq B_2)$$

Concept lattice of the context

- built from the hierarchy structure of concepts
- The greatest lower bound of the meet and the least upper bound of the join are defined for an index set T as

$$\bigwedge_{t \in T} (A_t, B_t) = \Big(\bigcap_{t \in T} A_t, \; (\bigcup_{t \in T} B_t)'' \Big)$$

$$\bigvee_{t \in T} (A_t, B_t) = \Big((\bigcup_{t \in T} A_t)'', \; \bigcap_{t \in T} B_t \Big)$$

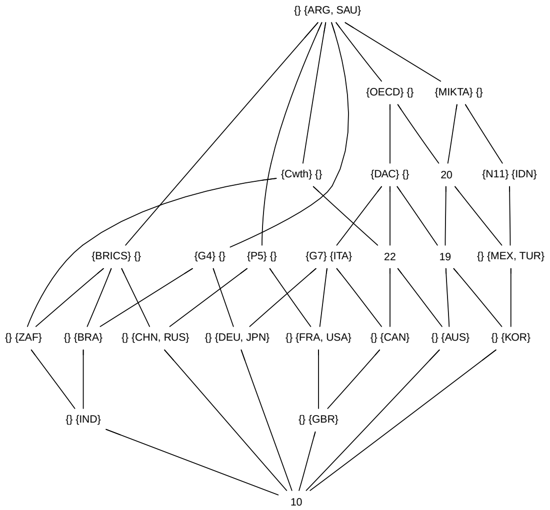
Partial order of concepts

```r
# construct hierarchy of concepts
g20gcpo <- partial.order(g20gc, type = "galois")
```

```
                {P5} {} {G4} {} {G7} {ITA} {BRICS} {} {MIKTA} {} {DAC} {} {OECD} {} {Cwth} {}
{P5} {}               1       0          0          0           0        0         0         0
{G4} {}               0       1          0          0           0        0         0         0
{G7} {ITA}            0       0          1          0           0        1         1         0
{BRICS} {}            0       0          0          1           0        0         0         0
{MIKTA} {}            0       0          0          0           1        0         0         0
{DAC} {}              0       0          0          0           0        1         1         0
{OECD} {}             0       0          0          0           0        0         1         0
{Cwth} {}             0       0          0          0           0        0         0         1
{N11} {IDN}           0       0          0          0           1        0         0         0
10                    1       1          1          1           1        1         1         1
{} {FRA, USA}         1       0          1          0           0        1         1         0
{} {CHN, RUS}         1       0          0          1           0        0         0         0
{} {GBR}              1       0          1          0           0        1         1         1
{} {DEU, JPN}         0       1          1          0           0        1         1         0
{} {BRA}              0       1          0          1           0        0         0         0
{} {IND}              0       1          0          1           0        0         0         1
{} {CAN}              0       0          1          0           0        1         1         1
{} {ZAF}              0       0          0          1           0        0         0         1
19                    0       0          0          0           1        1         1         0
20                    0       0          0          0           1        0         1         0
{} {AUS}              0       0          0          0           1        1         1         1
22                    0       0          0          0           0        1         1         1
{} {KOR}              0       0          0          0           1        1         1         0
{} {MEX, TUR}         0       0          0          0           1        0         1         0
{} {ARG, SAU}         0       0          0          0           0        0         0         0
```

Concept lattice of the context

```
# plot hierarchy of concepts as lattice diagram
diagram(g20gcpo)
```



Order Filters and Order Ideals

formal definition

- Let $(P, \leq)$ be an ordered set, and a, b are elements in P

- A non-empty subset U [resp. D] of P is an upset [resp. downset] called a *order filter* [resp. *order ideal*] if, for all $a \in P$ and $b \in U$ [resp. D]

$$b \leq a \quad \text{implies} \quad a \in U \qquad\qquad [\text{ resp. } a \leq b \quad \text{implies} \quad a \in D\ ]$$

- The upset $\uparrow x$ formed for all the upper bounds of $x \in P$ is called a *principal order filter* generated by x

- Dually, $\downarrow x$ is a *principal order ideal* with all the lower bounds of $x \in P$

☛ order filters and order ideals not coinciding with P are called *proper*

Order Filters and Order Ideals

```r
# find principal order filters in the partial order context
formals("fltr")
```

```
$x


$PO


$rclos
[1] TRUE

$ideal
[1] FALSE
```

Principal Order Filters

```r
# principal order filter of first concept
fltr(1, g20gcpo)
```

```
$`1`
[1] "{P5} {}"

$`25`
[1] "{} {ARG, SAU}"
```

```r
# another option is to use intent labels of different concepts
fltr(c("P5", "BRICS"), g20gcpo)
```

```
$`1`
[1] "{P5} {}"

$`4`
[1] "{BRICS} {}"

$`25`
[1] "{} {ARG, SAU}"
```

Principal Order Ideals

```r
# principal order ideal of the first concept in g20gcpo
fltr("P5", g20gcpo, ideal = TRUE)
```

```
$`1`
[1] "{P5} {}"

$`10`
[1] "10"

$`11`
[1] "{} {FRA, USA}"

$`12`
[1] "{} {CHN, RUS}"

$`13`
[1] "{} {GBR}"
```

$5b.$ Multilevel networks

Example 5: Group of Twenty

Network tie interlock concepts

- *Social structure*: in *simple* networks, configuration made of ties between actors

- *Positional system*: in *multilevel* networks, reduced structures of actors and events

- *Relational structure*: in *multiplex* networks, configuration made of interrelations between relations

- *Role structure*: relational system of aggregated relations

➟ *use of algebraic objects to represent relational and role structures*

➟ *apply relational and role structure notions to multilevel networks*

Multilevel network

A *multilevel network* X^{mlvl} in the context of social systems is

$$X^{mlvl} = \langle\, N,\ M,\ R_N,\ R_M,\ R_{N\times M}\, \rangle$$

for vertex sets N (domain) and M (codomain) that stand for n and m social entities, respectively.

➟ edge set R_N for relations on N (directed or not)

➟ edge set R_M for relations on M (undirected)

➟ *constitutive* relations $R_{N\times M}$ for the embeddedness of R_N in R_M (the two domains as levels in X^{mlvl})

- A *multiplex network* X^+ adds $r > 1$ types of relations R_N.
- A *valued* network $X^V = \langle N, R, V\rangle$ with V for weights
- An *affiliation network* X^B is a bipartite system with two domains N and M and constitutive relations $R_{N\times M}$ between domains

G20 affiliation network

```r
# object 'G20' is the formal context as a data frame
load("./data/G20.rda")
```

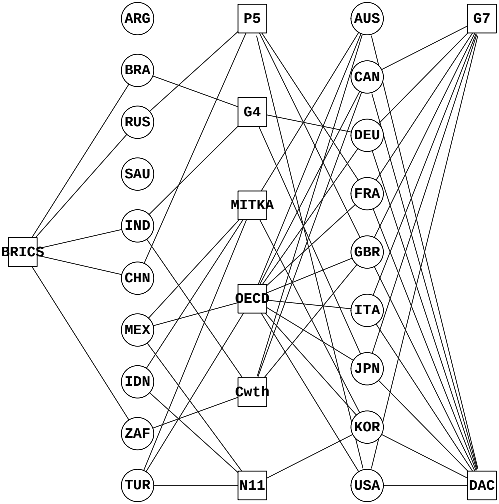
```
    P5 G4 G7 BRICS MITKA DAC OECD Cwth N11
ARG  0  0  0     0     0   0    0    0   0
AUS  0  0  0     0     1   1    1    1   0
BRA  0  1  0     1     0   0    0    0   0
CAN  0  0  1     0     0   1    1    1   0
CHN  1  0  0     1     0   0    0    0   0
DEU  0  1  1     0     0   1    1    0   0
FRA  1  0  1     0     0   1    1    0   0
GBR  1  0  1     0     0   1    1    1   0
IDN  0  0  0     0     1   0    0    0   1
IND  0  1  0     1     0   0    0    1   0
ITA  0  0  1     0     0   1    1    0   0
JPN  0  1  1     0     0   1    1    0   0
KOR  0  0  0     0     1   1    1    0   1
MEX  0  0  0     0     1   0    1    0   1
RUS  1  0  0     1     0   0    0    0   0
SAU  0  0  0     0     0   0    0    0   0
TUR  0  0  0     0     1   0    1    0   1
USA  1  0  1     0     0   1    1    0   0
ZAF  0  0  0     1     0   0    0    1   0
```

Group of Twenty (G20) affiliations

clustered bipartite graph

circles: *actors* in N

squares: *events* in M



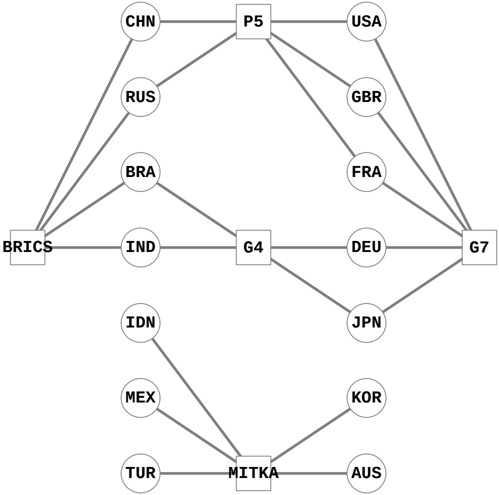
Classes of actors in G20 with "bridges"

1. G7
2. BRICS
3. MITKA

AUS	BRA	CHN	DEU	FRA	GBR	IDN	IND	JPN	KOR	MEX	RUS	TUR	USA
3	2	2	1	1	1	3	2	1	3	3	2	3	1

G20 with non-overlapping "bridge" organisations

clustered bipartite graph



Constructing G20 affiliation network with bridges

```r
# Option: P5 G4 MITKA none
acb <- factor(ac, levels = c("P5", "G4", "MITKA", "none"))
acb[which(G20[,1]==1)] <- "P5"    ; acb[which(G20[,2]==1)] <- "G4"
acb[which(G20[,5]==1)] <- "MITKA"; acb[which(is.na(acb))] <- "none"
```

```
[1] none MITKA G4 none P5 G4 P5 P5 MITKA G4 none G4 MITKA MITKA P5 none MITKA P5 none
Levels: P5 G4 MITKA none
```

```r
# bridge organisations
bridges <- which(acb!="none")

# G20 bridged affiliation network with a binomial projection
require("magrittr")
G20[bridges, c(1:5)] %>%
  bmgraph(layout = "force", seed = 321)
```

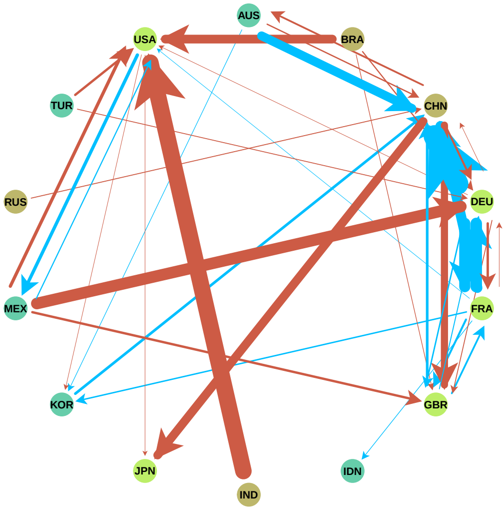
Plotting skeleton G20 trade network with bridges

```r
# load network data: G20net skeleton trade m&h
load(file = "./data/G20net.rda")

# to extract of the G20 m&h multiplex network (exclude)
club <- c(3, 2, 2, 1, 1, 1, 3, 2, 1, 3, 3, 2, 3, 1)

# define scopes and clustering information
scp <- list(cex = 4, pos = 0, fsize = 8, fstyle = "bold")
scpb <- list(vcol = c("#BCEE68","#BDB76B","#66CDAA"),
             ecol = c("#00BFFF","#CD5B45"))
```

```r
# plot graph and combine different types of scopes
require("magrittr")
G20net[bridges, bridges, ] %>%
  multigraph(valued = TRUE, scope = c(scp, scpb), clu = club)
```

G20 bridges trade network X^V_{G20b}

valued skeleton after triangle inequality



Directed edges E_N: blue for fresh milk (R_1), red for honey (R_2)

Co-membership multilevel valued graph

G20 multilevel structure with bridges

Plot with `mlgraph()`:

```
# default circular layout
require("magrittr")
mlvl(y = G20[bridges, c(1:5)], type = "cn") %>%
  mlgraph(valued = TRUE)
```

Plot valued graph with co-membership values with `multigraph()`:

```
# default circular layout
require("magrittr")
mlvl(y = G20[bridges, c(1:5)], type = "cn") %>%
  multigraph(valued = TRUE, values = TRUE, undRecip = TRUE)
```

Multilevel structure of G20 with bridges

actors co-affiliation

```r
# multilevel with co-affiliation of actors
require("magrittr")
mlvl(x = G20net[bridges,bridges,], y = G20[bridges,c(1:5)], type = "cn2") %>%
  mlgraph()
```

```
...

$lbs
$lbs$dm
 [1] "AUS" "BRA" "CHN" "DEU" "FRA" "GBR" "IDN" "IND" "JPN" "KOR" "MEX" "RUS" "TUR" "USA"

$lbs$cdm
[1] "P5"    "G4"    "G7"    "BRICS" "MITKA"


$modes
[1] "1M" "1M" "2M"

attr(,"class")
[1] "Multilevel" "cn2"
```

Actor co-affiliation

multilevel structure of G20 with bridges

Additional clustering information for events and `club` still for actors

```
# clustering information with events
club2 <- list(club, rep(1, ncol(G20[bridges, c(1:5)])) )
```

```
[[1]]
 [1] 3 2 2 1 1 1 3 2 1 3 3 2 3 1

[[2]]
[1] 1 1 1 1 1
```

```
# plot multilevel network with updated clustering info
require("magrittr")
mlvl(x = G20net[bridges,bridges,], y = G20[bridges,c(1:5)], type = "cn2") %>%
  mlgraph(valued = TRUE, scope = c(scp, scpb), clu = club2)
```

Binomial projection in multilevel structures

```r
# m&h trade network with G20 bridges
mlvl(x = G20net[bridges,bridges,], y = G20[bridges,c(1:5)], type = "bpn",
     lbs = c("M","H","O"))
```

```
$mlnet
, , M

      AUS BRA   CHN  DEU  FRA  GBR IDN IND JPN KOR  MEX RUS TUR USA P5 G4 G7 BRICS MITKA
AUS     0   0  5829    0    0    0   0   0   0 412    0   0   0   0  0  0  0     0     0
BRA     0   0     0    0    0    0   0   0   0   0    0   0   0   0  0  0  0     0     0
CHN     0   0     0    0    0    0   0   0   0   0    0   0   0   0  0  0  0     0     0
DEU     0   0 14417    0 7358  928   0   0   0   0    0   0   0   0  0  0  0     0     0
FRA     0   0  6714 7553    0 1345 443   0   0 921    0   0   0 254  0  0  0     0     0
GBR     0   0  1734  641 1170    0   0   0   0   0    0   0   0   0  0  0  0     0     0
IDN     0   0     0    0    0    0   0   0   0   0    0   0   0   0  0  0  0     0     0
...

, , O

      AUS BRA CHN DEU FRA GBR IDN IND JPN KOR MEX RUS TUR USA P5 G4 G7 BRICS MITKA
AUS     0   0   0   0   0   0   0   0   0   0   0   0   0   0  0  0  0     0     1
BRA     0   0   0   0   0   0   0   0   0   0   0   0   0   0  0  1  0     1     0
CHN     0   0   0   0   0   0   0   0   0   0   0   0   0   0  1  0  0     1     0
DEU     0   0   0   0   0   0   0   0   0   0   0   0   0   0  0  1  1     0     0
FRA     0   0   0   0   0   0   0   0   0   0   0   0   0   0  1  0  1     0     0
GBR     0   0   0   0   0   0   0   0   0   0   0   0   0   0  1  0  1     0     0
IDN     0   0   0   0   0   0   0   0   0   0   0   0   0   0  0  0  0     0     1
...
```

Multilevel structure with binomial projection

scope and clustering

Define additional scopes to handle events

```
# shapes and color of vertices for actors and events
scpm <- list(ecol = c("blue","red","orange"), pch = c(21,15), vcol0 = 8,
             vcol = c("#BCEE68","#BDB76B","#66CDAA","#838B8B","#FF7F00"))

# four classes of actors as factor with explicit levels
acc <- factor(ac, levels = c("A-G7", "Advanced", "E-BRICS", "Emerging"))
acc[which(G20[,3]==1)] <- "A-G7"; acc[which(G20[,4]==1)] <- "E-BRICS"

# clustering information for multilevel structure
cluml <- list(acc[bridges], rep(1, nrow(G20[bridges,c(1:5)])))
```

```
[[1]]
 [1] Advanced E-BRICS  E-BRICS  A-G7     A-G7     A-G7     Emerging E-BRICS  A-G7     Advanced
[11] Emerging E-BRICS  Emerging A-G7
Levels: A-G7 Advanced E-BRICS Emerging

[[2]]
 [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

Multilevel structure with binomial projection

plotting

Multilevel network with binomial projection (updated clustering information)

```
# plot with default circular layout and recycling scope
require("magrittr")
mlvl(x = G20net[bridges,bridges,], y = G20[bridges,c(1:5)], type = "bpn") %>%
  mlgraph(scope = c(scp, scpm), clu = cluml)

# clustering information in multilevel structure
nr <- c(rep(1,nrow(G20[bridges,c(1:5)])), rep(2,ncol(G20[bridges,c(1:5)])))
```

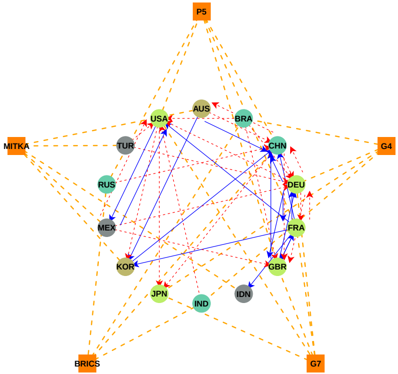
```
# [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2
```

Plot multilevel graph with binomial projection

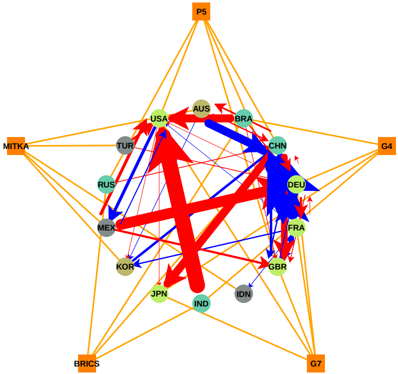
```
# concentric layout with two radii and recycled scopes
require("magrittr")
mlvl(x = G20net[bridges,bridges,], y = G20[bridges,c(1:5)], type = "bpn") %>%
  mlgraph(layout = "conc", nr = nr, scope = c(scp,scpm), clu = cluml, valued = TRUE)
```

Multilevel structure with binomial projection

concentric layout



valued



Multilevel positional systems for G20 bridges

Functions `reduc()` to reduce array structures

```
# positional system actors with clustering information
PSG20Ba <- reduc(G20net[bridges, bridges, ], valued = TRUE, clu = club,
                 lbs = c("G7.C","BRICS.C","MITKA.C"))
```

```
, , M

         G7.C BRICS.C MITKA.C
G7.C    19249   22865    3538
BRICS.C     0       0       0
MITKA.C   752    7572     412

, , H

         G7.C BRICS.C MITKA.C
G7.C     3050     296     293
BRICS.C 29577     584    1187
MITKA.C 13477     860       0
```

```
# positional system events ('row' option for two-mode networks)
PSG20Be <- reduc(G20[bridges,c(1:5)], valued = TRUE, clu = club, row = TRUE,
                 lbs = c("G7.C","BRICS.C","MITKA.C"))
```

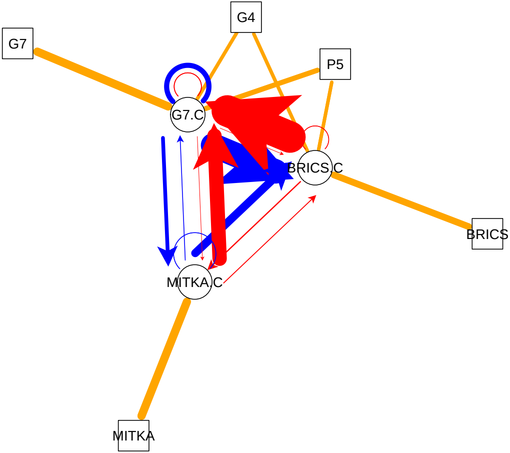
Positional system for G20 bridges with binomial projection

```r
# symmetrize co-domain
mlvl(x = PSG20Ba, y = PSG20Be, type = "bpn", symCdm = TRUE)
```

```
$mlnet
, , m

         G7.C BRICS.C MITKA.C P5 G4 G7 BRICS MITKA
G7.C    19249   22865    3538  0  0  0     0     0
BRICS.C     0       0       0  0  0  0     0     0
MITKA.C   752    7572     412  0  0  0     0     0
P5          0       0       0  0  0  0     0     0
G4          0       0       0  0  0  0     0     0

...


, , 3

        G7.C BRICS.C MITKA.C P5 G4 G7 BRICS MITKA
G7.C       0       0       0  3  2  5     0     0
BRICS.C    0       0       0  2  2  0     4     0
MITKA.C    0       0       0  0  0  0     0     5
P5         3       2       0  0  0  0     0     0
G4         2       2       0  0  0  0     0     0
G7         5       0       0  0  0  0     0     0
BRICS      0       4       0  0  0  0     0     0
MITKA      0       0       5  0  0  0     0     0
```

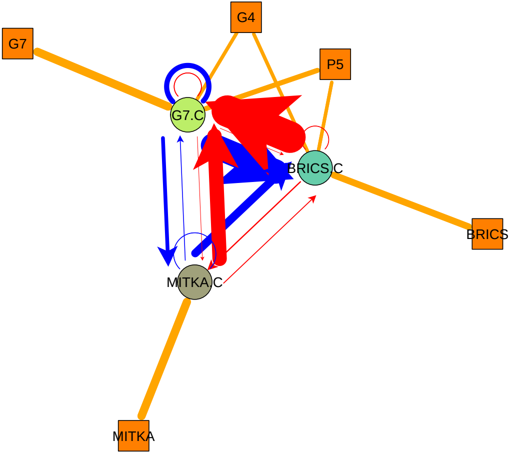
Valued multilevel positional systems G20 bridges

```
# plot multilevel graph of positional system
require("magrittr")
mlvl(x = PSG20Ba, y = PSG20Be, type = "bpn", symCdm = TRUE) %>%
  mlgraph(valued = TRUE, layout = "force", seed = 1, cex = 6, pos = 0,
          ecol = c("blue","red","orange"), fsize = 11)
```

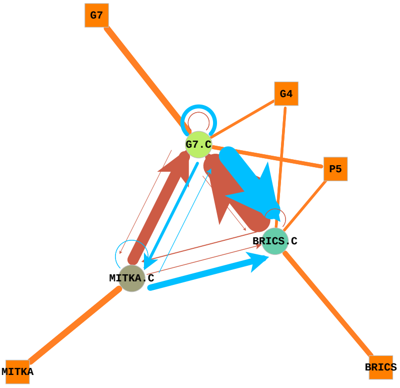


Valued multilevel positional system G20 bridges

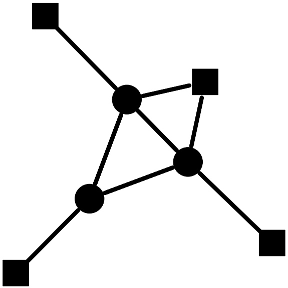
```
scpps <- list(cex=6, ecol=c("blue","red","orange"), clu=c(1:3, rep(4,5)),
              vcol=c("#BCEE68","#66CDAA","#A0A17B","#FF7F00"), pos=0, fsize=11)

require("magrittr")
mlvl(x = PSG20Ba, y = PSG20Be, type = "bpn", symCdm = TRUE) %>%
  mlgraph(layout = "force", seed = 1, valued = TRUE, scope = scpps)
```



Positional systems of X^{mlvl}_{G20b}



valued multilevel multigraph



skeleton with Structural equivalence applied

Multilevel positional system

Image matrices for X^{mlvl}_{G20b} made with X^{V}_{G20b} and class affiliations in X^{B}_{G20}

	G7.C	BRICS.C	MITKA.C
G7.C	19249	22865	3538
BRICS.C	0	0	0
MITKA.C	752	7572	412

Fresh Milk

	G7.C	BRICS.C	MITKA.C
G7.C	3050	296	293
BRICS.C	29577	584	1187
MITKA.C	13477	860	0

Honey

	P5	G4	G7	BRICS	MITKA
G7.C	3	2	5	0	0
BRICS.C	2	2	0	4	0
MITKA.C	0	0	0	0	5

Affiliation of classes to bridge organizations

Algebraic analysis of multilevel configuration

Role structure of G20 with bridges

G20 Countries network is a multilevel, multiplex and valued structure

➟ complex organisational network

☛ Multilevel version of G20 Countries network with bridges is represented as X_{G20b}^{mlvl}

Partially ordered semigroup of X^{mlvl}_{G20b} positional system

Cayley graph

Generators

m: milk

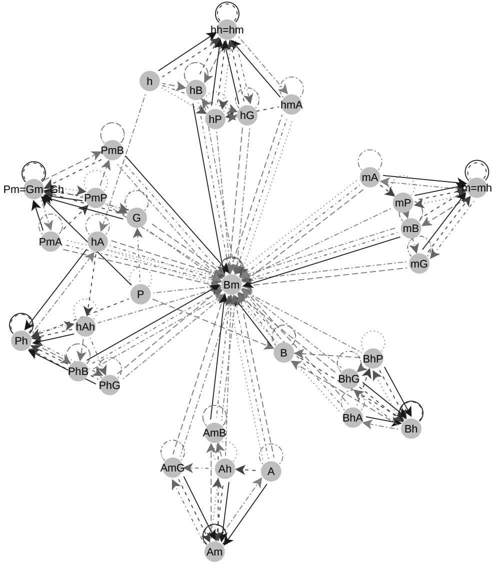
h: honey

G: G7

B: BRICS

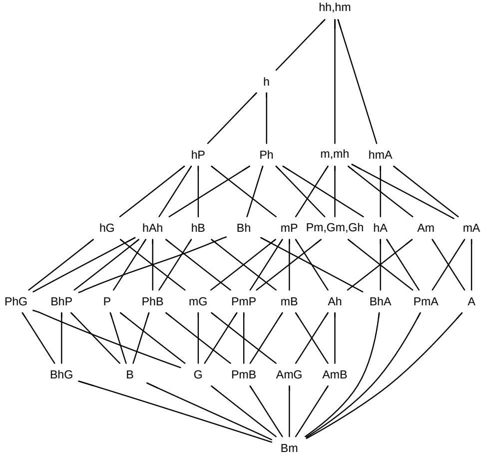
A: MITKA

P: P5 and G4



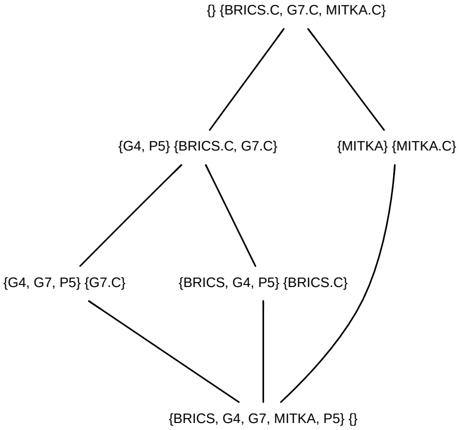
Partially ordered semigroup of X^{mlvl}_{G20b} positional system

Inclusion diagram



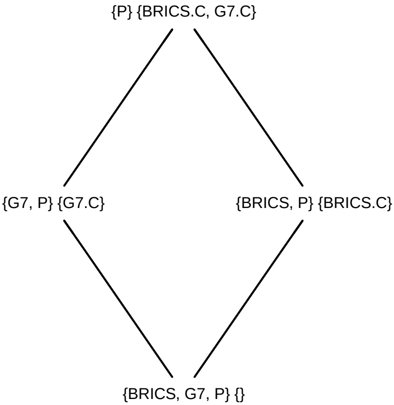
Concept diagram of X_{G20b}^{mlvl} positional system

full labeling

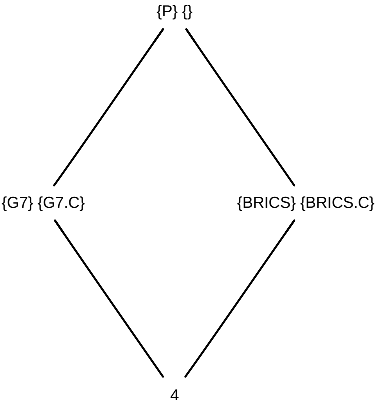


Concept diagrams of X^{mlvl}_{G20b}

with a two element positional system



full labeling



reduced labeling

+ Example 5: G20 Valued Trade network

Relational structure

Many-valued context

Relational structure in valued multiplex networks

- Assignments to labeled valued paths with the $\max - \min$ composition
  - ➟ minimum value of sending/receiving scores in nodes
  - ➟ then the maximum of these values

- For the G20 Trade network of milk (M) & honey (H)

$$\mathsf{M} \circ \mathsf{H} \;=\; \max_k\{\min\big(w_{\mathsf{M}}(i,k),\; w_{\mathsf{H}}(k,j)\big)\} \;=\; \mathsf{MH}$$

☛ `multiplex` supports valued networks

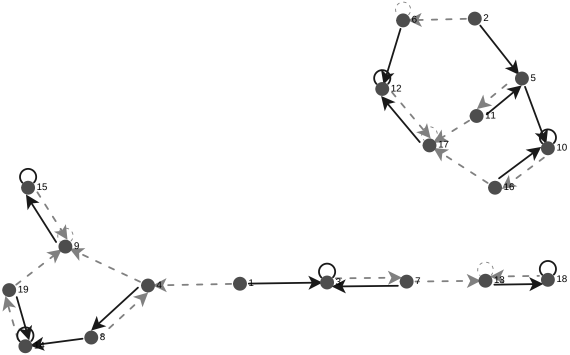
Relational structure in valued multiplex networks

```
load(file = "./data/vnet.rda")
```

```
vnet

, , M

      G7 BRICS MITKA
G7    19    23     4
BRICS  0     0     0
MITKA  1     8     0

, , H

      G7 BRICS MITKA
G7     3     0     0
BRICS 30     1     1
MITKA 13     1     0
```

Relational structure for valued multiplex networks

```
# semigroup with max-min product
semigroup(vnet, valued = TRUE)

# pipeing 'multiplex' and 'multigraph' functions
require("magrittr")
vnet %>% semigroup(valued = TRUE) %>% ccgraph(rot = 90, cex = 2, lwd = 2)
```



Many-valued context

(Wille, 1982)

- A *many-valued context*, $\mathbb{K}^V$ is defined as

$$\mathbb{K}^V = (\, G,\ M,\ W,\ I \,)$$

➟ G is the object set

➟ M is the many-valued attributes

➟ W are "weights" or attribute values

➟ I is the incidence relation.

☛ the context is said to be an k-*valued context* when W has k elements

Many-valued context of G20 network

economic and socio-demographic indicators

```r
# four-valued context
load(file = "./data/G20mv.rda")
```

```
    T  N  G  H  P  A
ARG vl vl l  l  vl l
AUS vl vl vh vh vl h
BRA vl l  l  l  l  h
CAN l  l  vh vh vl h
CHN vh vh vl vl vh h
DEU h  h  vh vh l  vl
FRA l  l  h  h  vl vl
GBR l  l  h  vh vl vl
IDN vl vl vl vl h  l
IND l  l  vl vl vh l
ITA l  l  h  h  vl vl
JPN h  h  h  h  l  vl
KOR l  vl h  h  vl vl
MEX l  vl vl l  l  l
RUS l  vl l  l  l  vh
SAU vl vl l  h  vl l
TUR vl vl vl l  vl vl
USA vh vh vh vh h  h
ZAF vl vl vl vl vl vl

T=trade, N=nom_GDP, G=GDP_PC; H=HDI, P=population, A=area
```

Conceptual scaling

many-valued context of G20 network

	very-high	*high*	*low*	*very-low*
very-high	1	0	0	0
high	0	1	0	0
low	0	0	1	0
very-low	0	0	0	1

Nominal $\mathbb{N}_n = \langle n, n, = \rangle$

$\leq$*very-high*	$\leq$*high*	$\geq$*low*	$\geq$*very-low*
1	1	0	0
0	1	0	0
0	0	1	0
0	0	1	1

Inter-ordinal $\mathbb{I}_n = \langle n, n, \leq \mid n, n, \geq \rangle$

Function `cscl()` uses a scaling matrix

```
# dot separation between extents and scale labels
cscl(G20mv, scl, sep = ".")
```

☛ then plot Concept lattice, and apply order filters and order ideals

Pathfinder semiring and Triangle inequality

one-mode valued networks

For the analysis of adjacency matrices:

- *Pathfinder semiring* for symmetric valued relations

  ➟ matrix reflects the "proximity" between pairs of network members

- *Triangle inequality* for asymmetric valued ties

  ➟ then *salient* structure of valued network

Use functions `pfvn()` and `ti()` from `multiplex`

References

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Thanks!

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