Models with Jumps - Levy Processes

Michael Miller

Introduction to Jump Processes

A pure jump process: - Begins at zero, - Has finitely many jumps in each finite time interval, - Is constant between jumps.

A change of measure with a jump process: - Adjusts the drift of the Brownian Motion, - Adjusts the intensity (average rate of jump arrival), - Adjusts the distribution of the jump sizes for the compound Poisson Process.

Poisson Process: Exponential Random Variables

Let τ be a random variable with density defined by:

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

where λ is a positive constant.

Expected Value of τ:

Using integration by parts,

$$E[\tau] = \int_0^\infty t f(t)\, dt = \lambda \int_0^\infty t e^{-\lambda t}\, dt = -t e^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}\, dt = 0 + \left(\frac{1}{\lambda} e^{-\lambda t}\right)\Big|_0^\infty = \frac{1}{\lambda}$$

In summary:

$$E[\tau] = \frac{1}{\lambda}$$

Cumulative distribution function:

$$F(t) = P(\tau \leq t) = \int_0^t \lambda e^{-\lambda u}\, du = -e^{-\lambda u}\Big|_0^t = 1 - e^{-\lambda t}, \text{ for } t \geq 0$$

Thus,

$$P(\tau > t) = e^{-\lambda t}, \text{ for } t \geq 0$$

For example, this event will occur with a mean of $\frac{1}{\lambda}$. If we already waited s time units, the probability of waiting t additional time units is the same as the probability $P(\tau > t + s | \tau > s) = \frac{P(\tau > t+s \text{ and } \tau > s)}{P(\tau > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}$ of having to wait t time units from a starting point of time 0. The condition that we have already waited s time units does not change the distribution of the remaining time. This is the **memorylessness property** of the Poisson Process.

Constructing a Poisson Process:

- Start with $\tau_1, \tau_2, \ldots$, a sequence of independent exponential random variables all with mean $\frac{1}{\lambda}$.
- The first jump occurs at τ_1, the second jump occurs τ_2 time intervals after τ_1, thus the second jump occurs at time $\tau_1 + \tau_2$, and so on, modeled by:

$$\text{Time of } n\text{th jump} = S_n = \sum_{k=1}^{n} \tau_k$$

- The Poisson process $N(t)$ counts the number of jumps that occur before time t:

$$N(t) = \begin{cases} 0 & \text{if } 0 \leq t < S_1 \\ 1 & \text{if } S_1 \leq t < S_2 \\ \vdots & \\ n & \text{if } S_n \leq t < S_{n+1} \\ \vdots & \end{cases}$$

  This process is forward continuous like time, thus the σ-algebra denotes the information acquired up to time s, $0 \leq s \leq t$ for $N(s)$.

- Expected time between jumps is $\frac{1}{\lambda}$, so jumps arrive at an average rate of λ per unit time (where λ is the intensity).
- S_n has a gamma density for $n \geq 1$:

$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s}, \quad s \geq 0$$

Proof by induction on n:

- For $n = 1$, $S_1 = \tau_1$ is exponential, with density:

$$g_1(s) = \lambda e^{-\lambda s}, \quad s \geq 0$$

- Assume S_n has density $g_n(s)$. To compute the density of S_{n+1}, knowing $S_{n+1} = S_n + \tau_{n+1}$ and both are independent, the density can be computed as follows:

$$\int_0^s g_n(v) f(s-v)\, dv = \int_0^s \frac{(\lambda v)^{n-1}}{(n-1)!} \lambda e^{-\lambda v} \lambda e^{-\lambda(s-v)}\, dv = \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda s} \int_0^s v^{n-1}\, dv$$
$$= \frac{\lambda^{n+1}}{n!} s^n e^{-\lambda s} = g_{n+1}(s)$$

Poisson process $N(t)$ with intensity λ has distribution:

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad \text{for } k = 0, 1, 2, \ldots$$

Proof: - For $k \geq 1$, $N(t) \geq k$ if and only if there are at least k jumps by time t (for instance, if $S_k \leq t$).

$$P(N(t) \geq k) = \int_0^t g_k(s)\, ds$$

- Similarly,

$$P(N(t) \geq k+1) = \int_0^t g_{k+1}(s)\, ds$$

- Integrate by parts:

$$P(N(t) = k) = \left(\frac{(\lambda t)^k}{k!} e^{-\lambda t} \right) - \left(\frac{(\lambda t)^{k+1}}{(k+1)!} e^{-\lambda t} \right)$$

- For $k = 0$,

$$P(N(t) = 0) = e^{-\lambda t}$$

Due to memorylessness,

$$N(t + s) - N(s)$$

, for the time step interval $(s, t + s]$, is independent of what happened previously and has the same distribution as $N(t)$. This property implies that the Poisson process, like Brownian motion, has stationary and independent increments.

Poisson Process Increment Distribution Defined by:

For $0 \leq s \leq t$ and $k = 0, 1, 2, \ldots$:

$$P(N(t) - N(s) = k) = \frac{\lambda^k (t - s)^k}{k!} e^{-\lambda(t-s)}$$

Exponential Power Series

The exponential power series is useful for understanding the expectation of a Poisson process increment:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=2}^{\infty} \frac{x^{k-2}}{(k-2)!}$$

Note that the summation of all increments' probabilities equals 1:

$$\sum_{k=0}^{\infty} P(N(t) - N(s) = k) = 1$$

Expected Value of Poisson Process Increment

The expected value of the increment of a Poisson process is:

$$E[N(t) - N(s)] = \lambda(t - s)$$

This implies that jumps arrive at an average rate of λ per unit time. Thus, the average number of jumps between times s and t is $\lambda(t - s)$.

Second Moment

$$E[(N(t) - N(s))^2] = \lambda^2 (t - s)^2 + \lambda(t - s)$$

Variance of Poisson Process

The variance of the increment of a Poisson process is:

$$\mathrm{Var}(N(t) - N(s)) = \lambda(t - s)$$

This shows that the mean and variance are equal.

Martingale Property for Compensated Poisson Process

Define:

$$M(t) = N(t) - \lambda t$$

Thus, $M(t)$ is a martingale because: - $(N(t) - N(s))$ is independent of the filtration $F(s)$, - $E[N(t) - N(s)] = \lambda(t - s)$, - $E[M(t)|F(s)] = M(s)$.

Construction of Compounded Poisson Process

- $N(t)$ represents a Poisson process.
- λ is the intensity or rate.
- $Y_1, Y_2, \ldots$ are i.i.d. random variables.
- Mean of jump sizes $\beta = E[Y_i]$.

Compound Poisson Process Defined by:

$$Q(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0$$

Jumps in $Q(t)$ and $N(t)$: - Occur at the same time, - $N(t)$ has size=1, while $Q(t)$ has random sizes.

Increment of Compound Poisson Process

$$Q(t) - Q(s) = Q(t - s) = \sum_{i=N(s)+1}^{N(t)} Y_i$$

Expected Value (Mean) of Compound Poisson Process

$$E[Q(t)] = \beta \lambda t$$

This reflects that on average there are λt jumps in the time interval $[0, t]$, the average jump size is β, and the number of jumps is independent of the size of the jumps.

To be a Martingale, the Compensated Compound Poisson Process is:

$$E[Q(t) - \beta \lambda t | F(s)] = Q(s) - \beta \lambda s$$

This is due to the increment $E[Q(t) - Q(s)|F(s)] = \beta \lambda (t - s)$, as it is independent of $F(s)$.

Moment-Generating Function of Poisson Process:

The explicit formula for the $Q(t-s)$ increment of the Poisson Process is too complicated to formulate the density or probability mass function. Thus, the moment-generating function (MGF) is used going forward.

Moment-generating function of a random variable Y_i defined by:

$$\varphi_Y(u) = E[e^{uY_i}]$$

which does not depend on i as all random variables have the same distribution.

Moment-generating function of the Compound Poisson Process $Q(t)$ defined by:

$$\varphi_{Q(t)}(u) = E[e^{uQ(t)}] = e^{\lambda t(\varphi_Y(u)-1)}$$

If Y_i are not random but are a series of constants with value y, then:

$$Q(t) = yN(t), \quad \varphi_Y(u) = e^{uy},$$

Moment Generating Function of Compound Poisson Process with constant y:

$$\varphi_{yN(t)}(u) = E[e^{uyN(t)}] = e^{\lambda t(e^{uy}-1)}$$

If $y = 1$, then:

$$\varphi_{N(t)}(u) = e^{\lambda t(e^{u}-1)}$$

Case: Y_i takes one of finitely many possible non-zero values such that $y_1, y_2, \ldots, y_M$ has probabilities $p(y_m) = P(Y_i = y_m)$ so $p(y_m) > 0$, and $\sum_{m=1}^{M} p(y_m) = 1$:

$$\varphi_{Q(t)}(u) = \prod_{m=1}^{M} e^{\lambda p(y_m)t(e^{uy_m}-1)}$$

This allows for the decomposition of the process where each jump is independent and the mean of its jump.

Two Equivalent Ways of Regarding Compound Poisson Process with Finite Jump Sizes:

First, is the thought that a single Poisson Process in which size-one jumps are replaced with jumps of random size.

Second, it is a sum of independent Poisson Processes in which the size-one jumps in each are replaced by jumps of a fixed size. This alternative is defined here:

$$Q(t) = \sum_{i=1}^{N(t)} Y_i$$

$$N(t) = \sum_{m=1}^{M} N_m(t)$$

$$Q(t) = \sum_{m=1}^{M} y_m N_m(t)$$

- where $N_m(t)$ represents the number of jumps in Q of size y_m up to and including time t,
- each $N_1, \ldots, N_M$ is independent,
- each N_m has intensity $\lambda p(y_m)$.

Integrals of Jump Processes

Brownian Motion and Poisson and Compound Poisson Process always have a single filtration associated with all of them.

Stochastic Integral of Process X:

$$X(t) = X(0) + I(t) + R(t) + J(t)$$

- $X(0)$ represents the nonrandom initial condition,
- $I(t)$ is the Ito integral,
- $R(t)$ is the Riemann integral,
- $J(t)$ is an adapted, right-continuous pure jump process with $J(0) = 0$.

$$J(t^-)$$

represents the value immediately before the jump,

$$J(t)$$

is the value immediately after the jump.

Assume J does not jump at time 0, has finitely many jumps on each finite time interval $(0, T]$, and is constant between jumps.

The constancy between jumps justifies calling $J(t)$ a pure jump process. A Poisson Process and a Compound Poisson Process have this property. A compensated Poisson Process does not because it decreases between jumps.

$$X(t^-) = X(0) + I(t) + R(t) + J(t^-)$$

Thus,

$$X(t) - X(t^-) = J(t) - J(t^-)$$

If $X(t)$ is continuous at t, then $\Delta_X(t) = 0$; if a jump occurs at time t, then $\Delta_X(t) =$ size of the jump.

Stochastic Integral of Phi (adapted process) with respect to X (jump process) is defined by:

$$\int_0^t \Phi(s)\, dX(s) = \int_0^t \Phi(s)\Gamma(s)\, dW(s) + \int_0^t \Phi(s)\Theta(s)\, ds + \sum_{0<s\leq t} \Phi(s)\Delta_J(s)$$

Differential notation:

$$\Phi(t)\, dX(t) = \Phi(t)\, dI(t) + \Phi(t)\, dR(t) + \Phi(t)\, dJ(t) = \Phi(t)\, dX^c(t) + \Phi(t)\, dJ(t),$$

where:

$$\Phi(t)\, dI(t) = \Phi(t)\Gamma(t)\, dW(t)$$
$$\Phi(t)\, dR(t) = \Phi(t)\Theta(t)\, dt$$

Thus,

$$\Phi(t)\, dX^c(t) = \Phi(t)\Gamma(t)\, dW(t) + \Phi(t)\Theta(t)\, dt$$

Example: $M(t) = N(t) - \lambda t$**, where** $M(t)$ **is a compensated Poisson Process**

- $J(t) = N(t)$,
- $\Phi(s) = \Delta N(s)$ where it takes the value of 1 if N has a jump at time s and 0 otherwise.

$$\int_0^t \Phi(s)\, dX^c(s) = 0$$
$$\int_0^t \Phi(s)\, dN(s) = N(t)$$

Thus,

$$\int_0^t \Phi(s)\, dM(s) = N(t)$$

We want a stochastic integral with respect to a martingale to be a martingale. However, that is not always the case, as $M(t)$ is a martingale but $N(t)$ is not because it can increase but cannot decrease.

An agent who invests in the compensated Poisson Process $M(t)$ by choosing his position according to the formula $\Phi(s) = \Delta N(s)$ has created an arbitrage. To do this, he holds a zero position at all times except at the jump times of $N(s)$, which are also the jump times of $M(s)$, at which times he holds a position of one. Because the jumps in $M(s)$ are always upwards and our investor holds a long position precisely at the jump times, he will reap the upside gain from all these jumps and have no possibility of loss.

In reality, the portfolio process $\Phi(s) = \Delta N(s)$ cannot be implemented because investors must take positions before jumps occur.

It is not enough for the strategy to be adapted (does not depend on the future of the path). It also needs to be left-continuous:

$$E\left[\int_0^t \Gamma^2(s)\Phi^2(s)\, ds\right] < \infty \text{ for all } t \geq 0.$$

Then, the stochastic integral $\int_0^t \Phi(s)\, dX(s)$ is also a martingale.

However, the integrator $X(t)$ is always right-continuous, so the integral $\int_0^t \Phi(s)\, dX(s)$ will be right-continuous at the upper limit of integration t. The integral jumps when both X jumps and Φ is not zero.

This binary value is shown here:

$$\Phi(s) = I_{[0,S_1]}(s)$$

Thus,

$$\int_0^t \Phi(s)\, dM(s) = I_{[S_1,\infty]}(t) - \lambda \min(t, S_1)$$

If strictly right-continuous, it would not be a martingale as the above would look like:

$$\int_0^t I_{[0,S_1]}(u)\, dM(u) = -\lambda \min(t, S_1)$$

When $s = 0$, you get:

$$E[-\lambda \min(t, S_1)] = e^{-\lambda t} - 1$$

which is strictly decreasing in t, thus not a martingale.

Quadratic Variation of Jump Processes

Brownian Motion's quadratic variation:

$$[X_1^c, X_2^c](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)\, ds$$

Extra jump term:

$$[J_1, J_2](T) = \Sigma_{0<s\leq T} \Delta J_1(s) \Delta J_2(s)$$

As time intervals approach zero, the Brownian Motion's quadratic variation approaches zero, whereas the jump term's quadratic variation converges to a finite number, not depending on the time intervals.

Differential Notation:

$$X_1(t) = X_1(0) + X_1^c(t) + J_1(t)$$
$$X_2(t) = X_2(0) + X_2^c(t) + J_2(t)$$

Then,

$$dX_1(t)\, dX_2(t) = dX_1^c(t)\, dX_2^c(t) + dJ_1(t)\, dJ_2(t)$$

Note:

$$dX_1^c(t)\, dJ_1(t) = 0$$
$$dX_2^c(t)\, dJ_2(t) = 0$$

Cross-variation between a continuous process and a pure jump process is zero. Thus, the cross-variation between Brownian Motion and a Poisson process is zero, as well as between Brownian Motion and a compensated Poisson Process. Only two terms with dW or two simultaneous jumps will have non-zero cross-variation.

Compensated Poisson Process:

$$M(t) = N(t) - \lambda t$$

Then,

$$[W, M](t) = 0, \quad t \geq 0$$

This implies W and M are independent, as well as W and N are independent (relative to the same filtration).

Example:

Assuming the process is adapted, right-continuous jump process $X_i(t)$; and $\Phi_i(s)$ is an adapted process.

$$\tilde{X}_i(t) = \tilde{X}_i(0) + \tilde{I}_i(t) + \tilde{R}_i(t) + \tilde{J}_i(t)$$

and,

$$[\tilde{X}_1, \tilde{X}_2](t) = [\tilde{X}_1^c, \tilde{X}_2^c](t) + [\tilde{J}_1, \tilde{J}_2](t) = \int_0^t \Phi_1(s)\Phi_2(s)\, d[X_1, X_2](s)$$

Thus, in differential notation:

$$d\tilde{X}_1(t)\, d\tilde{X}_2(t) = \Phi_1(t)\Phi_2(t)\, dX_1(t)\, dX_2(t)$$

Itô-Doeblin Formula for One Jump Process

$$df(X(s)) = f'(X(s))\,dX^c(s) + \frac{1}{2}f''(X(s))\,(dX^c(s))^2$$

where:

$$(dX^c(s))^2 = \Gamma^2(s)\,ds$$

$$dX^c(s) = \Gamma(s)\,dW(s) + \Theta(s)\,ds$$

When there is a jump in X, there is typically a jump in $f(X)$. When we integrate both sides from 0 to t, we must add all the jumps that occur between those two times.

Itô Formula:

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\,dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))\,(dX^c(s))^2 + \sum_{0<s\le t}[f(X(s)) - f(X(s-))]$$

The term $\int_0^t f'(X(s))\,dX^c(s)$ uses $dX^c(s)$ and not $dX(s)$, because if it did, the jump would not appear in the correct place.

If there is no jump at $\tau_n = t$, then the last term is zero.

It is not always possible to rewrite in differential form because it is not always possible to find a differential form for the sum of jumps.

One example where it can be done is a Geometric Poisson Process:

$$S(t) = S(0)e^{-\lambda\sigma t}(\sigma+1)^{N(t)}$$

where:

- $\sigma > -1$ is a constant,
- If $\sigma > 0$, the process jumps up and moves down between jumps.
- If $-1 < \sigma < 0$, the process jumps down and moves up between jumps.

To Show the Process is a Martingale:

Let $S(t) = S(0)f(X(t))$, where $f(x) = e^x$, and

$$X(t) = N(t)\log(\sigma+1) - \lambda\sigma t$$

- The continuous part $X^c(t) = -\lambda\sigma t$
- The pure jump part $J(t) = N(t)\log(\sigma+1)$

Itô's Formula for Jump Processes:

$$S(t) = S(0) - \lambda\sigma \int_0^t S(u)\,du + \sum_{0 \le u \le t} \big[S(u) - S(u^-)\big]$$

If a Jump Occurs at Time u:

$$S(u) = (\sigma + 1)S(u^-), \quad \text{thus} \quad S(u) - S(u^-) = \sigma S(u^-)$$

If No Jump Occurs at Time u:

$$S(u) - S(u^-) = 0$$

For Either Case, We Have:

$$S(u) - S(u^-) = \sigma S(u^-) dN(u)$$

Rewriting Itô's Formula:

$$S(t) = S(0) - \lambda\sigma \int_0^t S(u^-)\,du + \sigma \int_0^t S(u^-) dN(u) = S(0) + \sigma \int_0^t S(u^-) dM(u)$$

where $M(u) = N(u) - \lambda u$ represents the compensated Poisson Process (a martingale).

Itô Formula Differential:

$$dS(t) = \sigma S(t^-) dM(t) = -\lambda\sigma S(t)\,dt + \sigma S(t^-) dN(t)$$

This setup aims to establish $S(t)$ as a martingale through compensation of the Poisson process $N(t)$ by subtracting its expected growth rate λt, thus ensuring the mean value of the increments of $S(t)$ is zero, a key martingale property.

Proof: Independence of $W(t)$ and $N(t)$

Let u_1 and u_2 be real numbers.

Define $Y(t)$ as follows:

$$Y(t) = e^{u_1 W(t) + u_2 N(t) - \frac{1}{2}u_1^2 t - \lambda(e^{u_2} - 1)t}$$

Applying Itô's Formula to Show $Y(t)$ is a Martingale:

Define $X(s)$ as:

$$X(s) = u_1 W(s) + u_2 N(s) - \frac{1}{2}u_1^2 s - \lambda(e^{u_2} - 1)s$$

Let $f(x) = e^x$, so:

$$Y(s) = f(X(s))$$

Components of the process $X(s)$: - Itô integral: $u_1 W(s)$ - Riemann integral: $-\frac{1}{2}u_1^2 s - \lambda(e^{u_2} - 1)s$ - Pure Jump part: $u_2 N(s)$

The differential $dX^c(s)$ is given by:

$$dX^c(s) = u_1 dW(s) - \left(\frac{1}{2}u_1^2 + \lambda(e^{u_2} - 1)\right) ds$$

The quadratic variation of $X^c(s)$:

$$dX^c(s)dX^c(s) = u_1^2 ds$$

If Y has a jump at time s:

$$Y(s) = Y(s^-)e^{u_2}, \text{ thus } Y(s) - Y(s^-) = (e^{u_2} - 1)Y(s^-)dN(s)$$

Itô's Formula for Jump Processes Shows:

$$Y(t) = f(X(t)) = 1 + u_1 \int_0^t Y(s)dW(s) + (e^{u_2} - 1)\int_0^t Y(s^-)dM(s)$$

- where $M(s) = N(s) - \lambda s$ (compensated Poisson Process) - Y has finitely many jumps so $\int_0^t Y(s)\, ds = \int_0^t Y(s^-)\, ds$ - Y is a martingale

Since $Y(0) = 1$ and Y is a martingale, $\mathbb{E}[Y(t)] = 1$ for all t.

This obtains the moment generating function for both the Brownian Motion and the Poisson Process:

$$\mathbb{E}[e^{u_1 W(t) + u_2 N(t)}] = e^{\frac{1}{2}u_1^2 t + \lambda t(e^{u_2} - 1)}$$

- Brownian Motion term: $e^{\frac{1}{2}u_1^2 t}$ - Poisson Process term: $e^{\lambda t(e^{u_2}-1)}$

Since these are a product, the processes $W(t)$ and $N(t)$ are independent.

It follows that vectors of random variables for each of these processes are all independent.

Itô Formula for Multiple Jump Processes

Two-dimensional Itô Formula with Jumps:

Let $X_1(t)$ and $X_2(t)$ be jump processes.

Assume $f(t, x_1, x_2)$ has defined and continuous first and second derivatives.

Then $f(t, X_1(t), X_2(t))$ evolves as:

$$f(t, X_1(t), X_2(t)) = f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))\, ds$$

$$+ \int_0^t f_{x_1}(s, X_1(s), X_2(s))\, dX_1^c(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s))\, dX_2^c(s)$$

$$+\frac{1}{2}\int_0^t f_{x_1x_1}(s, X_1(s), X_2(s))\, (dX_1^c(s))^2$$

$$+ \int_0^t f_{x_1x_2}(s, X_1(s), X_2(s))\, dX_1^c(s)dX_2^c(s)$$

$$+\frac{1}{2}\int_0^t f_{x_2x_2}(s, X_1(s), X_2(s))\, (dX_2^c(s))^2$$

$$+\sum_{0<s\le t}\big[f(s,X_1(s),X_2(s))-f(s,X_1(s^-),X_2(s^-))\big]$$

Itô's Product Rule for Jump Processes

For $X_1(t)X_2(t)$:

$$X_1(t)X_2(t)=X_1(0)X_2(0)+\int_0^t X_2(s^-)\,dX_1(s)+\int_0^t X_1(s^-)\,dX_2(s)+[X_1,X_2](t)$$

Proof:

With derivatives:

$$f_{x_1}=x_2,\quad f_{x_2}=x_1,\quad f_{x_1x_1}=0,\quad f_{x_1x_2}=1,\quad f_{x_2x_2}=0$$

Applying the two-dimensional Itô formula:

$$X_1(t)X_2(t)=X_1(0)X_2(0)+\int_0^t X_2(s)\,dX_1^c(s)+\int_0^t X_1(s)\,dX_2^c(s)+\int_0^t 1\,dX_1^c(s)dX_2^c(s)$$
$$+\sum_{0<s\le t}\big[X_1(s)X_2(s)-X_1(s^-)X_2(s^-)\big]$$

$$[X_1^c,X_2^c](t)=\int_0^t 1\,dX_1^c(s)dX_2^c(s)$$

Girsanov's Theorem to Change the Measure Using Radon-Nikodym Derivative Process

$Z(t)$ is given by:

$$Z(t)=e^{-\int_0^t\Gamma(s)\,dW(s)-\frac{1}{2}\int_0^t\Gamma^2(s)\,ds}$$

This satisfies the differential equation:

$$dZ(t)=-\Gamma(t)Z(t)\,dW(t)=Z(t)\,dX^c(t)$$

Which can be rewritten as:

$$Z(t)=e^{X^c(t)-\frac{1}{2}[X^c,X^c](t)}$$

For jumps, the differential equation is:

$$dZ^X(s)=Z^X(t^-)\,dX(t)$$

When there is a jump in X, the jump size is:

$$dZ^X(s)=Z^X(s^-)\,dX(s)$$

thus,

$$Z^X(s)=Z^X(s^-)(1+dX(s))$$

Doleans-Dade Exponential of X Process

$Z^X(t)$ is defined as:

$$Z^X(t)=e^{X^c(t)-\frac{1}{2}[X^c,X^c](t)}\prod_{0<s\le t}(1+dX(s))$$

This is the solution to the differential equation with jumps above, with the initial condition $Z^X(0)=1$. Integral form:

$$Z^X(t)=1+\int_0^t Z^X(s^-)\,dX(s)$$

CHANGE OF MEASURE

- **Brownian Motion with Drift** becomes Brownian Motion without drift.
- **Poisson Process** — a change of measure alters the intensity.
- **Compound Poisson Process** — a change of measure adjusts both the intensity and the distribution of the jump sizes.

Change of Measure - Poisson Process

Under $\mathbb{P}$:

$$Z(t) = e^{(\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}$$

- $Z(T) > 0$
- $\mathbb{E}[Z(t)] = 1$

To verify the expected value is 1:

$$dZ(t) = \left(\frac{\tilde{\lambda} - \lambda}{\lambda}\right) Z(t^-) dM(t)$$

- $Z(t)$ is a martingale under $\mathbb{P}$
- $\mathbb{E}[Z(t)] = 1$

Proof:

Let

$$X(t) = \left(\frac{\tilde{\lambda} - \lambda}{\lambda}\right) M(t)$$

- $X(t)$ is a martingale.

Continuous Part:

$$X^c(t) = (\lambda - \tilde{\lambda})t$$

Pure Jump Part:

$$J(t) = \left(\frac{\tilde{\lambda} - \lambda}{\lambda}\right) N(t)$$

Quadratic Variation of Continuous Part:

$$[X^c, X^c](t) = 0$$

If there is a jump at time t:

$$dX(t) = \left(\frac{\tilde{\lambda} - lambda}{\lambda} \right)$$

Thus,

$$1 + dX(t) = \frac{\tilde{\lambda}}{\lambda}$$

The process is expressed as:

$$Z(t) = e^{X^c(t) - \frac{1}{2}[X^c, X^c](t)} \prod_{0<s\leq t} (1 + dX(s))$$

Here, $Z(t)$ is the Doléans-Dade exponential $Z^X(t)$:

$$Z(t) = 1 + \int_0^t Z(s^-) dX(s)$$

- X is a martingale.
- $Z(s^-)$ is left-continuous.
- $Z(t)$ is a martingale.
- $Z(0) = 1$.
- $\mathbb{E}[Z(t)] = 1$.

Use $Z(T)$ to change the measure:

$$\tilde{\mathbb{P}}(A) = \int_A Z(T) d\mathbb{P} \text{ for all } A \text{ within } \mathcal{F}$$

Change of Poisson Intensity

Under measure $\tilde{\mathbb{P}}$, the Poisson process has intensity $\tilde{\lambda}$.

Proof:

Moment-generating function of $N(t)$, the Poisson Process:

$$\mathbb{E}[e^{uN(t)} Z(t)] = e^{\tilde{\lambda} t (e^u - 1)}$$

Example: Stock Model - Geometric Poisson Process

$$S(t) = S(0) e^{\alpha t + N(t) \log(\sigma+1) - \lambda \sigma t} = S(0) e^{(\alpha - \lambda\sigma)t} (\sigma + 1)^{N(t)}$$

- $\sigma > -1$
- $\sigma \neq 0$
- $N(t)$ is a Poisson process with intensity λ under measure $\mathbb{P}$
- $S(t)$ has a mean rate of return of α

$$dS(t) = \alpha S(t)dt + \sigma S(t^-)dM(t)$$

- $M(t) = N(t) - \lambda t$ is the compensated Poisson process.

Change to Measure $\tilde{\mathbb{P}}$:

$$dS(t) = rS(t)dt + \sigma S(t^-)d\tilde{M}(t)$$

- r is the risk-free interest rate.
- $N(t)$ is a Poisson process with intensity $\tilde{\lambda}$ under $\tilde{\mathbb{P}}$.
- $\tilde{M}(t) = N(t) - \tilde{\lambda}t$ is the compensated Poisson process under $\tilde{\mathbb{P}}$.

Thus, under $\tilde{\mathbb{P}}$, the mean rate of return should equal the risk-free interest rate to qualify as a risk-neutral measure:

dt **term:**

$$(\alpha - \lambda\sigma)S(t)dt$$

$dM(t)$ **term:**

$$(r - \tilde{\lambda}\sigma)S(t)dt$$

Set the dt terms equal:

$$\tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}$$

$\tilde{\lambda} > 0$ must hold to maintain a risk-neutral stance:

$$\lambda > \frac{\alpha - r}{\sigma}$$

If this does not hold, arbitrage exists as follows:

- If $\sigma > 0$ and $\tilde{\lambda} \leq 0$:

$$S(t) \geq S(0)e^{rt}(\sigma + 1)^{N(t)} \geq S(0)e^{rt}$$

Borrow at r interest rate and invest in the stock.

- If $-1 < \sigma < 0$ and $\tilde{\lambda} \leq 0$:

$$S(t) \leq S(0)e^{rt}(\sigma + 1)^{N(t)} \leq S(0)e^{rt}$$

Short the stock to invest in a money market account.

Change of Measure - Compound Poisson Process

- $Y_1, Y_2, \ldots$ are a sequence of i.i.d. random variables under $\mathbb{P}$.

Compound Poisson Process:

$$Q(t) = \sum_{i=1}^{N(t)} Y_i$$

If N jumps at time t, then Q jumps at time t:

$$dQ(t) = Y_{N(t)}$$

A change of measure will affect both the intensity of $N(t)$ and the distribution of $Y_1, Y_2, \ldots$.

Case 1: Jump-size Random Variables Have Discrete Distribution

- Finitely possible nonzero values: $y_1, y_2, \ldots, y_M$.
- $p(y_m)$ is the probability that a jump is of size y_m.

$$p(y_m) = \mathbb{P}(Y_i = y_m)$$

$$p(y_m) > 0 \text{ for every } m$$

$$\sum_{m=1}^{M} p(y_m) = 1$$

- $N_m(t)$ is the number of jumps in $Q(t)$ of size y_m up to and including time t:

$$N(t) = \sum_{m=1}^{M} N_m(t)$$

$$Q(t) = \sum_{m=1}^{M} y_m N_m(t)$$

- $N_1, \ldots, N_M$ are independent Poisson processes.
- Each N_m has intensity $\lambda_m = \lambda \cdot p(y_m)$.

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} \left(\frac{\tilde{\lambda}_m}{\lambda_m} \right)^{N_m(t)}$$

$$Z(t) = \prod_{m=1}^{M} Z_m(t)$$

This formulation describes how the probability measure and the dynamics of the process change under the new measure, affecting both the arrival rate of the jumps and their size distribution within the compound Poisson framework.

Proof: $Z(t)$ is a Martingale and $\mathbb{E}[Z(t)] = 1$

The differential for $Z_m(t)$ is given by:

$$dZ_m(t) = \left(\frac{\tilde{\lambda}_m - \lambda_m}{\lambda_m} \right) Z_m(t^-) dM_m(t)$$

where $M_m(t) = N_m(t) - \lambda_m t$ is the compensated Poisson process:

- $M_m(t)$ is left-continuous.
- $M_m(t)$ is a martingale.

Thus, Z_m is also a martingale.

For $m \neq n$, the Poisson processes N_m and N_n have no simultaneous jumps, thus $[Z_m, Z_n] = 0$.

Using Itô's Product Rule:

For two processes, the rule implies:

$$d(Z_1(t)Z_2(t)) = Z_2(t^-)dZ_1(t) + Z_1(t^-)dZ_2(t)$$

Thus, $Z_1 Z_2$ is a martingale.

Expanding Further:

$$d(Z_1(t)Z_2(t)Z_3(t)) = Z_3(t^-)d(Z_1(t)Z_2(t)) + (Z_1(t^-)Z_2(t^-))dZ_3(t)$$

Thus, $Z_1 Z_2 Z_3$ is a martingale.

This pattern continues to conclude that:

$$Z(t) = Z_1(t)Z_2(t) \dots Z_m(t)$$

is a martingale.

For a Fixed $T > 0$, with $Z(T) > 0$ and $\mathbb{E}[Z(T)] = 1$, we use $Z(T)$ to change the measure defined by:

$$\tilde{\mathbb{P}}(A) = \int_A Z(T) d\mathbb{P} \text{ for all } A \text{ within } \mathcal{F}.$$

This formulation establishes the measure change technique using the martingale property of $Z(t)$, ensuring that under the new measure $\tilde{\mathbb{P}}$, the probability space behaves as expected with the new intensities and distributions.

Change of Compound Poisson Intensity and Jump Distribution for Finitely Many Jump Sizes

Under $\tilde{\mathbb{P}}$:

- $Q(t)$ is a compound Poisson process.

- The intensity is $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$.
- $Y_1, Y_2, \ldots$ are i.i.d. random variables.
- The random variables have probability $\tilde{\mathbb{P}}(Y_i = y_m) = \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}$.

Proof:

- $N_1, \ldots, N_M$ are independent under $\mathbb{P}$.

The moment generating function under $\tilde{\mathbb{P}}$ for $Q(t)$ is:

$$\tilde{\mathbb{E}}[e^{uQ(t)}] = e^{\tilde{\lambda} t \left(\sum_{m=1}^{M} \tilde{p}(y_m) e^{u y_m} - 1 \right)}$$

- The intensity is $\tilde{\lambda}$.
- The jump-size distribution is $\tilde{p}(y_m)$.

The Radon-Nikodym derivative process of $Z(t)$ is written as:

$$Z(t) = e^{\sum_{m=1}^{M} (\lambda_m - \tilde{\lambda}_m) t} \prod_{m=1}^{M} \left(\frac{\tilde{\lambda} \tilde{p}(y_m)}{\lambda p(y_m)} \right)^{N_m(t)} = e^{(\lambda - \tilde{\lambda}) t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda} \tilde{p}(Y_i)}{\lambda p(Y_i)}$$

If random variables Y_i are not discrete but instead continuous with density $f(y)$, then we could change the measure so that $Q(t)$ has intensity $\tilde{\lambda}$ and density $\tilde{f}(y)$ by using the Radon-Nikodym derivative process:

$$Z(t) = e^{(\lambda - \tilde{\lambda}) t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda} \tilde{f}(Y_i)}{\lambda f(Y_i)}$$

Assume $\tilde{f}(y) = 0$ whenever $f(y) = 0$, meaning under $\mathbb{P}$ and $\tilde{\mathbb{P}}$ those with probability $= 0$ will have the same probability under both measures.

Proof: $Z(t)$ is a Martingale, $\mathbb{E}[Z(t)] = 1$

Pure Jump process:

$$J(t) = \prod_{i=1}^{N(t)} \frac{\tilde{\lambda} \tilde{f}(Y_i)}{\lambda f(Y_i)}$$

At jump times of Q (also N and J):

$$J(t) = J(t^-) \frac{\tilde{\lambda} \tilde{f}(Y_{N(t)})}{\lambda f(Y_{N(t)})} = J(t^-) \frac{\tilde{\lambda} \tilde{f}(dQ(t))}{\lambda f(dQ(t))}$$

Thus,

$$dJ(t) = J(t) - J(t^-) = \left[\frac{\tilde{\lambda} \tilde{f}(dQ(t))}{\lambda f(dQ(t))} - 1 \right] J(t^-)$$

Compound Poisson Process:

$$H(t) = \sum_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$$

with

$$dH(t) = \frac{\tilde{\lambda}\tilde{f}(dQ(t))}{\lambda f(dQ(t))}$$

because

$$\mathbb{E}\left[\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}\right] = \frac{\tilde{\lambda}}{\lambda}$$

Compensated Poisson Process $H(t) - \tilde{\lambda}t$ is a martingale. We can rewrite $dJ(t)$ as:

[dJ(t) =

J(t^-) dH(t) - J(t^-) dN(t)]

Since all terms are $= 0$ if no jumps occur at time t, this holds at all times t.

Itô's product rule for $Z(t)$:

$$Z(t) = 1 + \int_0^t Z(s^-)d(H(s) - \tilde{\lambda}s) - \int_0^t Z(s^-)d(N(s) - \lambda s)$$

- $Z(t)$ is a martingale
- $Z(0) = 1$
- $\mathbb{E}[Z(t)] = 1$

Differential form:

$$dZ(t) = Z(t^-)dH(t) - Z(t^-)dN(t)$$

Then fix T and define:

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)d\mathbb{P} \text{ for all } A \text{ within } \mathcal{F}.$$

Change of Compound Poisson Intensity and Jump Distribution for a Continuum of Jump Sizes

Under measure $\tilde{\mathbb{P}}$:

- $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$.
- Jumps in $Q(t)$ are i.i.d with density $\tilde{f}(y)$.

Proof:

We need to show under $\tilde{\mathbb{P}}$ that $Q(t)$ has a moment-generating function for a compound Poisson Process:

$$\tilde{\mathbb{E}}[e^{uQ(t)}] = e^{\tilde{\lambda}t(\tilde{\varphi}_Y(u)-1)}$$

where

$$\tilde{\varphi}_Y(u) = \int_{-\infty}^{\infty} e^{uy}\tilde{f}(y)\,dy$$

Define,

$$X(t) = e^{uQ(t)-\tilde{\lambda}t(\tilde{\varphi}_Y(u)-1)}$$

and show that $X(t)Z(t)$ is a martingale under $\mathbb{P}$.

At jump times of Q:

$$X(t) = X(t^-)e^{udQ(t)}$$

Thus,

$$dX(t) = X(t) - X(t^-) = X(t^-)(e^{udQ(t)} - 1)$$

Compound Poisson Process:

$$V(t) = \sum_{i=1}^{N(t)} e^{uY_i}\left(\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}\right)$$

Since

$$\mathbb{E}\left[e^{uY_i}\left(\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}\right)\right] = \frac{\tilde{\lambda}}{\lambda}\tilde{\varphi}_Y(u)$$

Compensated compound Poisson Process:

$$V(t) - \tilde{\lambda}t\tilde{\varphi}_Y(u) \text{ is a martingale}$$

At jump times of Q:

$$dV(t) = e^{udQ(t)}dH(t)$$

$$[X,Z](t) = \sum_{0<s\leq t} X(s^-)Z(s^-)dV(s) - \sum_{0<s\leq t} X(s^-)Z(s^-)dH(s) - \sum_{0<s\leq t} X(s^-)Z(s^-)(e^{udQ(s)} - 1)$$

Since $dN(s) = \{1, 0\}$, thus,

$$(e^{udQ(s)} - 1)dN(s) = (e^{udQ(s)} - 1)$$

Itô's Product Rule for Jump Processes to write:

$$X(t)Z(t) = 1 + \int_0^t X(s^-)dZ(s) + \int_0^t Z(s^-)dX(s) + [X,Z](t)$$

This first term on the right-hand side we know is a martingale.

To show the second term is a martingale we show:

$$\int_0^t Z(s^-)dX(s) + [X,Z](t) = \int_0^t X(s^-)Z(s^-)d(V(s) - \tilde{\lambda}s\tilde{\varphi}_Y(u)) - \int_0^t X(s^-)Z(s^-)d(H(s) - \tilde{\lambda}s)$$

This is a martingale because $V(t) - \tilde{\lambda}t\tilde{\varphi}_Y(u)$ and $H(t) - \tilde{\lambda}t$ are martingales and are left-continuous.

We now prove that:

$$\tilde{\mathbb{E}}[e^{uQ(t)}] = \mathbb{E}[e^{uQ(t)}Z(t)]$$

$X(t)Z(t)$ has a constant expectation $= 1$, which implies:

$$1 = \mathbb{E}[X(t)Z(t)] = e^{-\tilde{\lambda}t(\tilde{\varphi}_Y(u)-1)}\mathbb{E}[e^{uQ(t)}Z(t)]$$

Change of Measure - Compound Poisson Process and a Brownian Motion

Itô's Product Rule helps combine these:

Continuous Part (Brownian Motion):

$$Z_1(t) = e^{-\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du}$$

Compound Jump Process:

$$Z_2(t) = e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$$

The Joint Process:

$$Z(t) = Z_1(t)Z_2(t)$$

$Z(t)$ is a martingale because:

$$Z(t) = Z_1(0)Z_2(0) + \int_0^t Z_1(s^-)\,dZ_2(s) + \int_0^t Z_2(s^-)\,dZ_1(s)$$

- Both integrals are martingales.
- $Z(0) = 1$.
- $\mathbb{E}[Z(t)] = 1$.

Fix T and define $\tilde{\mathbb{P}}(A) = \int_A Z(T)\,d\mathbb{P}$ for all A within $\mathcal{F}$.

Under measure $\tilde{\mathbb{P}}$:

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(s)\, ds$$

- $\tilde{W}$ is a Brownian Motion.
- $Q(t)$ is a compound Poisson Process.
- $\tilde{\lambda}$ is the intensity.
- Jump sizes are i.i.d. with density $\tilde{f}(y)$.
- $Q(t)$ and $\tilde{W}(t)$ are independent.

Proof:

Moment-generating function under $\tilde{\mathbb{P}}$:

$$\tilde{\mathbb{E}}[e^{u_1 \tilde{W}(t) + u_2 Q(t)}] = e^{\frac{1}{2}u_1^2 t} e^{\tilde{\lambda} t(\tilde{\varphi}_Y(u_2) - 1)}$$

This proves that we have the correct distributions under $\tilde{\mathbb{P}}$ and they are independent.

If $\Theta(t)$ is independent of $Q(t)$, then $Z_1(t)$ is independent of Q, thus:

$$\tilde{\mathbb{E}}[e^{u_1 \tilde{W}(t) + u_2 Q(t)}] = \mathbb{E}[e^{u_1 \tilde{W}(t)} Z_1(t)] \mathbb{E}[e^{u_2 Q(t)} Z_2(t)]$$

Girsanov's theorem shows the expectations are:

$$\mathbb{E}[e^{u_1 \tilde{W}(t)} Z_1(t)] = e^{\frac{1}{2}u_1^2 t}$$

$$\mathbb{E}[e^{u_2 Q(t)} Z_2(t)] = e^{\tilde{\lambda} t(\tilde{\varphi}_Y(u_2) - 1)}$$

Surprisingly this still holds if we make $\Theta(t)$ depend on $Q(t)$, by having $\Theta(t) = Q(t)$, the proof will still hold.

$$X_1(t) = e^{u_1 \tilde{W}(t) - \frac{1}{2}u_1^2 t}$$

$$X_2(t) = e^{u_2 Q(t) - \tilde{\lambda} t(\tilde{\varphi}_Y(u_2) - 1)}$$

We can show that $X_1(t)Z_1(t)$, $X_2(t)Z_2(t)$, and $X_1(t)Z_1(t)X_2(t)Z_2(t)$ are martingales under $\mathbb{P}$.

Itô Formula for Continuous Processes Implies:

$$dX_1(t) = u_1 X_1(t) dW(t) + u_1 \Theta(t) X_1(t) dt$$

$$dZ_1(t) = -\Theta(t) Z_1(t) dW(t)$$

Itô Product Rule Shows:

$$d(X_1(t)Z_1(t)) = (u_1 - \Theta(t))X_1(t)Z_1(t)dW(t)$$

Since there is no dt term, it is a martingale.

For $X_2(t)Z_2(t)$:

We know $X_2(t)Z_2(t)$ is a martingale, has no Itô integral part, and $X_1(t)Z_1(t)$ is continuous, then $[X_1(t)Z_1(t), X_2(t)Z_2(t)] = 0$.

Itô Product Rule Implies:

$$X_1(t)Z_1(t)X_2(t)Z_2(t) = 1 + \int_0^t X_1(s^-)Z_1(s^-)d(X_2(s)Z_2(s)) + \int_0^t X_2(s^-)Z_2(s^-)d(X_1(s)Z_1(s))$$

This implies it is a martingale, thus $\mathbb{E}[X_1(t)Z_1(t)X_2(t)Z_2(t)] = 1$.

For Discrete Jump Sizes:

If instead of a density $f(y)$ we have finitely many nonzero values $y_1, \dots, y_m$ with probability $p(y_m)$ that sum to 1, then $Z_2(t)$ will use $\frac{\tilde{p}(Y_i)}{p(Y_i)}$ instead of $\frac{\tilde{f}(Y_i)}{f(Y_i)}$.

The previous proof still applies under measure $\tilde{\mathbb{P}}(A) = \int_A Z(T)d\mathbb{P}$ for all A within $\mathcal{F}$, where jump sizes are now distributed by $\tilde{p}(y_m)$.

Pricing a European Call in a Jump Model

Two Cases:

1) The underlying asset is driven by a single Poisson Process - Market is complete in this case.
2) The underlying asset is driven by a Brownian Motion and a Compound Poisson Process - Market is incomplete in this case.

Asset Driven by a Poisson Process

$$S(t) = S(0)e^{(\alpha - \lambda\sigma)t}(\sigma + 1)^{N(t)}$$

$$dS(t) = \alpha S(t)dt + \sigma S(t^-)dM(t)$$

- $N(t)$ is a Poisson process.
- $M(t) = N(t) - \lambda t$ is a compensated Poisson process.
- Intensity $\lambda > 0$ on a measure $\mathbb{P}$.

European Call Payoff:

$$V(T) = (S(T) - K)^+$$

No-arbitrage Condition:

$$\lambda > \frac{\alpha - r}{\sigma}$$

Thus, $\tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}$ is positive.

Risk-Neutral Measure is Given by:

$$Z(t) = e^{(\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda} \right)^{N(t)}$$

Under the Risk-Neutral Measure, the Compensated Poisson Process is a Martingale:

$$\tilde{M}(t) = N(t) - \tilde{\lambda}t$$

$$dS(t) = rS(t)dt + \sigma S(t^-)d\tilde{M}(t)$$

Equivalent to:

$$d(e^{-rt}S(t)) = \sigma e^{-rt}S(t^-)d\tilde{M}(t)$$

This shows the discounted market price is a martingale under $\tilde{\mathbb{P}}$, and:

$$S(t) = S(0)e^{(r - \tilde{\lambda}\sigma)t}(\sigma + 1)^{N(t)}$$

European Call Risk-Neutral Price under $\tilde{\mathbb{P}}$:

$$e^{-rt}V(t) = \tilde{\mathbb{E}}[e^{-rT}V(T)|\mathcal{F}(t)] = \tilde{\mathbb{E}}[e^{-rT}(S(T) - K)^+|\mathcal{F}(t)]$$

Thus,

$$S(T) = S(t)e^{(r - \tilde{\lambda}\sigma)(T-t)}(\sigma + 1)^{N(T) - N(t)}$$

Thus,

$$V(t) = \tilde{\mathbb{E}}[e^{-r(T-t)}(S(t)e^{(r - \tilde{\lambda}\sigma)(T-t)}(\sigma + 1)^{N(T) - N(t)} - K)^+|\mathcal{F}(t)]$$

- $S(t)$ is $\mathcal{F}(t)$-measurable.
- $e^{(r - \tilde{\lambda}\sigma)(T-t)}(\sigma + 1)^{N(T) - N(t)}$ is independent of $\mathcal{F}(t)$.

Independence shows that $V(t) = c(t, S(t))$, where:

$$c(t,x) = \sum_{j=0}^{\infty} (xe^{-\tilde{\lambda}\sigma(T-t)}(\sigma+1)^j - Ke^{-r(T-t)})^+ \frac{(\tilde{\lambda}^j (T-t)^j)}{j!} e^{-\tilde{\lambda}(T-t)}$$

Computing the risk-neutral call price for $j = 0$:

$$c(t,x) = (xe^{-\tilde{\lambda}\sigma(T-t)} - Ke^{-r(T-t)})^+ e^{-\tilde{\lambda}(T-t)}$$

When $t = T$:

$$c(t,x) = (xe^{-\tilde{\lambda}\sigma(T-t)} - Ke^{-r(T-t)})^+ e^{-\tilde{\lambda}(T-t)} \to c(T,x) = (x-K)^+$$

This is the only nonzero term left and satisfies the terminal condition.

Derive the Partial Differential Equation that $c(t,x)$ Satisfies:

$$e^{-rt}V(t) = \tilde{\mathbb{E}}[e^{-rT}(S(T)-K)^+ | \mathcal{F}(t)]$$

Set the dt term to zero:

$$dS(t) = (r - \tilde{\lambda}\sigma)S(t)dt + \sigma S(t^-)dN(t)$$

Continuous Part:

$$dS^c(t) = (r - \tilde{\lambda}\sigma)S(t)dt$$

If the stock price jumps at time t, then:

$$dS(t) = S(t) - S(t^-) = \sigma S(t^-)$$

$$S(t) = (\sigma+1)S(t^-)$$

Itô Formula Implies:

$$e^{-rt}c(t,S(t)) = c(0,S(0)) + \int_0^t e^{-ru}[-rc(u,S(u)) + c_t(u,S(u)) + (r-\tilde{\lambda}\sigma)S(u)c_x(u,S(u))]du +$$

$$\int_0^t e^{-ru}[c(u,(\sigma+1)S(u^-)) - c(u,S(u^-))]d\tilde{M}(u)$$

Note:

$$\int_0^t e^{-ru}[c(u,(\sigma+1)S(u^-)) - c(u,S(u^-))]\tilde{\lambda}du = \int_0^t e^{-ru}[c(u,(\sigma+1)S(u)) - c(u,S(u))]\tilde{\lambda}du$$

So,

$$e^{-rt}c(t,S(t)) = c(0,S(0)) + \int_0^t e^{-ru}[-rc(u,S(u)) + c_t(u,S(u)) + (r-\tilde{\lambda}\sigma)S(u)c_x(u,S(u)) + \tilde{\lambda}(c(u,(\sigma+1)S(u)) - c(u,S(u)))]du$$
$$+ \int_0^t e^{-ru}[c(u,(\sigma+1)S(u^-)) - c(u,S(u^-))]d\tilde{M}(u)$$

- The last term is a martingale because $\tilde{M}(u)$ is a martingale and is left-continuous.
- $e^{-rt}c(t,S(t))$ is also a martingale.

Thus, testing the final two terms:

$$c(0,S(0)) + \int_0^t e^{-ru}[-rc(u,S(u)) + c_t(u,S(u)) + (r-\tilde{\lambda}\sigma)S(u)c_x(u,S(u)) + \tilde{\lambda}(c(u,(\sigma+1)S(u)) - c(u,S(u)))]du$$

Since this is a difference of two integrals, if $= 0$ then it is also a martingale:

$$[-rc(t,S(t)) + c_t(t,S(t)) + (r-\tilde{\lambda}\sigma)S(t)c_x(t,S(t)) + \tilde{\lambda}(c(t,(\sigma+1)S(t)) - c(t,S(t)))] = 0$$

Taking the differential with respect to a martingale integrator, $d\tilde{M}(t)$, and setting the dt term to zero. Note this does not hold for integrator $dN(t)$ as $N(t)$ is not a martingale.

The above formula is often called a differential-difference equation because it involves c at two different values of the stock price, for instance x and $(\sigma+1)x$.

Construct a hedge for a short position in a European Call Option

Using the following:

$$e^{-rt}c(t,S(t)) = c(0,S(0)) + \int_0^t e^{-ru}[c(u,(\sigma+1)S(u^-)) - c(u,S(u^-))]d\tilde{M}(u)$$

$$e^{-rT}(S(T)-K)^+ = e^{-rT}c(T,S(T)) = c(0,S(0)) + \int_0^t e^{-ru}[c(u,(\sigma+1)S(u^-)) - c(u,S(u^-))]d\tilde{M}(u)$$

Set $t = 0$, and initial capital $X(0) = C(0,S(0))$.

We want to invest in the stock and money market account so that $X(t) = c(t,S(t))$,

$$e^{-rt}X(t) = e^{-rt}c(t,S(t))$$

Match differentials of $X(t)$ and $e^{-rt}c(t,S(t))$:

$$d(e^{-rt}c(t,S(t))) = e^{-rt}[c(t,(\sigma+1)S(t^-)) - c(t,S(t^-))]d\tilde{M}(t)$$

$$dX(t) = \Gamma(t^-)dS(t) + r[X(t) - \Gamma(t)S(t)]dt$$

- $\Gamma(t)$ is the number of shares of stock held in the hedging portfolio, to avoid confusion with jump terms.

Thus,

$$d(e^{-rt}X(t)) = e^{-rt}\sigma\Gamma(t^-)S(t^-)d\tilde{M}(t)$$

We want $\Gamma(t^-)$ to determine the number of shares held just before any jump that may occur at time t to hedge:

$$\Gamma(t^-) = \frac{c(t,(\sigma+1)S(t^-)) - c(t,S(t^-))}{\sigma S(t^-)}$$

Hedge position to hold at all times, whether jumps occur or not is:

$$\Gamma(t) = \frac{c(t,(\sigma+1)S(t)) - c(t,S(t))}{\sigma S(t)}$$

Integration shows:

$$e^{-rt}X(t) = X(0) + \int_0^t e^{-ru}[c(u,(\sigma+1)S(u^-)) - c(u,S(u^-))]d\tilde{M}(u)$$

Compare with:

$$e^{-rt}c(t,S(t)) = c(0,S(0)) + \int_0^t e^{-ru}[c(u,(\sigma+1)S(u^-)) - c(u,S(u^-))]d\tilde{M}(u)$$

Shows:

$$X(t) = c(t,S(t)) \quad \text{and} \quad X(T) = (S(T)-K)^+$$

So the short position in the European Call has been hedged.

To show what happens when a jump occurs and when it does not:

If a jump occurs, the change in option price is:

$$c(t,(\sigma+1)S(t^-)) - c(t,S(t^-))$$

Change in hedging portfolio is:

$$\Gamma(t^-)(S(t) - S(t^-)) = \Gamma(t^-)\sigma S(t^-) = c(t,(\sigma+1)S(t^-)) - c(t,S(t^-))$$

which agrees with the change in the option price.

If no jump at time t, the stock price follows:

$$dS(t) = (r - \tilde{\lambda}\sigma)S(t)dt$$

Discounted option price differential is:

$$d(e^{-rt}c(t,S(t))) = -e^{-rt}\tilde{\lambda}[c(t,(\sigma+1)S(t)) - c(t,S(t))]dt$$

Differential of Discounted Portfolio Value at time t is:

$$d(e^{-rt}X(t)) = -e^{-rt}\tilde{\lambda}[c(t,(\sigma+1)S(t)) - c(t,S(t))]dt$$

Which shows again that the discounted portfolio value and the discounted option price are the same.

Completeness

The model is complete and the risk-neutral measure is unique if and only if every derivative security can be hedged. "Every" meaning also those that are path-dependent. They were not considered here, but can be hedged, thus is complete.

For a single Poisson Process, this is summarized by: payoff $h(S(T))$ at time T, one could replace the payoff by the function h, the differential-difference equation would still apply but with a terminal condition now of $c(T,x) = h(x)$, and the hedging formula would still be correct.

Asset Driven by a Brownian Motion and a Compound Poisson Process

- $(\Omega, \mathcal{F}, \mathbb{P})$ — Probability space
- Brownian Motion — $W(t)$, $0 \leq t \leq T$
- Independent Poisson Process — Number M, $N_1(t), \ldots, N_M(t)$, $0 \leq t \leq T$
- Filtration on Brownian Motion and Poisson Process — $\mathcal{F}(t)$, $0 \leq t \leq T$
- Intensity of m-th Poisson Process — $\lambda_m > 0$
- Nonzero random variables — $-1 < y_1 < \ldots < y_M$

Poisson Process:

$$N(t) = \sum_{m=1}^{M} N_m(t)$$

, with intensity $\lambda = \sum_{m=1}^{M} \lambda_m$

Compound Poisson Process:

$$Q(t) = \sum_{m=1}^{M} y_m N_m(t)$$

,

Rewrite this with Y_i denoting random variables for the size of the i-th jump in Q as:

$$Q(t) = \sum_{i=1}^{N(t)} Y_i$$

With each random variable having the probability of being a certain size written as:

$$P(Y_i = y_m) = p(y_m)$$

, where $p(y_m) = \frac{\lambda_m}{\lambda}$

Thus the expectation of the random variables of random jump sizes, denoted β follows:

$$\beta = E[Y_i] = \sum_{m=1}^{M} y_m p(y_m) = \frac{1}{\lambda} \sum_{m=1}^{M} \lambda_m y_m$$

The Martingale Property thus:

$$Q(t) - \beta\lambda t = Q(t) - t \sum_{m=1}^{M} \lambda_m y_m$$

is a martingale.

Stock Price Model by Stochastic Differential Equation with Jumps

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) + S(t^-)d(Q(t) - \beta\lambda t) = (\alpha - \beta\lambda)S(t)dt + \sigma S(t)dW(t) + S(t^-)dQ(t)$$

Under Measure $\mathbb{P}$:

- α — mean rate of return on the stock
- Assumption $y_i > -1$ for $i = 1, \ldots, M$ — guarantees that although the stock price can jump down, it cannot jump from a positive to a negative value or to zero
- $S(0)$ — initial stock price, and stock price is positive at all subsequent times, so if $S(0) = 0$ then $S(t) = 0$ for all t

The differential equation comes to:

$$S(t) = S(0)e^{\sigma W(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t} \prod_{i=1}^{N(t)} (Y_i + 1)$$

The continuous stochastic process portion is:

$$X(t) = S(0)e^{\sigma W(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t}$$

The jump process portion is:

$$J(t) = \prod_{i=1}^{N(t)} (Y_i + 1)$$

Thus to simplify:

$$S(t) = X(t)J(t)$$

Proof:

Itô Formula for Continuous Process:

$$dX(t) = (\alpha - \beta\lambda)X(t)\,dt + \sigma X(t)\,dW(t)$$

At Time of ith Jump:

$$J(t) = J(t^-)(Y_i + 1)$$

$$\Delta J(t) = J(t) - J(t^-) = J(t^-)Y_i = J(t^-)\Delta Q(t)$$

This also holds at non-jump times, with both sides equal to zero, $\Delta J(t) = J(t^-)\Delta Q(t)$, thus:

$$dJ(t) = J(t^-)\,dQ(t)$$

Itô's Product Rule for Jump Processes implies:

$$S(t) = X(t)J(t) = S(0) + \int_0^t X(s^-)\,dJ(s) + \int_0^t J(s)\,dX(s) + [X, J](t)$$

Since J is a pure jump process and X is continuous, $[X, J](t) = 0$, thus:

$$S(t) = X(t)J(t) = S(0) + \int_0^t X(s^-)J(s^-)\,dQ(s) + (\alpha - \beta\lambda)\int_0^t J(s)X(s)\,ds + \sigma\int_0^t J(s)X(s)\,dW(s)$$

Differential Form is:

$$dS(t) = d(X(t)J(t)) = X(t^-)J(t^-)\,dQ(t) + (\alpha - \beta\lambda)J(t)X(t)\,dt + \sigma J(t)X(t)\,dW(t)$$

$$= S(t^-)\,dQ(t) + (\alpha - \beta\lambda)S(t)\,dt + \sigma S(t)\,dW(t)$$

Construct Risk-Neutral Measure:

- θ –> constant
- $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$ –> positive constants
- If you make these both adapted stochastic processes, then additional risk-neutral measures can be created.

$$Z_0(t) = e^{-\theta W(t) - \frac{1}{2}\theta^2 t},$$

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} \left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}, \quad m = 1, \ldots, M$$

$$Z(t) = Z_0(t) \prod_{m=1}^{M} Z_m(t),$$

$$\tilde{P}(A) = \int_A Z(T)\, d\mathbb{P} \quad \text{for all } A \text{ within set } \mathcal{F}$$

Under Measure $\tilde{P}$:

- $\tilde{W}(t) = W(t) + \theta t$ –> Brownian Motion,
- N_m –> Poisson Process with intensity $\tilde{\lambda}_m$,
- $\tilde{W}$ and $N_1, \ldots, N_M$ are independent

$$\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$$

$$\tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\lambda_m}$$

Following measure P but under measure $\tilde{P}$, the following equations are similar:

$$N(t) = \sum_{m=1}^{M} N_m(t)$$

, with intensity $\tilde{\lambda}$

$$\tilde{P}(Y_i = y_m) = \tilde{p}(y_m)$$

, and $Q(t) - \tilde{\beta}\tilde{\lambda}t$ is a martingale.

$$\tilde{\beta} = \tilde{E}[Y_i] = \sum_{m=1}^{M} y_m \tilde{p}(y_m) = \frac{1}{\tilde{\lambda}} \sum_{m=1}^{M} \tilde{\lambda}_m y_m$$

$\tilde{P}$ is risk-neutral only if the mean rate of return for the stock under $\tilde{P}$ is the interest rate r, thus if and only if:

$$dS(t) = (\alpha - \beta\lambda)S(t)\, dt + \sigma S(t)\, dW(t) + S(t^-)\, dQ(t)$$

$$= rS(t)\, dt + \sigma S(t)\, d\tilde{W}(t) + S(t^-)\, d(Q(t) - \tilde{\beta}\tilde{\lambda}t)$$

Equivalent to the market price of risk equation for Jump-Diffusion Model:

$$\alpha - \beta\lambda = r + \sigma\theta - \tilde{\beta}\tilde{\lambda}$$

Considering β and $\tilde{\beta}$ we can rewrite as:

$$\alpha - r = \sigma\theta + \beta\lambda - \tilde{\beta}\tilde{\lambda}$$

$$= \sigma\theta + \sum_{m=1}^{M} (\lambda_m - \tilde{\lambda}_m) y_m$$

Multiple Unique Risk-Neutral Measures:

Because there is one equation with $M+1$ unknowns, $\theta, \lambda_1, \ldots, \lambda_M$—there are multiple risk-neutral measures.

Extra stocks would help determine a unique risk-neutral measure.

Example: with $M = 2$, 3 stocks, and 2 Poisson Processes (N_1 and N_2), one Brownian Motion — we can define 3 compound Poisson Processes:

$$Q_i(t) = y_{i,1} N_1(t) + y_{i,2} N_2(t), \quad i = 1, 2, 3$$

where $y_{i,m} > -1$ for $i = 1, 2, 3$ and $m = 1, 2$.

Set,

$$\beta_i = \frac{1}{\lambda}(\lambda_1 y_{i,1} + \lambda_2 y_{i,2}), \quad i = 1, 2, 3,$$

- λ_1 is intensity for N_1
- λ_2 is intensity for N_2

under the measure P.

For $i = 1, 2, 3$, the stock process is modeled by:

$$dS_i(t) = (\alpha_i - \beta_i\lambda) S_i(t)\, dt + \sigma_i S_i(t)\, dW(t) + S_i(t^-)\, dQ_i(t)$$

This creates three equations for the three unknowns $\theta, \tilde{\lambda}_1$, and $\tilde{\lambda}_2$—if a unique solution is found, then there is a unique risk-neutral measure. This would claim that the market is complete and free from arbitrage.

$$\alpha_1 - r = \sigma_1\theta + (\lambda_1 - \tilde{\lambda}_1) y_{1,1} + (\lambda_2 - \tilde{\lambda}_2) y_{1,2}$$

$$\alpha_2 - r = \sigma_2\theta + (\lambda_1 - \tilde{\lambda}_1) y_{2,1} + (\lambda_2 - \tilde{\lambda}_2) y_{2,2}$$

$$\alpha_3 - r = \sigma_3\theta + (\lambda_1 - \tilde{\lambda}_1) y_{3,1} + (\lambda_2 - \tilde{\lambda}_2) y_{3,2}$$

In a Single Stock Model:

- choosing some θ, and $\tilde{\lambda}_1, \dots, \tilde{\lambda}_M$ to satisfy the market price of risk equations

$$dS(t) = (r - \tilde{\beta}\tilde{\lambda})dt + \sigma S(t) d\tilde{W}(t) + S(t^-)dQ(t)$$

Which has a solution:

$$S(t) = S(0)e^{\sigma \tilde{W}(t) + (r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2)t} \prod_{i=1}^{N(t)} (Y_i + 1)$$

θ does not appear in this formula but will appear here:

$$\tilde{\beta}\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m y_m$$

Going forward we assume θ has been set to a chosen parameter:

Standard Black-Scholes-Merton call price on a geometric Brownian Motion:

$$C(t, x) = xN(d_+(t, x)) - Ke^{-rt}N(d_-(t, x)),$$

where

$$d_{\pm}(t, x) = \frac{1}{\sigma\sqrt{t}} \left[\log\left(\frac{x}{K}\right) + (r \pm \frac{1}{2}\sigma^2)t \right]$$

and the cumulative standard normal distribution function:

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}z^2}\, dz$$

- σ –> volatility
- current stock price –> x
- expiration date is –> t time units in the future
- interest rate –> r
- strike price –> K

Thus, we get the following where Y is a standard normal random variable under $\tilde{P}$:

$$C(t, x) = \tilde{E}\left[e^{-rt}(xe^{-\sigma\sqrt{t}Y + (r - \frac{1}{2}\sigma^2)t} - K)^+ \right]$$

Risk-Neutral Price of a Call Option

$$V(t) = C(t, S(t)) = \tilde{E}\left[e^{-r(T-t)}(S(T) - K)^+ \mid \mathcal{F}(t) \right]$$

$$C(t, x) = \sum_{j=0}^{\infty} e^{-\tilde{\lambda}(T-t)} \frac{\tilde{\lambda}^j (T-t)^j}{j!} \tilde{E}_{C(t,x)} \left[T - t, xe^{-\tilde{\beta}\tilde{\lambda}(T-t)} \prod_{i=1}^{j} (Y_i + 1) \right]$$

This notation formalizes the pricing formula using the risk-neutral measure $\tilde{E}$, and incorporates the probability of different numbers of jumps occurring over the interval $[t, T]$ using a Poisson process parameterized by $\tilde{\lambda}$.

Proof:

- t within $[0, T)$
- $\tau = T - t$

$$S(T) = S(t)e^{\sigma(\tilde{W}(T)-\tilde{W}(t))+(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2)\tau} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1)$$

- $S(t)$ is $\mathcal{F}(t)$-measurable
- The other term on the right-hand side is independent of $\mathcal{F}(t)$

Independence implies:

$$V(t) = C(t, S(t)) = \tilde{E}\left[e^{-r\tau}(S(T) - K)^+ \mid \mathcal{F}(t)\right]$$

where

$$c(t,x) = \tilde{E}\left[e^{-r\tau}\left(xe^{\sigma(\tilde{W}(T)-\tilde{W}(t))+(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2)\tau} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K\right)^+\right]$$

$$= \tilde{E}\left[\tilde{E}\left[e^{-r\tau}\left(xe^{\sigma(\tilde{W}(T)-\tilde{W}(t))+(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2)\tau} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K\right)^+ \mid \sigma\left(\prod_{i=N(t)+1}^{N(T)} (Y_i + 1)\right)\right]\right]$$

$$= \tilde{E}\left[\tilde{E}\left[e^{-r\tau}\left(xe^{-\tilde{\beta}\tilde{\lambda}\tau}e^{-\sigma\sqrt{\tau}Y+(r-\frac{1}{2}\sigma^2)\tau} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K\right)^+ \mid \sigma\left(\prod_{i=N(t)+1}^{N(T)} (Y_i + 1)\right)\right]\right]$$

where

$$Y = \frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{\tau}}$$

- is a standard normal random variable under $\tilde{P}$,
- filtration sigma-algebra is $\sigma\left(\prod_{i=N(t)+1}^{N(T)} (Y_i + 1)\right)$ generated from random variable: $\prod_{i=N(t)+1}^{N(T)} (Y_i + 1)$
- Random variable is filtration measurable
- Y is independent of the filtration,

Independence shows:

$$\tilde{E}\left[e^{-r\tau}\left(xe^{-\tilde{\beta}\tilde{\lambda}\tau}e^{-\sigma\sqrt{\tau}Y+(r-\frac{1}{2}\sigma^2)\tau}\prod_{i=N(t)+1}^{N(T)}(Y_i+1)-K\right)^+ \mid \sigma\left(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\right)\right]=\mathcal{K}(\tau, xe^{-\tilde{\beta}\tilde{\lambda}\tau}\prod_{i=N(t)+1}^{N(T)}(Y_i+1))$$

- $\mathcal{K}$ is the Black-Scholes-Merton model for Call Price

So,

$$C(t,x)=E\left[\mathcal{K}\left(\tau, xe^{-\tilde{\beta}\tilde{\lambda}\tau}\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\right)\right]$$

Conditioned on $N(T)-N(t)=j$, the random variable

has the same distribution as $\prod_{i=1}^{j}(Y_i+1)$

Thus, the probability is:

$$\mathbb{P}(N(T)-N(t)=j)=e^{-\tilde{\lambda}\tau}\frac{(\tilde{\lambda}\tau)^j}{j!}$$

Continuous Jump Distribution

Assume the jump sizes Y_i have a density $f(y)$ instead of a probability mass function $p(y_1),\dots,p(y_m)$, and this density is strictly positive on a subset $B\subset(-1,\infty)$ and zero elsewhere.

The following changes are made:

$$\beta=\mathbb{E}[Y_i]=\int_{-1}^{\infty}yf(y)\,dy$$

For the risk-neutral measure, the market price of risk equation is:

$$\alpha-r=\sigma\theta+\beta\lambda-\tilde{\beta}\tilde{\lambda}$$

where now,

$$\tilde{\beta}=\tilde{\mathbb{E}}[Y_i]=\int_{-1}^{\infty}y\tilde{f}(y)\,dy$$

Discrete Jump Size Model - Differential-Difference Equation Satisfied by a Call Price

$$C(t,x)=\sum_{j=0}^{\infty}e^{-\tilde{\lambda}(T-t)}\frac{\tilde{\lambda}^j(T-t)^j}{j!}\tilde{\mathbb{E}}_{C(t,x)}\left[T-t, xe^{-\tilde{\beta}\tilde{\lambda}(T-t)}\prod_{i=1}^{j}(Y_i+1)\right]$$

The call price $c(t,x)$ satisfies the equation:

$$-rc(t,x)+c_t(t,x)+(r-\tilde{\beta}\tilde{\lambda})xc_x(t,x)+\frac{1}{2}\sigma^2x^2c_{xx}(t,x)+\tilde{\lambda}\left[\sum_{m=1}^{M}\tilde{p}(y_m)c(t,(y_m+1)x)-c(t,x)\right]=0$$

for $0 \le t < T$, $x \ge 0$, and

terminal condition:

$$c(T,x)=(x-K)^+, \quad x\ge 0$$

Proof:

The continuous part of the stock price satisfies:

$$dS^c(t)=(r-\tilde{\beta}\tilde{\lambda})S(t)dt+\sigma S(t)d\tilde{W}(t)$$

Thus, the Itô formula implies:

$$e^{-rt}c(t,S(t))-c(0,S(0))=\int_0^t e^{-ru}\left[-rc(u,S(u))+c_t(u,S(u))+(r-\tilde{\beta}\tilde{\lambda})S(u)c_x(u,S(u))+\frac{1}{2}\sigma^2S(u)^2c_{xx}(u,S(u))\right]du$$
$$+\int_0^t e^{-ru}\sigma S(u)c_x(u,S(u))d\tilde{W}(u)$$
$$+\sum_{0<u\le t}e^{-ru}\left[c(u,S(u))-c(u,S(u-))\right]$$

The last term, the jump term, assume u is a jump time of the m-th Poisson process N_m, then the stock price satisfies $S(u)=(y_m+1)S(u-)$. Thus:

$$\sum_{0<u\le t}e^{-ru}\left[c(u,S(u))-c(u,S(u-))\right]=\sum_{m=1}^{M}\int_0^t e^{-ru}\left[c(u,(y_m+1)S(u-))-c(u,S(u-))\right]d(N_m(u)-\tilde{\lambda}_m u)$$
$$+\int_0^t e^{-ru}\tilde{\lambda}\left[\sum_{m=1}^{M}\tilde{p}(y_m)c(u,(y_m+1)S(u))-c(u,S(u))\right]du$$

Inserting this back into the initial Itô formula to get:

$$d(e^{-rt}c(t,S(t)))=e^{-rt}[-rc(t,S(t))+c_t(t,S(t))+(r-\tilde{\beta}\tilde{\lambda})S(t)c_x(t,S(t))+\frac{1}{2}\sigma^2S^2(t)c_{xx}(t,S(t))$$
$$+\tilde{\lambda}\sum_{m=1}^{M}[\tilde{p}(y_m)c(t,(y_m+1)S(t))-c(t,S(t))]]dt$$
$$+e^{-rt}\sigma S(t)c_x(t,S(t))d\tilde{W}(t)+\sum_{m=1}^{M}e^{-rt}\left[c(t,(y_m+1)S(t-))-c(t,S(t-))\right]d(N_m(t)-\tilde{\lambda}t)$$

All the terms are not martingales, and the dt term must be zero. Replace the price process $S(t)$ by the dummy variable x in the integrand of the dt term and we obtain:

$$
\begin{aligned}
d(e^{-rt}c(t,S(t))) = e^{-rt}\sigma S(t)c_x(t,S(t))d\tilde{W}(t) + e^{-rt}[c(t,S(t)) - c(t,S(t^-))]dN(t) \\
-e^{-rt}\tilde{\lambda}\left[\sum_{m=1}^{M}\tilde{p}(y_m)c(t,(y_m+1)S(t^-)) - c(t,S(t^-))\right]dt
\end{aligned}
$$

Proof:

Recall:

$N(t) = \sum_{m=1}^{M} N_m(t)$, and $\tilde{\lambda} = \sum_{m=1}^{M}\tilde{\lambda}_m$, and $\tilde{\lambda}\tilde{p}(y_m) = \tilde{\lambda}_m$

Continuous Jump Distribution Modifications - Jump sizes Y_i have a density $\tilde{f}(y)$ under Risk-Neutral Measure $\tilde{\mathbb{P}}$

Term:

$$
\sum_{m=1}^{M}\tilde{p}(y_m)c(t,(y_m+1)x)
$$

is replaced by

$$
\int_{-1}^{\infty} c(t,(y+1)x)\tilde{f}(y)\,dy
$$

In the call price differential we modify the last term so the full equation is:

$$
\begin{aligned}
d(e^{-rt}c(t,S(t))) = e^{-rt}\sigma S(t)c_x(t,S(t))d\tilde{W}(t) + e^{-rt}[c(t,S(t)) - c(t,S(t^-))]dN(t) \\
-e^{-rt}\tilde{\lambda}\left[\int_{-1}^{\infty} c(t,(y+1)S(t^-))\tilde{f}(y)\,dy - c(t,S(t^-))\right]dt
\end{aligned}
$$

Using the discounted call price differential:

$$
\begin{aligned}
d(e^{-rt}c(t,S(t))) = e^{-rt}\sigma S(t)c_x(t,S(t))d\tilde{W}(t) + e^{-rt}[c(t,S(t)) - c(t,S(t^-))]dN(t) \\
-e^{-rt}\tilde{\lambda}\left[\sum_{m=1}^{M}\tilde{p}(y_m)c(t,(y_m+1)S(t^-)) - c(t,S(t^-))\right]dt
\end{aligned}
$$

Consider hedging a short position in a European Call whose discounted price satisfies the equation above. Begin with a short call position and hedging portfolio with initial capital $X(0) = c(0,S(0))$.

We compare the discounted call price differential and the discounted hedging portfolio value differential:

If $\Gamma(t)$ shares of stock are held by the hedging portfolio at each time t, then

$$
dX(t) = \Gamma(t^-)dS(t) + r[X(t) - \Gamma(t)S(t)]dt
$$

and

$$d(e^{-rt}X(t)) = e^{-rt}\left[\Gamma(t)\sigma S(t)d\tilde{W}(t) + \Gamma(t^-)S(t^-)\sum_{m=1}^{M} y_m(dN_m(t) - \tilde{\lambda}dt)\right]$$

Delta Hedging Strategy

Let $\Gamma(t) = c_x(t, S(t))$. This choice equates the $d\tilde{W}(t)$ terms in the formulas for $d(e^{-rt}c(t, S(t)))$ and $d(e^{-rt}X(t))$, providing a perfect hedge against the risk introduced by the Brownian motion. However, it leaves us with:

$$d(e^{-rt}c(t,S(t)) - e^{-rt}X(t)) = \sum_{m=1}^{M} e^{-rt}\left[c(t,(y_m+1)S(t^-)) - c(t,S(t^-)) - y_m S(t^-)c_x(t,S(t^-))\right](dN_m(t) - \tilde{\lambda}dt)$$

Since $c(t, x)$ is strictly convex in x due to the strict convexity in the Black-Scholes-Merton (BSM) model, strict convexity gives:

$$c(t, x_2) - c(t, x_1) > (x_2 - x_1)c_x(t, x_1)$$

for all $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 \neq x_2$. Thus:

$$c(t,(y_m+1)S(t^-)) - c(t,S(t^-)) > y_m S(t^-)c_x(t,S(t^-))$$

Given that each $y_m > -1$ and $\neq 0$, it follows that between jumps:

$$d(e^{-rt}c(t,S(t)) - e^{-rt}X(t)) < 0$$

Between jumps, the hedging portfolio outperforms the option. At jump times, the option outperforms the hedging portfolio.

Since both are martingales under the risk-neutral measure, so is their difference. At the initial time, their difference is zero: $c(0, S(0)) - X(0) = 0$.

Therefore, the expected value of the difference is always zero:

$$\mathbb{E}_Q[e^{-rt}c(t,S(t))] = \mathbb{E}_Q[e^{-rt}X(t)], \quad \text{for } 0 \leq t \leq T$$

The delta hedging strategy hedges the option on an **average basis**. Should we choose $\tilde{\lambda}_m = \lambda_m$, then the jumps average under the risk-neutral measure will also be the same average under the actual probability measure.

Continuous Jump Distribution - Delta Hedging Strategy

$$d(e^{-rt}c(t,S(t)) - e^{-rt}X(t)) = e^{-rt}\left[c(t,S(t)) - c(t,S(t^-)) - (S(t) - S(t^-))c_x(t,S(t^-))\right]dN(t)$$
$$-e^{-rt}\tilde{\lambda}\int_{-1}^{\infty}\left[c(t,(y+1)S(t^-)) - c(t,S(t^-)) - yS(t^-)c_x(t,S(t^-))\right]\tilde{f}(y)dy\,dt$$

This can be interpreted similarly to the discrete model where, due to strict convexity, the value is < 0 between jumps, meaning the hedging portfolio outperforms the option between jumps. At jump times, the option outperforms the hedging portfolio.

On average under the risk-neutral measure, these effects cancel out.

Summary

The fundamental pure jump process is the Poisson Process, which is Markov but not a martingale.

The Poisson Process only jumps upwards and remains constant between jumps. To convert it into a martingale, one must subtract the mean of the Poisson Process, resulting in a Compensated Poisson Process.

All jumps of a Poisson Process are of size one. To make the jumps random sizes, we use a Compounded Poisson Process. Like a Poisson Process, it is Markov, generally not a martingale but can be made into a martingale through the Compensated Compounded Poisson Process, where the mean of the process is subtracted.

For a finite number (M) of jump sizes, it can be decomposed into a summation of M independent scaled Poisson Processes.

A Jump Process has four components:

1) Initial Condition
2) Itô integral
3) Riemann integral
4) Pure Jump Process

The first three components together make up the continuous part of the jump process, similar to the Brownian Motion model that underpins the Black-Scholes Model.

A pure jump process is defined as a right-continuous process with finitely many jumps in each finite time interval and is constant between jumps.

The quadratic variation of a pure jump process is the sum of the squares of the jumps within that time interval.

Poisson Processes and Brownian Motions are independent of each other and under the same filtration. Similarly, Poisson Processes compared to another Poisson Process are independent as long as no two jumps occur simultaneously.

Integrating this adapted process does not lead to a martingale. It would need to be left-continuous to be a martingale.

The Compound Poisson Process can change measure to obtain an arbitrary positive intensity (average rate of jump arrival) and arbitrary distribution of jump sizes. If there are M possible jump sizes, then there are M-1 degrees of freedom in assigning probabilities to these jump sizes (sum to 1). A complete market under this scenario requires a money market and as many nonredundant securities as there are uncertainties (each possible jump size represents an uncertainty). This implies that jump-diffusion models are generally incomplete and have multiple risk-neutral measures.

In practice, one considers a parametrized class of such measures and then calibrates the model to market prices to determine values for the parameters. Using the risk-neutral pricing formula to price derivatives securities will not work under the hedging argument but will instead be an interpolation by which prices of nontraded securities are computed based on prices of traded ones. This formula can be used to examine the effectiveness of various hedging techniques, as demonstrated with the delta-hedging rule.