Model with Jumps

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Overview of Jump Processes in Financial Modeling

- **Fundamental Characteristics:**
  - **Poisson Process:** Markov but not a martingale; only jumps upwards and remains constant between jumps.
  - **Compensated Poisson Process:** Converts the Poisson Process into a martingale by subtracting its mean.
- **Compounded Processes:**
  - **Compounded Poisson Process:** Introduces random-sized jumps, still Markov but generally not a martingale unless compensated.
  - **Decomposition:** For finite number of jump sizes, it decomposes into a summation of independent scaled Poisson Processes.
- **Components of a Jump Process:**
  - **Continuous Part:** Comprises the initial condition, Itô integral, and Riemann integral, akin to Brownian Motion in the Black-Scholes Model.
  - **Pure Jump Process:** Right-continuous with finitely many jumps in each interval, constant between jumps.
- **Quadratic Variation and Independence:**
  - The quadratic variation is the sum of the squares of the jumps.
  - Poisson Processes and Brownian Motions are independent under the same filtration; Poisson Processes are independent unless jumps coincide.
- **Measure Change and Market Completeness:**
  - **Compound Poisson Process:** Allows change of measure for any positive intensity and jump size distribution.
  - **Market Completeness:** Requires a security for each type of uncertainty (jump size); typically leads to market incompleteness with multiple risk-neutral measures.

Introduction to Poisson Process

Jump processes are modeled using Poisson distributions. The following is a brief reintroduction to Poisson distributions, prior to diving into Jump Processes.

- **Definition**: Random variable τ with density $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$ (where λ is a positive constant) and $f(t) = 0$ for $t < 0$.
- **Expected Value ($E[\tau]$)**: $E[\tau] = \frac{1}{\lambda}$
- **Cumulative Distribution Function (CDF)**: $F(t) = 1 - e^{-\lambda t}$ for $t \geq 0$.
- **Memorylessness**: The probability of waiting an additional t time units does not change, regardless of how long has already been waited. This property defines the Poisson process as memoryless, expressed as $P(\tau > t + s \mid \tau > s) = e^{-\lambda t}$.

Constructing a Poisson Process

- **Basic Construction**:
  - Sequence of independent exponential random variables $\tau_1, \tau_2, \ldots$ with mean $\frac{1}{\lambda}$.
  - Time of nth jump: $S_n = \sum_{k=1}^{n} \tau_k$.
- **Poisson Process Definition**:
  - $N(t)$: Counts the number of jumps before time t.
- **Properties**:
  - **Time Between Jumps**: Expected time is $\frac{1}{\lambda}$, with jumps arrive at a rate of λ per unit time.
  - **Gamma Density for S_n**: $g_n(s) = \frac{\lambda^n s^{n-1} e^{-\lambda s}}{(n-1)!}$, proving S_{n+1} also follows gamma distribution.
  - **Distribution of $N(t)$**: $P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$ for $k = 0,1,2, \ldots$.
- **Memorylessness and Stationarity**:
  - **Increment Independence**: $N(t+s) - N(s)$ is independent of prior events and follows the same distribution as $N(t)$.
  - **Expected Value of Increment**: $E[N(t) - N(s)] = \lambda(t-s)$, showing the average number of jumps is proportional to time interval.
  - **Variance**: $\mathrm{Var}\big(N(t) - N(s)\big) = \lambda(t-s)$, equal to the mean.

Comparison of Poisson Processes

- **Poisson Process**:
  - **Definition**: Counts the number of events happening at a constant average rate (λ).
  - **Key Formula**: $N(t) =$ number of jumps by time t.
  - **Jumps**: Fixed size (typically size = 1).
  - **Distribution**: Exponential inter-jump times with density $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.
  - **Mean & Variance**: Mean and variance of increments are $\lambda(t - s)$.
  - **Increment Formula**: $P(N(t) - N(s) = k) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$.
- **Compensated Poisson Process**:
  - **Definition**: Adjusted Poisson process to maintain martingale properties by subtracting λt from $N(t)$. This eliminates drift in modeling.
  - **Key Formula**: $M(t) = N(t) - \lambda t$.
  - **Mean & Variance**: Mean is 0, variance is $\lambda(t - s)$.
  - **Increment Formula**: $M(t)$ is a martingale, so $E[M(t)|F(s)] = M(s)$.
- **Compound Poisson Process**:
  - **Definition**: Sum of random variables Y_i associated with each event in $N(t)$, each Y_i represents the size of the i-th jump.
  - **Key Formula**: $Q(t) = \sum_{i=1}^{N(t)} Y_i$.
  - **Jumps**: Random sizes Y_i, varying impact per event.
  - **Distribution**: Y_i are i.i.d with average jump size β, and λt number of jumps in time interval [0,t].
  - **Mean & Variance**: Mean of $Q(t)$ is $\beta\lambda t$.
  - **Usage**: Suitable for modeling financial returns with varying jump sizes.
  - **Increment Formula**: Increment $Q(t) - Q(s) = \sum_{i=N(s)+1}^{N(t)} Y_i$.
- **Compensated Compound Poisson Process**:
  - **Definition**: Adjusts the compound Poisson process to be a martingale by subtracting expected jump totals over time. Adjusts compound process for risk-neutral pricing and hedging in finance.
  - **Key Formula**: $Q(t) - \beta\lambda t$ where $\beta = E[Y_i]$.
  - **Jumps**: Same random sizes as Compound, but adjusted for expected total impact.
  - **Distribution**: Ensures $E[Q(t)|F(s)] = Q(s)$ for all $s < t$.
  - **Mean & Variance**: Compensates to align mean with expected value over time.
  - **Increment Formula**: $E[Q(t) - \beta\lambda t|F(s)] = Q(s) - \beta\lambda s$.
- **Memorylessness**: Classic Poisson processes (including compounded and compensated) retain the property where the future probability distribution depends only on the present, not on how the process arrived there.
- **Martingale Property**: Compensated processes adjust to ensure that the expected value of future increments, given the past, equals the current value, crucial for financial derivatives pricing.

Moment-Generating Functions (MGF) of Poisson Processes

The compound poisson process increment density function is too complex to solve explicitly, therefore the moment-generating function is used.

- **Random Variable MGF**:
  - Y_i as:

$$\varphi_Y(u) = E[e^{uY_i}]$$

- **Compound Poisson Process MGF**:
  - For $Q(t)$ where jumps are random:

$$\varphi_{Q(t)}(u) = e^{\lambda t(\varphi_Y(u)-1)}$$

- For constant jump size y, $Q(t) = yN(t)$:

$$\varphi_{yN(t)}(u) = e^{\lambda t(e^{uy}-1)}$$

- For $y = 1$ (standard Poisson Process): $\varphi_{N(t)}(u) = e^{\lambda t(e^u-1)}$.
- **Finitely Many (M) Jump Sizes MGF**:
  - For jumps taking finitely many values $y_1, y_2, \ldots, y_M$ with probabilities $p(y_m)$:
  - $\varphi_{Q(t)}(u) = \prod_{m=1}^{M} e^{\lambda p(y_m)t(e^{uy_m}-1)}$
- **Interpretations of Compound Poisson Process**:
  - **Viewpoint 1**: A single Poisson process with size-one jumps replaced by random-sized jumps. We have already demonstrated this.
  - **Viewpoint 2**: A sum of independent Poisson processes, where size-one jumps in each are replaced by fixed-size jumps. Constuction of this viewpoint is as follows:

$$Q(t) = \sum_{m=1}^{M} y_m N_m(t)$$

where $N_m(t)$ is the number of jumps of size y_m up to time t, each N_m independent with intensity $\lambda p(y_m)$.

Introduction to Stochastic Process with Jumps

- **Components of Stochastic Process** $X(t)$:
  - **Initial Condition**: $X(0)$ - nonrandom.
  - **Ito Integral**: $I(t)$ - captures continuous stochastic changes.
  - **Riemann Integral**: $R(t)$ - accounts for deterministic changes.
  - **Jump Process**: $J(t)$ - adapted, right-continuous, pure jump process with $J(0) = 0$.
- **Jump Process Properties**:
  - Does not jump at time 0.
  - Finitely many jumps in each finite interval $(0, T]$.
  - Constant between jumps, embodying a pure jump process.
- **Process Dynamics**:
  - $X(t) = X(0) + I(t) + R(t) + J(t)$.
  - $X(t-) = X(0) + I(t) + R(t) + J(t-)$.
  - Change at jump: $\Delta X(t) = J(t) - J(t-)$. Where $J(t-)$ is the time immediately before a jump occurs, and $J(t)$ is the time immediately after the jump occurs.
- **Stochastic Integral of $\Phi(s)$ with respect to $X(t)$**:
  - Defined as: $\int_0^t \Phi(s)dX(s) = \int_0^t \Phi(s)\Gamma(s)dW(s) + \int_0^t \Phi(s)\Theta(s)ds + \sum_{0<s\leq t} \Phi(s)\Delta J(s)$.
  - Differentiates between continuous and jump changes.
- **Example: Compensated Poisson Process $M(t) = N(t) - \lambda t$**:
  - $J(t) = N(t)$, and $\Phi(s) = \Delta N(s)$ takes value 1 if N jumps at s.
  - Integrating over $M(t)$ creates a scenario where an investor could theoretically achieve an arbitrage by exploiting jumps, although not feasible in reality due to the requirement of anticipating jumps.
- **Martingale Consideration**:
  - A stochastic integral aims to be a martingale, which is contingent on the integrand and the integrator properties. The integral $\int_0^t \Phi(s)dX(s)$ is right-continuous but may not always form a martingale due to the dynamics at jump times and the adaptiveness requirements of $\Phi(s)$.

Quadratic Variation in Jump Processes

- **Key Components**:
  - **Brownian Motion**: Quadratic variation $[X_c^1, X_c^2](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds$.
  - **Jump Processes**: Additional term for jumps, $[J_1, J_2](T) = \sum_{0<s\leq T} \Delta J_1(s)\Delta J_2(s)$.
- **Behavior Over Time**:
  - Brownian motion's quadratic variation approaches zero as time intervals shrink.
  - Quadratic variation from jumps converges to a finite number, independent of the time intervals.
- **Differential Notation**:
  - $X_1(t) = X_1(0) + X_c^1(t) + J_1(t)$
  - $X_2(t) = X_2(0) + X_c^2(t) + J_2(t)$
  - $dX_1(t)dX_2(t) = dX_c^1(t)dX_c^2(t) + dJ_1(t)dJ_2(t)$
- **Cross-Variation Insights**:
  - Cross-variation between continuous and pure jump processes is zero.
  - For Brownian motion and Poisson processes, including compensated Poisson, the cross-variation is also zero.
  - Non-zero cross-variation occurs only with simultaneous jumps or with two dW terms.
- **Independence and Compensation**:
  - Compensated Poisson Process: $M(t) = N(t) - \lambda t$, $[W, M](t) = 0$ showing independence.
  - This implies stochastic independence between W and M as well as between W and N within the same filtration.
- **Example of Process Adaptation**:
  - For an adapted, right-continuous jump process $\tilde{X}_i(t)$, quadratic variation is: $[\tilde{X}_1, \tilde{X}_2](t) = \int_0^t \Phi_1(s)\Phi_2(s)d[X_1, X_2](s)$.

Itô-Doeblin Formula: Application to Jump Processes

- **Formula Overview**:
  - Continuous changes: $df\big(X(s)\big) = f'\big(X(s)\big)dX_c(s) + \frac{1}{2}f''\big(X(s)\big)\big(dX_c(s)\big)^2$
  - Jump adjustments: $\sum_{0<s\leq t}\big[f\big(X(s)\big) - f\big(X(s-)\big)\big]$
  - Integration of changes from 0 to t: $f\big(X(t)\big) = f\big(X(0)\big) + \int_0^t f'\big(X(s)\big)dX_c(s) + \frac{1}{2}\int_0^t f''\big(X(s)\big)\big(dX_c(s)\big)^2 + \sum_{0<s\leq t}\big[f\big(X(s)\big) - f\big(X(s-)\big)\big]$
- **Key Components**:
  - $\big(dX_c(s)\big)^2 = \Gamma^2(s)ds$
  - $dX_c(s) = \Gamma(s)dW(s) + \Theta(s)ds$
  - Ensures all jumps are captured in the model.
- **Geometric Poisson Process Example**:

$$S(t) = S(0)e^{-\lambda\sigma t}(\sigma+1)^{N(t)}$$

where: - $\sigma > -1$ is a constant, - If $\sigma > 0$, the process jumps up and moves down between jumps. - If $-1 < \sigma < 0$, the process jumps down and moves up between jumps.

Itô's Formula for Jump Processes:

$$S(t) = S(0) - \lambda\sigma\int_0^t S(u)\,du + \sum_{0\leq u\leq t}[S(u) - S(u^-)]$$

If a Jump Occurs at Time u:

$$S(u) = (\sigma+1)S(u^-), \quad \text{thus} \quad S(u) - S(u^-) = \sigma S(u^-)$$

If No Jump Occurs at Time u:

$$S(u) - S(u^-) = 0$$

For Either Case, We Have:

$$S(u) - S(u^-) = \sigma S(u^-)dN(u)$$

- $X(t) = N(t)\log(\sigma+1) - \lambda\sigma t$
- Martingale transformation using the compensated Poisson process $M(u) = N(u) - \lambda u$:

$$S(t) = S(0) - \lambda\sigma\int_0^t S(u-)du + \sigma\int_0^t S(u-)dN(u)$$

- **Differential Formulation**:
- $dS(t) = \sigma S(t-)dM(t) = -\lambda\sigma S(t)dt + \sigma S(t-)dN(t)$

Itô Formula for Multi-Dimensional Jump Processes

- **Framework**:
  - Analyze two jump processes $X_1(t)$ and $X_2(t)$ with continuous derivatives.
  - $f\big(t, X_1(t), X_2(t)\big)$ evolves based on a combination of drift, diffusion, and jump components.
- **Formula Evolution**:
  - Initial value: $f\big(0, X_1(0), X_2(0)\big)$
  - Drift component: $\int_0^t \frac{\partial f}{\partial t}\big(s, X_1(s), X_2(s)\big)\, ds$
  - Diffusion components:
    - $\int_0^t \frac{\partial f}{\partial x_1}\big(s, X_1(s), X_2(s)\big)\, dX_c^1(s)$
    - $\int_0^t \frac{\partial f}{\partial x_2}\big(s, X_1(s), X_2(s)\big)\, dX_c^2(s)$
  - Mixed partial derivatives:
    - $\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_1^2}\big(s, X_1(s), X_2(s)\big)\, \big(dX_c^1(s)\big)^2$
    - $\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_2^2}\big(s, X_1(s), X_2(s)\big)\, \big(dX_c^2(s)\big)^2$
    - $\int_0^t \frac{\partial^2 f}{\partial x_1 \partial x_2}\big(s, X_1(s), X_2(s)\big)\, dX_c^1(s) dX_c^2(s)$
  - Jump adjustments: $\sum_{0<s\leq t}\big[f\big(s, X_1(s), X_2(s)\big) - f\big(s, X_1(s-), X_2(s-)\big)\big]$
- **Itô's Product Rule**:
  - Applied for the product $X_1(t)X2(t)$:
    - $X_1(t)X_2(t) = X_1(0)X_2(0) + \int_0^t X_2(s-)dX_1(s) + \int_0^t X_1(s-)dX_2(s) + [X_1, X_2](t)$
  - Cross-variation: $[X_1, X_2](t) = \int_0^t 1\ dX_c^1(s) dX_c^2(s)$

Reintroduction to Girsanov's Theorem and Radon-Nikodym Derivatives

- **Definition of $Z(t)$**:
  - **Without Jumps**: $Z(t) = e^{-\int_0^t \Gamma(s)dW(s) - \frac{1}{2}\int_0^t \Gamma^2(s)d}$
  - Represents the Radon-Nikodym derivative used to change probability measures.
- **Differential Equation**:
  - $dZ(t) = -\Gamma(t)Z(t)dW(t)$
  - Can be expressed in terms of the continuous process: $Z(t) = e^{X_c(t) - \frac{1}{2}[X_c, X_c](t)}$
- **Handling Jumps**:
  - For jump processes, $dZ^X(s) = Z^X(s-)dX(s)$
  - Adjusts the measure to account for jumps, where: $Z^X(s) = Z^X(s-)\big(1 + dX(s)\big)$
- **Doleans-Dade Exponential of X Process**:
  - Defines the exponential formula incorporating jumps:
  - $Z^X(t) = e^{X_c(t) - \frac{1}{2}[X_c, X_c](t)} \prod_{0<s\le t}\big(1 + dX(s)\big)$
  - Integral form for $Z^X(t)$:
  - $Z^X(t) = 1 + \int_0^t Z^X(s-)dX(s)$
  - This form is the solution to the differential equation with initial condition $Z^X(0) = 1$.

SUMMARY OF CHANGE OF MEASURE

- **Brownian Motion with Drift** becomes Brownian Motion without drift.
- **Poisson Process** — a change of measure alters the intensity.
- **Compound Poisson Process** — a change of measure adjusts both the intensity and the distribution of the jump sizes.

Change of Measure in Poisson Processes

- **Change of Measure Formula**:
  - $Z(t) = e^{(\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}$
  - This transformation adjusts the probability measure for the Poisson process from λ to $\tilde{\lambda}$.
- **Martingale Property and Expected Value**:
  - $Z(t)$ is a martingale under original measure P.
  - Expected value $E[Z(t)] = 1$ is maintained, ensuring the measure change is valid.
- **Differential Equation**:
  - $dZ(t) = \left(\frac{\tilde{\lambda} - \lambda}{\lambda}\right) Z(t-) dM(t)$
  - Where $M(t)$ is a compensated Poisson process, and $dM(t)$ denotes the martingale part of the change.
- **Components of the Change**:
  - **Continuous Part**: $X_c(t) = (\lambda - \tilde{\lambda})t$
  - **Jump Part**: $J(t) = \left(\frac{\tilde{\lambda} - \lambda}{\lambda}\right) N(t)$
- **Quadratic Variation**:
  - For the continuous part: $[X_c, X_c](t) = 0$
  - Reflects the absence of variability in the rate change part of the process.
- **Application to Change Measure**:
  - New probability measure $\tilde{P}$ uses $Z(T)$ as the Radon-Nikodym derivative:
  - $\tilde{P}(A) = \int_A Z(T) dP \quad$ for all $A \in \mathcal{F}$
  - Under $\tilde{P}$, the Poisson process operates at a new intensity $\tilde{\lambda}$.

Change of Poisson Intensity in Financial Modeling

- **Intensity Under Measure $\tilde{P}$**:
  - Under the new measure $\tilde{P}$, the Poisson process has an intensity $\tilde{\lambda}$.
  - Moment-generating function of $N(t)$: $E\left[e^{uN(t)}Z(t)\right] = e^{\tilde{\lambda}t(e^u-1)}$.
- **Example: Geometric Poisson Process in Stock Modeling**:
  - Stock price model: $S(t) = S(0)e^{\alpha t + N(t)\log(\sigma+1) - \lambda}$.
  - Modified to: $S(t) = S(0)e^{(\alpha-\lambda\sigma)t}(\sigma+1)^{N(t)}$.
  - Under P: $N(t)$ is Poisson with intensity λ, under $\tilde{P}$: intensity changes to $\tilde{\lambda}$.
- **Risk-Neutral Measure Adjustment**:
  - Adjusted stock dynamics under $\tilde{P}$: $dS(t) = rS(t)dt + \sigma S(t-)d\tilde{M}(t)$ where $\tilde{M}(t) = N(t) - \tilde{\lambda}t$.
  - Change in rate due to measure change ensures $S(t)$ aligns with risk-free rate r.
- **Condition for Risk-Neutrality**:
  - $\tilde{\lambda} = \lambda - \frac{\alpha-r}{\sigma}$ ensures the mean rate under $\tilde{P}$ matches r.
  - $\tilde{\lambda} > 0$ is necessary for a valid risk-neutral measure.
  - $\lambda > \frac{\alpha-r}{\sigma}$
- **Arbitrage Opportunities**:
  - If $\sigma > 0$ and $\tilde{\lambda} \leq 0$: $S(t) \geq S(0)e^{rt}$, suggesting borrow at rate r and invest in stock.
  - If $-1 < \sigma < 0$ and $\tilde{\lambda} \leq 0$: $S(t) \leq S(0)e^{rt}$, suggesting short the stock and invest in risk-free assets.

Change of Measure in Compound Poisson Processes with Discrete Jumps

- **Overview of Compound Poisson Process**:
  - **Process Definition**: $Q(t) = \sum_{i=1}^{N(t)} Y_i$ where Y_i are i.i.d random variables.
  - **Jump Mechanism**: If $N(t)$ jumps at time t, $Q(t)$ jumps by $Y_{N(t)}$.
- **Changing Measure**:
  - Influences both the intensity of $N(t)$ and the distribution of Y_i.
  - **Discrete Jump-Size Distribution**:
    - Jump sizes $y_1, y_2, \dots, y_M$ with probabilities $p(y_m) = P(Y_i = y_m)$.
    - $N(t) = \sum_{m=1}^{M} N_m(t)$ where each $N_m(t)$ is an independent Poisson process for jumps of size y_m.
    - Each N_m has intensity $\lambda_m = \lambda \cdot p(y_m)$.
- **Measure Change**:
  - $Z_m(t)$ for each jump size: $Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} \left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}$
  - $Z(t) = \prod_{m=1}^{M} Z_m(t)$
- **Properties and Implications**:
  - $Z(t)$ product rule allows this to expand as follows: $Z(t) = Z_1(t)Z_2(t) \dots Z_m(t)$
  - Cross-variation between different $Z_m(t)$ and $Z_n(t)$ is zero due to independence.
  - Under the new measure $\tilde{P}$, $N_m(t)$ has adjusted intensity $\tilde{\lambda}_m$, and the overall process $Q(t)$ reflects these changes.

Change of Measure in Compound Poisson Process for Finitely Many Jump Sizes

- **Overview of Measure Change**:
  - **Process Definition**: $Q(t)$ is a compound Poisson process under $\tilde{P}$ with intensity $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$.
  - **Jump Size Distribution**: Under $\tilde{P}$, $\tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}$ determines the new probability of each jump size.
- **Radon-Nikodym Derivative Process**:
  - **Formula**:
  - $Z(t) = e^{(\lambda-\tilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}$
  - Adjusts the measure to reflect the new intensity $\tilde{\lambda}$ and jump distribution $\tilde{p}(y_m)$.
- **Moment Generating Function Under $\tilde{P}$**:
  - $\tilde{E}\left[e^{uQ(t)}\right] = e^{\tilde{\lambda}t(\sum_{m=1}^{M} \tilde{p}(y_m)e^{uy_m}-1)}$
  - Highlights the change in distribution under the new measure.
- **Application and Implications**:
  - **Pure Jump Process Adjustment**:
  - $dJ(t) = J(t-)\left(\frac{\tilde{\lambda}\tilde{f}(dQ(t))}{\lambda f(dQ(t))} - 1\right)$
  - **Compensated Poisson Process**:
  - $H(t) = \sum_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$
  - with $dH(t) = \frac{\tilde{\lambda}\tilde{f}(dQ(t))}{\lambda f(dQ(t))}$, making $H(t) - \tilde{\lambda}t$ a martingale.

Change of Measure in Compound Poisson Process with Continuous Jump Sizes

- **Overview of Process Under $\tilde{P}$**:
  - **Compound Poisson Process**: $Q(t)$ with intensity $\tilde{\lambda}$.
  - **Jump Density**: Jumps in $Q(t)$ are i.i.d. with density $\tilde{f}(y)$.
- **Moment-Generating Function**:
  - Expected moment-generating function under $\tilde{P}$:
  - $\tilde{E}\left[e^{uQ(t)}\right] = e^{\tilde{\lambda}t(\tilde{\varphi}_Y(u)-1)}$
  - Where $\tilde{\varphi}_Y(u) = \int_{-\infty}^{\infty} e^{uy}\, \tilde{f}(y)\, dy$.
- **Martingale Property and Verification**:
  - Define $X(t) = e^{uQ(t)-\tilde{\lambda}t(\tilde{\varphi}_Y(u)-1)}$.
  - **Martingale Verification**: Show that $X(t)Z(t)$ is a martingale under P, where $Z(t)$ adjusts for the measure change.
- **Jump Behavior and Compound Process**:
  - At jump times: $dX(t) = X(t-)\left(e^{udQ(t)} - 1\right)$.

$$V(t) = \sum_{i=1}^{N(t)} e^{uY_i}\left(\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}\right)$$

- **Compensated Compound Poisson Process**

$$V(t) - \tilde{\lambda}t\tilde{\varphi}_Y(u) \text{ is a martingale}$$

$$Z(t) = 1 + \int_0^t Z(s-)dX(s) + \int_0^t X(s-)dZ(s) + [X,Z](t)$$

$$\tilde{E}\left[e^{uQ(t)}\right] = E\left[e^{uQ(t)}Z(t)\right]$$

Compound Poisson Process and Brownian Motion Under Change of Measure

- **Component Processes**:
  - **Brownian Motion Part**: $Z_1(t) = e^{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du}$
  - **Compound Poisson Part**: $Z_2(t) = e^{(\lambda - \tilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$
- **Combined Process**:
  - **Joint Process**: $Z(t) = Z_1(t)Z_2(t)$
  - $Z(t)$ is a martingale, ensuring $E[Z(t)] = 1$ and $Z(0) = 1$.
- **Martingale Verification**:
  - Integrals involving $Z_1(s-)dZ_2(s)$ and $Z_2(s-)dZ_1(s)$ are martingales.
  - This confirms the validity of $Z(t)$ as a combined measure change tool.
- **Change of Measure Under $\tilde{P}$**:
  - Adjusted Brownian motion: $\widetilde{W}(t) = W(t) + \int_0^t \Theta(s)ds$, with $\widetilde{W}$ maintaining Brownian motion properties under $\tilde{P}$.
  - The compound Poisson process $Q(t)$ retains intensity $\tilde{\lambda}$ and adopts jump density $\tilde{f}(y)$.
- **Independence and Moment-Generating Function**:
  - **Independence Proof**: $\tilde{E}\left[e^{u_1\widetilde{W}(t) + u_2 Q(t)}\right] = e^{\frac{1}{2}u_1^2 t + \tilde{\lambda}t(\tilde{\phi}_Y(u_2) - 1)}$, confirming $Q(t)$ and $\widetilde{W}(t)$ are independent under $\tilde{P}$.
  - Interaction via $\Theta(t)$ potentially depending on $Q(t)$ does not affect the martingale property of combined processes.

Pricing a European Call in a Jump Model

Two Cases:

1) The underlying asset is driven by a single Poisson Process - Market is complete in this case.
2) The underlying asset is driven by a Brownian Motion and a Compound Poisson Process - Market is incomplete in this case.

Pricing European Call Options - Jump Model by single Poisson Process

- **Asset Price Dynamics**:
  - **Formula**: $S(t) = S(0)e^{(\alpha-\lambda\sigma)t}(\sigma+1)^{N(t)}$
  - **Differential**: $dS(t) = \alpha S(t)dt + \sigma S(t-)dM(t)$
  - Where $M(t) = N(t) - \lambda t$ is a compensated Poisson process.
- **No-Arbitrage and Risk-Neutral Measure**:
  - **No-Arbitrage Condition**: $\lambda > \frac{\alpha-r}{\sigma}$
  - **Risk-Neutral Intensity**: $\tilde{\lambda} = \lambda - \frac{\alpha-r}{\sigma}$ ensures $\tilde{\lambda}$ is positive.
  - **Radon-Nikodym Derivative**: $Z(t) = e^{(\lambda-\tilde{\lambda})t}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}$
- **Risk-Neutral Asset Dynamics**:
  - Under $\tilde{P}$, $dS(t) = rS(t)dt + \sigma S(t-)d\tilde{M}(t)$
  - Shows discounted stock price $e^{-rt}S(t)$ is a martingale.
- **Option Pricing Under $\tilde{P}$**:
  - **Payoff at Expiration**: $V(T) = (S(T) - K)^+$
  - **Risk-Neutral Valuation**:
  - $V(t) = \tilde{E}\left[e^{-r(T-t)}\left(S(t)e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)} - K\right)^+ \mid F(t)\right]$
  - **Price Function**:
  - $c\big(t, S(t)\big) = \sum_{j=0}^{\infty}\left(S(t)e^{-\tilde{\lambda}\sigma(T-t)}(\sigma+1)^j - Ke^{-r(T-t)}\right)^+ \frac{\left(\tilde{\lambda}(T-t)\right)^j}{j!}e^{-\tilde{\lambda}(T-t)}$
    - Captures the impact of both the drift change under $\tilde{P}$ and the jump distribution in option pricing.
- **Terminal Condition Satisfaction**:
  - At $t = T$, the payoff simplifies to $c\big(T, S(T)\big) = (S(T) - K)^+$, fulfilling the European call option payoff requirement.

Hedging a European Call Option in a Jump-Driven Model

- **Option Pricing Dynamics**:
  - Option price under risk-neutral measure:
  - $e^{-rt}c\big(t,S(t)\big) = c\big(0,S(0)\big) + \int_0^t e^{-ru}\left[c\big(u,(\sigma+1)S(u-)\big) - c\big(u,S(u-)\big)\right]d\widetilde{M}(u)$
  - At maturity T:
  - $e^{-r}\ \ (S(T)-K)^+ = c\big(0,S(0)\big) + \int_0^t e^{-ru}\left[c\big(u,(\sigma+1)S(u-)\big) - c\big(u,S(u-)\big)\right]d\widetilde{M}(u)$
- **Hedging Strategy**:
  - **Portfolio Value**: $X(t) = c\big(t,S(t)\big)$, ensuring the initial capital $X(0) = c\big(0,S(0)\big)$.
  - **Differential Matching**:
  - $dX(t) = \Gamma(t-)dS(t) + r[X(t) - \Gamma(t)S(t)]dt$
- $\Gamma(t)$ is the number of shares of stock held in the hedging portfolio
  - **Hedge Ratio** ($\Gamma(t)$):
  - $\Gamma(t) = \frac{c\big(t,(\sigma+1)S(t)\big) - c\big(t,S(t)\big)}{\sigma S(t)}$
  - Ensures $d\big(e^{-rt}X(t)\big) = e^{-rt}\sigma\Gamma(t-)S(t-)d\widetilde{M}(t)$ aligns with changes in option value.
- **Integration and Equivalence**:
  - The discounted portfolio value $e^{-r}\ \ X(t)$ replicates the discounted option value, validating the hedge.
  - Adjusts dynamically to jumps and continuous movements in $S(t)$.
- **Impact of Jumps**:
  - Change in option value due to a jump:
  - $c\big(t,(\sigma+1)S(t-)\big) - c\big(t,S(t-)\big)$
  - Corresponding change in the hedging portfolio value:
  - $\Gamma(t-)\big(S(t) - S(t-)\big) = \Gamma(t-)\sigma S(t-)$
  - Ensures the hedge adjusts appropriately at each jump, maintaining alignment with the option value.

Completeness

The model is complete and the risk-neutral measure is unique if and only if every derivative security can be hedged. “Every” meaning also those that are path-dependent. They were not considered here, but can be hedged, thus is complete.

For a single Poisson Process, this is summarized by: payoff $h\big(S(T)\big)$ at time T, one could replace the payoff by the function h, the differential-difference equation would still apply but with a terminal condition now of $c(T, x) = h(x)$, and the hedging formula would still be correct.

Asset Dynamics in a Mixed Brownian and Jump Process Environment

- **Probabilistic Framework**:
  - **Probability Space**: $(\Omega, \mathcal{F}, P)$
  - **Brownian Motion**: $W(t)$, with $0 \leq t \leq T$
  - **Poisson Processes**: $N_m(t)$, independent, each with intensity $\lambda_m > 0$
- **Compound Poisson Process**:
  - **Total Jumps**: $N(t) = \sum_{m=1}^{M} N_m\,(t)$, total intensity $\lambda = \sum_{m=1}^{M} \lambda_m$
  - **Jump Sizes**: Y_i with $P(Y_i = y_m) = \frac{\lambda_m}{\lambda}$, ensuring $\sum_{m=1}^{M} p\,(y_m) = 1$
  - **Expected Jump Size** (β): $\beta = \frac{1}{\lambda} \sum_{m=1}^{M} \lambda_m\, y_m$
- **Asset Price Dynamics**:
  - **SDE with Jumps**:
  - $dS(t) = (\alpha - \beta\lambda) S(t) dt + \sigma S(t) dW(t) + S(t\,-) dQ(t)$
  - **Martingale Adjustment**:
  - $Q(t) = \sum_{m=1}^{M} y_m\, N_m(t), \quad Q(t) - \beta\lambda t$ is a martingale
- **Model Decomposition**:
  - **Continuous Part**:
  - $X(t) = S(0) e^{\sigma W(t) + \left(\alpha - \beta\lambda - \frac{1}{2}\sigma^2\right) t}$
  - **Jump Part**:
  - $J(t) = \prod_{i=1}^{N(t)} (Y_i + 1)$
  - **Combined Stock Price**:
  - $S(t) = X(t) J(t)$

Construction of a Risk-Neutral Measure for a Jump-Diffusion Model

- **Foundational Components**:
  - **Brownian Motion Adjustment**:
  - $Z_0(t) = e^{-\theta W(t) - \frac{1}{2}\theta^2 t}$
  - **Compound Poisson Adjustment**:
  - $Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} \left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}, \quad m = 1, \ldots, M$
  - **Combined Risk-Neutral Measure**:
  - $Z(t) = Z_0(t) \prod_{m=1}^{M} Z_m(t)$
  - Change measure $\tilde{P}(A) = \int_A Z(T) dP$ for all A within $\mathcal{F}$.
- **Adjusted Dynamics Under $\tilde{P}$**:
  - **Adjusted Brownian Motion**:
  - $\widetilde{W}(t) = W(t) + \theta t$
  - **Adjusted Poisson Processes**:
  - $N_m(t)$ has intensity $\tilde{\lambda}_m$ under $\tilde{P}$
  - Independence between $\widetilde{W}$ and $N_1, \ldots, N_M$.
- **Risk-Neutral Conditions**:
  - **Total Adjusted Intensity**:
  - $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$
  - **Jump Size Probabilities**:
  - $\tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\lambda_m}$
  - **Expected Jump Size Under $\tilde{P}$**:
  - $\tilde{\beta} = \frac{1}{\tilde{\lambda}} \sum_{m=1}^{M} \tilde{\lambda}_m y_m$
- **Equivalence for Risk-Neutrality**:
  - **Required Return Adjustment**:
  - $\alpha - \beta\lambda = r + \sigma\theta - \tilde{\beta}\tilde{\lambda}$
  - $= \sigma\theta + \sum_{m=1}^{M} (\lambda_m - \tilde{\lambda}_m) y_m$
  - This ensures that under $\tilde{P}$, the mean rate of return on the stock is equivalent to the risk-free interest rate r.

Multiple Unique Risk-Neutral Measures

- **Example Setup:**
  - Consider a scenario with 3 stocks influenced by 2 Poisson processes (N_1 and N_2) and one Brownian Motion.
  - Define 3 compound Poisson Processes:
  - $Q_i(t) = y_{i,1}N_1(t) + y_{i,2}N_2(t)$, for $i = 1,2,3$
  - Parameters $y_{i,m} > -1$, ensuring well-defined processes.
  - Each stock $S_i(t)$ evolves according to:
  - $dS_i(t) = (\alpha_i - \beta_i\lambda)S_i(t)dt + \sigma_i S_i(t)dW(t) + S_i(t^-)dQ_i(t)$
  - β_i defined as:
  - $\beta_i = \frac{1}{\lambda}\left(\lambda_1 y_{i,1} + \lambda_2 y_{i,2}\right)$
  - λ_1 and λ_2 are intensities for N_1 and N_2 respectively.
- **Implications for Risk-Neutral Measure:**
  - For $i = 1,2,3$:
  - $\alpha_i - r = \sigma_i\theta + \left(\lambda_1 - \tilde{\lambda}_1\right)y_{i,1} + \left(\lambda_2 - \tilde{\lambda}_2\right)y_{i,2}$
  - The system yields three equations linking the unknowns $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, and θ.

Single Stock Model with Jumps

- **Model Setup:**
  - In the single stock model, parameters such as θ and $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$ are chosen to meet the market price of risk equations.
  - The stock price dynamics are modeled as:
  - $dS(t) = \left(r - \tilde{\beta}\tilde{\lambda}\right)dt + \sigma S(t)d\widetilde{W}(t) + S(t^{-})dQ(t)$
  - Solution to this model:
  - $S(t) = S(0)e^{\sigma\widetilde{W}(t)+\left(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2\right)t}$
  - θ is indirectly related through $\tilde{\beta}\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m\, y_m$.
- **Black-Scholes-Merton Call Pricing:**
  - Standard formula under geometric Brownian motion:
  - $C(t, x) = xN\left(d_{+}(t, x)\right) - Ke^{-rt}N\left(d_{-}(t, x)\right)$
  - Where:
  - $d_{\pm}(t, x) = \frac{1}{\sigma\sqrt{t}}\left(\log\left(\frac{x}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)t\right)$
  - $N(y)$ represents the cumulative standard normal distribution function.
  - Parameters:
    - σ (volatility)
    - x (current stock price)
    - t (time until expiration)
    - r (interest rate)
    - K (strike price)
  - Expected value under the risk-neutral measure $\tilde{P}$:
  - $C(t, x) = \tilde{E}\left[e^{-rt}\left(xe^{-\sigma\sqrt{t}Y+\left(r-\frac{1}{2}\sigma^2\right)t} - K\right)^{+}\right]$
  - Y is a standard normal random variable under $\tilde{P}$.

Risk-Neutral Pricing of Call Options with Jump Diffusion

- **Call option price with Discrete Jumps:**

$$V(t) = C\big(t, S(t)\big) = \tilde{E}\left[e^{-r(T-t)}(S(T)-K)^{+} \mid \mathcal{F}(t)\right]$$

$$C(t,x) = \sum_{j=0}^{\infty} e^{-\tilde{\lambda}(T-t)} \frac{\tilde{\lambda}^{j}(T-t)^{j}}{j!} \tilde{E}_{C(t,x)}\left[T-t, x e^{-\tilde{\beta}\tilde{\lambda}(T-t)} \prod_{i=1}^{j}(Y_i+1)\right]$$

- **Stock price at time T with Jumps:**
- $S(T) = S(t) e^{\sigma(\widetilde{W}(T)-\widetilde{W}(t)) + \left(r-\tilde{\beta}\tilde{\lambda}-\frac{1}{2}\sigma^2\right)(T-t)} \prod_{i=N(t)+1}^{N(T)}(Y_i+1)$
- **Independence and Filtration:**
  - Independence of Y from the filtration $\sigma\left(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\right)$ leads to:

$$V(t) = \tilde{E}\left[e^{-r(T-t)}(S(T)-K)^{+} \mid \sigma\left(\prod_{i=N(t)+1}^{N(T)}(Y_i+1)\right)\right]$$

$$C(t,x) = \tilde{E}\left[\mathcal{K}\left((T-t), x e^{-\tilde{\beta}\tilde{\lambda}(T-t)} \prod_{i=N(t)+1}^{N(T)}(Y_i+1)\right)\right]$$

- $\mathcal{K}$ represents the Black-Scholes-Merton model for the call price.
- **Discounted Call Price with Discrete Jumps:**

$$d\left(e^{-rt}c\big(t,S(t)\big)\right) = e^{-rt}\sigma S(t) c_x\big(t,S(t)\big) d\widetilde{W}(t) + e^{-rt}\left[c\big(t,S(t)\big) - c\big(t,S(t^{-})\big)\right] dN(t) - e^{-rt}\tilde{\lambda}\left[\sum_{m=1}^{M} \tilde{p}\,(y_m) c\big(t,(y_m+1)S(t^{-})\big) - c\big(t,S(t^{-})\big)\right] dt$$

Continuous Jump Distribution Modifications in Option Pricing

- **Jump Size Distribution:**
  - Assume jump sizes, Y_i, follow a continuous distribution with density function $f(y)$, strictly positive on subset $B \subset (-1, \infty)$ and zero elsewhere.
- **Expected Jump Size:**
  - The expected value of Y_i under the probability measure:
  - $\beta = E[Y_i] = \int_{-1}^{\infty} y\, f(y) dy$
- **Risk-Neutral Measure:**
  - Under the risk-neutral measure, expected value changes:
  - $\tilde{\beta} = \tilde{E}[Y_i] = \int_{-1}^{\infty} y\, \tilde{f}(y) dy$
  - Adjustments to the market price of risk equation:
  - $\alpha - r = \sigma\theta + \beta\lambda - \tilde{\beta}\tilde{\lambda}$
- **Modification in Option Pricing Formula:**
  - Replaces discrete summation with continuous integration:
  - $\int_{-1}^{\infty} c\,(t, (y+1)x)\tilde{f}(y)\, dy$
  - For a jump size Y_i with density $\tilde{f}(y)$ under the risk-neutral measure $\widetilde{\mathbb{P}}$.
- **Adjusted Differential Equation:**
  - The adjusted call price differential is:
  - $d\left(e^{-r}\ c\big(t, S(t)\big)\right) = e^{-r}\ \sigma S(t) c_x\big(t, S(t)\big) d\widetilde{W}(t) + e^{-rt}\big[c\big(t, S(t)\big) - c\big(t, S(t^-)\big)\big] dN(t) - e^{-r}\ \tilde{\lambda}\big[\int_{-1}^{\infty} c\,\big(t, (y+1)S(t^-)\big)\tilde{f}(y)\, dy - c\big(t, S(t^-)\big)\big] dt$
- **Hedging Strategy:**
  - Hedge a short position in a European Call:
  - $dX(t) = \Gamma(t^-) dS(t) + r[X(t) - \Gamma(t)S(t)]dt$
  - Differential of the discounted hedging portfolio value:
  - $d\big(e^{-r}\ X(t)\big) = e^{-r}\ \big[\Gamma(t)\sigma S(t) d\widetilde{W}(t) + \Gamma(t^-)S(t^-) \sum_{m=1}^{M} y_m \left(dN_m(t) - \tilde{\lambda} dt\right)\big]$

Delta Hedging Strategy for Options with Jumps

- **Delta Hedging Mechanics:**
  - Delta ($\Gamma(t)$) is set to the derivative of the call option price with respect to the stock price ($c_x\big(t, S(t)\big)$), equating the $d\widetilde{W}(t)$ terms in the formulas for the option and the hedging portfolio, targeting a perfect hedge against continuous price changes.
- **Risk from Jumps:**
  - The residual risk relates to jump sizes:
  - $d\left(e^{-rt}c\big(t,S(t)\big) - e^{-rt}X(t)\right) = \sum_{m=1}^{M} e^{-r}\ \left[c\big(t,(y_m+1)S(t^-)\big) - c\big(t,S(t^-)\big) - y_m S(t^-)c_x\big(t,S(t^-)\big)\right]\big(dN_m(t) - \tilde{\lambda}dt\big)$
  - Due to the strict convexity of $c(t,x)$, the hedging portfolio outperforms the option value between jumps, while the option value outperforms at jump times.
- **Overall Performance:** strategy hedges the option on an **average basis**

$$\mathbb{E}_Q\left[e^{-r}\ c\big(t,S(t)\big)\right] = \mathbb{E}_Q\left[e^{-r}\ X(t)\right], \quad \text{for } 0 \le t \le T$$

- **Continuous Jump Distribution Adaptation:**

$$d\left(e^{-rt}c\big(t,S(t)\big) - e^{-r}\ X(t)\right)$$

$$= e^{-rt}\left[c\big(t,S(t)\big) - c\big(t,S(t^-)\big) - \big(S(t) - S(t^-)\big)c_x\big(t,S(t^-)\big)\right]dN(t) - e^{-r}\ \tilde{\lambda}\int_{-1}^{\infty}\left[c\big(t,(y+1)S(t^-)\big) - c\big(t,S(t^-)\big) - yS(t^-)c_x\big(t,S(t^-)\big)\right]\tilde{f}(y)dy\,dt$$

References

Shreve, Steven E. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer, 2004. Springer Science + Business Media, New York.

Appendix

See “Models with Jumps – Levy Processes” by Michael Miller

See “Simulations for Models with Jumps” by Michael Miller