ADAPTIVE LOCATION AND SCALE ESTIMATION WITH KERNEL WEIGHTED AVERAGES

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Supplementary Material

This supplementary documents contains additional figures, tables and proofs referred to in the paper.

S1 Technical Appendix

S1.1 Location and Scale Functionals

Let $\big(f(x|\mu,\sigma)\big)_{(\mu,\sigma)}$ be a location-scale family defined as

$$f(x|\mu,\sigma) = \frac{1}{\sigma} f_0\Big(\frac{x-\mu}{\sigma}\Big) \quad \text{for } x \in \mathbb{R} \tag{S1.1}$$

with location and scale parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, respectively, and a baseline probability density function (pdf) $f_0(\cdot)$ assumed symmetric around 0. Let X be a square-integrable random variable with a probability density $f(x)$ from the family in Equation (S1.1).

Consider $\mu = \int x f^2(x)\mathrm{d}x$ and $\sigma = \bigg(\int (x-\mu)^2 f^2(x)\mathrm{d}x \bigg)^{1/2}$ as alternative functionals for location and scale of $f(x)$, respectively. We apply the transformation theorem to compute the former two integrals. Let $z = \frac{x-\mu}{\sigma}$, then $x = z\sigma + \mu$ and $\mathrm{d}z = \frac{1}{\sigma}\mathrm{d}x$. Hence, the integrals for μ and σ can be expressed in terms of the baseline pdf $f_0(\cdot)$.

The integral for μ is given by

$$\begin{aligned} \int x f^2(x)\mathrm{d}x &= \frac{1}{\sigma} \int (z\sigma + \mu) f_0^2(z)\mathrm{d}z \\ &= \frac{\mu}{\sigma} \int f_0^2(z)\mathrm{d}z + \sigma^2 \int z f_0^2(z)\mathrm{d}z \end{aligned}$$

$$= \frac{\mu}{\sigma} \int f_0^2(z) \mathrm{d}z.$$

Solve for the location parameter μ to obtain

$$\begin{aligned} \mu &= \sigma \frac{\int x f^2(x) \mathrm{d}x}{\int f_0^2(z) \mathrm{d}z} \\ &= \frac{\int x f^2(x) \mathrm{d}x}{\int f^2(x) \mathrm{d}x}. \end{aligned}$$

The integral for the scale parameter expressed in terms of the baseline pdf is given by

$$\begin{aligned} \int (x-\mu)^2 f^2(x) \mathrm{d}x &= \int \sigma^2 z^2 \frac{1}{\sigma^2} f_0^2(z) \sigma \mathrm{d}z \\ &= \sigma \int z^2 f_0^2(z) \mathrm{d}z. \end{aligned}$$

Solving for σ, we get

$$\sigma = \frac{\int (x-\mu)^2 f^2(x) \mathrm{d}x}{\int z^2 f_0^2(z) \mathrm{d}z}.$$

Invoking the identity from Ahmad (1982)

$$\sigma = \frac{\int f_0^2(x)\, \mathrm{d}x}{\int f^2(x) \mathrm{d}x},$$

we obtain

$$\begin{aligned} \sigma &= \frac{\int (x-\mu)^2 f^2(x) \mathrm{d}x}{\int z^2 f_0^2(z) \mathrm{d}z} \times \frac{\int f^2(x) \mathrm{d}x}{\int f^2(x) \mathrm{d}x} \\ &= \frac{\int (x-\mu)^2 f^2(x) \mathrm{d}x}{\int z^2 f_0^2(z) \mathrm{d}z} \times \frac{1}{\sigma^2} \frac{\int f_0^2(z) \sigma \mathrm{d}z}{\int f^2(x) \mathrm{d}x} \\ &= \frac{\int (x-\mu)^2 f^2(x) \mathrm{d}x}{\int z^2 f_0^2(z) \mathrm{d}z} \times \frac{1}{\sigma} \frac{\int f_0^2(z) \mathrm{d}z}{\int f^2(x) \mathrm{d}x}. \end{aligned}$$

Multiplying both sides of the equation by σ

$$\begin{aligned} \sigma^2 &= \frac{\int (x-\mu)^2 f^2(x) \mathrm{d}x}{\int z^2 f_0^2(z) \mathrm{d}z} \times \frac{\int f_0^2(z) \mathrm{d}z}{\int f^2(x) \mathrm{d}x} \\ &= \frac{\int (x-\mu)^2 f^2(x) \mathrm{d}x}{\int f^2(x) \mathrm{d}x} \times \frac{\int f_0^2(z) \mathrm{d}z}{\int z^2 f_0^2(z) \mathrm{d}z} \end{aligned}$$

and taking the square root on both sides of the equation, we arrive at

$$\sigma = \left(\frac{\int z^2 f_0^2(z) \mathrm{d}z}{\int f_0^2(z) \mathrm{d}z} \right)^{-1/2} \left(\frac{\int (x-\mu)^2 f^2(x) \mathrm{d}x}{\int f(x) \mathrm{d}x} \right)^{1/2}.$$

S2 Supplemental Figures

S2.1 Optimal Bandwidth Selection for KWA Location and Scale Estimators

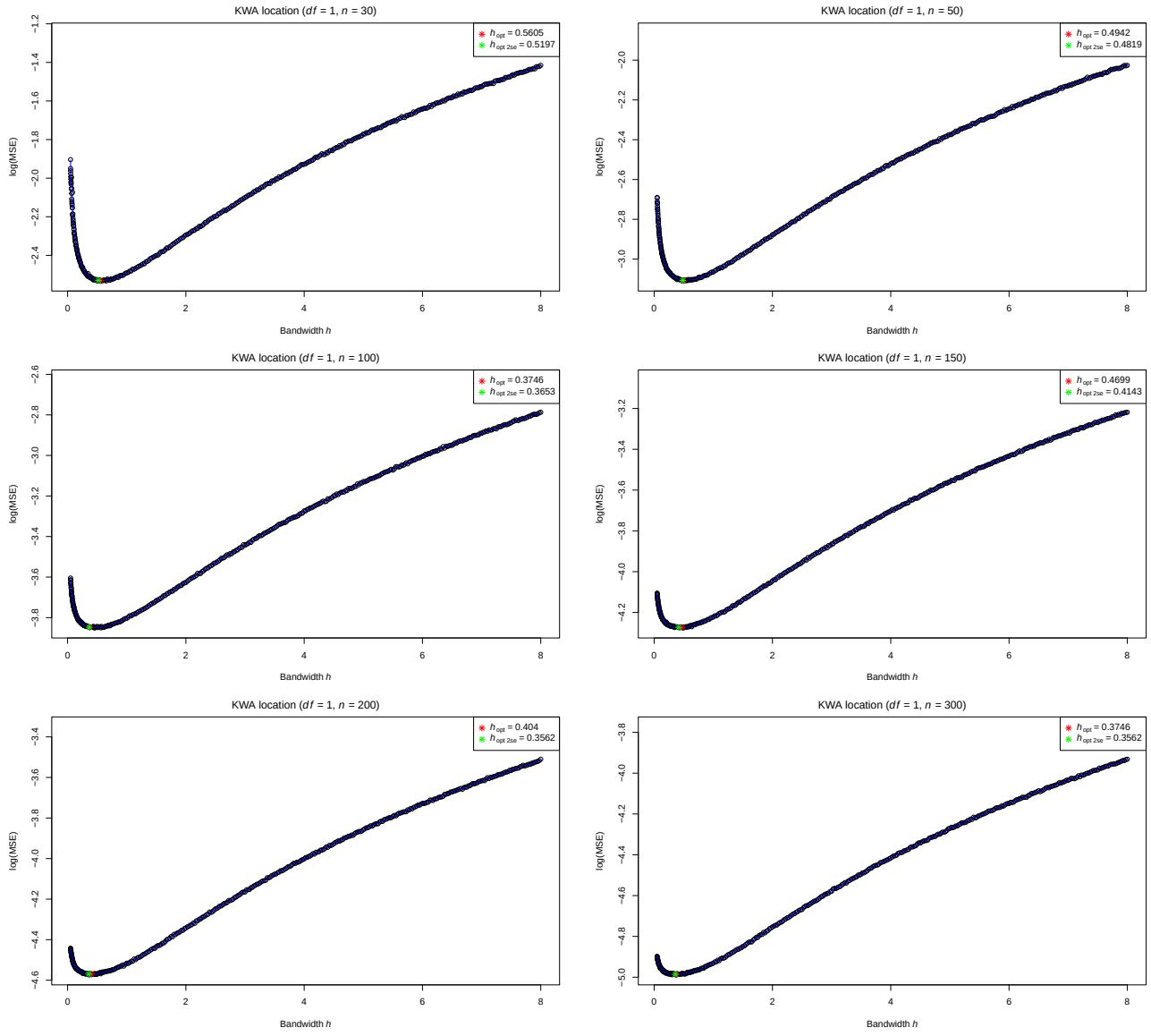

Figure S(1): KWA location estimation errors and optimal bandwidths for df $\nu = 1$ across various sample sizes n.

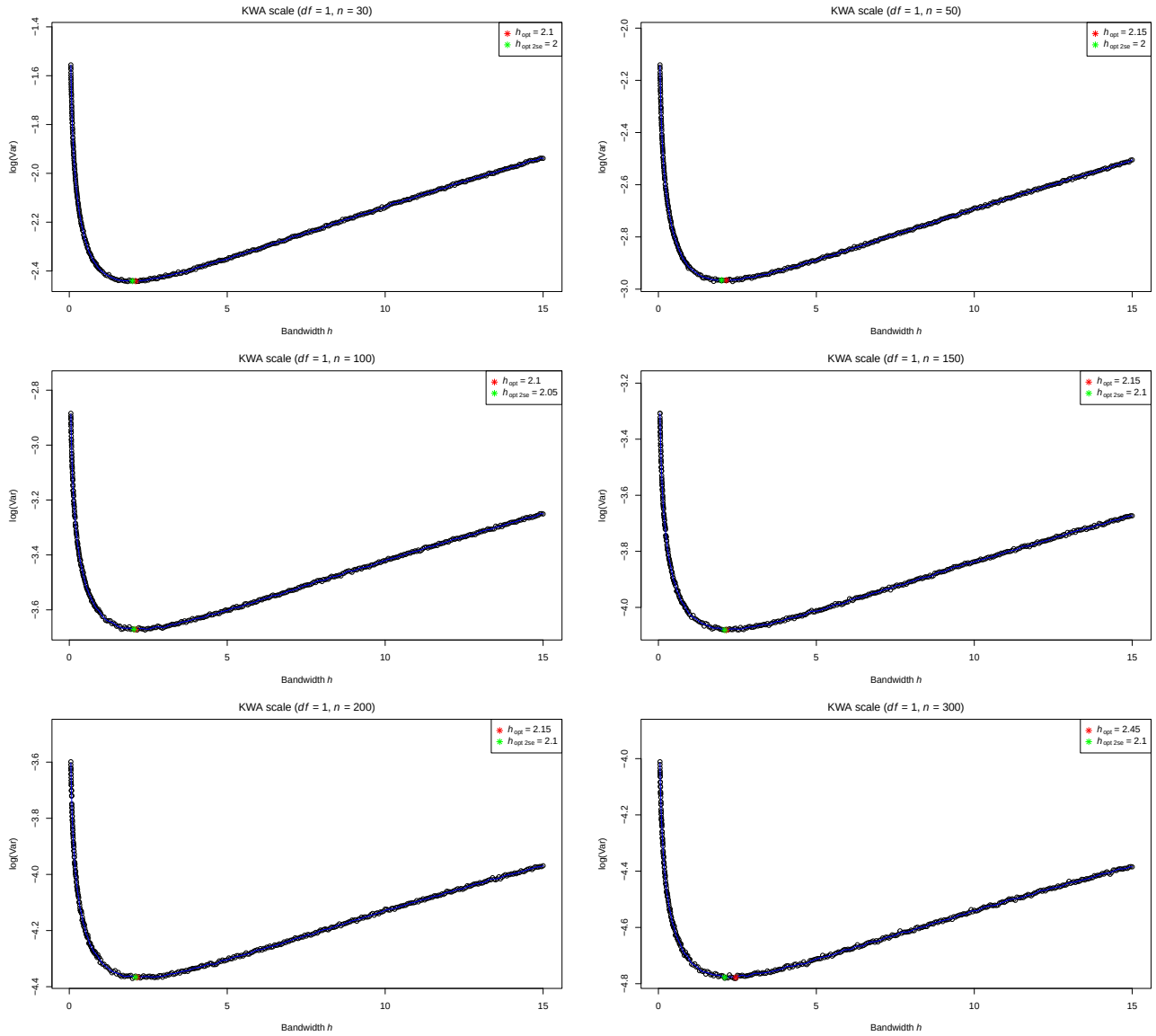

Figure S(2): KWA scale location estimation errors and optimal bandwidths for df $\nu = 1$ across various sample sizes n.

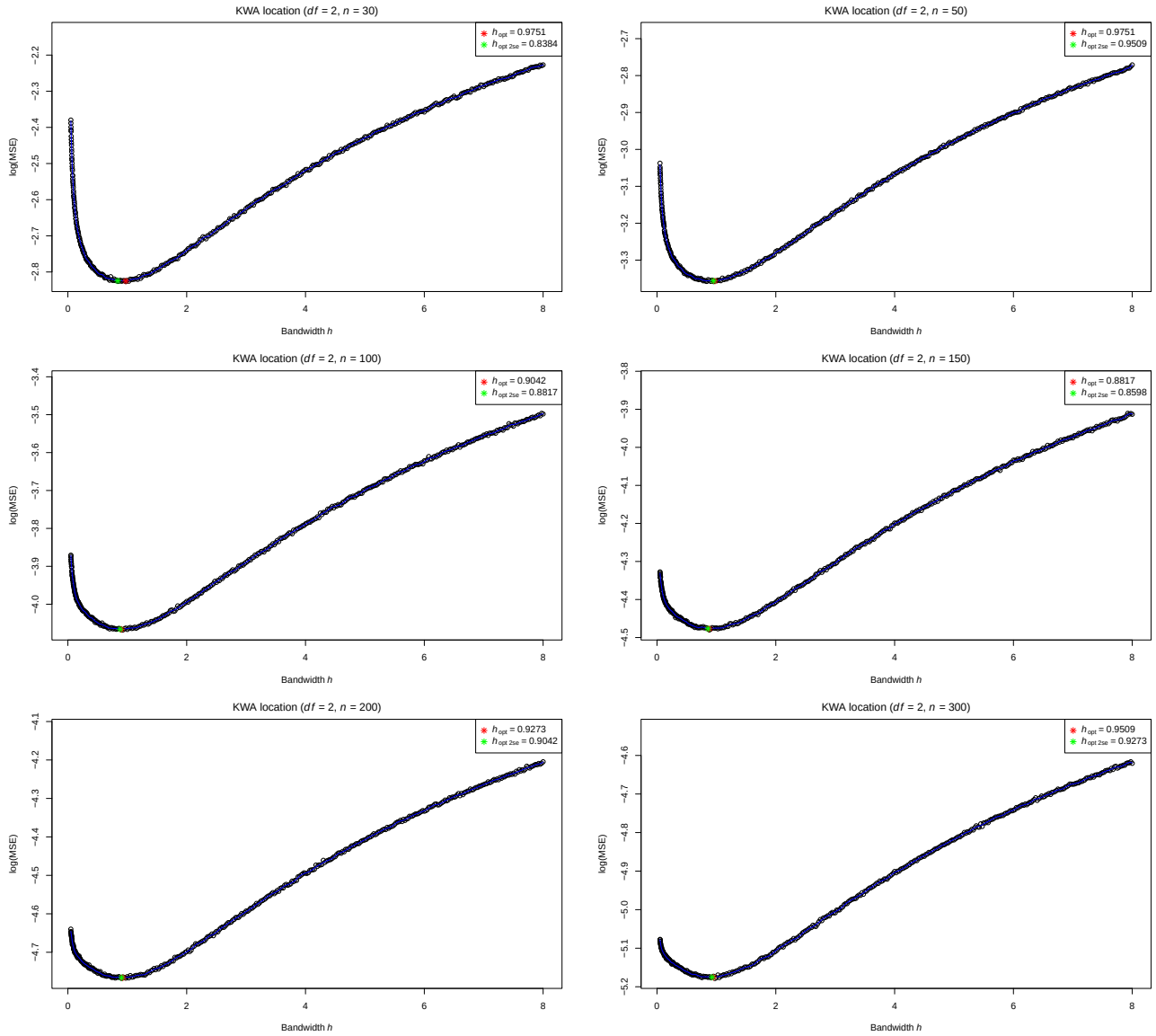

Figure S(3): KWA location estimation errors and optimal bandwidths for df $\nu = 2$ across various sample sizes n.

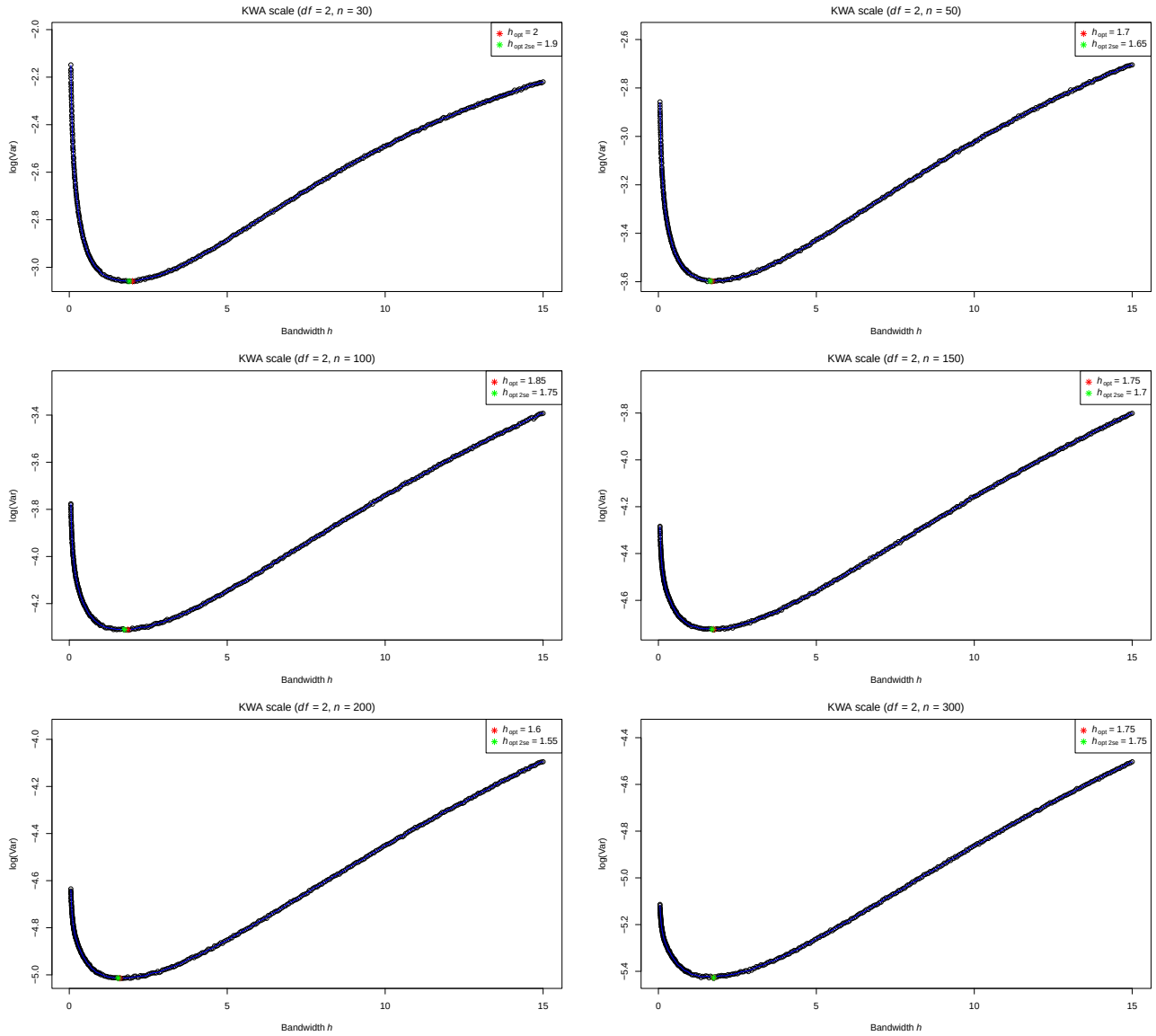

Figure S(4): KWA scale estimation errors and optimal bandwidths for df $\nu = 2$ across various sample sizes n.

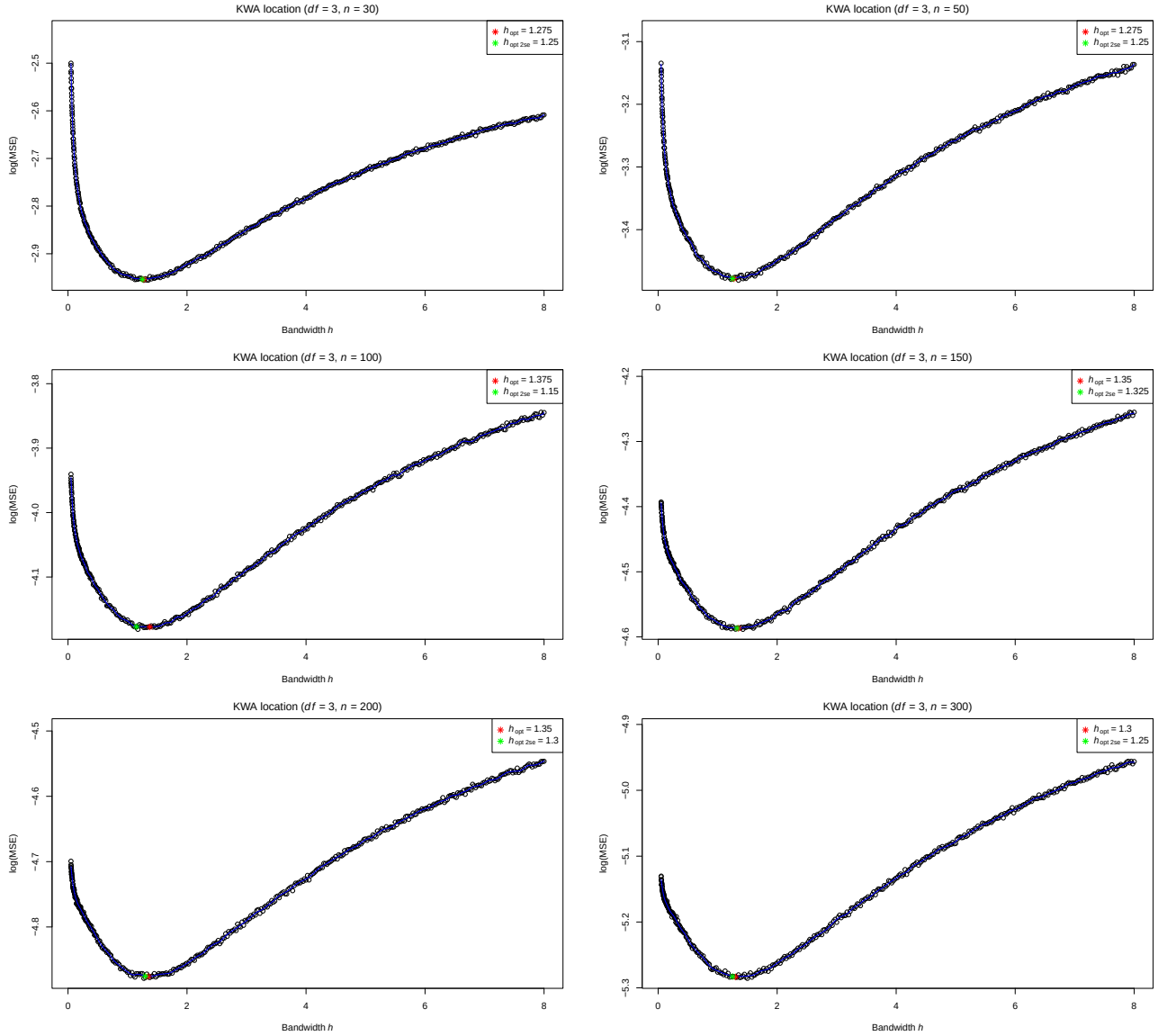

Figure S(5): KWA location estimation errors and optimal bandwidths for df $\nu = 3$ across various sample sizes n.

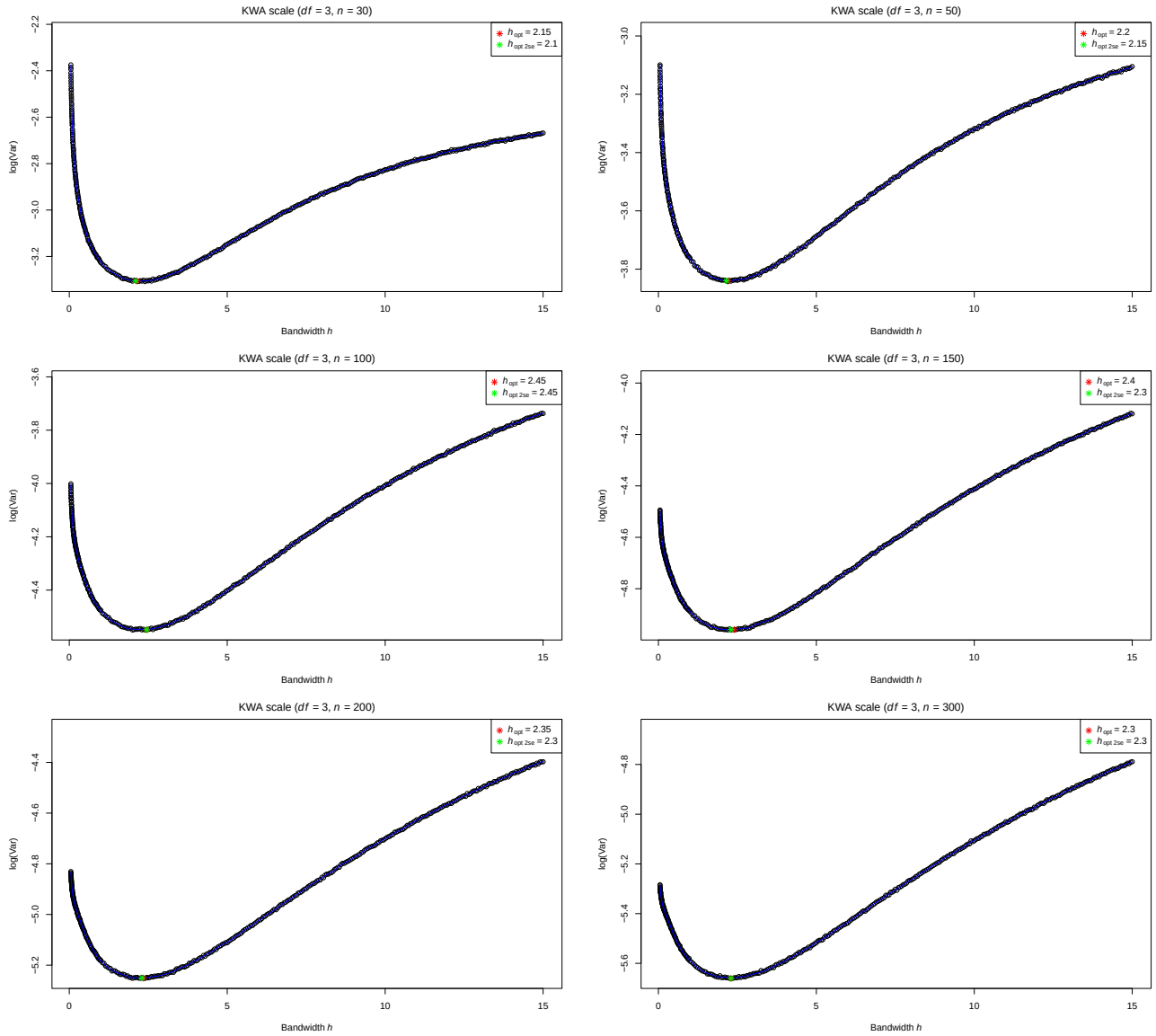

Figure S(6): KWA scale estimation errors and optimal bandwidths for df $\nu = 3$ across various sample sizes n.

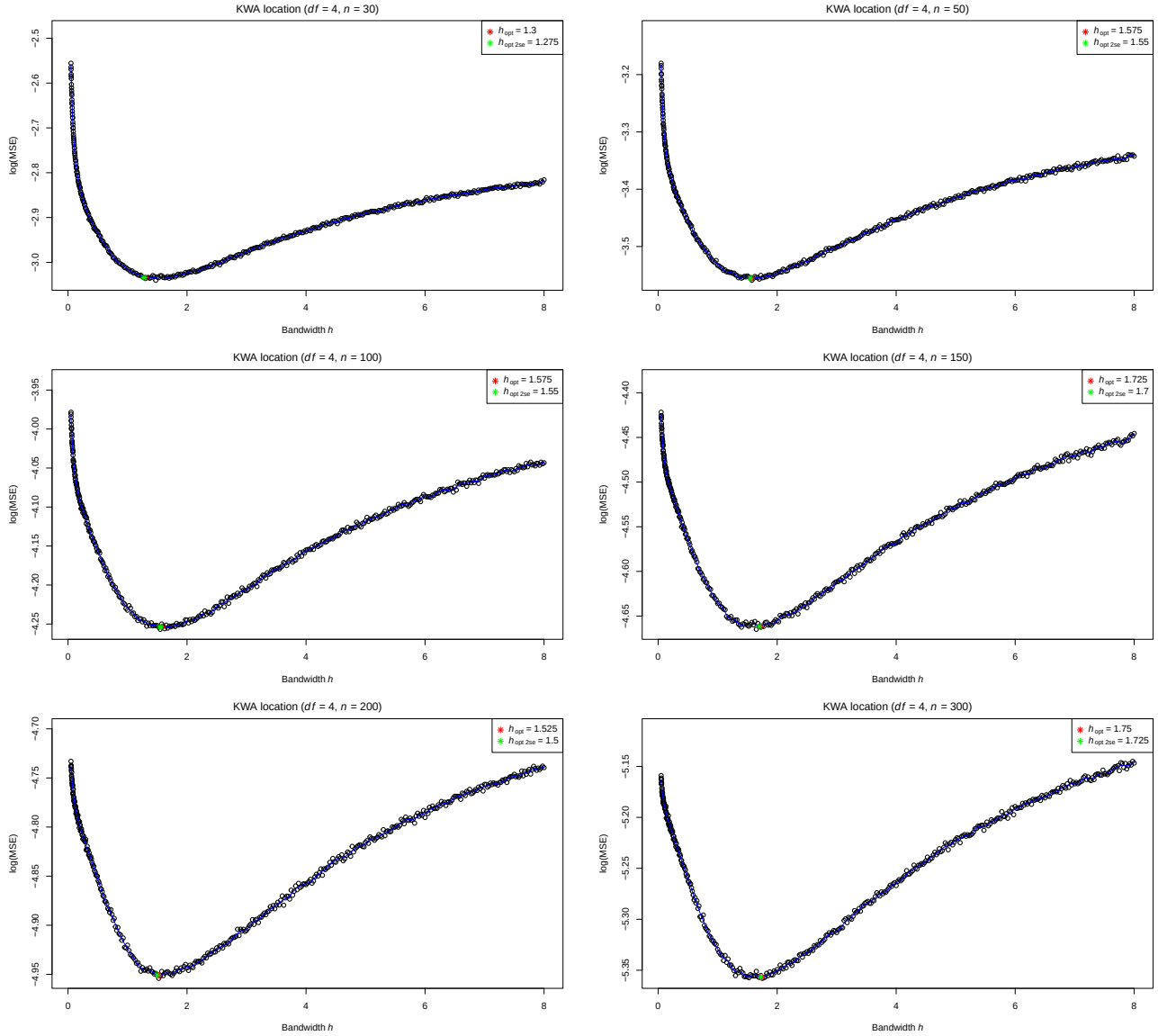

Figure S(7): KWA location estimation errors and optimal bandwidths for df $\nu = 4$ across various sample sizes n.

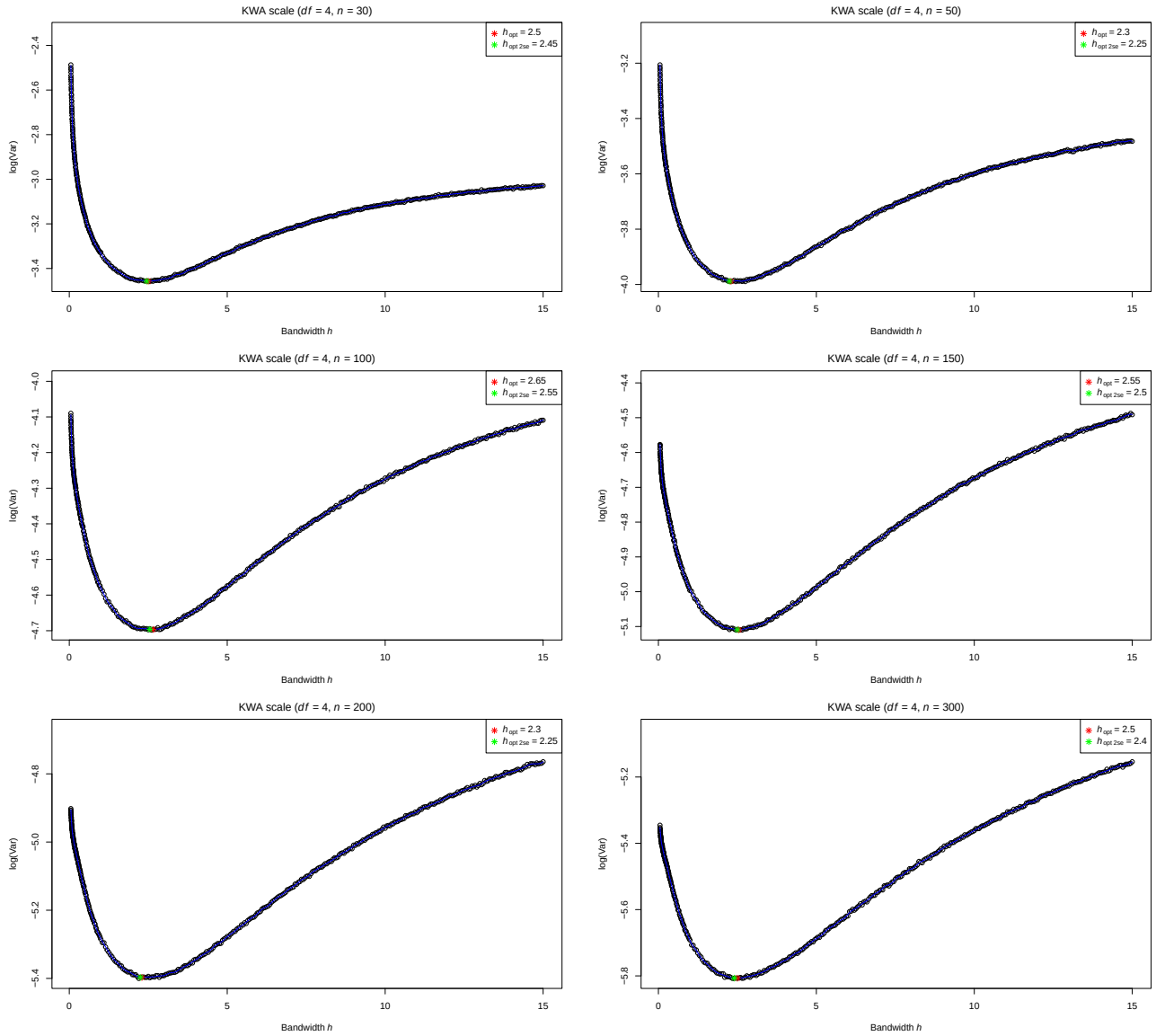

Figure S(8): KWA scale estimation errors and optimal bandwidths for df $\nu = 4$ across various sample sizes n.

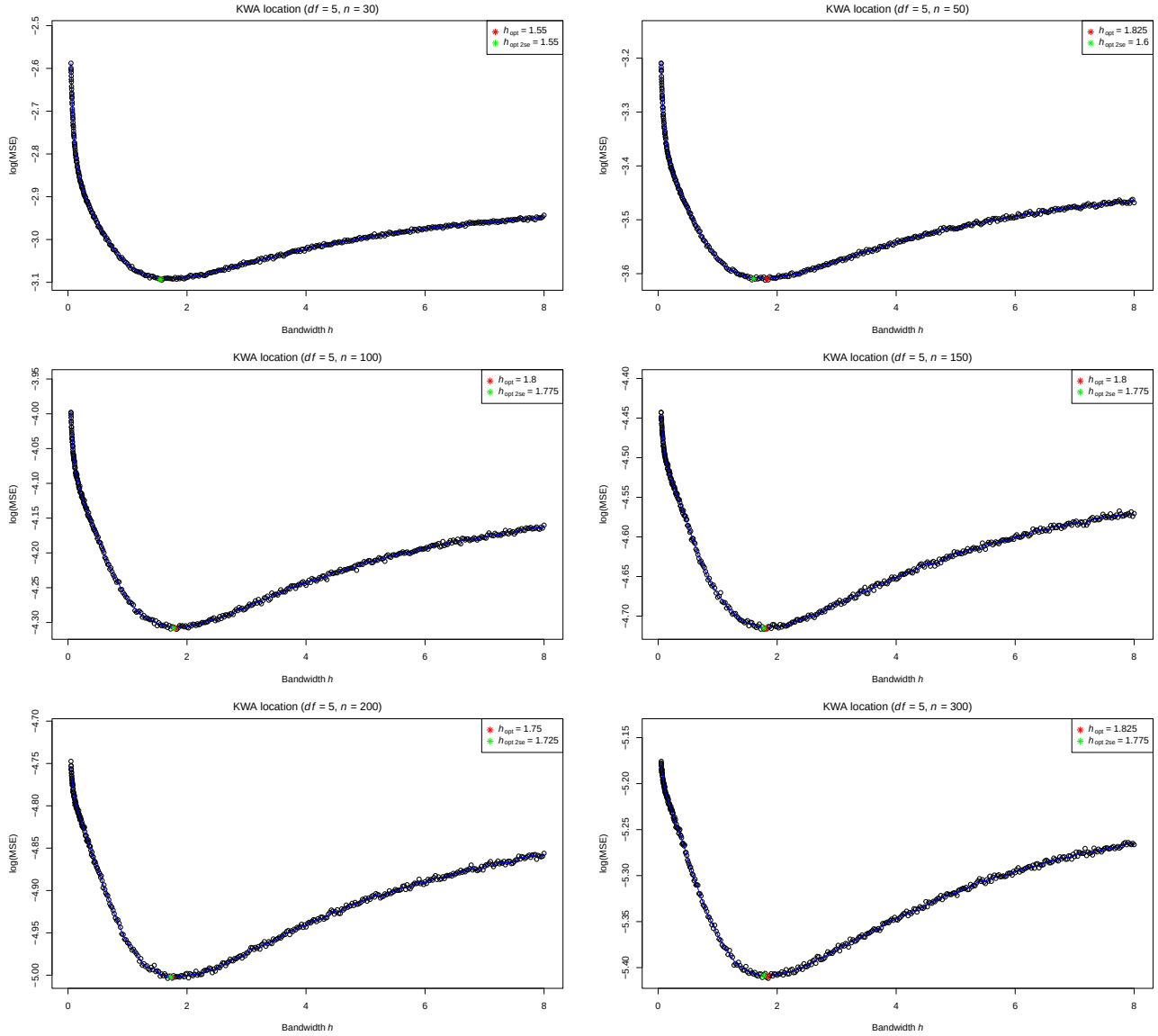

Figure S(9): KWA location estimation errors and optimal bandwidths for df $\nu = 5$ across various sample sizes n.

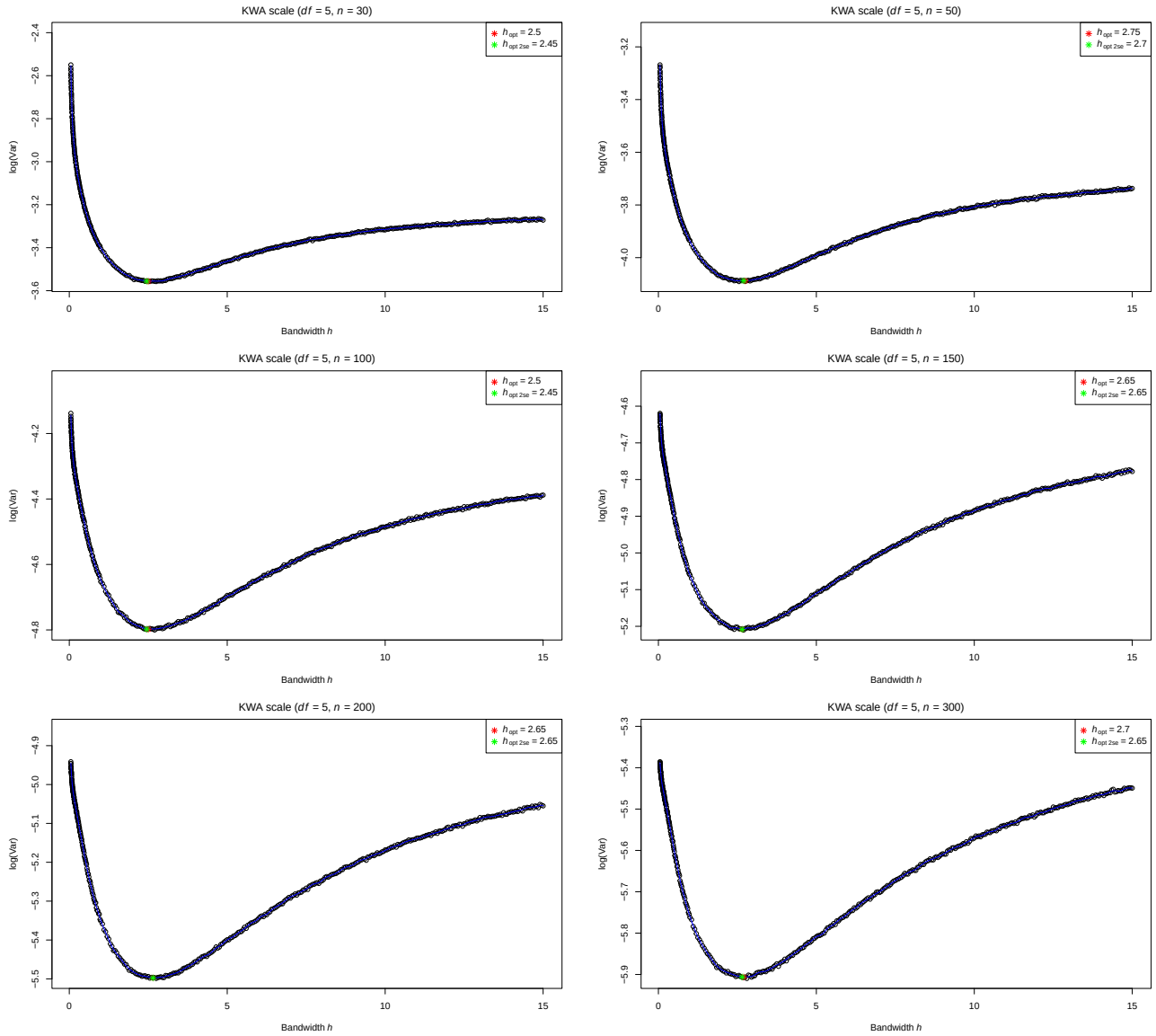

Figure S(10): KWA scale estimation errors and optimal bandwidths for df $\nu = 5$ across various sample sizes n.

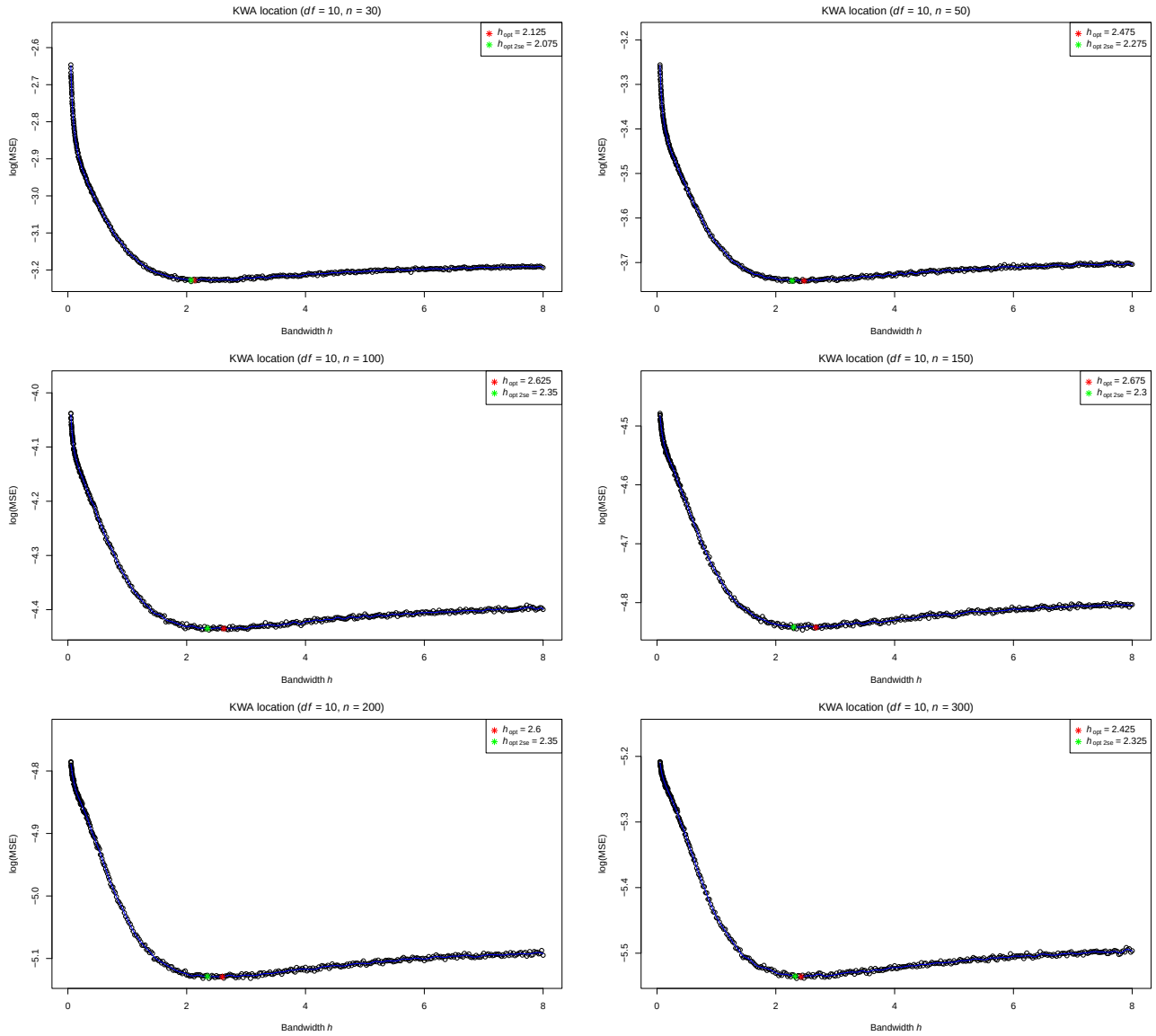

Figure S(11): KWA location estimation errors and optimal bandwidths for df $\nu = 10$ across various sample sizes n.

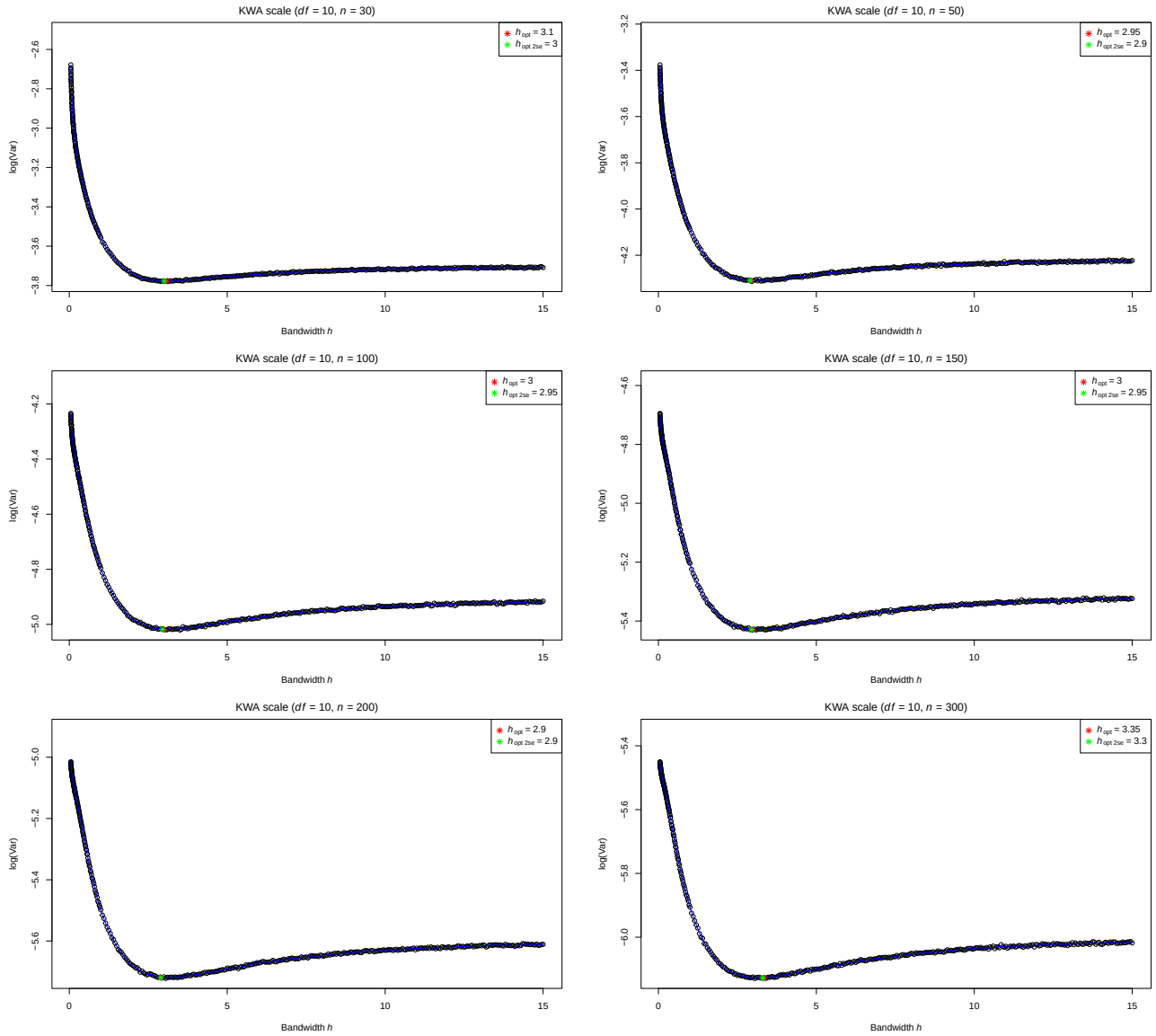

Figure S(12): KWA scale estimation errors and optimal bandwidths for df $\nu = 10$ across various sample sizes n.

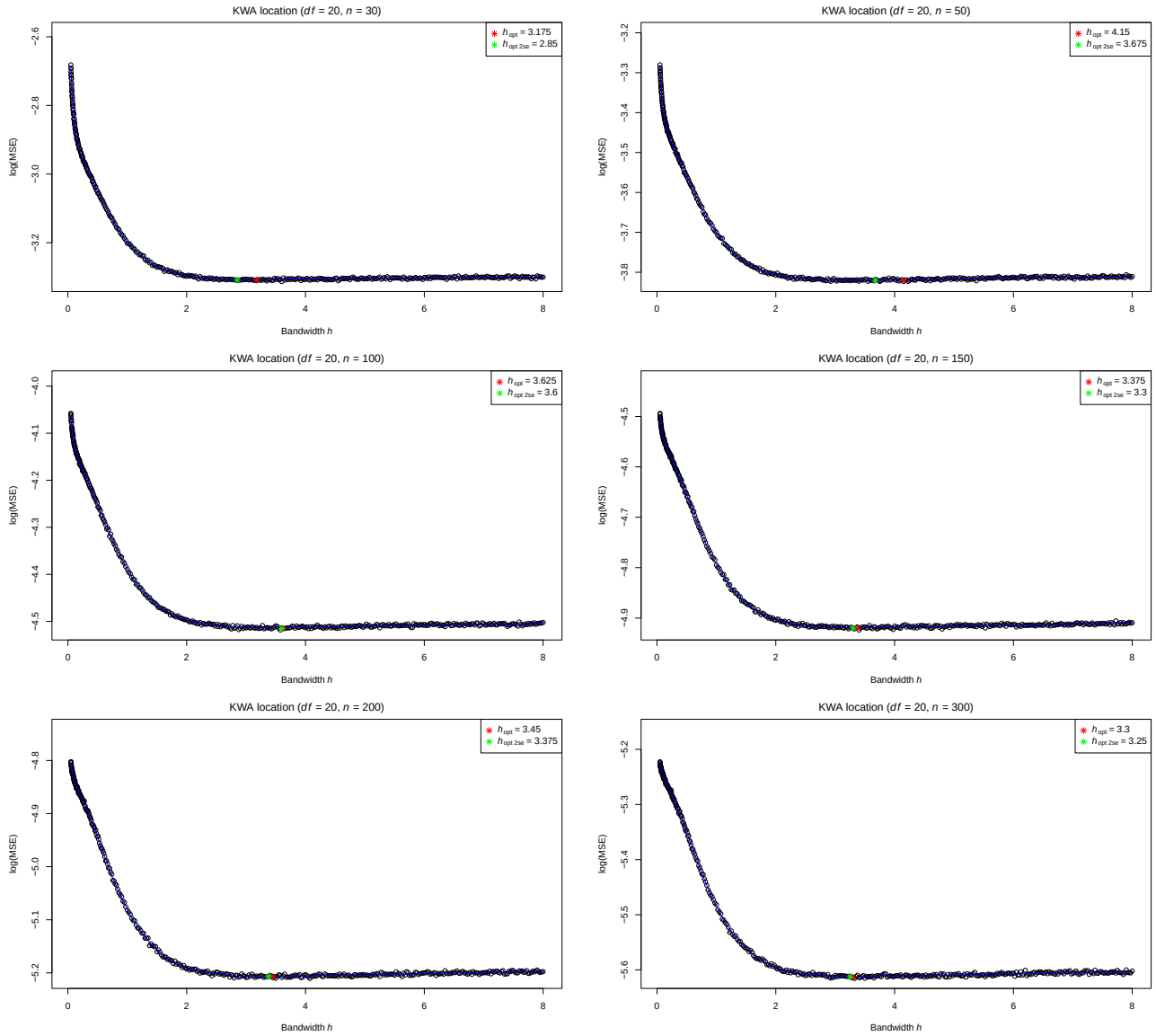

Figure S(13): KWA location estimation errors and optimal bandwidths for df $\nu = 20$ across various sample sizes n.

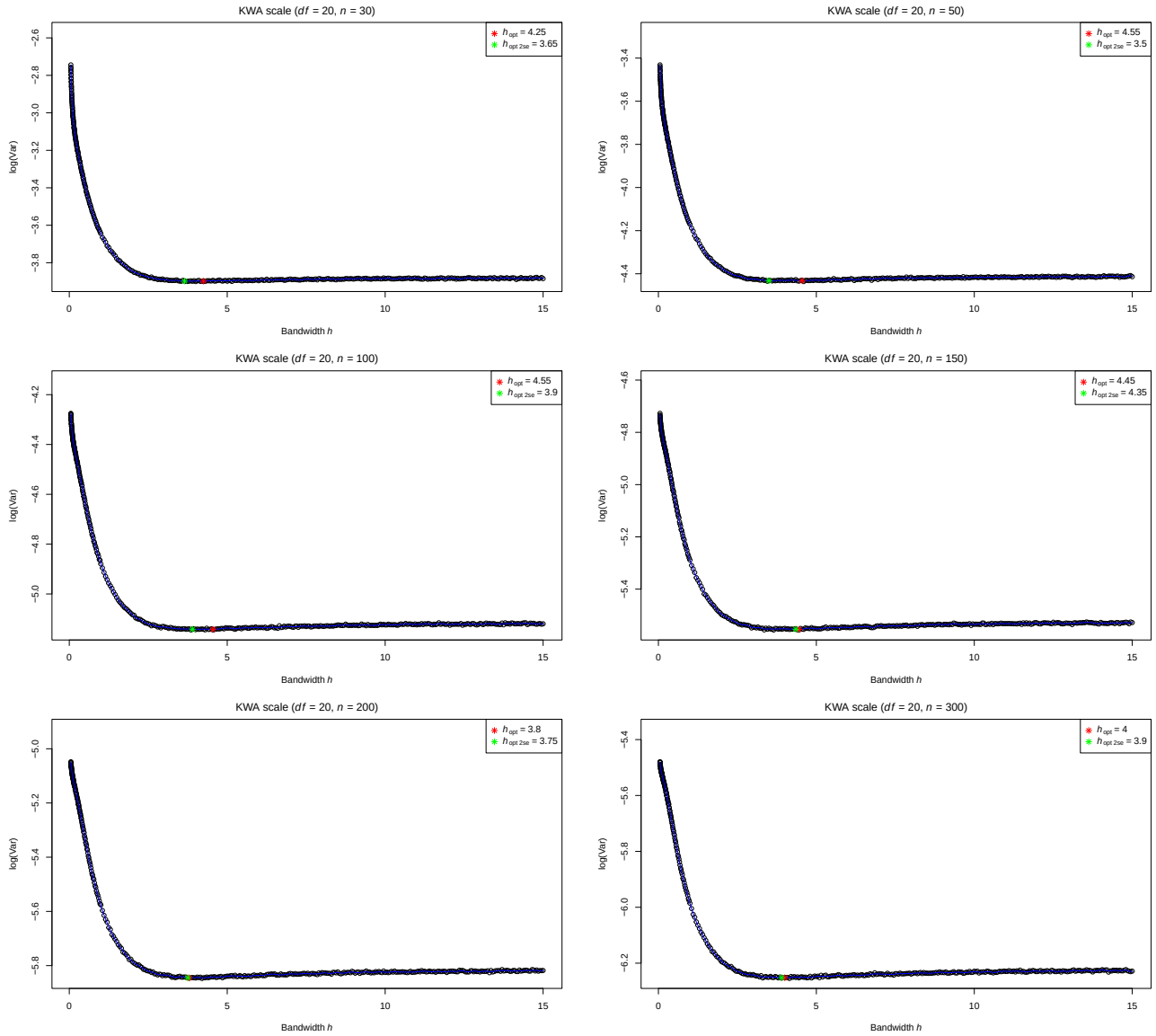

Figure S(14): KWA scale estimation errors and optimal bandwidths for df $\nu = 20$ across various sample sizes n.

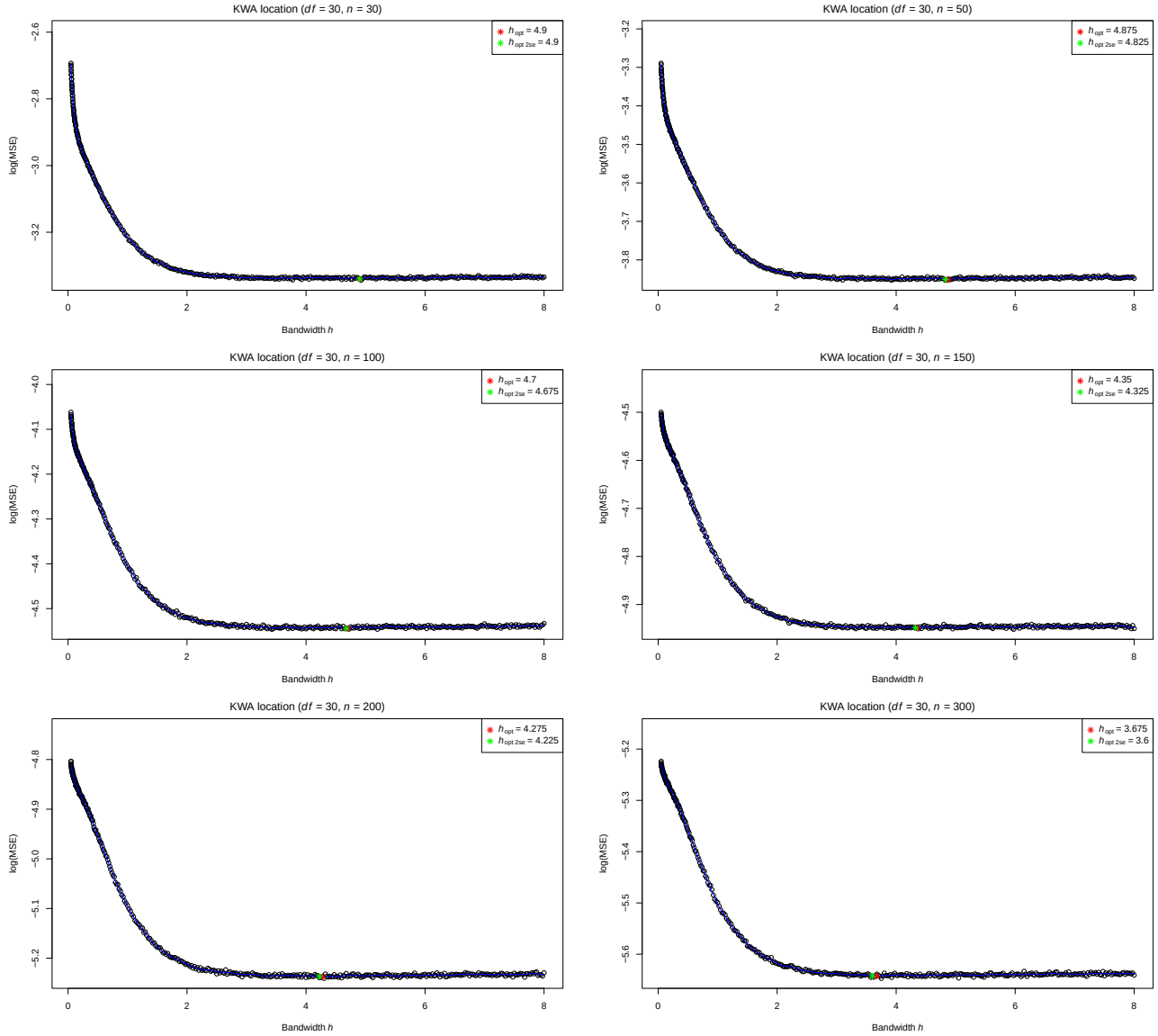

Figure S(15): KWA location estimation errors and optimal bandwidths for df $\nu = 30$ across various sample sizes n.

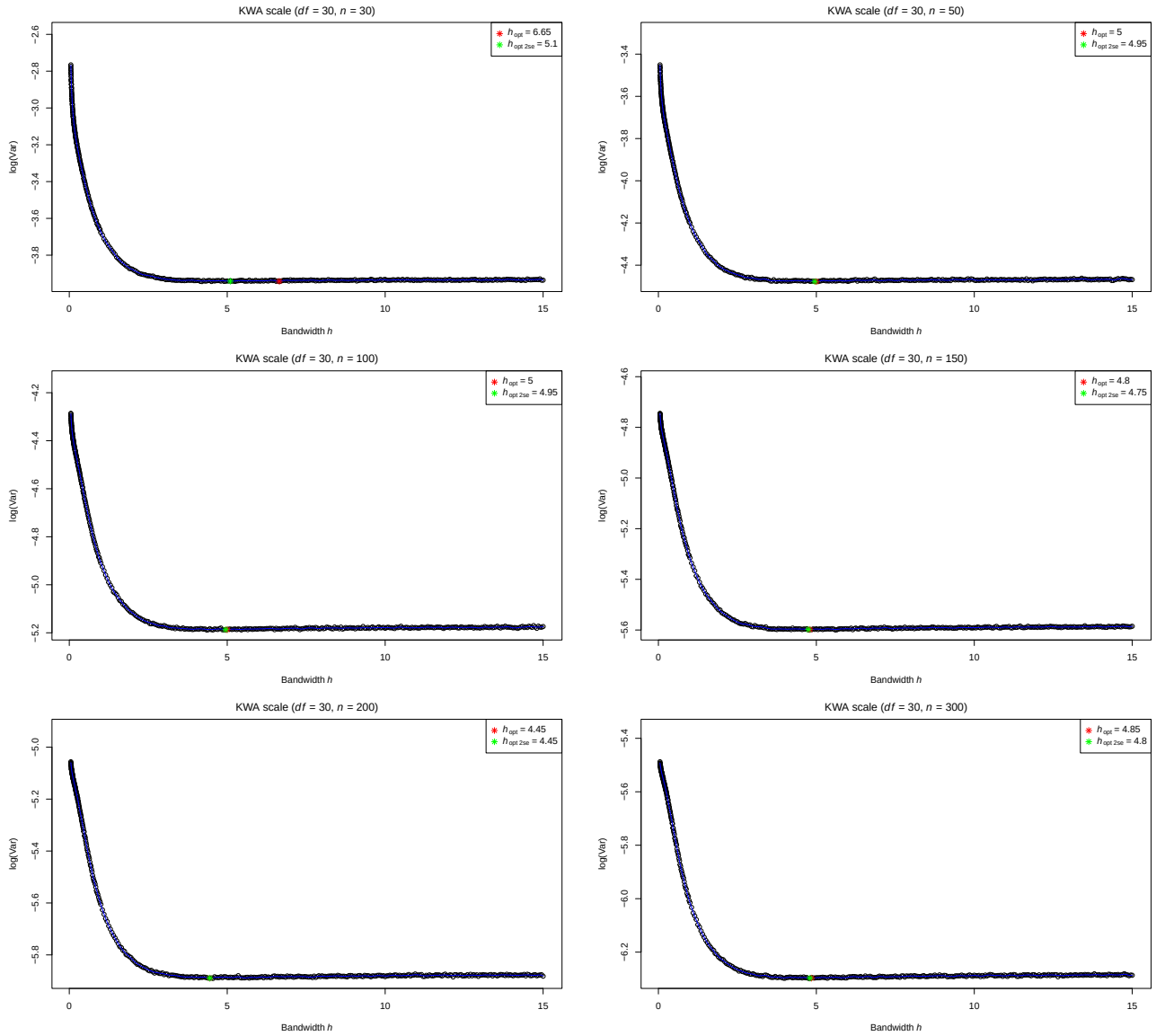

Figure S(16): KWA scale estimation errors and optimal bandwidths for df $\nu = 30$ across various sample sizes n.

S2.2 Asymptotically Optimal Bandwidths for KWA Location and Scale Estimators

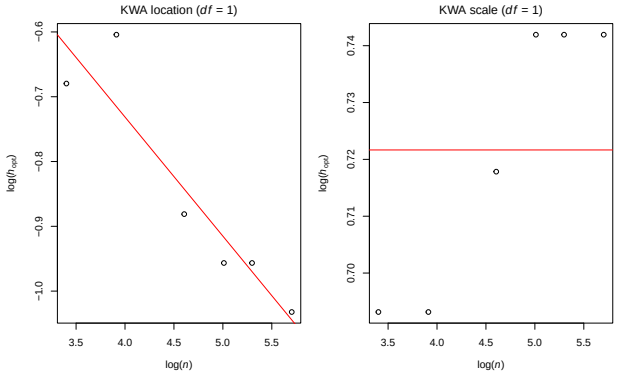

Figure S(17): Asymptotically optimal bandwidths for KWA location and scale estimators for df $\nu = 1$.

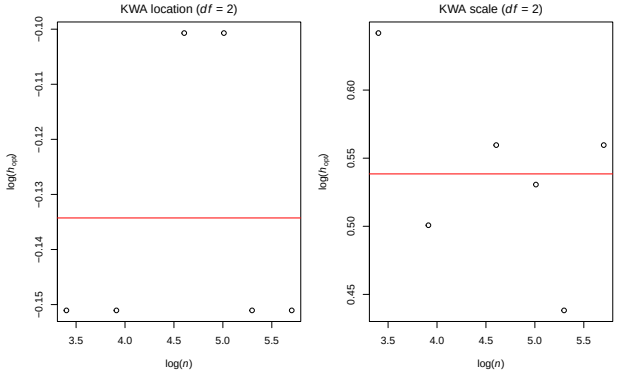

Figure S(18): Asymptotically optimal bandwidths for KWA location and scale estimators for df $\nu = 2$.

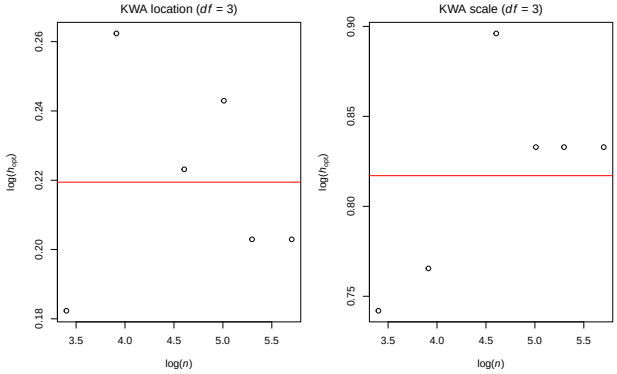

Figure S(19): Asymptotically optimal bandwidths for KWA location and scale estimators for df $\nu = 3$.

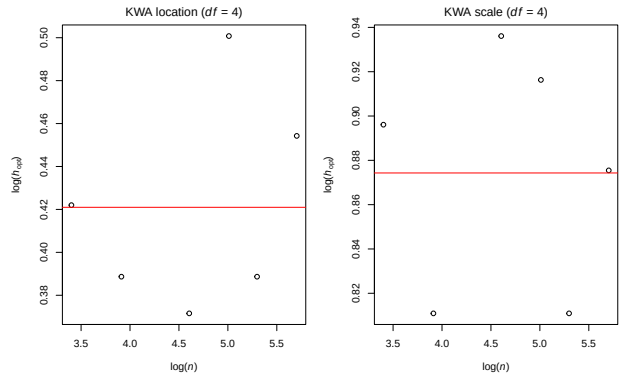

Figure S(20): Asymptotically optimal bandwidths for KWA location and scale estimators for df $\nu = 4$.

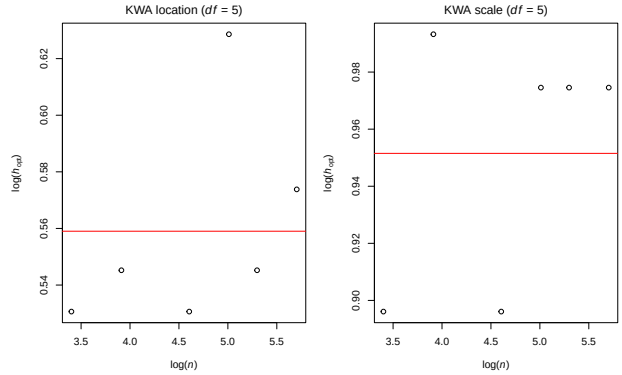

Figure S(21): Asymptotically optimal bandwidths for KWA location and scale estimators for df $\nu = 5$.

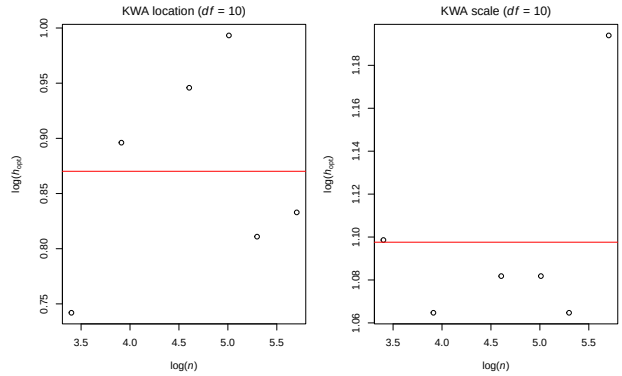

Figure S(22): Asymptotically optimal bandwidths for KWA location and scale estimators for df $\nu = 10$.

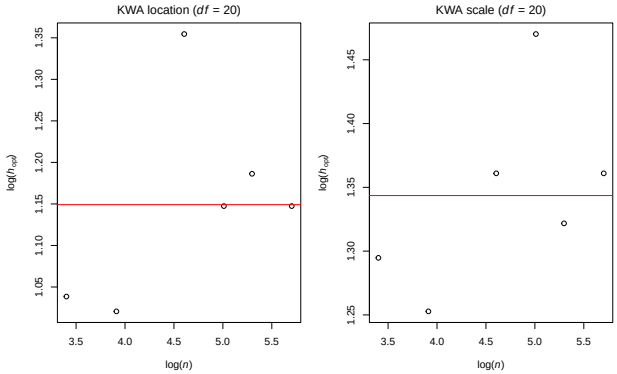

Figure S(23): Asymptotically optimal bandwidths for KWA location and scale estimators for df $\nu = 20$.

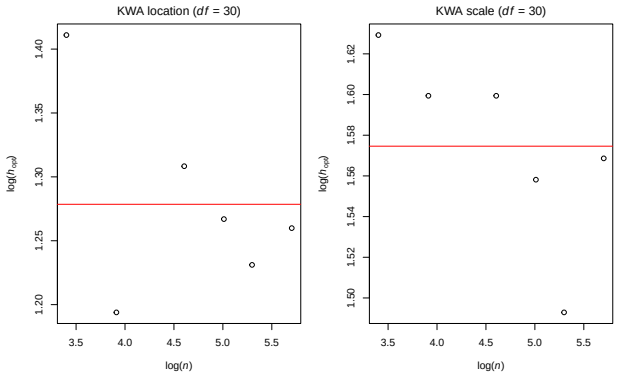

Figure S(24): Asymptotically optimal bandwidths for KWA location and scale estimators for df $\nu = 30$.

S2.3 Estimated KWA Scale Bias

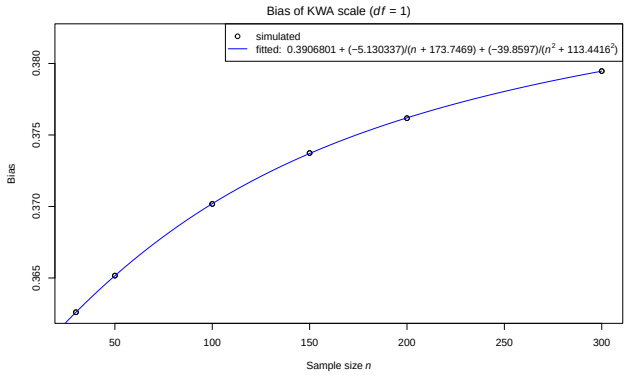

Figure S(25): Estimated bias vs sample size n for unscaled KWA scale estimator for df $\nu = 1$.

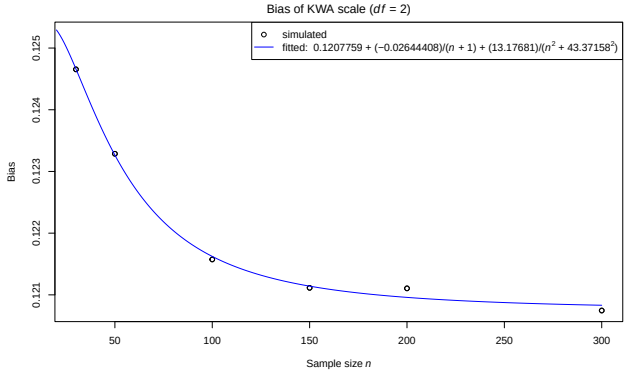

Figure S(26): Estimated bias vs sample size n for unscaled KWA scale estimator for df $\nu = 2$.

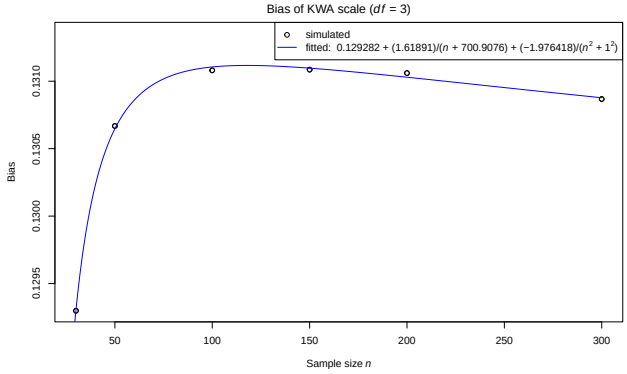

Figure S(27): Estimated bias vs sample size n for unscaled KWA scale estimator for df $\nu = 3$.

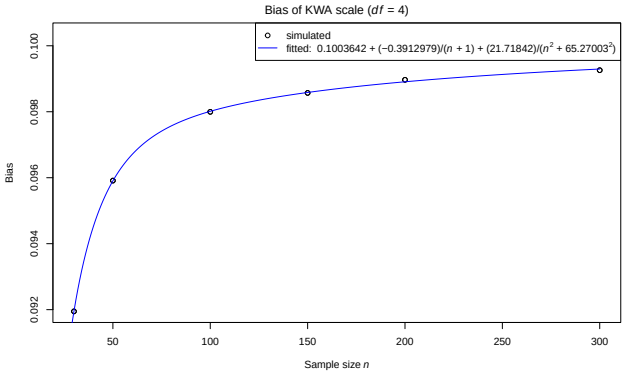

Figure S(28): Estimated bias vs sample size n for unscaled KWA scale estimator for df $\nu = 4$.

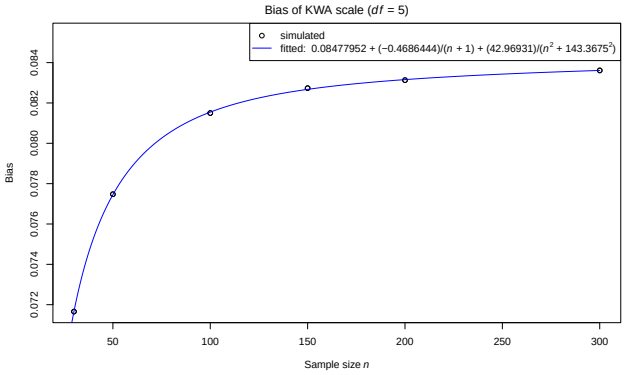

Figure S(29): Estimated bias vs sample size n for unscaled KWA scale estimator for df $\nu = 5$.

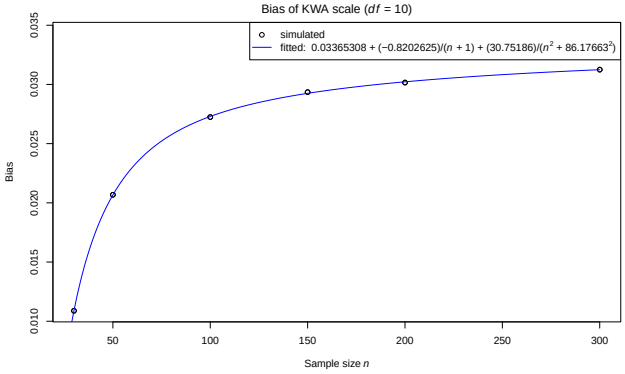

Figure S(30): Estimated bias vs sample size n for unscaled KWA scale estimator for df $\nu = 10$.

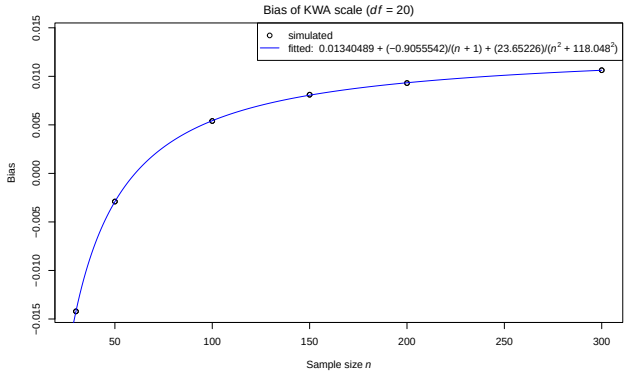

Figure S(31): Estimated bias vs sample size n for unscaled KWA scale estimator for df $\nu = 20$.

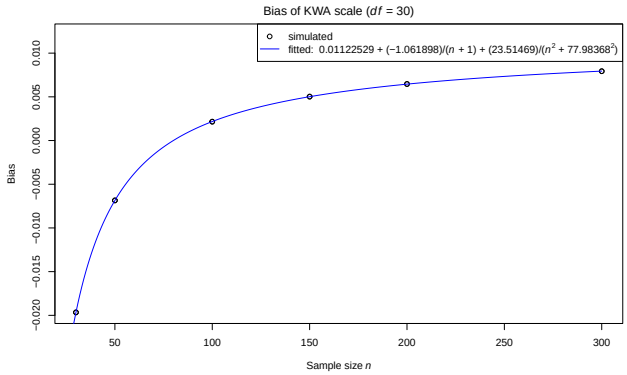

Figure S(32): Estimated bias vs sample size n for unscaled KWA scale estimator for df $\nu = 30$.

S2.4 Simulated Location and Scale Estimation Errors

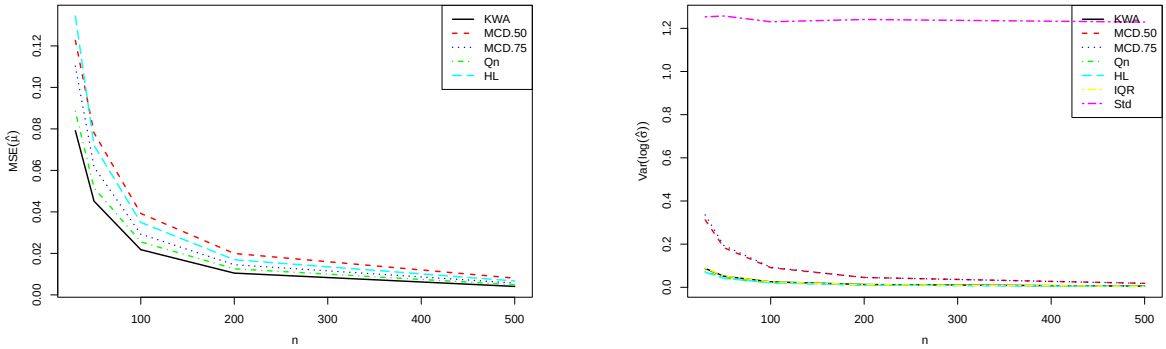

Figure S(33): Simulated location and scale estimation errors for df $\nu = 1$.

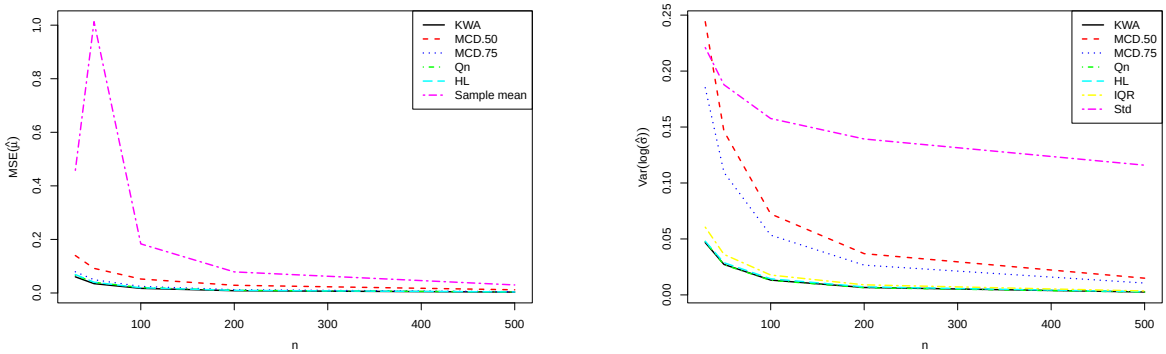

Figure S(34): Simulated location and scale estimation errors for df $\nu = 2$.

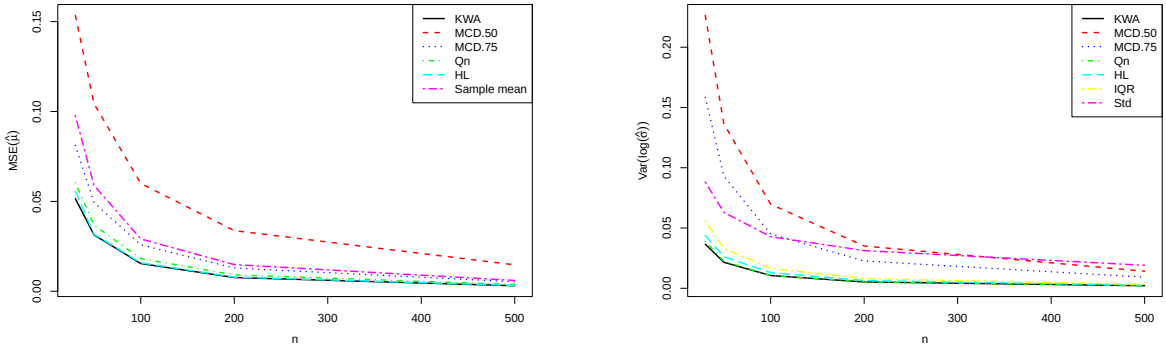

Figure S(35): Simulated location and scale estimation errors for df $\nu = 3$.

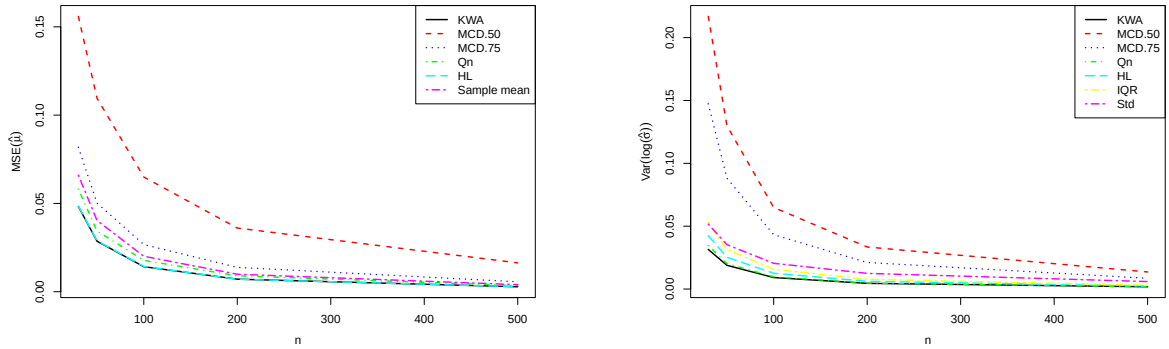

Figure S(36): Simulated location and scale estimation errors for df $\nu = 4$.

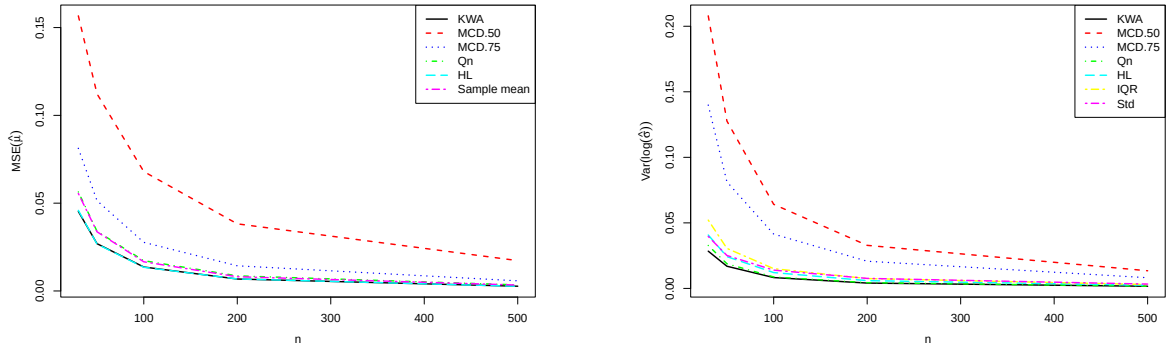

Figure S(37): Simulated location and scale estimation errors for df $\nu = 5$.

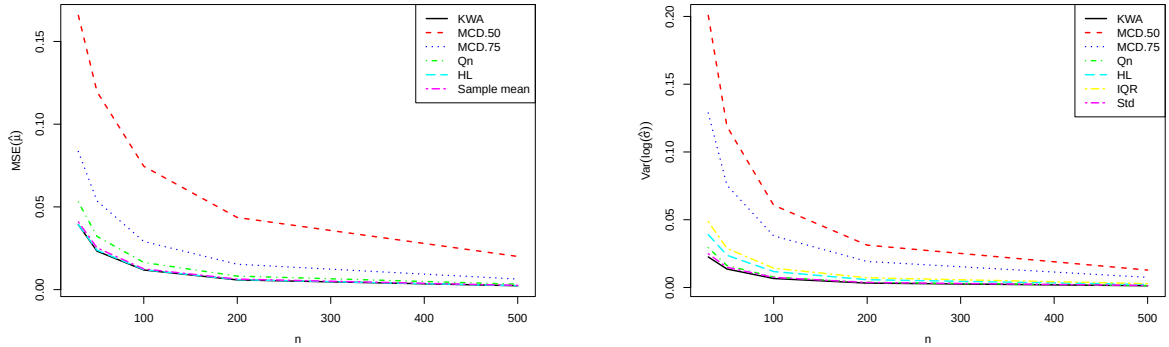

Figure S(38): Simulated location and scale estimation errors for df $\nu = 10$.

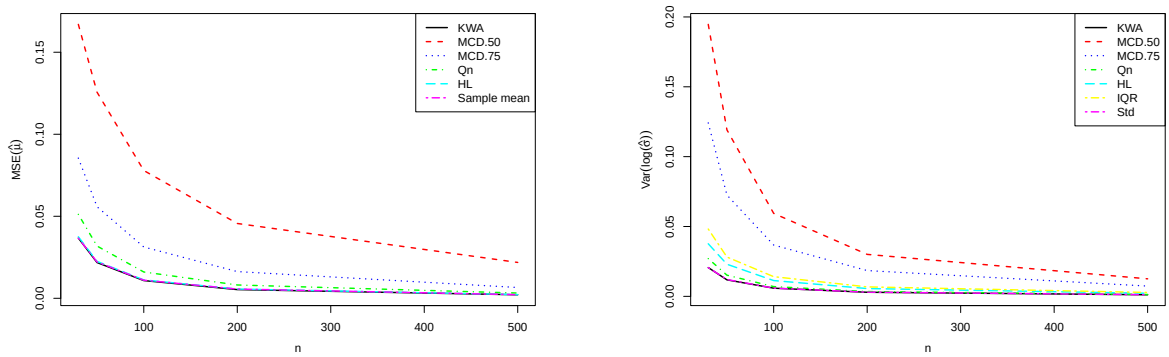

Figure S(39): Simulated location and scale estimation errors for df $\nu = 20$.

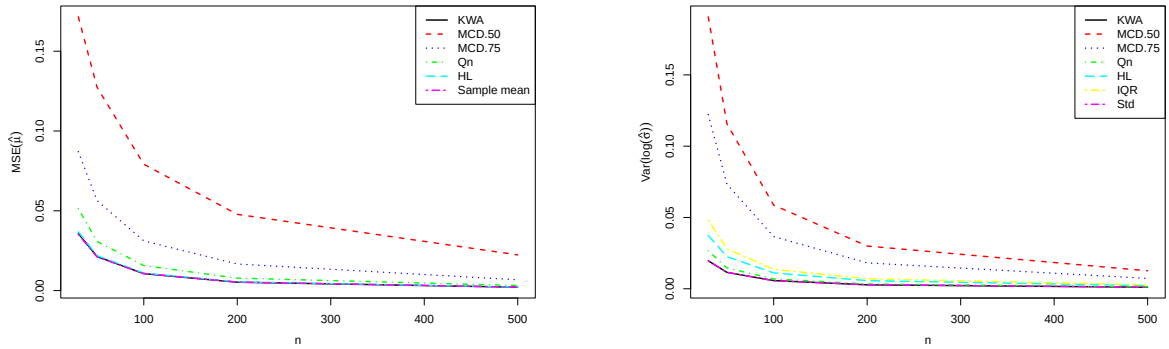

Figure S(40): Simulated location and scale estimation errors for df $\nu = 30$.

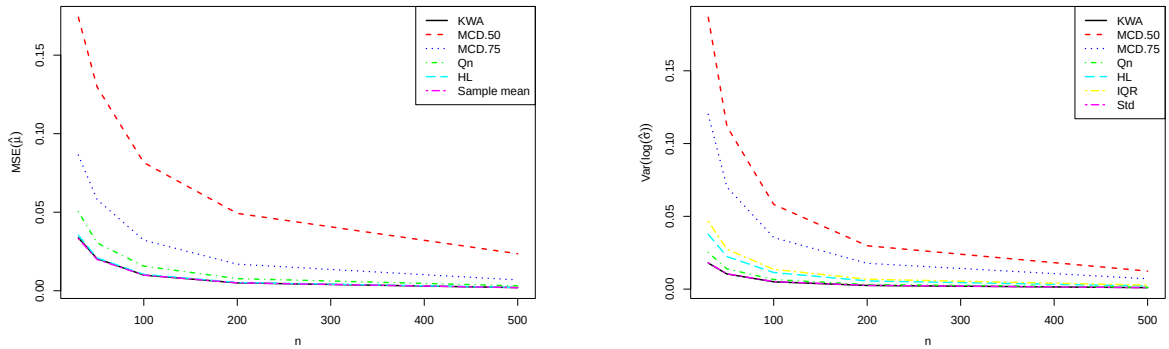

Figure S(41): Simulated location and scale estimation errors for df $\nu = \infty$.

S3 Supplemental Tables

S3.1 Optimal Bandwidths for KWA Location and Scale Estimators

	n					
df ν	30	50	100	150	200	300
1	0.5197	0.4819	0.3653	0.4143	0.3562	0.3562
2	0.8384	0.9509	0.8817	0.8598	0.9042	0.9273
3	1.2500	1.2500	1.1500	1.3250	1.3000	1.2500
4	1.2750	1.5500	1.5500	1.7000	1.5000	1.7250
5	1.5500	1.6000	1.7750	1.7750	1.7250	1.7750
10	2.0750	2.2750	2.3500	2.3000	2.3500	2.3250
20	2.8500	3.6750	3.6000	3.3000	3.3750	3.2500
30	4.9000	4.8250	4.6750	4.3250	4.2250	3.6000

Table S(1): Optimal bandwidths for KWA location estimator.

	n					
df ν	30	50	100	150	200	300
1	2.00	2.00	2.05	2.10	2.10	2.10
2	1.90	1.65	1.75	1.70	1.55	1.75
3	2.10	2.15	2.45	2.30	2.30	2.30
4	2.45	2.25	2.55	2.50	2.25	2.40
5	2.45	2.70	2.45	2.65	2.65	2.65
10	3.00	2.90	2.95	2.95	2.90	3.30
20	3.65	3.50	3.90	4.35	3.75	3.90
30	5.10	4.95	4.95	4.75	4.45	4.80

Table S(2): Optimal bandwidths for KWA scale estimator.

S3.2 Simulated Location and Scale Estimation Errors

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	Sample mean
30	0.079146	0.122696	0.110245	0.088538	0.134402	18919.835009
50	0.045182	0.078084	0.061621	0.051601	0.072004	5804.388665
100	0.021794	0.039267	0.029291	0.025537	0.034942	196452.213607
200	0.010495	0.019966	0.014537	0.012577	0.016953	28087.526728
500	0.004083	0.00805	0.005697	0.004881	0.006723	6130.523776

Table S(3): Simulated location estimation errors for df $\nu = 1$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	Sample mean
30	0.060345	0.139854	0.079173	0.066169	0.068922	0.45811
50	0.034886	0.091949	0.049504	0.039818	0.040255	1.012998
100	0.017271	0.052199	0.024542	0.020391	0.019535	0.183346
200	0.008482	0.028821	0.012465	0.009931	0.009774	0.078763
500	0.003376	0.011908	0.004984	0.004013	0.003926	0.030024

Table S(4): Simulated location estimation errors for df $\nu = 2$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	Sample mean
30	0.051548	0.153485	0.081115	0.060354	0.055435	0.097735
50	0.03138	0.104352	0.049227	0.036795	0.031718	0.058908
100	0.015416	0.059982	0.025839	0.018156	0.015825	0.029105
200	0.007638	0.033731	0.012925	0.009125	0.00788	0.01483
500	0.003069	0.014695	0.005257	0.003756	0.003244	0.006014

Table S(5): Simulated location estimation errors for df $\nu = 3$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	Sample mean
30	0.048278	0.15593	0.081809	0.058111	0.048492	0.066037
50	0.028611	0.109548	0.050175	0.034364	0.028828	0.040322
100	0.014182	0.06496	0.026697	0.017741	0.014324	0.020054
200	0.007108	0.036063	0.013795	0.009007	0.007058	0.009896
500	0.002805	0.01634	0.005619	0.003504	0.002841	0.003999

Table S(6): Simulated location estimation errors for df $\nu = 4$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	Sample mean
30	0.045266	0.156666	0.081216	0.056406	0.045897	0.055482
50	0.026925	0.11224	0.051395	0.033966	0.02671	0.033694
100	0.013639	0.06799	0.027682	0.017237	0.013489	0.016607
200	0.006802	0.038222	0.014328	0.008598	0.006852	0.008106
500	0.0027	0.017285	0.005763	0.003448	0.002676	0.003368

Table S(7): Simulated location estimation errors for df $\nu = 5$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	Sample mean
30	0.039275	0.165784	0.083471	0.053006	0.03927	0.040913
50	0.023249	0.119632	0.053585	0.032182	0.023869	0.025241
100	0.011868	0.074562	0.029095	0.016406	0.012045	0.012521
200	0.005864	0.043593	0.015333	0.008175	0.006026	0.006341
500	0.002337	0.020046	0.006382	0.003289	0.002385	0.00256

Table S(8): Simulated location estimation errors for df $\nu = 10$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	Sample mean
30	0.036724	0.166987	0.085443	0.051168	0.037528	0.036654
50	0.021797	0.12564	0.055824	0.031861	0.022503	0.022193
100	0.010799	0.077905	0.031274	0.016035	0.0112	0.011053
200	0.005303	0.045611	0.016283	0.008122	0.005644	0.005637
500	0.002148	0.021856	0.006579	0.003196	0.002249	0.002203

Table S(9): Simulated location estimation errors for df $\nu = 20$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	Sample mean
30	0.035786	0.171574	0.087063	0.051409	0.036856	0.035138
50	0.02135	0.127478	0.056382	0.030839	0.021943	0.021085
100	0.010599	0.079162	0.031132	0.015683	0.010957	0.010691
200	0.00524	0.047784	0.016661	0.007809	0.005458	0.005322
500	0.002116	0.022364	0.006762	0.003179	0.00218	0.002176

Table S(10): Simulated location estimation errors for df $\nu = 30$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	Sample mean
30	0.033722	0.174123	0.086383	0.050351	0.035366	0.033169
50	0.02046	0.129851	0.058119	0.030471	0.020915	0.01994
100	0.009975	0.081601	0.032176	0.015657	0.010433	0.00995
200	0.005005	0.049253	0.01692	0.007715	0.005238	0.005039
500	0.001998	0.023553	0.006892	0.003134	0.002104	0.001977

Table S(11): Simulated location estimation errors for df $\nu = \infty$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	IQR	Std deviation
30	0.087222	0.312537	0.335167	0.074862	0.069838	0.089263	1.253379
50	0.05135	0.18379	0.193483	0.042726	0.041819	0.052057	1.257105
100	0.025203	0.091021	0.09316	0.020892	0.021082	0.025268	1.230457
200	0.012828	0.045485	0.045395	0.010266	0.010359	0.012375	1.241047
500	0.005228	0.018473	0.018336	0.004132	0.004157	0.004851	1.229025

Table S(12): Simulated scale estimation errors for df $\nu = 1$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	IQR	Std deviation
30	0.046734	0.244129	0.185097	0.047713	0.047953	0.060513	0.220974
50	0.027269	0.146298	0.109509	0.026948	0.028767	0.036216	0.187815
100	0.013249	0.072321	0.053395	0.013021	0.01442	0.017749	0.157598
200	0.006631	0.036851	0.026598	0.006419	0.007182	0.008997	0.139315
500	0.002598	0.014929	0.010591	0.00252	0.00284	0.003511	0.115892

Table S(13): Simulated scale estimation errors for df $\nu = 2$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	IQR	Std deviation
30	0.036615	0.226445	0.158338	0.038879	0.043792	0.055995	0.088163
50	0.021533	0.136326	0.092949	0.022346	0.026065	0.033101	0.062994
100	0.010571	0.069703	0.045078	0.010607	0.013024	0.016518	0.042812
200	0.005242	0.035056	0.022652	0.005298	0.006407	0.008053	0.03126
500	0.002098	0.014101	0.009205	0.002109	0.002609	0.003228	0.01905

Table S(14): Simulated scale estimation errors for df $\nu = 3$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	IQR	Std deviation
30	0.031519	0.216891	0.147361	0.034568	0.042368	0.052912	0.051619
50	0.01889	0.129814	0.088188	0.020146	0.025252	0.031411	0.035257
100	0.009144	0.06489	0.043218	0.00972	0.012512	0.015631	0.02047
200	0.004516	0.033438	0.021133	0.004697	0.006163	0.007732	0.012459
500	0.001807	0.013574	0.008562	0.001844	0.00245	0.003066	0.005948

Table S(15): Simulated scale estimation errors for df $\nu = 4$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	IQR	Std deviation
30	0.028415	0.207927	0.139829	0.032762	0.040739	0.052114	0.039657
50	0.016911	0.127654	0.081258	0.018795	0.024255	0.030522	0.025119
100	0.008267	0.064143	0.041446	0.008755	0.012081	0.015078	0.013951
200	0.004066	0.032851	0.020744	0.004376	0.005944	0.007503	0.00764
500	0.001644	0.013458	0.008159	0.001676	0.002449	0.002954	0.003304

Table S(16): Simulated scale estimation errors for df $\nu = 5$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	IQR	Std deviation
30	0.022561	0.2008	0.128759	0.029404	0.039091	0.048643	0.0249
50	0.01366	0.118874	0.075484	0.016226	0.023788	0.028858	0.014964
100	0.006635	0.060713	0.038086	0.007749	0.011756	0.014243	0.007448
200	0.003253	0.031258	0.019221	0.003781	0.005755	0.007177	0.003702
500	0.001303	0.012869	0.007502	0.001475	0.002281	0.002858	0.001507

Table S(17): Simulated scale estimation errors for df $\nu = 10$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	IQR	Std deviation
30	0.020608	0.194687	0.124113	0.02702	0.037637	0.048194	0.020546
50	0.011816	0.11916	0.072491	0.015152	0.022989	0.028145	0.012052
100	0.005846	0.059391	0.036729	0.006986	0.011302	0.014092	0.005854
200	0.002922	0.030042	0.018455	0.003442	0.00558	0.006778	0.003003
500	0.001132	0.012539	0.00735	0.001338	0.002235	0.002786	0.001181

Table S(18): Simulated scale estimation errors for df $\nu = 20$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	IQR	Std deviation
30	0.019652	0.190727	0.122163	0.02661	0.037434	0.048058	0.019493
50	0.01149	0.115693	0.073383	0.014579	0.022508	0.02804	0.011559
100	0.005747	0.058651	0.036579	0.006893	0.011132	0.013627	0.005614
200	0.002757	0.029969	0.018145	0.003343	0.005705	0.007024	0.002849
500	0.001093	0.012621	0.007273	0.001278	0.002218	0.00275	0.001114

Table S(19): Simulated scale estimation errors for df $\nu = 30$.

n\method	KWA	MCD(0.50)	MCD(0.75)	Q_n	HL	IQR	Std deviation
30	0.017928	0.186783	0.120059	0.025171	0.037695	0.04664	0.018018
50	0.010393	0.111799	0.070096	0.013929	0.022369	0.027395	0.010533
100	0.005018	0.058211	0.035439	0.006557	0.011389	0.013563	0.005176
200	0.002513	0.029778	0.017722	0.003144	0.005627	0.006791	0.002568
500	0.00101	0.012369	0.007101	0.001236	0.002199	0.002716	0.001014

Table S(20): Simulated scale estimation errors for df $\nu = \infty$.

References

Ahmad, I. A. (1982). Nonparametric estimation of the location and scale parameters based on density estimation. *Annals of the Institute of Statistical Mathematics 34* (1), 39–53.