KdV, KPII and Kuramoto-Sivashinsky equations: from shallow water waves to chaos

Matteo Polimeno

San Diego State University

05/01/2018

Korteweg - de Vries Equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

- Named after Diederik Korteweg and Gustav de Vries (1895), but first introduced by Boussinesq in his *Essai su la theorie de eaux courantes* (1877)
- Describes weakly non-linear shallow water waves
- Admits *interacting solitary waves* solutions (solitons)

- Pseudospectral Methods, see [6]
- Time stepper: Fourth-Order Runge Kutta with integrating factor (explicit scheme)
- Initial data:

$$u_0 = 3A^2 sech^2(.5(A(x+2))) + 3B^2 sech^2(.5(B(x+1)))$$

$$u_t + \left(\frac{1}{2}u^2\right)_x + u_{xxx} = 0 \qquad (2)$$

- Fourier transform equation
- Integrating factor to solve linear part and find "linear operator"
- Change of variables to get rid of stiffness
- $\hat{U}_t + \frac{i}{2}e^{-ik^3t}k\mathcal{F}(real(\mathcal{F}^{-1}(e^{ik^3t}\hat{U}))^2) = 0$

KdV — Code

```
set up periodic grid and n-soliton initial data
FFT initial data
set up k-vector and Linear operator
for time=1:Number of iterations
   define integrating factor
   %apply 4th-order Runge Kutta with integrating factor
     t=n*dt;g=-.5i*dt*k;%kdv
     E=exp(dt*Lop/2); E2=E.^2;
     a=g.*fft(real(ifft(w)).^2);
     b=g.*fft(real(ifft(E.*(w+a/2))).^2);
     c=g.*fft(real(ifft(E.*w+b/2)).^2);
     d=g.*fft(real(ifft(E2.*w+E.*c)).^2);
     w=E2.*w+(E2.*a+2*E.*(b+c)+d)/6;
    IFFT solution
end
plot results
```

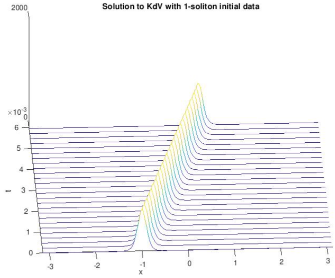

Figure 1: Solution for A=0 and B=16. Tmax=0.006. Computation involved 655 steps and an took an average of 0.16 seconds (10 simulations)

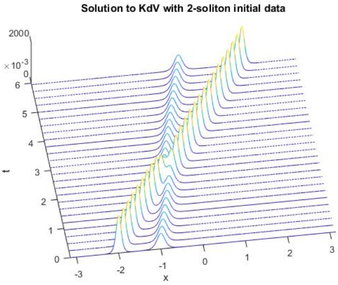

Figure 2: Solution for A=25 and B=16. Tmax=0.006.Computation involved 983 steps and took an average of 0.14 seconds (10 simulations)

KP

$$(u_t + uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0 \tag{3}$$

$$\Rightarrow \partial_t u + \partial_x u^2 + \partial_x^3 u + \sigma^2 \partial_x^{-1} \partial_y^2 u = 0 \tag{4}$$

- Introduced by Boris Kadomtsev and Vladimir Petiashvili in 1970
- 2D Generalization of KdV
- $\sigma^2 = 1 \Rightarrow KPII$, $\sigma^2 = -1 \Rightarrow KPI$

- Periodic domain $[-L, L] \times [-L, L]$
- $\widehat{(\partial_x^{-1})} = \frac{L}{i\pi k}$
- Time stepper: fourth order Runge Kutta with integrating factor

KPII — Function call

```
function J = rk4exp2(w,dt,g,E,KT)
E2=E.^2;
g=1;
a=g.*reshape((fft2(real(ifft2(reshape(w.',KT,KT).')).^2))
    ',KT^2,1); %you have to reshape w to ifft2
b=g.*reshape((fft2(real(ifft2(reshape((E.*(w+a/2)).',KT,KT
    ).')).^2))',KT^2,1);
c=g.*reshape((fft2(real(ifft2(reshape((E.*w+b/2).',KT,KT)
    .')).^2))',KT^2,1);
d=g.*reshape((fft2(real(ifft2(reshape((E2.*w+E.*c).',KT,KT
    ).')).^2))',KT^2,1);

J=E2.*w+(E2.*a+2*E.*(b+c)+d)/6;
end
```

KPII — code

```
%initialize parameters and initial data
clf,drawnow

Dds = 1i.*pi/L*[0:KT/2 -KT/2+1:-1]';
Dy=kron(Dds,ones(KT,1));
Dx=kron(ones(KT,1),Dds);
Dx3=Dx.^3;
Dy2=6.*Dy.^2;
Dds1 = length(Dds(2:end));
b = ones(Dds1,1)./Dds(2:end);
Dxn1sb = [0; b];
Dxn1=kron(ones(KT,1),Dxn1sb);
Dx=(3/2).*Dx;
Lop=Dx3+(Dy2.*Dxn1);
g=1;
E=exp(dt.*Lop./2);
%solve PDE and plot

w = reshape(fft2(u0)',KT^2,1);
```

Pseudocode

```
for n=1:nmax
    t=n*dt;
    w = rk4exp2(w,dt,g,E,KT);
end
wnp1 = (real(ifft2(reshape(w.',KT,KT).'))).';
plot
```

Pseudocode

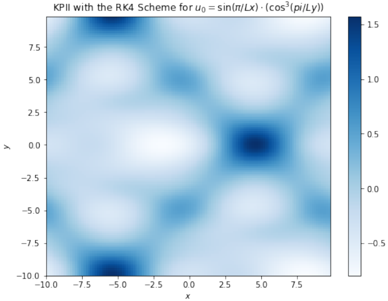

Figure 3: Colormap of solution to KPII. Tmax=10, modes N=64, dt=0.01

Kuramoto-Sivashinsky Equation

$$u_t + (u^2)_x + u_{xx} + u_{xxxx} = 0 \tag{5}$$

- Introduced by Yoshiki Kuramoto and Gregory Sivashinsky (mid 1970's)
- Reaction-Diffusion systems
- Chaotic behavior
- $\hat{U}_t + \frac{i}{2}e^{(-k^2+k^4)t}k\mathcal{F}(real(\mathcal{F}^{-1}(e^{(k^2-k^4)t}\hat{U}))^2) = 0$

Plots

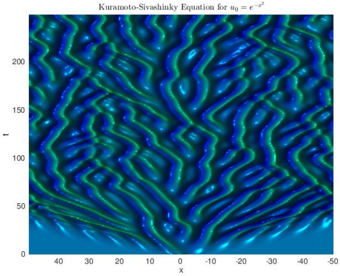

Figure 4: KS - Chaotic evolution

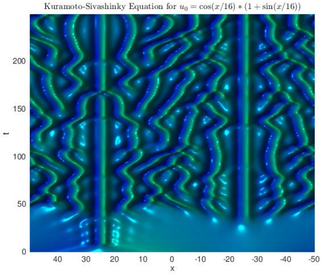

Figure 5: KS - Chaotic Evolution

Final Remarks

- Even simple nonlinearity, when combined with stiffness, raises interesting problems
- Water waves, plasmas and chaos
- Implement different numerical schemes
- Comparison

- *Fourth-Order time stepping for STIFF PDE's* Aly-Khan Kassam, Lloyd N. Trefethen. SIAM, 2005
- Kenneth L. Jones. *Three dimensional Korteweg - de Vries equations and traveling wave solution*. Hindawi Publishing, 1999
- Kundu P., Cohen I, Dowling D. *Fluid Mechanics*. Elsevier, 2016
- J.Nathan Kutz. *Data-Driven Modeling & Scientific Computation Methods for Complex Systems and Big Data*. Oxford University Press, 2013.
- John C. Strikwerda. *Finite Difference Schemes and Partial Differential Equations*. SIAM, 2004
- Lloyd N. Trefethen *Spectral Methods in Matlab*. SIAM, 2000


That's all Folks!