San Diego State University

Department of Mathematics and Statistics
Math 639
Nonlinear Waves

Project Proposal:

Shallow Waves and the Advection-Diffusion Equation:
A computational approach in two dimensions

Bin Hoang,
Horacio Lopez,
Matteo Polimeno

Professor:
Dr. Christopher Curtis

March 29^{th}, 2018

Contents

1 Introduction

In this project we will implement a numerical scheme using the Fast Fourier Transform aimed to solve and analyze the shallow-water wave equations and compare it to the LU decomposition method and the Crank-Nicolson scheme.

2 Background and Motivation

The shallow-wave equations have a mathematical definition that relies on the technique of separation of scales, which makes the approximation reasonable in a wide variety of physical settings. From atmospheric dynamics to ocean waves, there are several fields of study that involves the aforementioned equations, that rely on the assumption that the wavelengths are much larger than the upper-layer end of the fluid. Thus, for instance, in a oceanic context, the upper layer might have a thickness of about $50m$, while we might be interested in much longer interfacial waves. Letting the height be H and the length L, we have the relation:

$$\delta = \frac{H}{L} << 1,$$

which reduces the Navier-Stokes equations to the shallow-water description.

2.1 Brief Mathematical Framework

Using conservation of mass and momentum with the assumptions of incompressibility and constant fluid height, the governing equation of shallow waves with diffusion is given by [3], (page 609):

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega$$

$$\nabla^2 \psi = \omega$$

$$\frac{\partial \omega}{\partial t} + [\psi, \omega] = 0$$

with

$$[\psi, \omega] = \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x},$$

where ν is a parameter related to the diffusion present in the fluid, v is the velocity vector and ω is the vorticity:

$$\mathbf{v} = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

and

$$\omega_z = \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

3 Presented Work

We will be solving for the streamline function, $\psi(x, y, t)$, and the vorticity function, $\omega(x, y, t)$, with a generalized procedure from [3]:

1. Solve the Elliptic problem

$$\nabla^2 \psi = \omega_0$$

to find the streamfunction at $t = 0$, $\psi_0 = \psi(x, y, 0)$.

2. Using both ω_0 and ψ_0, we will solve the advection-diffusion equation using the Crank-Nicolson scheme and the Fast Fourier Transform method.

3. With the updated values of ω and ψ, the process is repeated in a loop for calculation of future values.

The governing equation is given by

$$\omega_t + [\psi, \omega] = \nu \nabla^2 \omega,$$

where $[\psi, \omega] = \psi_x \omega_y - \psi_y \omega_x$, $\nabla^2 = \partial^2 x + \partial^2 y$ and the streamfunction satisfies $\nabla^2 = \omega$, from [3].

We will assume a Gaussian-shaped mound of initial vorticity $\omega(x, y, 0)$ for a given initial amplitude. The diffusion parameter ν will be chosen to be positive, but much smaller than 1. Assuming periodic boundary conditions for both the streamfunction and the vorticity, we will proceed to numerically-integrate our equations.

Finally, we will present our results in terms of the computational speed and accuracy for each method chosen.

3.1 Novelty

In addition to solving the streamline functions by the IBVP specified above, we will modify our codes to generate multiple vortices with variable strength at any position of choice on a periodic domain. From this freedom of choice, we can generate interesting configurations such as vortex dipoles and quadrupoles, and compare them to the available analytical solutions. Our goal is both to study the dynamics of these configurations, as well as comparing the solutions of the numerical and analytical methods implemented.

4 Bibliography

References

[1] Cohen, I.M. and Kundu, P.K. *Fluid Mechanics*. Elsevier Science, 2007.

[2] Garcia, A. *Numerical Methods for Physics*. CreateSpace Independent Publishing Platform, 2015.

[3] Kutz, J.N. *Data-Driven Modeling & Scientific Computation Methods for Complex Systems and Big Data*. Oxford University Press, 2013.

[4] Pironneau, O. *Finite Element Methods for Fluids*. Chichester etc., John Wiley & Sons, Paris etc., Masson 1989.

[5] Strikwerda, J.C. *Finite Difference Schemes and Partial Differential Equations*. Society for Industrial and Applied Mathematics, 2004.