Instructions. Identify the group of 2 classmates with whom you will complete this assignment. Then, for each of the homework problems below,

1. Please have the group write up one complete solution in LaTeX.
2. Document the contributions of each member of the group in solving this problem.
3. Include sufficient written discussion to demonstrate your understanding of the problem.

Please submit a PDF file of your homework solutions by **March 20** before 11:59 pm.

Homework Problems.

1. Consider the following method for solving the heat equation $u_t = u_{xx}$:

$$U_i^{n+2} = U_i^n + \frac{2k}{h^2}(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}).$$

   (a) Determine the order of accuracy of this method (in both space and time).

   (b) Suppose we take $k = \alpha h^2$ for some fixed $\alpha > 0$ and refine the grid. For what values of α (if any) will this method be Lax-Richtmyer stable and hence convergent?

   Hint: Consider the MOL interpretation and the stability region of the time-discretization being used.

   (c) Is this a useful method?

2. The m-file `heat_CN.m` solves the heat equation $u_t = \kappa u_{xx}$ using the Crank-Nicolson method. Run this code, and by changing the number of grid points, confirm that it is second-order accurate. (Observe how the error at some fixed time such as $T = 1$ behaves as k and h go to zero with a fixed relation between k and h, such as $k = 4h$.)

   You might want to use the function `error_table.m` to print out this table and estimate the order of accuracy, and `error_loglog.m` to produce a log-log plot of the error vs. h. See `bvp_2.m` for an example of how these are used.

3. Modify `heat_CN.m` to produce a new m-file `heat_trbdf2.m` that implements the TR-BDF2 method on the same problem. Test it to confirm that it is also second order accurate. Explain how you determined the proper boundary conditions in each stage of this Runge-Kutta method.

4. Modify `heat_CN.m` to produce a new m-file `heat_FE.m` that implements the forward Euler explicit method on the same problem. Test it to confirm that it is $O(h^2)$ accurate as $h \to 0$

provided when $k = 24h^2$ is used, which is within the stability limit for $\kappa = 0.02$. Note how many more time steps are required than with Crank-Nicolson or TR-BDF2, especially on finer grids.

5. Modify `heat_CN.m` to solve the heat equation for $-1 < x < 1$ with step function initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

With appropriate Dirichlet boundary conditions, the exact solution is

$$u(x,t) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{4\kappa t}}\right)$$

where erfc is the complementary error function

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} \mathrm{d}z.$$

(a) Test this routine $m = 39$ and $k = 4h$. Note that there is an initial rapid transient decay of the high wave numbers that is not captured well with this size time step.

(b) How small do you need to take the time step to get reasonable results? For a suitably small time step, explain why you get much better results by using $m = 38$ than $m = 39$. What is the observed order of accuracy as $k \to 0$ when $k = \alpha h$ with α suitably small and m even?