Instructions. Identify the group of 2 classmates with whom you will complete this assignment. Then, for each of the homework problems below,

1. Please have the group write up one complete solution in LaTeX.
2. Document the contributions of each member of the group in solving this problem. Be sure to include any references you have used to arrive at this solution, *e.g.* web pages, texts, etc.
3. Include sufficient written discussion to demonstrate your understanding of the problem.

Please submit your homework solutions by **March 12** before 11:59 pm.

Homework Problems.

In this assignment we will consider the solution of an elliptic PDE in a inhomogeneous medium. This problem appears in several applications : it is for example the equation that describes the distribution of temperature in a medium with inhomogeneous thermal conductivity.

Assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded domain with smooth boundary $\Gamma = \partial\Omega$, $f \in L^2(\Omega)$, $k \in L^\infty(\Omega)$, and there exist $\alpha > 0$ such that

$$k(x) \geq \alpha, \;\; \forall \; x \in \Omega \tag{1}$$

1. We want to find the temperature u in a material with thermal conductivity $k(x)$,

$$\begin{cases} -div\,(k(x)\nabla u(x)) = f(x) & \text{in } \Omega \\ \\ u = 0 & \text{on } \Gamma \end{cases} \tag{2}$$

   (a) Write problem (2) in a weak form

$$a(u, v) = L(v) \;\; \forall \; v \in V \tag{3}$$

2. Assume that Ω consists of two sub-domains $\Omega = \Omega_1 \cup \Omega_2$, and that the thermal conductivity is constant in each subdomain

$$k(x) = \begin{cases} k_1 & \text{in } \Omega_1 \\ \\ k_2 & \text{in } \Omega_2 \end{cases} \tag{4}$$

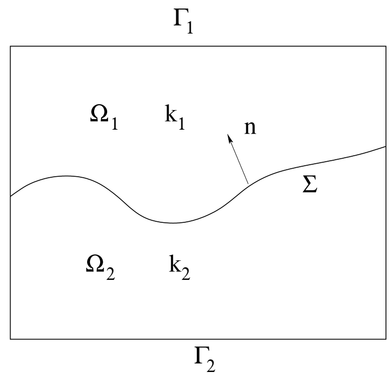

Figure 1: Schematic representation of the problem $\Omega = \Omega_1 \cup \Omega_2$

Denote by Σ the boundary between Ω_1 and Ω_2, $\Sigma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, $\Gamma_j = \Gamma \cap \partial\Omega_j$, $j = 1, 2$.

Take $\Omega = [-1, 1]^2$, $\Sigma = [-1, 1] \times \{y = 0\}$, $\Omega_1 = \Omega \cap \{y > 0\}$ and $\Omega_2 = \Omega \cap \{y < 0\}$.

We want to approximate (3) using Q^1 finite elements. Construct a discretization of Ω using squares of size h, $\Omega = \cup_{l=1}^{N_{el}} T_l$. Let N_{el} be the total number of finite elements, T_l, $l = 1, \ldots, N_{el}$ and denote $(M_i)_{i=1..N_s}$ the degrees of freedom (d.o.f.) associated to the vertices of the finite elements, N_s being the total number of d.o.f. We can write $N_s = N_i + N_d$, with N_i the interior points that do not belong to the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and N_d the the d.o.f on Γ with the boundary condition Dirichlet.

(a) Construct a discretization of Ω, so that Σ coincides with some finite elements edges (see Figure 2).

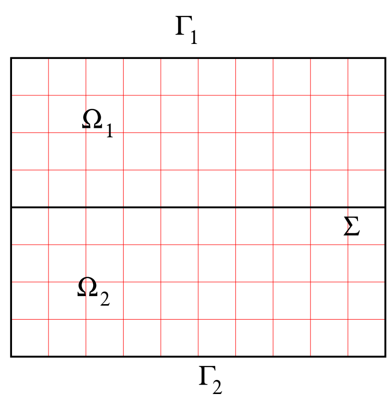

Figure 2: Discretization example

It would be convenient to define :

(i) N_{el}, N_i, N_d, N_s

(ii) $coor(m, i)$, the ith coordinate of the mth point ($1 \le m \le N_s, 1 \le i \le 2$).

(iii) $lg(l, i)$ the global number of the ith degree of freedom of the lth finite element ($1 \le l \le N_{el}, 1 \le i \le 4$).

(iv) $ref(l)$ a pointer depending on the finite element l, such that

$$ref(l) = \begin{cases} 1 & \text{if } T_l \subset \Omega_1 \\ 2 & \text{if } T_l \subset \Omega_2 \end{cases}$$

The discretization space is V_h,

$$V_h = \left\{ v \in C^0(\bar{\Omega}), v_{/T_l} \in Q^1(T_l), \forall T_l, \right\}$$

and $(\varphi_J)_{J=1,N_s}$, the basis functions,

$$\varphi_J(M_I) = \delta_{IJ} \;\; 1 \le I, J \le N_s$$

We seek an approximate solution of the form

$$u_h(x) = \sum_{J=1}^{N_s} u_J \varphi_J(x)$$

(b) Show that the discrete problem can be writtem a linear system of the form :

$$KU = F \tag{5}$$

with $U = (u_J)_{J=1,N_s}$ and $u_J = 0$ if M_J is on the boundary Γ. Compute the rigidity matrix K. Don't forget to take into account the Dirichlet boundary condition.

To compute the second member we can approximate f by f_h,

$$f_h = \Pi_h f = \sum_{J=1}^{N_i} f_J \varphi_J$$

with $f_J = f(M_J)$. What is the resulting second member of our linear system?

(c) Write a Matlab code to solve this problem.

(d) First test your program for k =cste. Use known exact solutions to check your code. For example for $k = 1$ and $f(x, y) = -2\left(\left(x^2 - 1\right) + \left(y^2 - 1\right)\right)$ the exact solution is $u(x, y) = (x^2 - 1)(y^2 - 1)$.

(e) Check the error $u - u_h$ in $L^2(\Omega)$ and $H^1(\Omega)$. You may use a quadrature formula to compute the integrals

$$\int_T f(x,y)dxdy = mes(T)\sum_{k=1}^{n} f(x_k, y_k)\omega_k$$

with $M_k(x_k, y_k)$ the quadrature points and ω_k the corresponding weights (e.g. you may use the Gauss Lobatto quadrature formula) :

- $M_1(0,0),\ M_2(1,0),\ M_3(1,1),\ M_4(0,1),$
  $M_5(1/2,0),\ M_6(1,1/2),\ M_7(1/2,1),\ M_8(0,1/2),\ M_9(1/2,1/2)$
  and $\omega_i = 1/36,\ i = 1,\dots 4,\ \omega_i = 1/9, i = 5,\dots,8$ and $\omega_9 = 4/9$.

(f) Using succesive refinements ($h = 1/10,\ h = 1/20,\ h = 1/30,\dots$) find the order of the method (for $k = 1$ and using the exact solution given in 2-(d)).

(g) Use your program for different values of $k(x)$

$$(k_1, k_2) = (1,1)\ ,\ (1,2)\ ,\ (1,10)\ ,\ (2,1)\ ,\ (10,1)$$

and $f(x,y) = -2\left(\left(x^2-1\right)+\left(y^2-1\right)\right)$. What do you observe?