Instructions. Identify the group of 2 classmates with whom you will complete this assignment. Then, for each of the homework problems below,

1. Please have the group write up one complete solution in LaTeX.
2. Document the contributions of each member of the group in solving this problem.
3. Include sufficient written discussion to demonstrate your understanding of the problem.

Please submit the PDF and mfiles of your homework solutions by **April 9** before 11:59 pm.

Homework Problems.

1. Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}). \tag{1}$$

for the advection equation $u_t + au_x = 0$ on $0 \le x \le 1$ with periodic boundary conditions.

(a) This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix A? Don't forget the boundary conditions.

(b) Suppose we want to fix the Courant number ak/h as $k,\ h \to 0$. For what range of Courant numbers will the method be stable if $a > 0$? If $a < 0$? Justify your answers in terms of eigenvalues of the matrix A from part (a) and the stability regions of the trapezoidal method.

(c) Apply von Neumann stability analysis to the method (1). What is the amplification factor $g(\xi)$?

(d) For what range of ak/h will the CFL condition be satisfied for this method (with periodic boundary conditions)?

(e) Suppose we use the same method (1) for the initial-boundary value problem with $u(0,t) = g_0(t)$ specified. Since the method has a one-sided stencil, no numerical boundary condition is needed at the right boundary (the formula (1) can be applied at x_{m+1}). For what range of ak/h will the CFL condition be satisfied in this case? What are the eigenvalues of the A matrix for this case and when will the method be stable?

2. Determine the modified equation on which the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}).$$

from Problem 1 is second order accurate. Is this method predominantly dispersive or dissipative?

3. The m-file `advection_LW_pbc.m` implements the Lax-Wendroff method for the advection equation on $0 \leq x \leq 1$ with periodic boundary conditions.

   (a) Modify the m-file to create a version `advection_lf_pbc.m` implementing the leapfrog method and verify that this is second order accurate. Note that you will have to specify two levels of initial data. For the convergence test set $U_j^1 = u(x_j, k)$, the true solution at time k.

   (b) Modify `advection_lf_pbc.m` so that the initial data consists of a wave packet

   $$\eta(x) = \exp(-\beta(x-0.5)^2)\sin(\xi x) \tag{2}$$

   Work out the true solution $u(x,t)$ for this data. Using $\beta = 100$, $\xi = 80$ and $U_j^1 = u(x_j, k)$, test that your code still exhibits second order accuracy for k and h sufficiently small.

   (c) Using $\beta = 100$, $\xi = 150$ and $U_j^1 = u(x_j, k)$, estimate the group velocity of the wave packet computed with leapfrog using $m = 199$ and $k = 0.4h$. How well does this compare with the value (10.52) predicted by the modified equation?