A Proof for the commutativity in the diagram in Figure 2.

Proof. Let $h_{\sigma_A} : \sigma_A \longrightarrow R_{\sigma_A}$, $h_{\sigma_B} : \sigma_B \longrightarrow R_{\sigma_B}$, $\rho_A : R_{\sigma_A} \longrightarrow V_{\sigma_A}$ and $\rho_B : R_{\sigma_B} \longrightarrow V_{\sigma_B}$ be monomial homomorphisms. By hypothesis, the lattice cones σ_A and σ_B are isomorphic, that is, $\sigma_A \approx \sigma_B$. Then there exists an unimodular linear transformation given by, L, such that $L(\sigma_A) = \sigma_B$ and vice versa $L^{-1}(\sigma_B) = \sigma_A$. This linear transformation is well defined through the Hilbert basis. With this fact, we build the monomial homomorphisms, as follows,

$h_{\sigma_A}(a) = \sum \lambda_a t^a \ \in \ R_{\sigma_A}$ and $a \ \in \ supp(h_{\sigma_A}) \ \subset \ \sigma_A$,

$h_{\sigma_B}(b) = \sum \lambda_b t^b \ \in \ R_{\sigma_B}$ and $b \ \in \ supp(h_{\sigma_B}) \ \subset \ \sigma_B$,

$\Psi(h_{\sigma_B}) = \sum \lambda_B t^{L(\sigma_A)} \ \in \ R_{\sigma_B}$ and $L(\sigma_A) = \sigma_B \ \in \ supp(\Psi) \ \subset \ \sigma_B$,

$\Psi^{-1}(h_{\sigma_A}) = \sum \lambda_A t^{L^{-1}(\sigma_B)} \ \in \ R_{\sigma_A}$ and $L^{-1}(\sigma_B) = \sigma_A \ \in \ supp(\Psi^{-1}) \ \subset \sigma_A$.
We also chose the prime generators $t^a \ \in \ R_{\sigma_A}$, $a \in \sigma_A$, $t^b \ \in \ R_{\sigma_B}$, and $b \ \in \ \sigma_B$. and We define the following maps:

$\rho_A(t^a) = < t^a > \in \ V_{\sigma_A} = Spec(R_{\sigma_A})$,

$\rho_B(t^b) = < t^b > \in \ V_{\sigma_B} = Spec(R_{\sigma_B})$,

$\Phi(< t^b >) = < t^{L^{-1}(b)} > \in \ V_{\sigma_A} = Spec(R_{\sigma_A})$,

$\Phi^{-1}(< t^a >) = < t^{L(a)} > \in \ V_{\sigma_B} = Spec(R_{\sigma_B})$.

Here, the $< t^a >$ and $< t^{L^{-1}(b)} >$ are prime ideals, from the spectrum of coordinate rings, of Laurent´s formal power series R_{σ_A} and R_{σ_B} respectively. We can see without lost of generality that the monomial homomorphisms fulfill the following identities, we can take, $\lambda_a ==_b= 1$, hence, $L \circ L^{-1} = id_{\sigma_B}$, $L^{-1} \circ L = id_{\sigma_A}$, $\Psi \circ \Psi^{-1} = id_{R_{\sigma_A}}$, $\Psi^{-1} \circ \Psi = id_{R_{\sigma_B}}$, and $\Phi \circ \Phi^{-1} = id_{V_{\sigma_A}}$, $\Phi^{-1} \circ \Phi = id_{V_{\sigma_B}}$. We remember that the maps, L, Ψ, and Φ are isomorphisms for, lattice cones, algebras of coordinate rings, and toric morphisms. The two first morphisms are well defined, one is an unimodular transformation, second one is an algebra homomorphism. Only we need to show that Φ is a toric morphism. We define the monomial homomorphism $\Phi^* : C^n \longrightarrow C^n$ such that, $\Phi^*(< t^b >) =< t^a > \ni \ \Phi^*(V_{\sigma_B}) \ \subset \ V_{\sigma_A}$ this homomorphism induces the morphism Φ it which is bijective, since that for the generator, $t^0 = id_{V_{\sigma_B}}$, with lattice vector $a = 0 \ \in \sigma_A$, $\Phi(t^0) = t^{L^{-1}(0)} = id_{V_{\sigma_A}}$. Therefore Φ is injective, Now we take the generator $t^b \ \in \ V_{\sigma_B}$, thus $L(a) = b$, $\Longrightarrow \ \exists \ t^a \ \in V_{\sigma_A} \ \ni$ $\Phi(t^b) = t^{L^{-1}(b)} = t^a$, therefore Φ is sobrejective. We can notice that we can write, $\Phi = \Phi^*|_{V_{\sigma_B}}$. At the same time, we can notice that Φ^{-1} is also a toric morphism. For the map Ψ, we can see easily that, $\Psi(h_{\sigma_B}(b)) = h_{\sigma_A}(L^{-1}(b))$ and $\Psi(i_B(< t^b >)) = i_A(< t^{L^{-1}(b)} >)$. The proof for the isomorphisms $V_{\sigma_B} \approx V_{\sigma_A}$ implies the isomorphism $\sigma_B \approx \sigma_A$, see Ewald [37]. Together, all those facts proved the commutativity of the diagram in Figure 2, **q.e.d.**