Hull-White Model and Calibration
Perfect Fit of the Term Structure

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Introduction

- Hull-White model: $dr_t = [\theta(t) - ar_t]dt + \sigma dW_t$
- Since the Hull-White model is inhomogenous we can exactly fit the term structure
- In order to achieve the perfect fit we construct a trinomial tree in two consecutive steps

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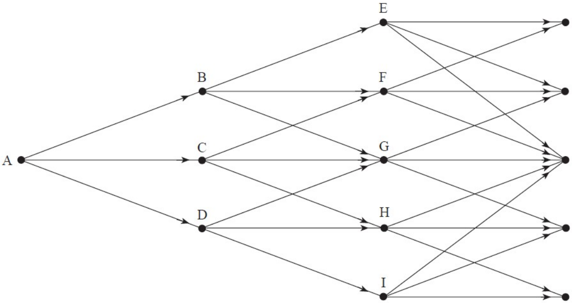
Trinomial tree according to Hull and White step one

Options, Futures and other Derivatives by Hull and White

Figure 30.8 Tree for R^* in Hull–White model (first stage).

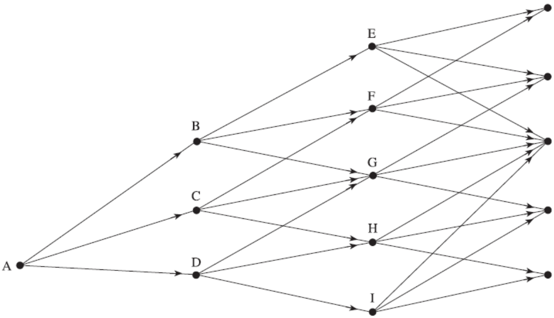


Dynamics: $dR_t^* = \sigma dW_t$, assuming $a = 0$

Trinomial tree according to Hull and White step two

Options, Futures and other Derivatives by Hull

Figure 31.9 Tree for R in Hull–White model (the second stage).



Dynamics: $dR_t = \theta(t)dt + \sigma dW_t$, assuming $a = 0$

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Trinomial tree according to Hull and White step one

- During the first step the branching probabilities for going up, down and remaining constant are calculated
- The difference $R^*(t+\Delta t) - R^*$ is normally distributed with mean $-aR^*\Delta t$ and variance $\sigma^2 \Delta t$
- We have to equate the theoretical mean and variance with the one in the model

$$p_u \Delta R - p_u \Delta R = -aj\Delta R \Delta t$$

$$p_u \Delta R^2 + p_d \Delta R^2 = \sigma^2 \Delta t + a^2 j^2 \Delta R^2 \Delta t^2$$

$$p_u + p_m + p_d = 1$$

Trinomial tree according to Hull and White step one

This leads to the following probabilities:

$$p_u = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - aj\Delta t)$$
$$p_m = \frac{2}{3} + a^2 j^2 \Delta t^2$$
$$p_u = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + aj\Delta t)$$

Trinomial tree according to Hull and White step two

We have to shift the nodes in order to fit the term structure

- We define $Q_{i,j}$ as the present value of a security that pays off 1 EUR at node (i,j)
- We initialize $Q_{0,0} = 1$
- We calcuate α_0 such that the right price of the given zero coupond bond at time Δt is perfectly met
- Next we calculate the prices $Q_{1,j}$ at time 0 for all nodes j as the multiplication of the respective probabilities and the zero coupond bonds according to α_0
- We can iteratively calculate all α_i and $Q_{i,j}$ to exactly fit the term structure with the following formulas

$$\alpha_m = \frac{ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R \Delta t} - ln P_{m+1}}{\Delta t}$$

$$Q_{m+1,j} = sum_k Q_{m,k} q(k,j) exp[-(\alpha_m + k\Delta R)\Delta t]$$

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Trinomial tree according to Hull and White step two

We demonstrate how well the term structure is fit and how arbitrary cash flows and options on cash flows can be valued!
Have fun:)