

# Hull-White model and Calibration

## Perfect Fit of the Term Structure

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# Inhaltsverzeichnis

- 1 Introduction
- 2 Trinomial tree according to Hull
- 3 Calculations
  - Calculation of the probabilities
  - Perfect fit of the term structure
- 4 Coding - Have Fun

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# Introduction

- Since the Hull-White model is inhomogenous we can exactly fit the term structure
- In order to achieve the perfect fit we construct a trinomial tree in two consecutive steps

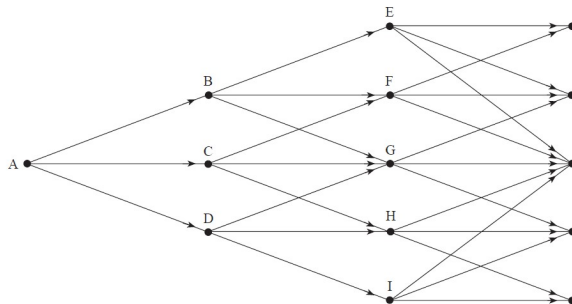
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# Trinomial tree according to Hull and White step one

## Options, Futures and other Derivatives by Hull and White

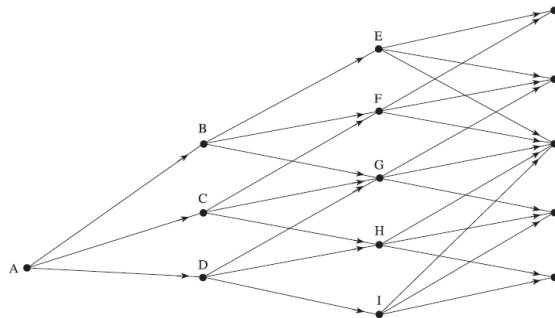
**Figure 30.8** Tree for  $R^*$  in Hull–White model (first stage).



# Trinomial tree according to Hull and White step two

## Options, Futures and other Derivatives by Hull

Figure 31.9 Tree for  $R$  in Hull-White model (the second stage).



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# Trinomial tree according to Hull and White step one

- During the first step the branching probabilities for going up, down and remaining constant are calculated
- The difference  $R^*(t + \Delta t) - R^*$  is normally distributed with mean  $-aR^*\Delta t$  and variance  $\sigma^2\Delta t$
- We have to equate the theoretical mean and variance with the one in the model

$$p_u\Delta R - p_d\Delta R = -aj\Delta R\Delta t$$

$$p_u\Delta R^2 + p_d\Delta R^2 = \sigma^2\Delta t + a^2j^2\Delta R^2\Delta t^2$$

$$p_u + p_m + p_d = 1$$

# Trinomial tree according to Hull and White step one

This leads to the following probabilities:

$$p_u = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - aj \Delta t)$$

$$p_m = \frac{2}{3} + a^2 j^2 \Delta t^2$$

$$p_d = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + aj \Delta t)$$

# Trinomial tree according to Hull and White step two

We have to shift the nodes in order to fit the term structure

- We define  $Q_{i,j}$  as the present value of a security that pays off 1 EUR at node  $(i,j)$
- We initialize  $Q_{0,0} = 1$
- We calculate  $\alpha_0$  such that the right price of the given zero coupon bond at time  $\Delta t$  is perfectly met
- Next we calculate the prices  $Q_{1,j}$  at time 0 for all nodes  $j$  as the multiplication of the respective probabilities and the zero coupon bonds according to  $\alpha_0$
- We can iteratively calculate all  $\alpha_i$  and  $Q_{i,j}$  to exactly fit the term structure with the following formulas

$$\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R\Delta t} - \ln P_{m+1}}{\Delta t}$$

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp[-(\alpha_m + k\Delta R)\Delta t]$$

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# Trinomial tree according to Hull and White step two

We demonstrate how well the term structure is fit and how arbitrary cashflow and options on cashflows can be valued!

Have fun:)