Introduction to Recursion

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Today's lecture

In this first lecture we will discuss the following topics:

- Recursion: informal concept and (more) formal definition
- Examples
- Recursion in Programming
- Recursion in Data Structures
- Summary/Overview

The goal for today's lecture is to introduce you to the topic of *Recursion*, and review its main applications in programming and data structures.
Please stop me if you have a question.

Examples and codes from slides, available at

`https://github.com/mponce0/UTSC-CS_lectures`

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More formally, a class of objects or methods exhibits *recursive* behavior when it can be defined by two properties:

- **A base rule**: A simple base case (or cases) – a terminating scenario that does not use recursion to produce an answer.
- **A recursive step** – a set of rules that reduces all successive cases toward the base case.

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Sourdough fermentation

Credit: *Janus Sandsgaard* (wikipedia)

Examples

- Factorial:

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\boldsymbol{n! \equiv \begin{cases} 0! = 1! = 1 & n = 0,1 \\ 1 \times 2 \times \ldots \times (n-1) \times n = \prod_{1}^{n} i & n > 1 \end{cases}}
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  '''Function to compute factorial using a recursive implementation'''

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  else:
    return num * factorialRecursive(num-1)
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Examples

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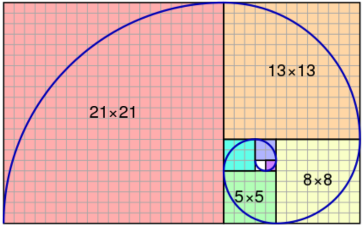
$$\begin{cases} F_0 = 0, F_1 = 1 \\ F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \end{cases}$$



Credit *Jahobr* (wikipedia)

Examples

- Fibonacci numbers

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```python
def Fibonacci(n):
  '''Recursive definition of Fibonacci numbers'''

  if n >= 2:
    # recursive rule
    return (Fibonacci(n-1)+Fibonacci(n-2))
  elif ( n==0 or n==1 ):
    # base rule
    return n
  else:
    print("Error: Fibonacci nbrs defined for 
        positive integers!")


def FibonacciSeq(n):
  '''Function to generate Fibonacci series'''

  for i in range(0,n+1):
    print(Fibonacci(i))
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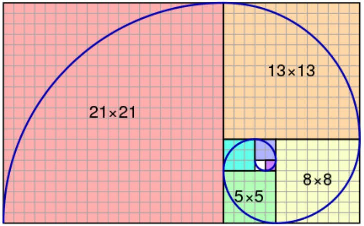

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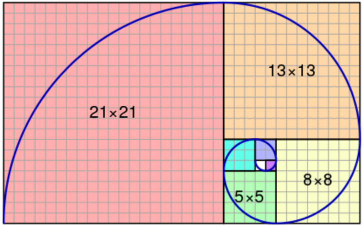

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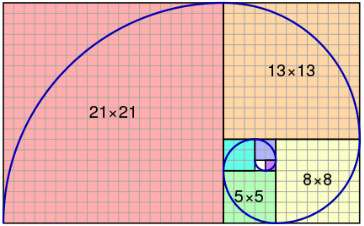

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- Fibonacci nbrs – multiple applications in CS: GCD, merge-sort, Fibonacci trees, PRN,...
- Mathematics
  - Fractals
  - Proof by induction

Recursive Algorithms

The Three “Laws” of Recursion

- A recursive algorithm must have a *base case –rule–*.
- A recursive algorithm must change its *state* and “move” toward the base case – **converge**.
- A recursive algorithm must call itself, *recursively*.

Recursion vs Iteration

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Let's go back to our factorial example,

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```python
def factorialIteractive(num):
  '''Function to compute factorial
      using the iterative product
      implementation'''

  fact = 1

  # iteration to compute product
  # i.e. n! = 1*2*3*...*(n-1)*n
  for n in range(2, num + 1):
    fact = fact * n

  return fact
```

Recursion vs Iteration: Performance

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```
factorialRecursive(3)           # 1st call with 3
3 * factorialRecursive(2)       # 2nd call with 2
3 * 2 * factorialRecursive(1) # 3rd call with 1
3 * 2 * 1           # return from 3rd call as num=1
3 * 2               # return from 2nd call
6                   # return from 1st call
```

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3 * 2              # return from 2nd call
6                  # return from 1st call
```

Implem.	Fn.calls	cum.Time
Iterative	17	26 msec
Recursive	10119	40 msec
math.factorial	17	28 msec
numpy	12682	262 msec

Recursion vs Iteration: Stack

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Stack Function Calls

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Stack Function Calls

A *stack* is a data structure that operates on a **Last In/First Out** (LIFO) basis.

An item is "pushed" onto a stack to add to it, and an item is "popped" off the stack to remove it.

Using a stack is a method of ordering certain operations for execution.

Recursion vs Iteration: Stack

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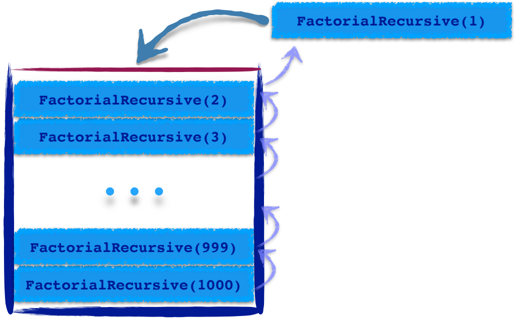
Using a stack is a method of ordering certain operations for execution.

A recursive function, calls itself until a "base condition" is met and then its execution stops.
While the condition is not met, we will keep placing *execution contexts* on top of the stack.
In some cases, this could lead to a **stack overflow**: *run out of "space" to hold items in the stack.*



Python specific limits on Recursion

- Every recursive function must have a base condition that stops the recursion or else the function calls itself infinitely.
- The Python interpreter limits the depths of recursion to help avoid infinite recursions, resulting in stack overflows.
- By default, the maximum depth of recursion is 1000. If the limit is crossed, it results in **RecursionError**.

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```
>>> from factorials import *

>>> factorialRecursive(3)
6

>>> factorialRecursive(999)
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
  File code/factorials.py", line 28, in factorialRecursive
    return num * factorialRecursive(num-1)
   ...
RecursionError: maximum recursion depth exceeded in comparison

>>> factorialRecursive(998)
4027900501272209945382406745976015873066815457564711036474447...
```

For more details, see the GitHub repository for this lecture.

Quick Sort

- It's an in-place sorting algorithm with worst-case time complexity $\sim \mathcal{O}(n^2)$
- It's a *divide-and-conquer* algorithm
- It works by selecting a **'pivot'** element from the array and partitioning the other elements into two sub-arrays, according to whether they are less than or greater than the pivot. The sub-arrays are then sorted *recursively*.

```
('Original Array: ', [4, 2, 7, 3, 1, 6])
([1, 2, 3], 4, [7, 6])
([], 1, [2, 3])
([], 2, [3])
([6], 7, [])
('Sorted Array: ', [1, 2, 3, 4, 6, 7])
```

```python
def QuickSort(arr, verbose=False):

    elements = len(arr)

    if elements < 2:  # Base case
        return arr

    current_position = 0  #Position of the partitioning element

    for i in range(1, elements):  #Partitioning loop
        if arr[i] <= arr[0]:
            current_position += 1
            temp = arr[i]
            arr[i] = arr[current_position]
            arr[current_position] = temp

    temp = arr[0]
    arr[0] = arr[current_position]
    arr[current_position] = temp  #Brings pivot to it's appropriate position

    if verbose:
        print(arr[0:current_position],arr[current_position],
              arr[current_position+1:elements])

    left = QuickSort(arr[0:current_position],verbose) #Sorts the elements to the l
    right = QuickSort(arr[current_position+1:elements],verbose) #Sorts the element

    arr = left + [arr[current_position]] + right #Merging everything together

    return arr
```

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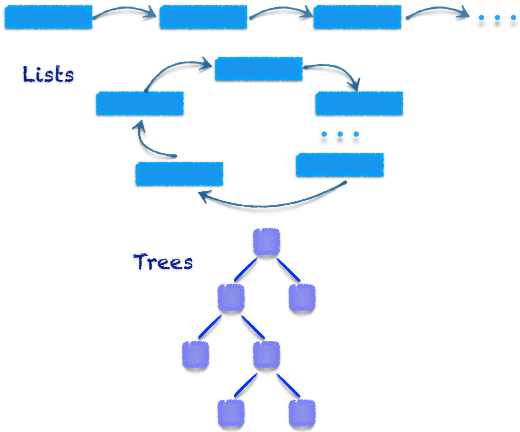

For instance,

- Lists, sets, dictionaries
- Nested lists
- Trees
- File-system: folders contain other folders can contain other folders, until finally at the bottom of the recursion are plain (non-folder) files.

Recursive lists

```
def addList(lst, newElement, head=True):
  """function to add elements to a list
      recursively"""
  # elements can be added at the
      beginning or the end of the list
  if head:
    lst = [newElement] + lst
  else:
    lst = lst + [newElement]
  return lst

def nestedList(lst, newElement):
  """function to generate nested lists of
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```python
mylist=[0]
mynestedlist=[0]
for i in range(1,7):
  # create a list by appending elements
  mylist = addList(mylist,i)
  # create a nested list
  mynestedlist = nestedList(
      mynestedlist,i)

# display resulting lists
print("incremental list: ", mylist)
print("Nested list: ", mynestedlist)

# combine list
print("append list+nested list: ",
    addList(mylist,mynestedlist))
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```

```
incremental list:  [6, 5, 4, 3, 2, 1, 0]
Nested list:  [[[[[[[0], 1], 2], 3], 4], 5], 6]
append list+nested list:  [[[[[[[[0], 1], 2], 3], 4], 5], 6], 6, 5, 4, 3, 2, 1, 0]
nest list+nested list:  [[6, 5, 4, 3, 2, 1, 0], [[[[[[[0], 1], 2], 3], 4], 5], 6]]
```

Trees

```
# binary tree
class bTree:
    def __init__(self):
        self.left = None
        self.right = None
        self.data = None

# example
root = bTree()
root.data = "root"
root.left = bTree()
root.left.data = "left"
root.right = bTree()
root.right.data = "right"
```

```
# arbitrary childs
class aTree:
    def __init__(self, data):
        self.children = []
        self.data = data

# example
left = aTree("left")
middle = aTree("middle")
right = aTree("right")
root = aTree("root")
root.children = [left, middle, right]
```

```
# tree using dictionaries
tree = {
    "a": ["b", "c"],
    "b": ["d", "e"],
    "c": [None, "f"],
    "d": [None, None],
    "e": [None, None],
    "f": [None, None],
}
```

Closing Remarks

Reentrant Code

- Recursion is a special case of a more general situation known as *reentrancy*.
- Reentrant code can be safely re-entered, meaning that it can be *called again* even while a call to it is underway.
- Direct recursion is one way that reentrancy can happen.
- Mutual recursion between two or more functions is another way this can happen – A calls B, which calls A again...
- Direct mutual recursion is virtually always intentional and designed by the programmer. But unexpected mutual recursion can lead to bugs.
- Reentrancy is also related to *concurrency*: in a concurrent program, a method may be called at the same time by different parts of the program that are running concurrently.

Summary

Recursion

- Elegant and clean programming approach
- Powerful due to its simplicity and direct "translation"
- In some cases it can be a direct implementation from the problem
- Recursion $\leftrightarrow$ Iteration
- Be aware of corner/edge cases ("base rule") or infinite recursions, sometimes logic can be complex
  - implement test cases
- Be aware of limitations and pitfalls (i.e. performance and stack utilization)
- Relevant also in data structures

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Q&A

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What does the following function do?

```
def to_string(n, b, convTab='0123456789abcdef'):
    # base case
    if n < b:
        return convTab[n]

    # // -> integer division ; % -> reminder of the division
    return to_string(n // b, b, convTab) + convTab[n % b]
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```python
print(to_string(1453, 16))  # => 5Ad
```