3.5 - Implicit differentiation (an application of the chain rule)

Recall that a **function** is a rule that assigns a unique output to each input.

Consider the following:

$$3x^2 + y - 2 = 0$$

and

$$y = 2 - 3x^2$$

Notice anything? Are the equations related?

They're the same equation

Sometimes the rule is *direct* or **explicit**

$$y = 2 - 3x^2$$ looks like a function

Sometimes the rule is *indirect* or **implicit**:

$$3x^2 + y - 2 = 0$$ doesn't "look" like a function

Implicit differentiation

When the rule is **explicit**, we take the derivative in the *usual* way...

$$y = 2 - 3x^2$$

$$y' = -6x$$

When the rule is **implicit**, keeping in mind that the equation does define y as a function of x, we can take the derivative of both sides...

$$\frac{d}{dx}\left(3x^2 + \underline{y(x)} - 2\right) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(3x^2)\ \frac{d}{dx}(y(x))\ \frac{d}{dx}(-2) = \frac{d}{dx}(0)$$

$$6x + y' = 0$$

$$y' = -6x$$

This process is called **implicit differentiation** and it is useful in situations where we have an equation that defines y as a function of x but it's not "convenient" or *sometimes* even *possible* to solve for y in terms of x.

Example: This equation

$$x^2 + y^2 = 25$$

describes the circle of radius 5 centered at the origin $(0, 0)$.

$x^2+y^2=25$

$y^2=25-x^2$

$y=\sqrt{25-x^2}$

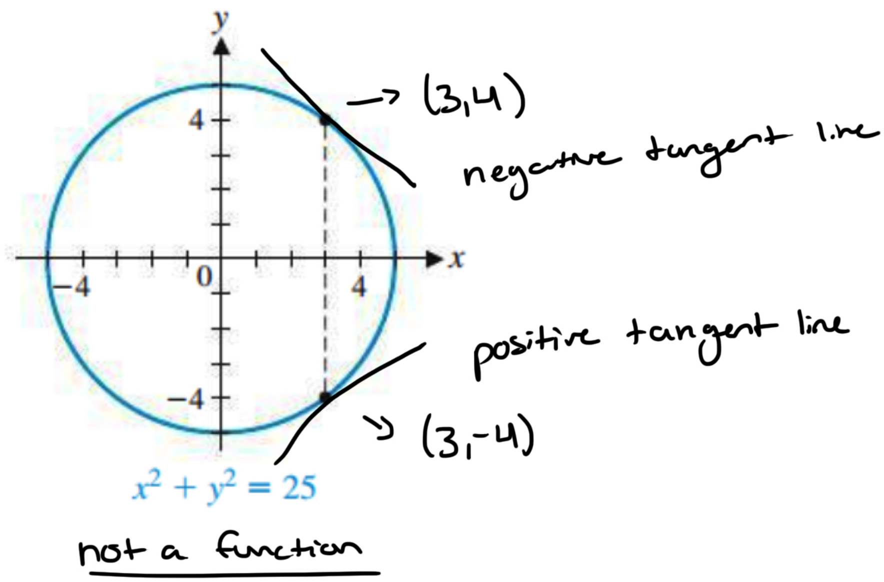
$y=\pm 5-x$

upper tangent line:

$y=5-x$

lower-tangent line:

$y=-5-x$



Goal: Find the slope of the graph when $x = 3$.

First, there are actually two such points and for one the slope will be negative and for the other it will be positive!

$x^2+y^2=25$

$3^2+y^2=25$

$9+y^2=25$

$y^2=16$

$y=\pm\sqrt{16}$

$y=\pm 4$

coordinates: $(3,-4)$
$(3,4)$

To find the slope when $x = 3$, we could go the **explicit route**...

moving y to one side

$$\begin{cases} y = \sqrt{25-x^2} \\ y = -\sqrt{25-x^2} \end{cases}$$

Or we could go the **implicit route**...

$$x^2+y^2=25$$

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25)$$

use the chain rule

$$2x+2y\cdot y'=0$$

$$2y\cdot y'=-2x$$

$$y'=\frac{-2x}{2y}$$

$$\boxed{y'=-\frac{x}{y}}$$

Goal: Find the slope when $x=3$

$$y'\Big|_{(3,4)}=-\frac{3}{4}$$

$$y'\Big|_{(3,-4)}=-\frac{3}{(-4)}=\frac{3}{4}$$

Example: Find the slope of the graph of

$$y - xy^2 + x^2 + 1 = 0$$

when $x = 1$.

$$\frac{d}{dx}(y - xy^2 + x^2 + 1) = \frac{d}{dx}(0)$$

$$y' - (x(2y \cdot y') + 1(y^2)) + 2x = 0$$

Solve for y' $\longrightarrow$

$$y' - (x(2y \cdot y') + y^2) + 2x = 0$$

$$y' - 2xy(y') - y^2 + 2x = 0$$

$$y' - 2xy(y') = y^2 - 2x$$

$$\frac{y'(1-2xy)}{1-2xy} = \frac{y^2-2x}{1-2xy}$$

$$\boxed{y' = \frac{y^2 - 2x}{1 - 2xy}}$$

$x = 1$

$$y - xy^2 + x^2 + 1 = 0$$

$$y - (1)y^2 + (1)^2 + 1 = 0$$

$$y - y^2 + 2 = 0$$ FACTOR

$$y^2 - y - 2$$

$$0 = (y-2)(y+1)$$

$y = 2 \quad y = -1$

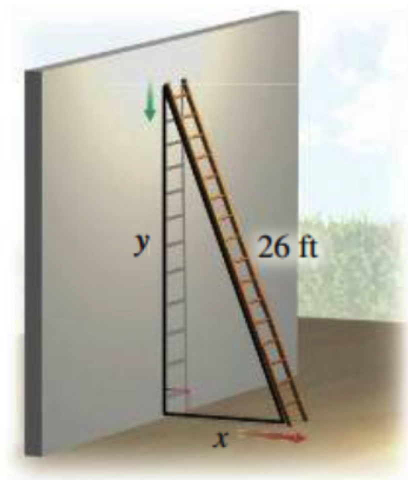
$(1,-1)$ or $(1,2)$

$$y' = \frac{(-1)^2 - 2(1)}{1 - 2(1)(-1)} \qquad y' = \frac{(2)^2 - 2(1)}{1 - 2(1)(2)}$$

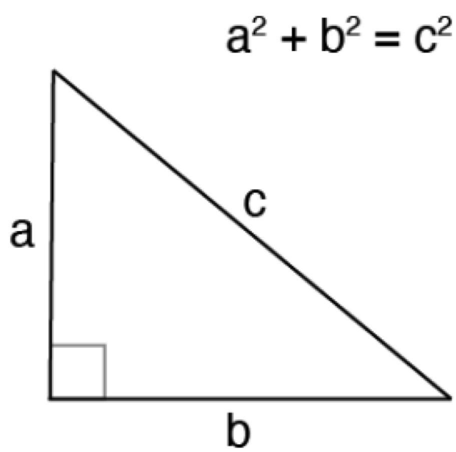
$$\boxed{y'\Big|_{(1,-1)} = -\frac{1}{3} \qquad y'\Big|_{(1,2)} = -\frac{2}{3}}$$

An application of implicit differentiation: Related rates (3.6)

Example: A 26-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?



Recall: The figure forms a right triangle so we can use the Pythagorean theorem!



So $x^2 + y^2 = 26^2$.

Key point: Think carefully about the problem description. What's really behind the changes described? Also, think about the rate given, "2 feet *per second*."

So we're thinking of x and y in terms of time t and we want to find dx/dt when $x = 10$ and $dy/dt = -2$ (Why -2?)

$$x^2 + y^2 = 26^2$$

Example: Suppose that two motorboats leave from the same point at the same time. If one travels north at 15 miles per hour and the other travels east at 20 miles per hour, how fast will the distance between them be changing after 2 hours

Example: Suppose that for a company manufacturing flash drives, the cost, revenue, and profit equations are given by

$$C = 5000 + 2x$$

$$R = 10000x - 0.001x^2$$

where the production output in 1 week is x flash drives. If production is increasing at the rate of 500 flash drives per week when production is 2000 flash drives, find the rate of increase in cost and the rate of increase in revenue.