Steps for constructing a sign chart

Constructing a sign chart for f

1. Find all partition numbers for f
2. Plot the partition numbers on the number line
3. Pick a test number in each interval (between partition numbers) and determine the sign for that interval. Do this for all such intervals.

Example: Construct a sign chart for $f(x) = \dfrac{x-2}{x^2-2x-3}$.

$$f(x) = \frac{(x-2)}{(x-3)(x+1)}$$

$x-3=0 \quad x+1=0$

$x=3 \quad x=-1$

partition numbers:

$x-3=0$, $x=3$ $\qquad$ $x+1=0$, $x=-1$ $\qquad$ $x-2=0$, $x=2$

$-\quad + \quad - \quad +$

$-1 \quad 2 \quad 3$

$$f(x) = \frac{x-2}{(x+1)(x-3)}$$

$$f(-10) = \frac{-10-2}{(-10+1)(-10-3)} = \frac{-12}{(-11)(-13)}$$ negative

$$f(0) = \frac{0-2}{(0+1)(0-3)} = \frac{-2}{(1)(-3)}$$ positive

$$f(2.5) = \frac{2.5-2}{(2.5+1)(2.5-3)} = \frac{0.5}{(3.5)(-0.5)}$$ negative

$$f(4) = \frac{4-2}{(4+1)(4-3)} = \frac{2}{(5)(1)}$$ positive

Example: Construct a sign chart for $f(x) = 2x - 4$. $= 0$

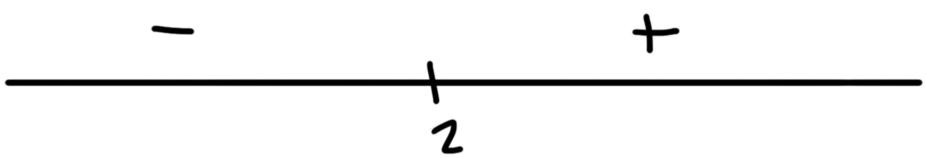
Partition Numbers

$2x-4=0$

$x=2$

No numbers of discontinuity

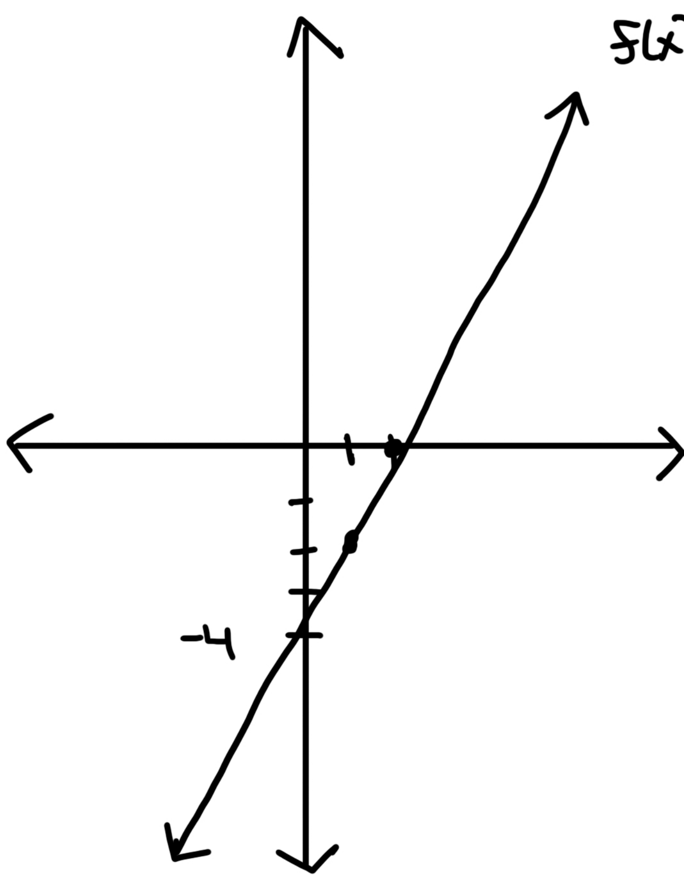
$x=2$



$f(0)=2(0)-4$ negative

$f(3)=2(3)-4=6-4$ positive

$f(x)=2x-4$ $m=\frac{2}{1}$



Section 2.4 - The derivative

As mentioned previously, the concept of limits allowed mathematicians to study and answer two major questions:

1. If an object is traveling along a path, what is its rate of change (i.e. velocity) *at a specific moment in time*?
2. How to we find the "slope" of a graph at a given point?

We'll look at both ideas carefully...

Rate of change

$$\text{distance} \;=\; \text{rate} \;\times\; \text{time}$$

$$\text{rate} \;=\; \frac{\text{distance}}{\text{time}}$$

Example: If you pass mile marker 120 at 8am and mile marker 300 at 11am, then you travelled 180 miles in 3 hours or 180 miles/3 hr = 60 miles per hour.

Of course, this is only your *average* rate of change over the 3 hours of travel.

$\Longrightarrow$ It's likely the case that you went faster than 60 mph at times and slower than 60 mph at times. $\Longleftarrow$

The more useful question: *How do we calculate your rate of change at any given moment of time?*

Formula for the average rate of change

For $y = f(x)$, the **average rate of change from** $x = a$ **to** $x = a + h$ is

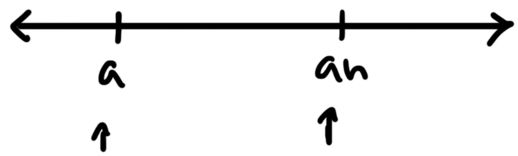
$$\underbrace{f(a+h) - f(a)}_{\text{Distance traveled}} \Big/ \underbrace{(a+h) - a}_{\text{time elapsed}}$$

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}, h \neq 0$$

Distance traveled

time elapsed

$R = \frac{D}{T}$

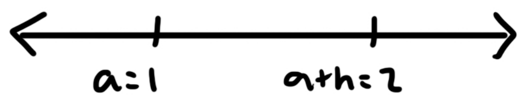


Example: A small steel ball dropped from a tower will fall a distance of y feet in x seconds where

$$y = f(x) = 16x^2$$

(a) Find the average velocity from $x = 1$ seconds to $x = 2$ seconds.

$$\frac{f(2)-f(1)}{1} = \frac{64-16}{1} = 48 \text{ ft/s}$$



(b) Find the average velocity from $x = 2$ seconds to $x = 3$ seconds.

$$\frac{f(3)-f(2)}{1} = \frac{144-14}{1} = 80 \text{ ft/s}$$

(c) Find the average velocity from $x = 2$ seconds to $x = 2 + h$ seconds.

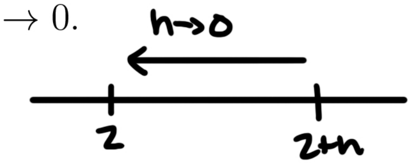
$$\frac{f(2+h)-f(2)}{h} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16((2+h)^2 - 2^2)}{h}$$

$$\frac{16(h^2+4h\cancel{+4-4})}{h} = \frac{16(4h+h^2)}{h} = \frac{16\cancel{h}(4+h)}{\cancel{h}} = 16(4+h)$$

$$(2+h)(2+h) = 4+2h+2h+h^2 = h^2+4h+4$$

$$\boxed{\text{Average velocity from 2 seconds to } 2+h \text{ seconds is } 64+16h \text{ ft/s}}$$

(d) Find the limit of your expression in (c) as $h \to 0$.



64+16h ft/s

$\lim_{h\to 0} f(x) = 64+16h = 64+16(0)$

$\boxed{\lim_{h\to 0} f(x) = 64 \text{ ft/s}}$

(e) What does that limit represent? How can you interpret that limit?

$\lim_{h\to 0} 64+16h$ = the rate of change of the object at x=2 seconds; the instantaneous rate of change.

For $y = f(x)$, the **instantaneous** **rate of change at** $x = a$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Slope of a graph

The graph of a linear function is a straight line and the slope of that line gives us a lot of useful information about that line.

Example: Sketch a graph of $f(x) = \frac{1}{2}x + 4$.

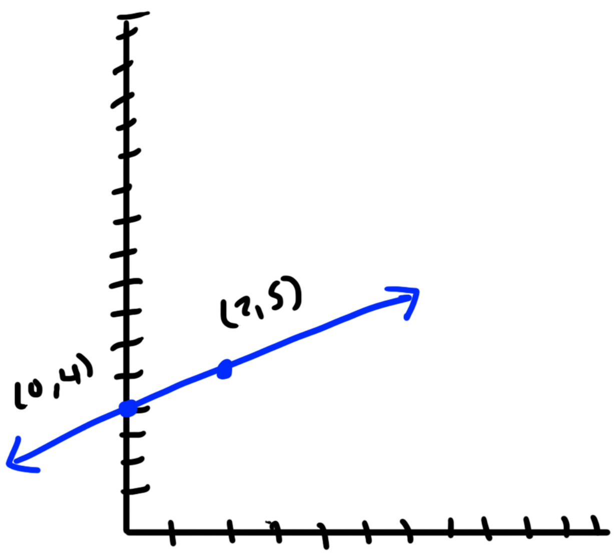
POI

(0,4)

(2,5)

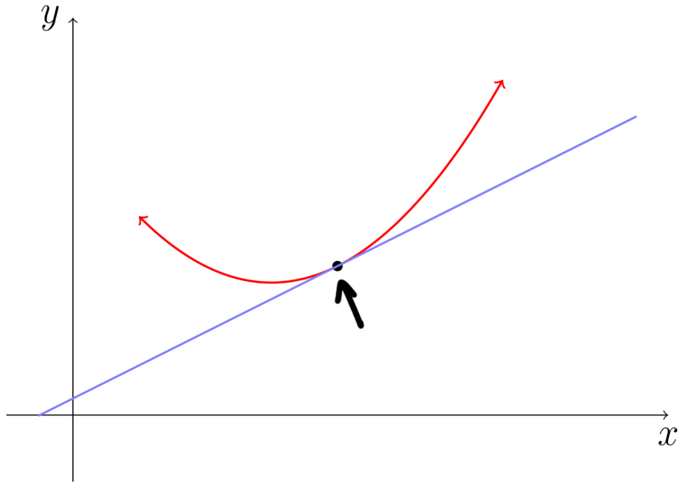
$f(0) = \frac{1}{2}(0) + 4 = 4 \quad (0,4)$

$f(2) = \frac{1}{2}(2) + 4 = 1 + 4 = 5 \quad (2,5)$



Capitalizing on that idea: Tangent lines

It turns out that if we zoom in on the graph of a given function, the graph will appear to be *close* to being a straight line. Mathematicians capitalized on this idea by focusing on the idea of a *tangent line*; i.e. the line *that "touches" the graph at a given point*. [In other words, the tangent line and the zoomed-in graph are very *close* to each other.]

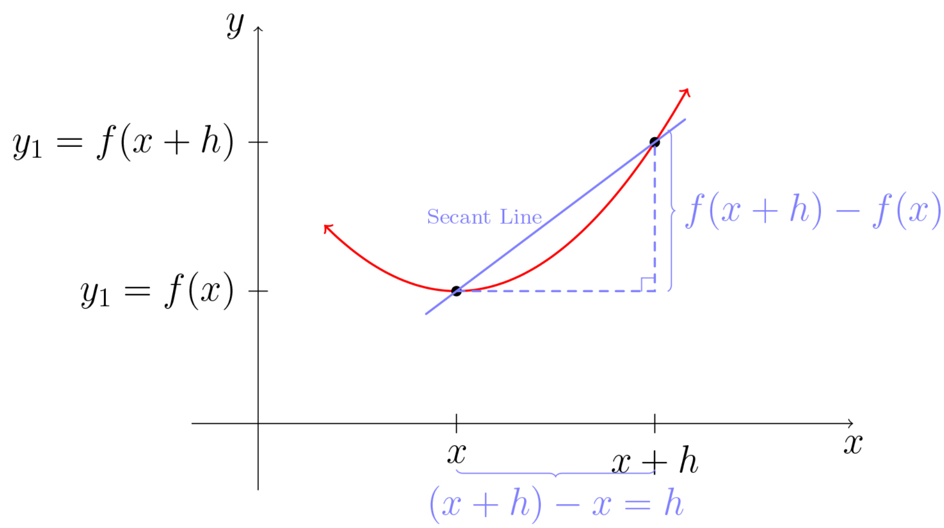


Definition: The **slope of the graph at that point** is the slope of the tangent line at that point.

Question: How do we find the slope of the tangent line?

Answer: It turns out that the process is very similar to the last example (involving instantaneous rate of change).

Recall: To calculate the slope of a line we need *two points*.



The slope of the secant line is

$$m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

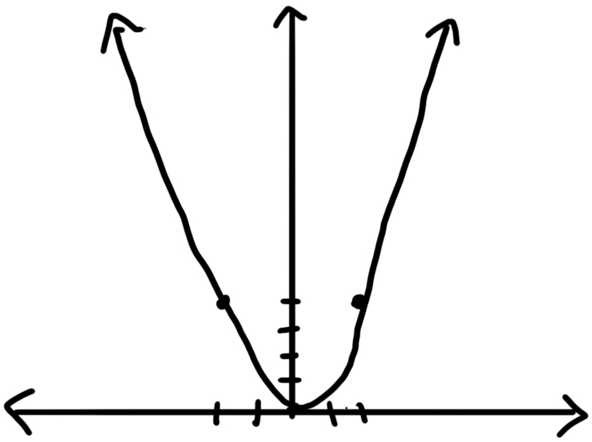
and the slope of the tangent line is

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: What is the slope of the line tangent to the graph of $f(x) = x^2$ at the point $(2, 4)$?

$$\frac{f(x+h)-f(x)}{h}$$

$$\underset{x \to 0}{m} \frac{f(x+h)-f(x)}{h}$$



Slope of tangent line $f(x)=x^2 = x=4$

$$m = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2-2^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4}+4h+h^2-\cancel{4}}{h}$$

$$= \lim_{h \to 0} \frac{4h+h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (4+h) = 4$$

The derivative

For $y = f(x)$, we define the **derivative of f at x**, denoted $f'(x)$, by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ if the limit exists}$$

Interpretations of the derivative
The derivative of a function f is a new function f'.

1. *Slope of the tangent line* - $f'(x)$ is the slope of the line tangent to the graph of f at the point $(x, f(x))$.
2. *Instantaneous rate of change* - $f'(x)$ is the instantaneous rate of change of $y = f(x)$ with respect to x.
3. *Velocity* - If $f(x)$ is the position of a moving object at time x, then $v = f'(x)$ is the velocity of the object at that time.

Example: Find $f'(x)$, for $f(x) = 3x - x^2$.

$$f'(x) = \lim_{h \to 0} = \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h) - (x+h)^2 - (3x - x^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x} + 3h - \cancel{x^2} - 2xh - h^2 - \cancel{3x} + \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{3h - 2xh - h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(3 - 2x - h)}{\cancel{h}} = \lim_{h \to 0} (3 - 2x - h)$$

$$\boxed{= 3 - 2x}$$

Example: Find the slope of the graph of $f(x) = 3x - x^2$ at $x = 0, x = 1$, and $x = 3$.