Chapter 5 - Integration

There are two major Calculus questions that we cover in this chapter:

- One big question is, "Can we calculate the area of a region defined by the graph of a function?"

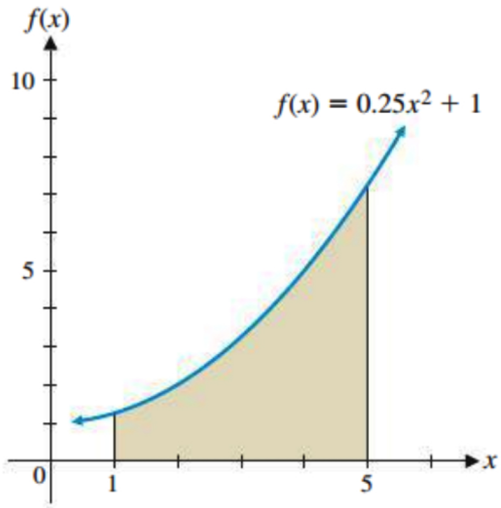

Figure 1 What is the shaded area?

- Also, in math, we like to "reverse" or "undo" processes and another big question is, "Is a given function the *derivative* of another function?"

$$f(x) = x^2 \text{ is the derivative of } F(x) = \frac{1}{3}x^3$$

It turns out that the two questions are related!

Antiderivatives

A function F is an **antiderivative** of a function f if

$$F'(x) = f(x)$$

Example: $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.

$F'(x) = x^2$ ✓

$= x^2 + 3x + 2$

Example: Is $(x+1)(x+2)$ an antiderivative of $f(x) = 2x + 3$?

$\frac{d}{dx}\left((x+1)(x+2)\right) = 1 \cdot (x+2) + (x+1) \cdot 1 = 2x + 3$

Example: How many antiderivatives does a function have?

(a) $F(x) = \frac{1}{2}x^2$ is an antiderivative of what function?

$F'(x) = f(x) = x$

(b) $F(x) = \frac{1}{2}x^2 + 5$ is an antiderivative of what function?

$F'(x) = f(x) = x$

(c) $F(x) = \frac{1}{2}x^2 - 2$ is an antiderivative of what function?

$F'(x) = f(x) = x$

(d) $F(x) = \frac{1}{2}x^2 + C$ (*where C is any constant*) is an antiderivative of what function?

$F'(x) = f(x) = x$

Theorem 1 - Antiderivatives If the derivatives of two functions are equal on an open interval (a, b), then the functions differ by at most a constant.

Symbolically, if F and G are differentiable functions on the interval (a, b) and $F'(x) = G'(x)$ for all x in (a, b), then $F(x) = G(x) + k$ for some constant k.

Conclusion: If you know one antiderivative, *you know them all!*

Example: Find all antiderivatives of $f(x) = 3x^2$

$F(x) = x^3 + C$

$F'(x) = 3x^2$

Example: Find all antiderivatives of $f(x) = x^3$

$F(x) = \frac{1}{4}x^4 + C$

$F'(x) = x^3$

Example: Find all antiderivatives of $f(x) = x^n$ for any $n \neq -1$.

$$F(x) = \frac{1}{n+1} x^{n+1} + C$$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

Example: What if $n = -1$?

$$f(x) = x^{-1} = \frac{1}{x}$$

$$F(x) = \ln|x| + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

Example: Find all antiderivatives for $f(x) = e^x$.

$$F(x) = e^x + C$$

Indefinite Integrals

The symbol

$$\int f(x)dx$$

is used to represent the family of all antiderivatives of $f(x)$, and we write

$$\int f(x)dx = F(x) + C \quad \text{if } F'(x) = f(x)$$

$\int$ is called the **integral sign**

$f(x)$ is called the **integrand**

dx indicates that the antidifferentiation is done *with respect to* x.

C is called the **constant of integration.**

Examples:

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

$$\int x^3 dx = \frac{1}{4}x^4 + C$$

$$\int x^4 dx = \frac{1}{5}x^5 + C$$

$$\int x^n dx = \frac{1}{n} x^{n+1} + C$$

$$\int e^x dx = e^x + C$$

Formulas for finding antiderivatives

1. $\displaystyle\int x^n dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$

2. $\displaystyle\int e^x dx = e^x + C$

3. $\displaystyle\int \frac{1}{x} dx = \ln|x| + C, x \neq 0$

4. For k a constant, $\displaystyle\int kf(x)dx = k\int f(x)dx$ - pull the constant out and take the derivative

5. $\displaystyle\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$ - term by term

Find each indefinite integral

Example: $\int 5dx$

$$\int 5dx = 5\int 1\cdot dx = 5\int x^0 dx = 5x + C$$

$$\frac{d}{dx} = 5 \checkmark$$

Example: $\int 5xdx$

$$\int 5xdx = 5\int x^1 dx = 5\left(\frac{1}{2}x^2\right) + C = \frac{5}{2}x^2 + C$$

$$\frac{d}{dx} = 5x \checkmark$$

Example: $\int 5x^2 dx$

$$\int 5x^2 dx = 5\int x^2 dx = 5\left(\frac{1}{3}x^3\right) + C = \frac{5}{3}x^3 + C$$

$$\frac{d}{dx} = 5x^2 \checkmark$$

Example: $\int \frac{1}{3}e^x dx$

$$\int \frac{1}{3}e^x dx = \frac{1}{3}\int e^x dx = \frac{1}{3}e^x + C$$

$$\frac{d}{dx} = \frac{1}{3}e^x \checkmark$$

Example: $\int \frac{5}{x} dx$

$$\int \frac{5}{x} dx = 5\int \frac{1}{x} dx = 5\ln|x| + C$$

Other antiderivative rules...

We can *reverse* any of our derivative rules to create an antiderivative rule.

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$$

$$\implies \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1$$

$$\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$$

$$\implies \int e^{f(x)}f'(x)dx = e^{f(x)} + C$$

$$\frac{d}{dx}\ln f(x) = \frac{1}{f(x)}f'(x)$$

$$\implies \int \frac{1}{f(x)}f'(x)dx = \ln|f(x)| + C$$

Summary:

1. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1$
2. $\int e^{f(x)}f'(x)dx = e^{f(x)} + C$
3. $\int \frac{1}{f(x)}f'(x)dx = \ln|f(x)| + C$

Find each indefinite integral

Example: $\int 2(1+2x)^5 dx = \frac{(1+2x)^6}{6} + C$

$\frac{d}{dx}(1+2x) = 2$

Example: $\int 5e^{5x} dx = e^{5x} + C$

$\frac{d}{dx} 5x = 5$

Example: $\int \frac{2x}{1+x^2} dx = \ln|1+x^2| + C$

$\frac{d}{dx}(1+x^2) = 2x$

Example: $\int x e^{x^2+1} dx$

Example: Find the equation of the curve that passes through $(2, 5)$ if the slope of the curve is given by $\frac{dy}{dx} = 2x$ at any point x.

Example: If the marginal cost of producing x units of a commodity is given by

$$C'(x) = 0.3x^2 + 2x$$

and the fixed cost is \$2,000, find the cost function $C(x)$ and the cost of producing 20 units.