From last time...

Example: For $f(x) = 6x - x^2$.

(a) Find $f'(x)$ $\boxed{6-2x}$

(b) Find the slope of the graph at $x = 2$ $6-2(2)=6-4=2$ $\boxed{m=2}$

(c) Find the slope of the graph at $x = 4$ $6-2(4)=6-8=-2$ $\boxed{m=-2}$

(d) Find the equation of the tangent line at $(2, 8)$

$y-y_1=m(x-x_1)$

$y-8=2(x-2)$

$y=2x-4+8$

$\boxed{y=2x+4}$

(e) Find the value(s) of x for which the tangent is horizontal

$\boxed{x=3}$

Where $m=0$

Set $f'(x)=0$ and solve

$f'(x)=6-2x$

$0=6-2x$

$2x=6$

$x=3$

2.6 - Increments and differentials

Difference quotients

From last section: For two points $(x, f(x))$ and $(x+h, f(x+h))$, we calculate the **difference quotient** (slope of the secant line) as

$$\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

There are also other forms of the difference quotient:

If the two points are labelled $(x, f(x))$ and $(x+\Delta x, f(x+\Delta x))$:

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Δx = change in x

If the two points are labelled $(x_1, f(x_1))$ and $(x_2, f(x_2))$:

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Recall our definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Using the other versions of the difference quotient...

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Motivation for what follows

Fact: For a line, the rise-over-run interpretation of slope is useful

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{3}{2} = \frac{\text{rise}}{\text{run}}$

$m = \frac{3}{2}, (1,-1)$

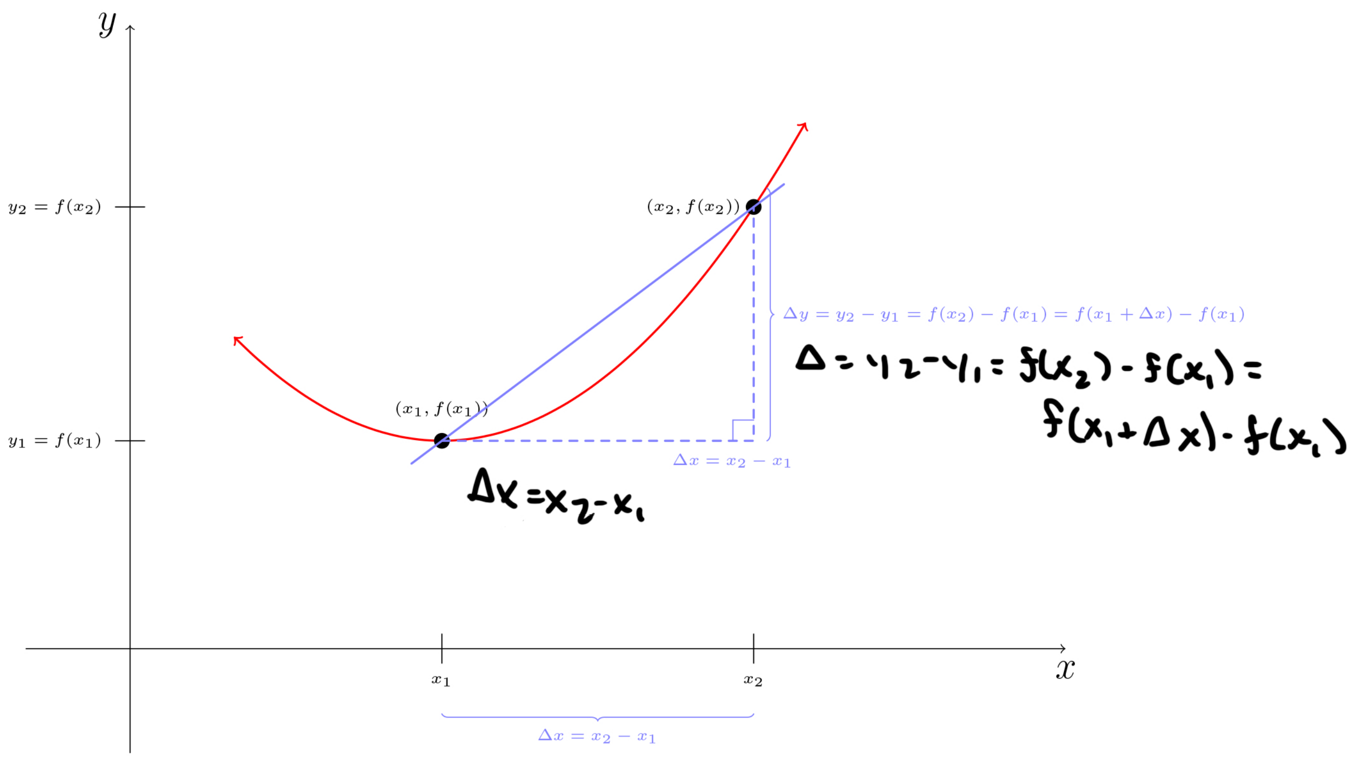
$m = \frac{\Delta y}{\Delta x}$

★ $m \Delta x = \Delta y$ ★

Fact: For a line, we can use the slope to quickly compute Δy for a given Δx.

The goal of this section is to answer the question, "Can we use this same idea for other functions?"

Increments



Δx is called the **increment** in x and denotes the *actual* change in x; i.e.

$$\Delta x = x_2 - x_1$$

Δy is called the **increment** in y and denotes the *actual* change in y that corresponds to the change in x; i.e.

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1) = f(x_1 + \Delta x) - f(x_1)$$

Key points to note:

- Δx is the *actual* change in x
- Δy is the *actual* change in y

Example: Let $f(x) = x^2 + 1$.

(A) Find Δx, Δy, and $\Delta y/\Delta x$ for $x_1 = 2$ and $x_2 = 3$.

$\Delta x = x_2 - x_1$

$\Delta x = 3 - 2$

$\Delta x = 1$

$\Delta y = f(x_2) - f(x_1)$

$\Delta y = (3^2+1) - (2^2+1)$

$\Delta y = 10 - 5$

$\Delta y = 5$

(B) Find $\dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$ for $x_1 = 1$ and $\Delta x = 2$.

$f(x) = x^2 + 1$

$\Delta x = 2$

$$\frac{((1+2)^2+1) - ((1)^2+1)}{2}$$

$$\frac{(3^2+1) - (1+1)}{2}$$

$$\frac{10-2}{2} = \frac{8}{2} = \boxed{4}$$

Differentials

Since

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Δy
change in y

$\Longrightarrow$ For small Δx the difference quotient $\Delta y/\Delta x$ is close to $f'(x)$; i.e.

$$\frac{\Delta y}{\Delta x} \approx f'(x)$$

for small Δx

$$\boxed{\Longrightarrow \Delta y \approx f'(x)\Delta x}$$

the differential

In words, we can approximate the change in y using the derivative when Δx is small.

We call this approximation the **differential** dy and

$$\boxed{dy = f'(x)dx \text{ where } dx = \Delta x}$$

Example: Find dy for $f(x) = x^2 + 3x$. Evaluate dy for $x = 2$ and $dx = 0.1$.

differential

$f'(x) = 2x + 3$

$\boxed{dy = (2x+3)\,dx}$

$\boxed{dy = 0.7}$

Change in y when $x = 2$ and $\Delta x = 0.1$

$$
\begin{aligned}
dy &= (2(2)+3)(0.1) \\
&= (4+3)(0.1) \\
&= (7)(0.1) \\
&\boxed{= 0.7}
\end{aligned}
$$

Example: Let $y = f(x) = 6x - x^2$. $x(6-x)$

(A) Find Δy and dy when $x = 2$.

$f'(x) = 6 - 2x$

$dy = f'(x)dx$

$dy = (6-2x)dx$

$dy(6-2(2))dx$

$dy(6-4)dx$

$\boxed{dy = (2)dx}$

(B) Find Δy and dy from part (A) for $\Delta x = 0.1, 0.2$, and 0.3.

$f(x) = 6x - x^2$

Δx	Δy	dy
0.1	0.19	0.2
0.2	0.36	0.4
0.3	0.51	0.6

☆ $dy = f'(x)dx$ ☆

$$\begin{cases} \Delta y = f(2.1) - f(2) \\ dy = 2(0.1) \end{cases}$$

$m = \dfrac{\Delta y}{\Delta x}$ so $m\Delta x = \Delta y$

Example: Let $y = f(x) = 6x - x^2$.

(A) Find Δy and dy when $x = 4$.

$f'(x) = 6 - 2x$

$dy = f'(x)\,dx$

$dy = (6-2x)dx$

$dy = (6-2(4))dx$

$dy = (6-8)dx$

$\boxed{dy = -2dx}$

(B) Find Δy and dy from part (A) for $\Delta x = -0.1, -0.2$, and -0.3.

$x = 4 \qquad f(x) = 6x - x^2$

Δx	Δy	dy
-0.1	0.19	0.2
-0.2	0.36	0.4
-0.3	0.51	0.6

$f(4) - f(3.9) = 0.19$

$f(4) - f(3.8) = 0.36$

$f(4) - f(3.7) = 0.51$

$dy = -2(-0.1) = 0.2$

$dy = -2(-0.2) = 0.4$

$dy = -2(-0.3) = 0.6$

$\Delta y = f(x) - f(x + \Delta x)$

$dy = -2\,\Delta x$

Example: A company manufactures and sells x microprocessors per week. If the weekly cost and revenue equations are

$$C(x) = 5,000 + 2x \text{ and } R(x) = 10x - x^2, 0 \leq x \leq 8000,$$

then use differentials to approximate the changes in revenue and profit if production is increased from $2,000$ to $2,010$ units per week.