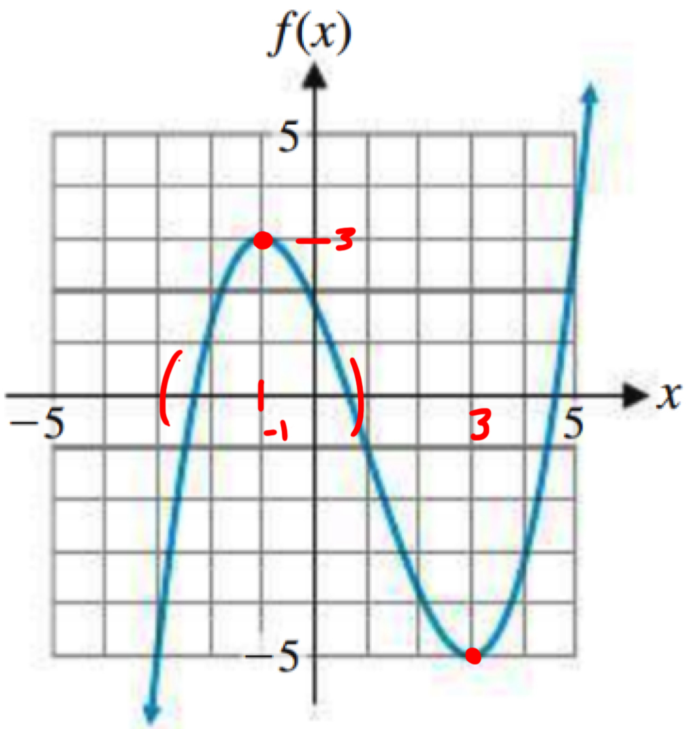
Local extrema

We say that $f(c)$ is a **local maximum** if there is an interval (a, b) containing c such that
$$f(x) \leq f(c)$$
for all x in (a, b).

We say that $f(c)$ is a **local minimum** if there is an interval (a, b) containing c such that
$$f(x) \geq f(c)$$
for all x in (a, b).

Let's look at the following graph.

$f(-1) = 3$



Question: What do you notice about the graph of f "near" the local extrema? *Think about the tangent lines!*

local maximum: $(-1, 3)$

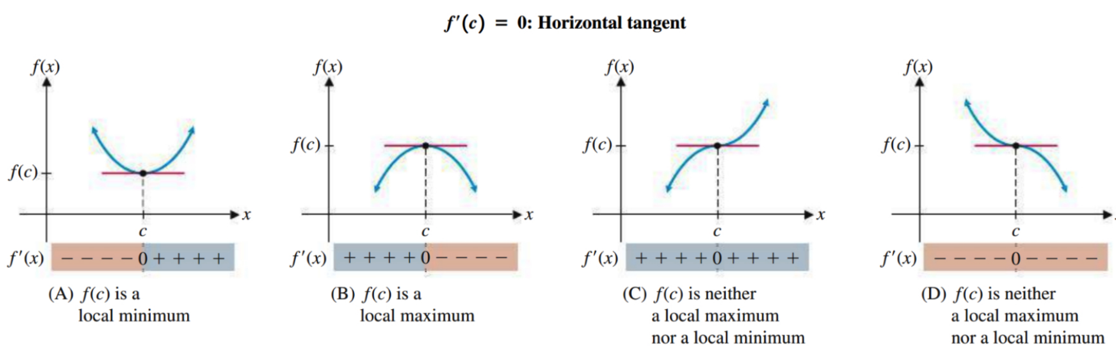
local minimum: $(3, -5)$

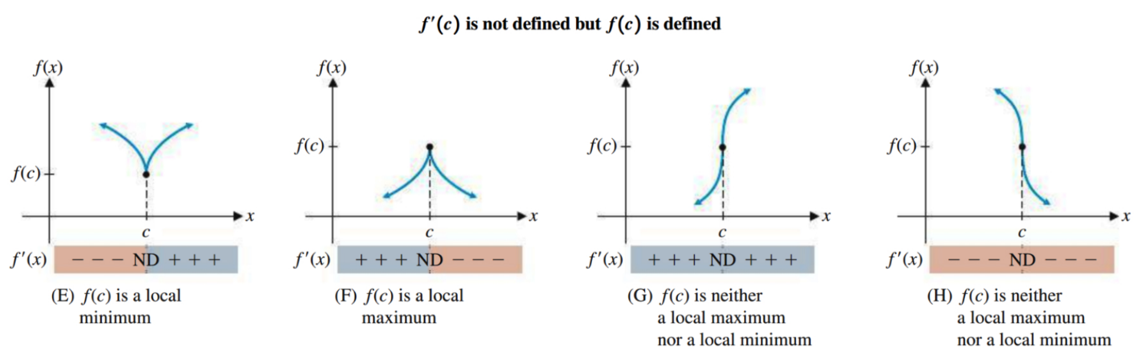
The first derivative test for local extrema

Theorem 2: If $f(c)$ is a local extremum of f, then c is a critical number of f.

First derivative test for local extrema: Let c be a critical number of f and construct a sign chart for f' around c.

- If the sign of f' changes from $-$ to $+$ at c, $f(c)$ is a *local minimum.*
- If the sign of f' changes from $+$ to $-$ at c, $f(c)$ is a *local maximum.*
- If the sign of f' does not change at c, then $f(c)$ is neither a local maximum nor local minimum.





Example: Let $f(x) = x^3 - 6x^2 + 9x + 1$

(a) Find the critical numbers of f. $x = 1, 3$

(b) Find the local extrema for f.

(c) Sketch the graph of f.

$f'(x) = 3x^2 - 12x + 9$

$= 3(x-3)(x-1)$

$x = 3$

$x = 1$

$f'(0) = 3(0)^2 - 12(0) + 9$

$f'(0) = 9$

$f'(2) = 3(2)^2 - 12(2) + 9$

$= 3(4) - 24 + 9$

$= 12 - 24 + 9$

$= 21 - 24$

$= -3$

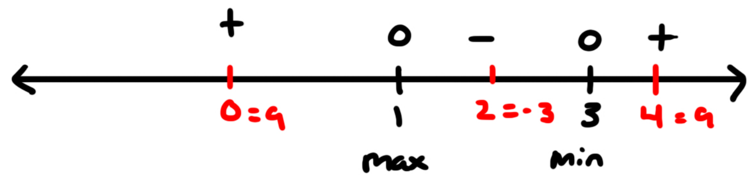
$f'(4) = 3(4)^2 - 12(4) + 9$

$= 3(16) - 48 + 9$

$= 48 - 48 + 9$

$= 9$

x-intercept: $x = 1$

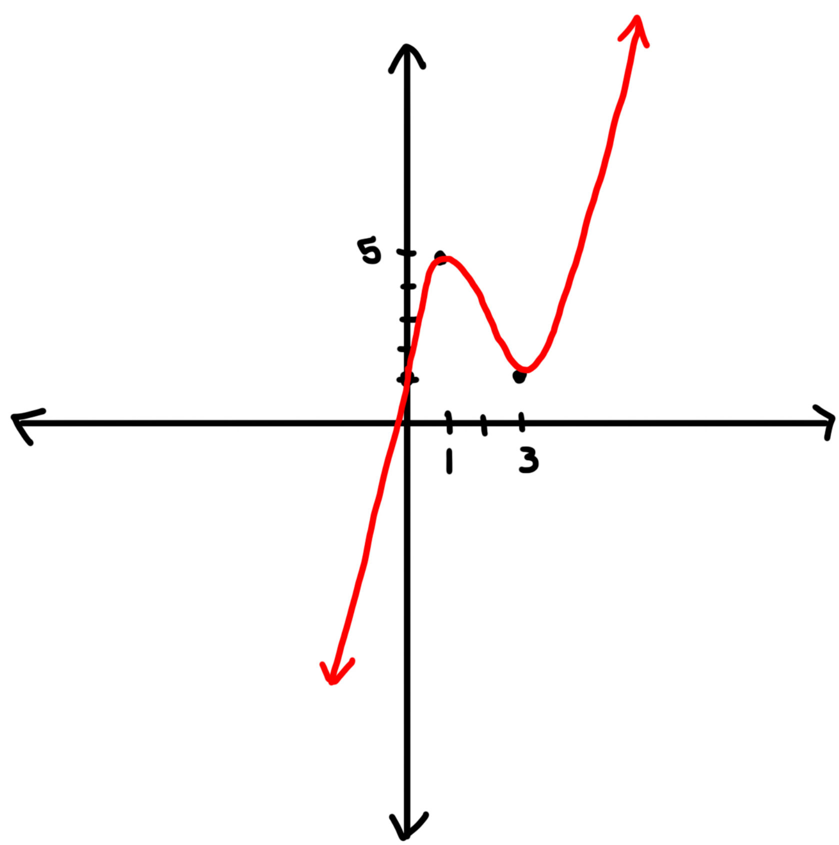


local maximum: $f(1) = 5$

local minimum: $f(3) = 1$

increasing on $(-\infty, 1)$ and $(3, \infty)$

decreasing on $(1, 3)$



Example: Let $f(x) = x^3 - 9x^2 + 24x - 10$

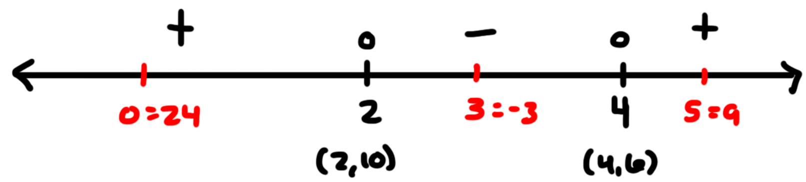
$f(2) = (2)^3 - 9(2)^2 + 24(2) - 10$
$= 10$
$f(4) = (4)^3 - 9(4)^2 + 24(4) - 10$
$= 6$

(a) Find the critical numbers of f.

(b) Find the local extrema for f.

(c) Sketch the graph of f.

$f'(x) = 3x^2 - 18x + 24$
$= 3(x-4)(x-2)$
$x = 4$
$x = 2$

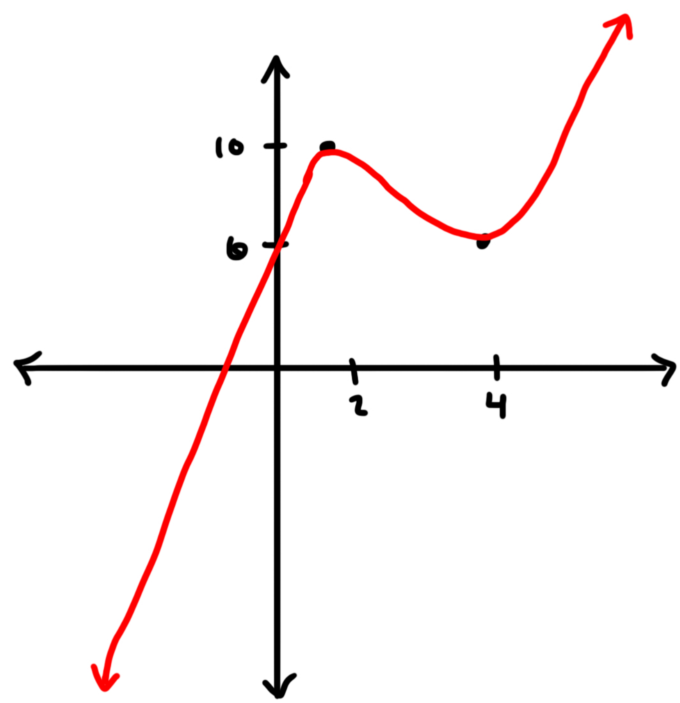


$f'(0) = 3(0)^2 - 18(0) + 24$
$= 24$

$f'(3) = 3(3)^2 - 18(3) + 24$
$= 3(9) - 54 + 24$
$= 27 - 54 + 24$
$= 51 - 54$
$= -3$

$f'(5) = 3(5)^2 - 18(5) + 24$
$= 3(25) - 90 + 24$
$= 75 - 90 + 24$
$= 99 - 90$
$= 9$

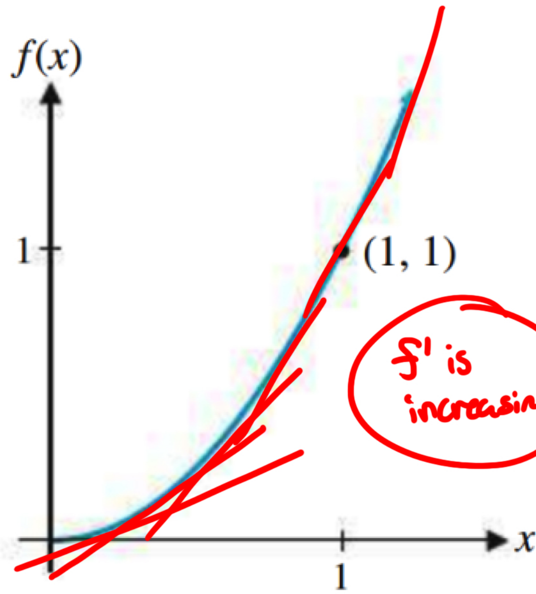
local Maximum: $f(2) = 10$
local Minimum: $f(4) = 6$
increasing on $(-\infty, 2)$ and $(4, \infty)$
decreasing on $(2, 4)$



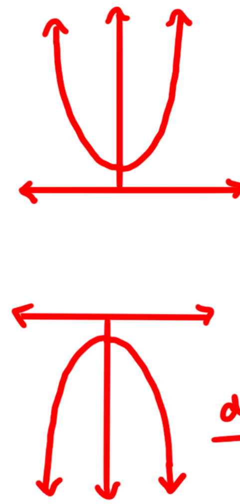
4.2 - Second derivative and graphs

We use the term *concave upward* to describe a graph that opens *upward.*

concavity



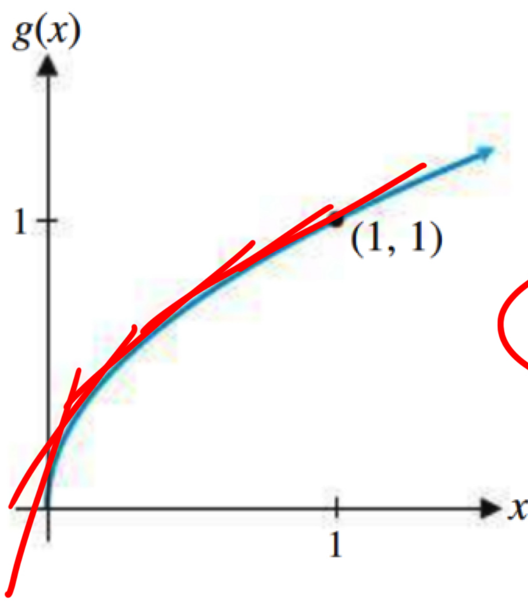
f' is increasing



concave upward

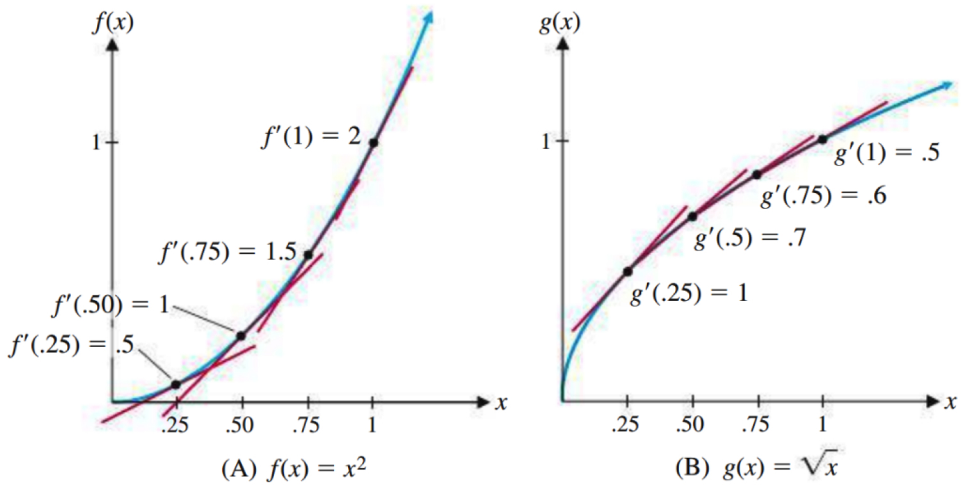
concave downward

We use the term *concave downward* to describe a graph that opens *downward.*



f' is decreasing

Definition: The graph of function f is **concave upward** on (a,b) if $f'(x)$ is *increasing* on (a,b) and is **concave downward** on (a,b) if $f'(x)$ is *decreasing* on (a,b).



Geometrically speaking, the graph is concave upward if it lies above its tangent lines and concave downward if it lies below its tangent lines.

Question: How do we determine the intervals where f' is increasing and the intervals where f' is decreasing?

Answer: We create a sign chart for the *derivative of the derivative*!

Notation: For $y = f(x)$, the **second derivative** of f, provided that it exists, is

$$f''(x) = \frac{d}{dx} f'(x)$$

Other notations:

$$\frac{d^2y}{dx^2} \text{ and } y''$$

Example: Let $f(x) = x^2$. Find $f''(x)$.

$f'(x) = 2x$

$f''(x) = 2$

Example: Let $f(x) = \sqrt{x}$. Find $f''(x)$.

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$

Simplified

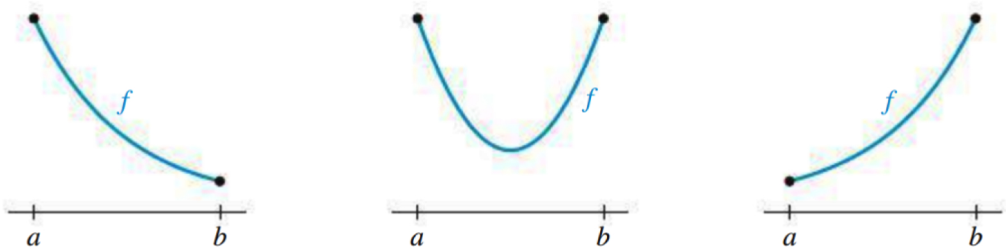
$f''(x) = -\frac{1}{4}x^{-3/2}$ or $-\frac{1}{4\sqrt{x^3}}$

Concavity

For the interval (a, b), if $f'' > 0$, f is concave upward, and if $f'' < 0$, f is concave downward.

Conclusion: We use a sign chart for f'' to determine concavity.

$f''(x) > 0$ on (a, b)
Concave upward

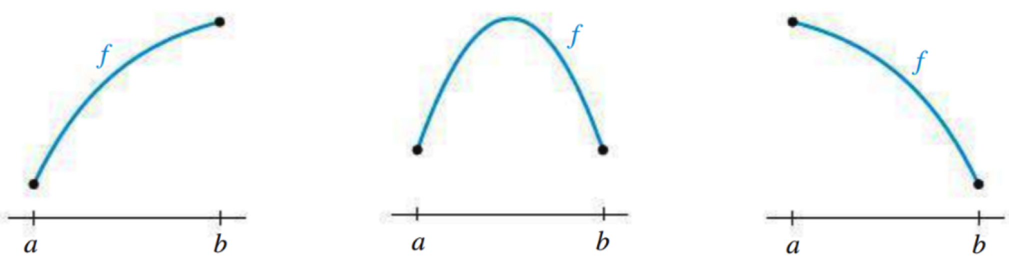


(A) $f'(x)$ is negative and increasing. Graph of f is falling.

(B) $f'(x)$ increases from negative to positive. Graph of f falls, then rises.

(C) $f'(x)$ is positive and increasing. Graph of f is rising.

$f''(x) < 0$ on (a, b)
Concave downward



(D) $f'(x)$ is positive and decreasing. Graph of f is rising.

(E) $f'(x)$ decreases from positive to negative. Graph of f rises, then falls.

(F) $f'(x)$ is negative and decreasing. Graph of f is falling.

Example: Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward.

$$f(x) = x^4 - 24x^2$$

$f'(x) = 4x^3 - 48x$

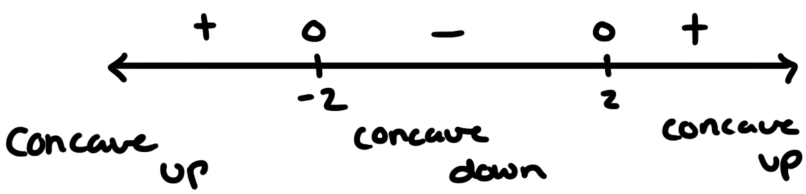
$f''(x) = 12x^2 - 48$

$0 = 12x^2 - 48$

$0 = 12(x^2 - 4)$

$0 = (x-2)(x+2)$

$x = \pm 2$

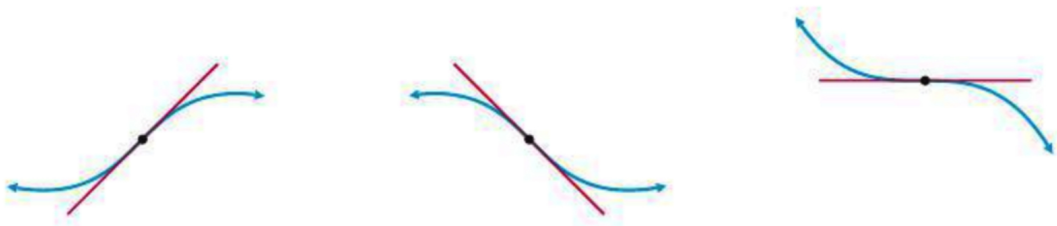


Example: Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward.

$$f(x) = x^3 - 3x^2 + 7x + 2$$

Inflection points

Definition: An **inflection point** is a *point on the graph* of a function where the function is continuous and the concavity changes.



Theorem: If $(c, f(c))$ is an inflection point of f, then c is a partition number for f''.

PROCEDURE Testing for Inflection Points

\# 1: Find all partition numbers c of f'' such that f is continuous at c.

\# 2: For each of these partition numbers c, construct a sign chart of f'' near $x = c$.

\# 3: If the sign chart of f'' changes sign at c, then $(c, f(c))$ is an inflection point of f. If the sign chart does not change sign at c, then there is no inflection point at $x = c$.

Example: Find the inflection points of

$$f(x) = x^3 - 6x^2 + 9x + 1$$

Example: Find the inflection points of

$$f(x) = x^4 - 24x^2$$

Curve sketching

We can combine all of the info that we get from f, f' and f'' to produce a pretty accurate sketch of the graph of a given function!

Curve sketching procedure:

1: *Analyze* $f(x)$. Find the domain and intercepts.

2: *Analyze* $f'(x)$. Find the partition numbers for f' and critical numbers for f. Construct a sign chart for f' and determine intervals of increase/decrease and all local extrema.

3: *Analyze* $f''(x)$. Find the partition numbers for f''. Construct a sign chart for f'' and determine intervals of concavity and all inflection points.

4: *Sketch the graph of* $f(x)$. Compile all of the above into a sketch of the graph. Plot additional points as needed.

Examples: Graph the function

$$f(x) = x^4 + 4x^3$$

Examples: Graph the function

$$f(x) = 3x^{2/3} - x$$