3.7 - The elasticity of demand

Recurring (and useful!) theme in Calculus:

The derivative provides a tool that we can use to study how a function changes; in particular, the rate at which it changes.

In business models, we know revenue, price, and demand are all related.

Changing the price of a product affects demand and revenue.

The concept of *elasticity of demand* provides a way to analyze these relationships.

Key question: Will an increase in price increase or decrease revenue?

Relative and percentage rates of change

Which is better?

could be
$2 to $12
600% increase

- I tell you that the price of an item increased by $10.
- I tell you that the price of a $100 item increased by $10.

Compared to
$100 to $110
10% increase

Relative rate of change

The **relative rate of change** of a function $f(x)$ is

$$\frac{f'(x)}{f(x)}$$

The idea is that

- The numerator is the rate of change part
- The denominator provides *context*

The **percentage (relative) rate of change** of a function $f(x)$ is

$$100 \times \frac{f'(x)}{f(x)}$$

The *percentage* rate of change provides a *standard* context; i.e. it frames the change in "terms of 100."

Example: Find the relative rate of change of $f(x) = 500 - 6x$ when $x = 40$.

$$\frac{f'(x)}{f(x)} \qquad \frac{-6}{500-6x} \qquad f'(x) = -6$$

When $x=40$

$$\frac{-6}{500-6(40)} = \boxed{-\frac{6}{260}}$$

Example: Find the relative rate of change of $f(x) = 500 - 6x$ when $x = 75$.

When $x=75$ $\qquad \frac{-6}{500-6x}$

$$\frac{-6}{500-6(75)} = \boxed{-\frac{6}{50}}$$

Elasticity of demand

In business models, we want to know how a change in price will affect demand (which, in turn, affects *revenue*).

Using the *relative* idea, economists compare the relative rate of change of demand and the relative range of change of price.

Let the price p and demand x for a product be related by a price-demand equation of the form $x = f(p)$. ← Determine demand given price

The **elasticity of demand** at price p, denoted by $E(p)$, is

$$E(p) = -\frac{\text{relative rate of change of demand}}{\text{relative rate of change of price}}$$
$$= -\frac{pf'(p)}{f(p)}$$

$$E(p) = \frac{-pf'(p)}{f(p)}$$

Example: The price p and the demand x for a product are related by the price–demand equation

$$x + 500p = 10,000$$

Find the elasticity of demand, $E(p)$, and interpret for each of the following:

(A) $E(4)$

(B) $E(16)$

(C) $E(10)$

$x = 10000 - 500p$

$E(p) = \frac{-p\,f'(x)}{f(x)}$

$x = f(p) = 10000 - 500p$

$f'(p) = -500$

$\frac{-p(-500)}{10000 - 500p}$

(A) $E(4)\ \frac{(-4)(-500)}{10000 - 500(4)} = \frac{2000}{8000} = 0.25$

(B) $E(16)\ \frac{(-16)(-500)}{10000 - 500(16)} = \frac{8000}{2000} = 4$

(C) $E(10)\ \frac{(-10)(-500)}{10000 - 500(10)} = \frac{-5000}{5000} = 1$

$$\boxed{\begin{aligned} E(4) &= 0.25 \\ E(16) &= 4 \\ E(10) &= 1 \end{aligned}}$$

To interpret, since

$$E(p) = -\frac{\text{relative rate of change of demand}}{\text{relative rate of change of price}}$$

That means

$$-(\text{relative rate of change of demand}) = E(p)(\text{relative rate of change of price})$$

In a loose sense, a "small" $E(p)$ means a change in price has little impact on the demand and a "big" $E(p)$ means that a change in price has a big impact on the demand.

For the interpretation, assume the price changes by 10%.

$E(4) = 0.25$

$-$(Relative rate of change of decimal) $= 0.25 \times 10\% = 0.025 = 2.5\%$

demand change by 2.5%

$E(16) = 4$

$E(p) \cdot (10\%) = 4 \cdot 10\% = 40\%$

demand change by 40%

$E(10) = 1$

$E(p) \cdot (10\%) = 1 \cdot 10\% = 10\%$

demand change by 10%

Summary

Note: What we really care about is *revenue*!

$E(p)$	Demand	Interpretation	Revenue
$0 < E(p) < 1$	Inelastic	Demand is not sensitive to changes in price; that is, percentage change in price produces a smaller percentage change in demand.	A price increase will increase revenue.
$E(p) > 1$	Elastic	Demand is sensitive to changes in price; that is, a percentage change in price produces a larger percentage change in demand.	A price increase will decrease revenue.
$E(p) = 1$	Unit	A percentage change in price produces the same percentage change in demand.	

In summary, if demand is inelastic, then a price increase will increase revenue. But if demand is elastic, then a price increase will decrease revenue.

Example: A manufacturer of sunglasses currently sells one type for \$15 a pair. The price p and the demand x for these glasses are related by

$$x = f(p) = 9500 - 250p$$

If the current price is increased, will revenue increase or decrease?

$E(p) = \frac{-p(-250)}{9500-250p}$ $\quad f'(p) = -250$ $\quad E(p) = \frac{-p f'(p)}{f(p)}$

$E(15) = \frac{(-15)(-250)}{9500-250(15)} = \frac{3750}{5750} = 0.6521 = 65.21\%$

$E(16) = \frac{(-16)(-250)}{9500-250(16)} = \frac{4000}{5500} = 0.7272 = 72.72\%$

If the price is increased, the revenue will increase by 7.51%

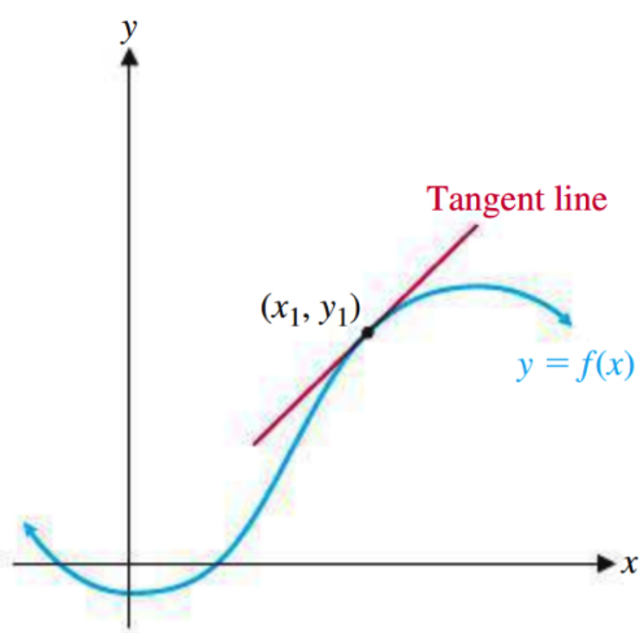
Recall: The derivative has two key interpretations

Rate of change: The derivative of a function at a specific point represents the instantaneous rate at which the function is changing at that point.

Here are some examples where we've applied this interpretation:

- Speed/velocity
- Differentials, $dy = f'(x)dx$
- Marginal cost, etc.
- Related rates problems
- Elasticity of demand

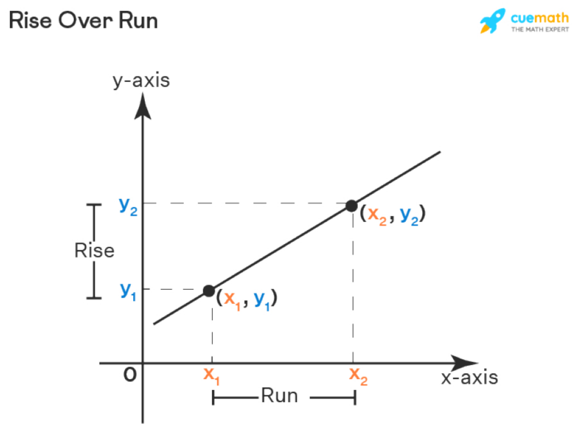
Slope: $f'(c)$ is the slope of the graph of f at $x = c$; in particular, $f'(c)$ is the slope of the line tangent to the graph of f at $x = c$.



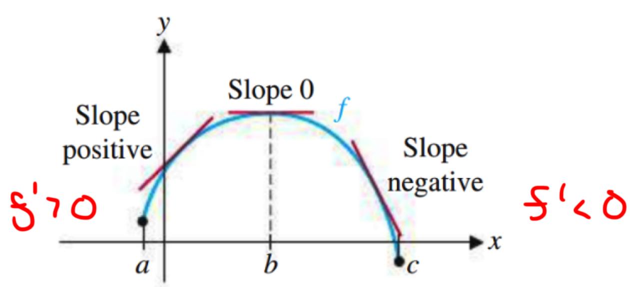
4.1 - First derivative and graphs

Slope (review):

$$\text{Slope } m = \frac{\text{rise}}{\text{run}}$$



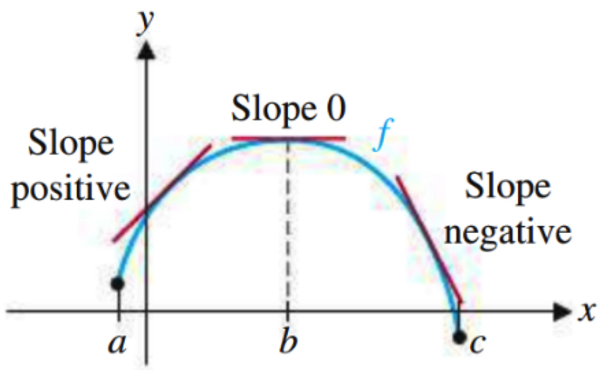
The following picture sums up this entire section. We can use *slope* to determine where a graph rises (increases), falls (decreases), and locate high and low points of the graph (extrema).



In this chapter, we'll use the slope interpretation of the derivative as a tool to:

- Sketch graphs
- Solve optimization problems

Finding intervals of increase or decrease



Theorem 1: For the interval (a, b), if $f' > 0$, then f is increasing (i.e. rises), and if $f' < 0$, then f is decreasing (i.e. falls).

Conclusion and key idea: We can determine the intervals of increase and decrease for our function f by creating a sign chart for its derivative, f'.

Example: Let $f(x) = x^2 - 6x + 10$.

$f'(x) = 2x - 6$

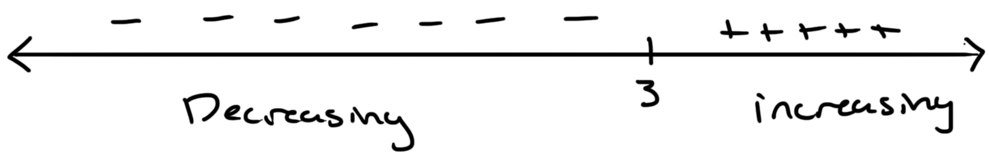
$2x - 6 = 0$

$2x = 6$

$x = 3$ ↖ only partition number

(a) For what interval(s) is f increasing? Decreasing?

Sign chart for $f'(x)$:



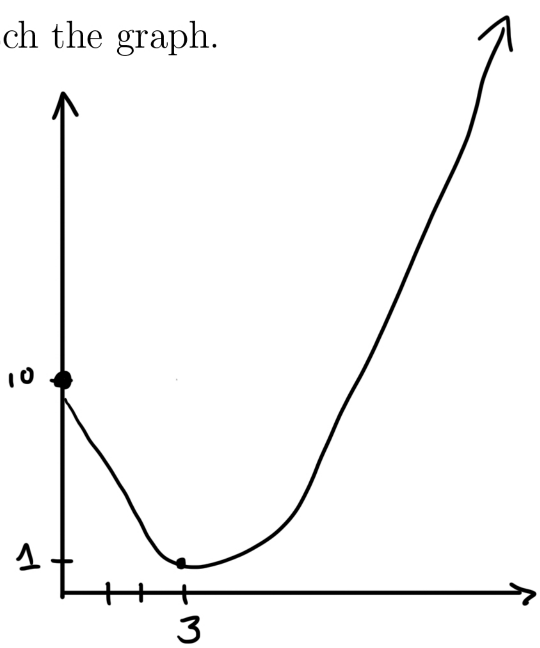
(b) For what values of x does the graph of f have a horizontal tangent line? What does that tell us about the graph?

Set the derivative $f'(x) = 0$ to solve

$2x - 6 = 0$

$x = 3$

(c) Sketch the graph.



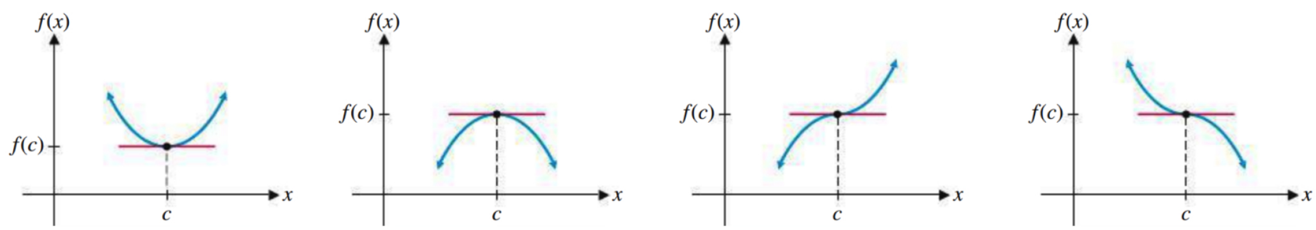
Critical numbers

As we saw in the last example, the sign chart for f' is a useful tool for analyzing the graph of f. This means that the *partition numbers* for f' (used to create the sign chart for f') are key to understanding the behavior of the function f.

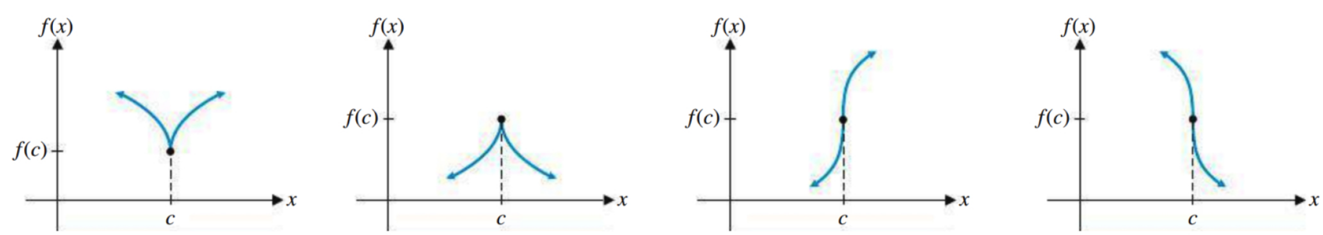
We call the partition numbers of f' the *critical numbers* of f (as long as they are in the domain of f).

Critical numbers: A real number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ doesn't exist is called a **critical number** of f.

$f'(c) = 0$: Horizontal tangent



$f'(c)$ is not defined but $f(c)$ is defined



Example: Find the critical numbers for $f(x) = x^2$, the intervals on which f is increasing, and those on which f is decreasing.

Example: Find the critical numbers for $f(x) = x^3$, the intervals on which f is increasing, and those on which f is decreasing.

Example: Find the critical numbers for $f(x) = \sqrt[3]{x}$, the intervals on which f is increasing, and those on which f is decreasing.

Example: Find the critical numbers for $f(x) = \dfrac{1}{x}$, the intervals on which f is increasing, and those on which f is decreasing.

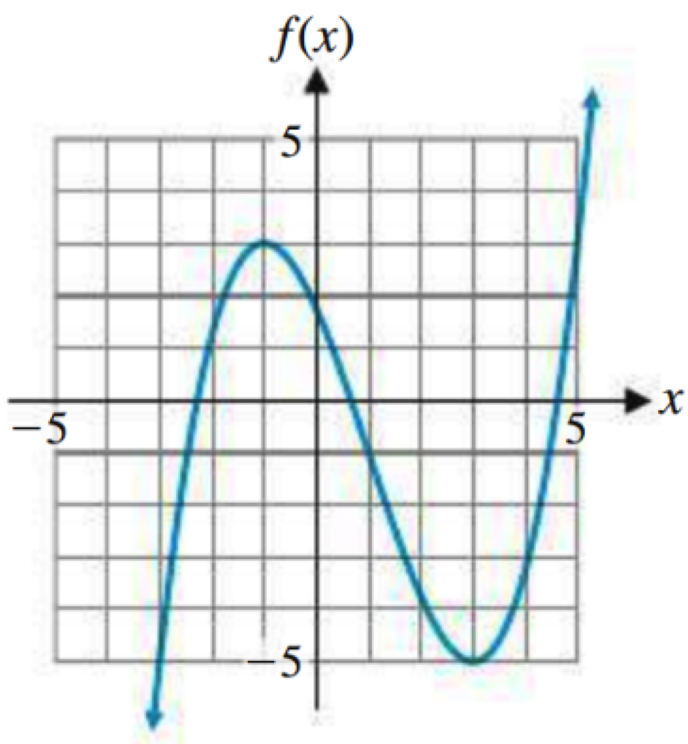
Example: Find the critical numbers for $f(x) = 8\ln x - x^2$, the intervals on which f is increasing, and those on which f is decreasing.

Local extrema

We say that $f(c)$ is a **local maximum** if there is an interval (a, b) containing c such that
$$f(x) \leq f(c)$$
for all x in (a, b).

We say that $f(c)$ is a **local minimum** if there is an interval (a, b) containing c such that
$$f(x) \geq f(c)$$
for all x in (a, b).

Let's look at the following graph.



Question: What do you notice about the graph of f "near" the local extrema? *Think about the tangent lines!*

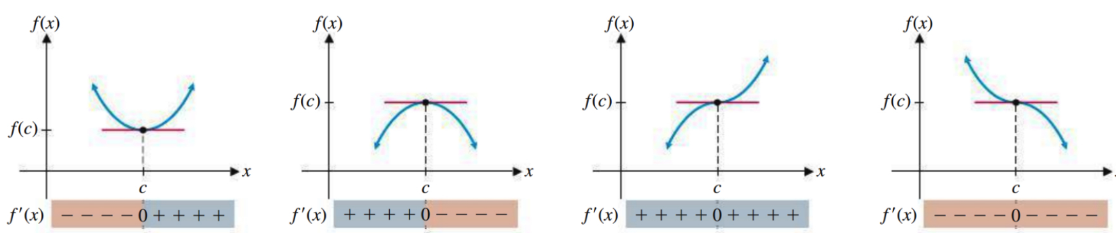
The first derivative test for local extrema

Theorem 2: If $f(c)$ is a local extremum of f, then c is a critical number of f.

First derivative test for local extrema: Let c be a critical number of f and construct a sign chart for f' around c.

- If the sign of f' changes from $-$ to $+$ at c, $f(c)$ is a *local minimum.*
- If the sign of f' changes from $+$ to $-$ at c, $f(c)$ is a *local maximum.*
- If the sign of f' does not change at c, then $f(c)$ is neither a local maximum nor local minimum.

$f'(c) = 0$: Horizontal tangent



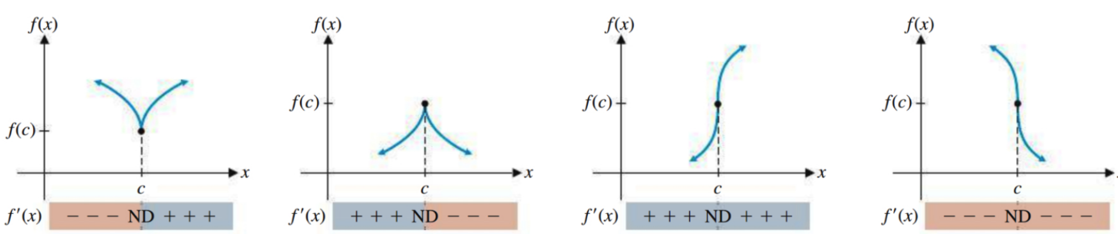
(A) $f(c)$ is a local minimum

(B) $f(c)$ is a local maximum

(C) $f(c)$ is neither a local maximum nor a local minimum

(D) $f(c)$ is neither a local maximum nor a local minimum

$f'(c)$ is not defined but $f(c)$ is defined



(E) $f(c)$ is a local minimum

(F) $f(c)$ is a local maximum

(G) $f(c)$ is neither a local maximum nor a local minimum

(H) $f(c)$ is neither a local maximum nor a local minimum

Example: Let $f(x) = x^3 - 6x^2 + 9x + 1$

(a) Find the critical numbers of f.

(b) Find the local extrema for f.

(c) Sketch the graph of f.

Example: Let $f(x) = x^3 - 9x^2 + 24x - 10$

(a) Find the critical numbers of f.

(b) Find the local extrema for f.

(c) Sketch the graph of f.