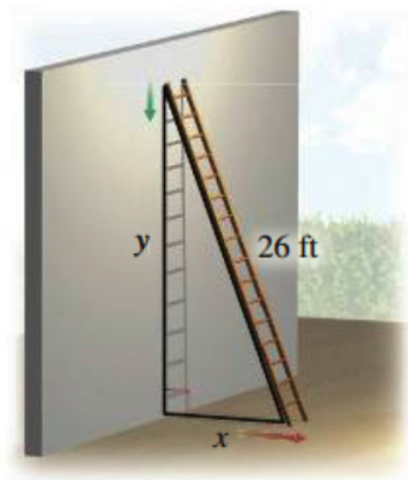
An application of implicit differentiation: Related rates (3.6)

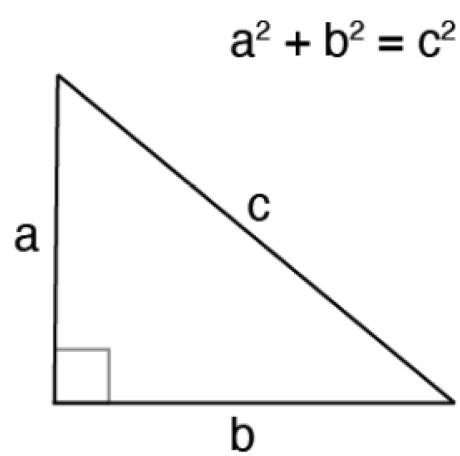
Example: A 26-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?

decrease because it's sliding down

$\frac{dy}{dt} = -2$



Recall: The figure forms a right triangle so we can use the Pythagorean theorem!



$26^2 = x^2 + y^2$
$26^2 = 10^2 + y^2$
$676 = 100 + y^2$
$\sqrt{576} = y^2$
$24 = y$

in respect to time

So $x^2 + y^2 = 26^2$. $(x(t))^2 + (y(t))^2 = 26^2$

Key point: Think carefully about the problem description. What's really behind the changes described? Also, think about the rate given, "2 feet *per second*."

time

So we're thinking of x and y in terms of time t and we want to find dx/dt when $x = 10$ and $dy/dt = -2$ (Why -2?)

$$x^2 + y^2 = 26^2$$

What rate is the bottom of the ladder moving away from the wall when $x=10$

$y=24$ solved for on last page

$$(x(t))^2 + (y(t))^2 = 26^2$$

$$\frac{d}{dt}\left((x(t))^2 + (y(t))^2\right) = \frac{d}{dt}(26^2)$$

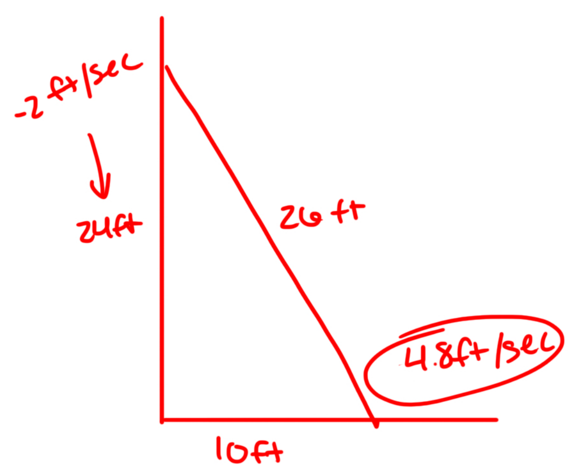
$$2\underbrace{x(t)}_{x=10} \cdot \frac{dx}{dt} + 2\underbrace{y(t)}_{y=24} \cdot \underbrace{\frac{dy}{dt}}_{-2} = 0$$

$$2 \cdot 10 \frac{dx}{dt} + 2 \cdot 24 \cdot (-2) = 0$$

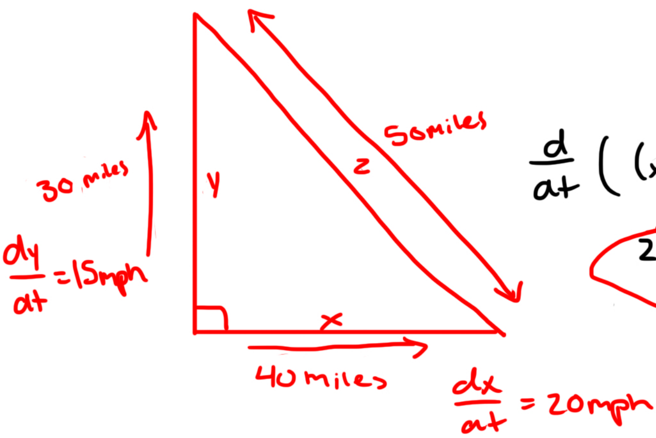
$$20 \frac{dx}{dt} - 96 = 0$$

$$20 \frac{dx}{dt} = 96$$

$$\frac{dx}{dt} = 4.8 \frac{ft}{sec}$$



Example: Suppose that two motorboats leave from the same point at the same time. If one travels north at 15 miles per hour and the other travels east at 20 miles per hour, how fast will the distance between them be changing after 2 hours



$t=2$

$\frac{dz}{dt}$ is what we're solving for

$$x^2+y^2=z^2$$

$$\frac{d}{dt}\left((x(t))^2 + (y(t))^2 = (z(t))^2 \right)$$

$$2x(t)\cdot\frac{dx}{dt} + 2y(t)\cdot\frac{dy}{dt} = 2z(t)\cdot\frac{dz}{dt}$$

$$30^2+40^2=z^2$$

$$z=50$$

$$2(40)(20)+2(30)(15)=2(50)\cdot\frac{dz}{dt}$$

$$2(800) + 60(15) = 100$$

$$1600 + 900 = 100\frac{dz}{dt}$$

$$2900 = 100\frac{dz}{dt}$$

$$25=\frac{dz}{dt}$$

$$\frac{dz}{dt}=25\text{mph}$$

Example: Suppose that for a company manufacturing flash drives, the cost, revenue, and profit equations are given by

$$C = 5000 + 2x$$

$$R = 10000x - 0.001x^2$$

where the production output in 1 week is x flash drives. If production is increasing at the rate of 500 flash drives per week when production is 2000 flash drives, find the rate of increase in cost and the rate of increase in revenue.

$$C(t) = 5000 + 2\,x(t)$$

Find $\frac{dC}{dt}$

$$\frac{dC}{dt} = 0 + 2\left(\frac{dx}{dt}\right) \qquad \frac{dx}{dt} = 500$$

$$\frac{dC}{dt} = 2(500)$$

$$\frac{dC}{dt} = \$1000$$

increasing at a rate of \$1000 per week

$$R(t) = 10000\,x(t) - 0.001\,(x(t))^2 \qquad x = 2000$$

$$\frac{dR}{dt} = 10000\left(\frac{dx}{dt}\right) - 0.001\,(2\,x(t))\frac{dx}{dt} \qquad \frac{dx}{dt} = 500$$

$$\frac{dR}{dt} = 10000(500) - 0.002(2000)(500)$$

$$\frac{dR}{dt} = 5{,}000{,}000 - 0.002(1{,}000{,}000)$$

$$= 5{,}000{,}000 - 2000$$

$$\frac{dR}{dt} = 4{,}998{,}000$$