Getting [redacted]. . .

started

Example: Find $\lim_{x\to 1} \frac{x+1}{x^2-1}$ DNE

Example: Find $\lim_{x\to 1} \frac{x-1}{x^2-1}$ indeterminate form $\left(\frac{0}{0}\right)$

$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to} \frac{\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}$

$\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$

Example: Find $\lim_{x\to\infty} \frac{x+1}{x^2-1} = 0$

degrees: $\frac{1}{2}$

Rule: $=0$

Limits at infinity (...or it's all in how you look at it!)

Example: For $p(x) = 5x^3 - 2x^2 + x - 7$. Find $\lim_{x\to\infty} p(x)$ and $\lim_{x\to-\infty} p(x)$.

$\lim_{x\to\infty}(5x^3-2x^2+x-7) = \lim_{x\to\infty} 5x^3\left(1-\frac{2}{5x}-\frac{1}{5x^2}-\frac{7}{5x^3}\right)$ all values approach 0

$= \lim_{x\to\infty} 5x^3 = \infty$

$2x^2 = 2\left(\frac{5}{5}\right)x^2\left(\frac{x^3}{x^3}\right) = 5x^3\left(\frac{2}{5x}\right)$

$\lim_{x\to-\infty}(5x^3-2x^2+x-7) = \lim_{x\to-\infty} 5x^3$

For a polynomial function

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$$

and

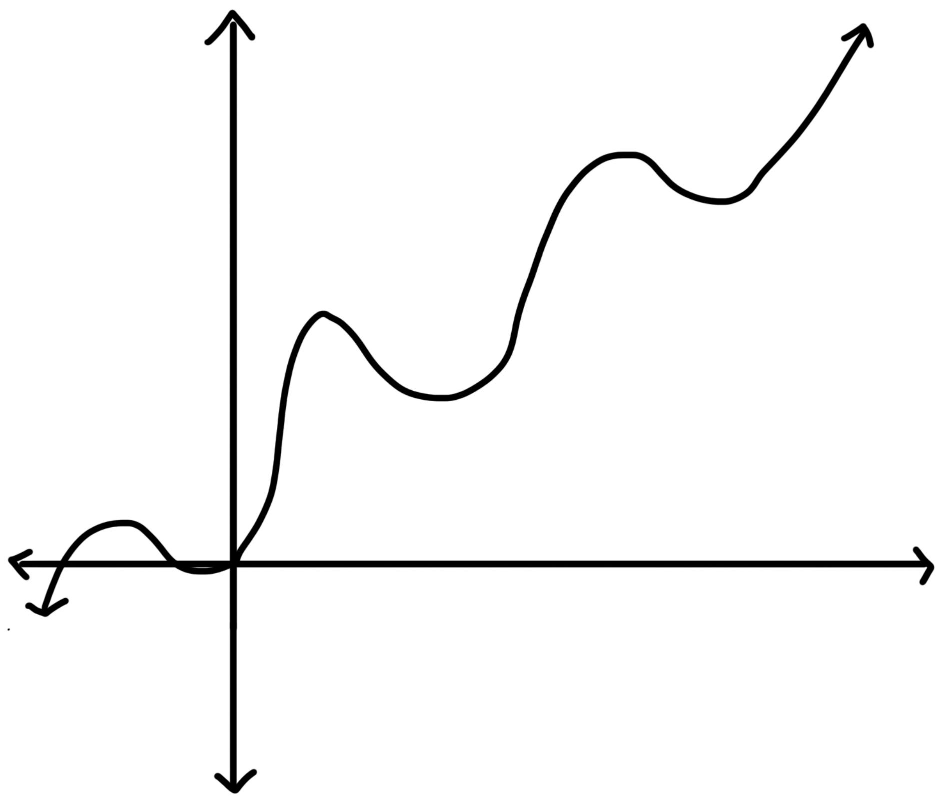
$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$$

2.3 - Continuity

Two key ideas that we'll cover in this section:

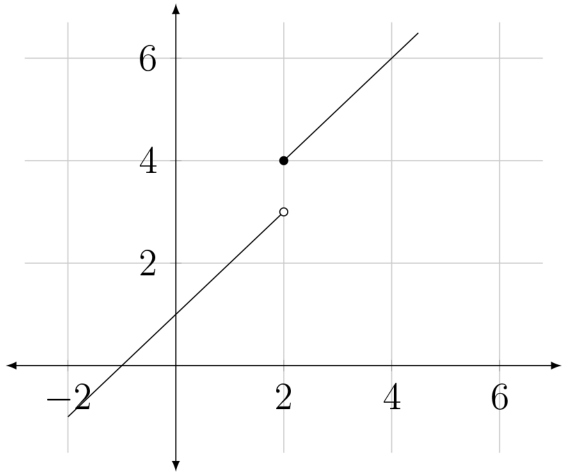
- the concept of *continuity* (for functions)
- the process for creating a *sign chart* for a function

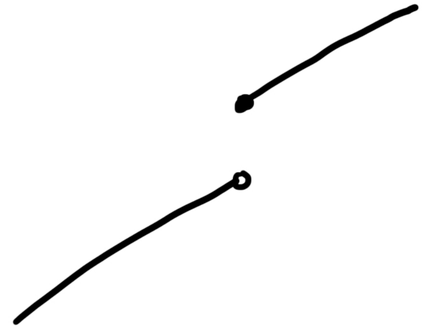
In a loose sense, a function is *continuous (over an interval)* if its graph can be drawn (over that interval) without lifting your pen.



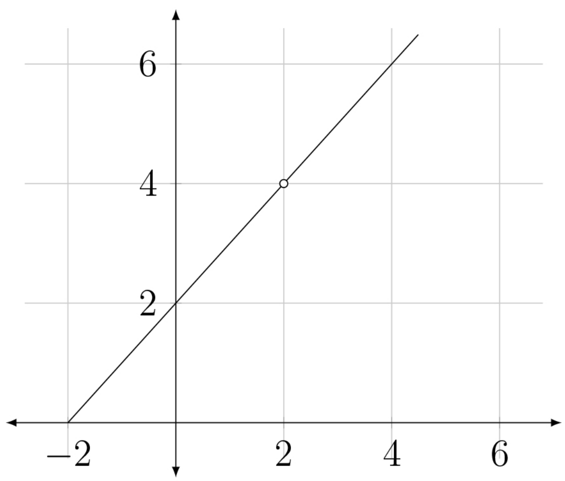
Types of discontinuities

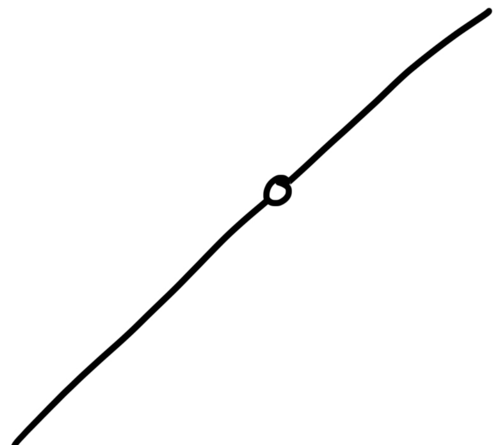
Jump discontinuity



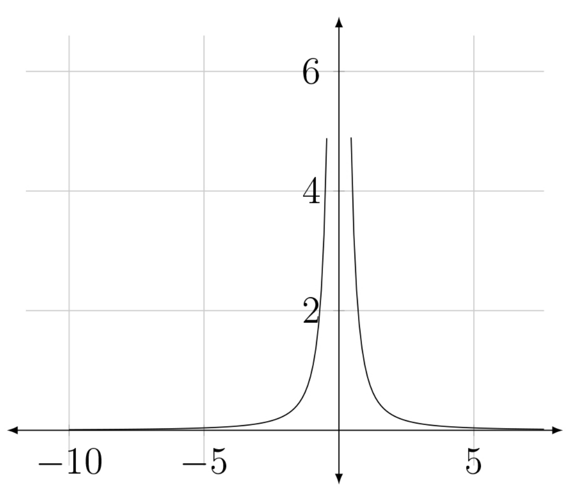


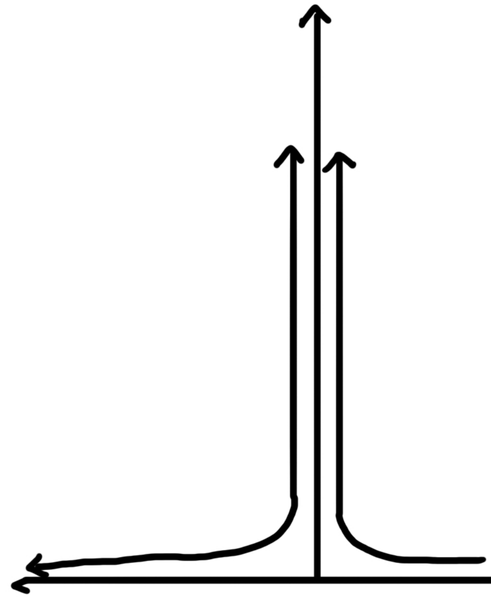
Removable discontinuity





Infinite discontinuity





A mathematical definition for continuity

A function is **continuous at the point** $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

$\to$ *Note that there are three parts to check for this statement!*

A function is **continuous on the open interval** (a, b) if it is continuous at each point on the interval.

A function is **continuous on the closed interval** $[a, b]$ if it is continuous at each point on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a), \text{ and}$$

$$\lim_{x \to b^-} f(x) = f(b)$$

Example: True/False: The function $f(x) = x^2 - 2x + 1$ is continuous over $(-\infty, \infty)$. True

polynomial function results in all real numbers

Example: Fill in the blank: The function $f(x) = \dfrac{x+1}{x-2}$ is continuous over $(-\infty, \underline{2}) \cup (\underline{2}, \infty)$

$(-\infty, 2) \cup (2, \infty)$

Example: Fill in the blank: The function $f(x) = \sqrt{x-1}$ is continuous over $[\underline{1}, \infty)$

$[1, \infty)$

Some common examples:

- A constant function $f(x) = k$ is continuous for all x.
- A constant function $f(x) = x^n$ (n is a positive integer) is continuous for all x.
- A polynomial function is continuous for all x.
- A rational function is continuous for all x except those values that make the denominator 0.
- If n is a positive odd integer, $n \geq 1$, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous.
- If n is a positive even integer, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and nonnegative.

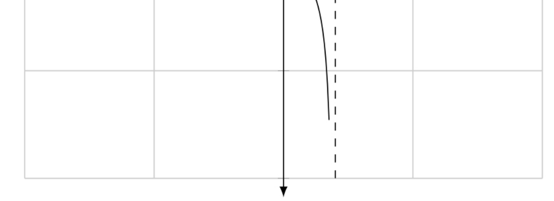
Partition numbers and sign charts

Fact: If f is continuous and $f(x) \neq 0$ on the interval (a, b), then $f(x)$ **does not** change sign on (a, b).

Said another way... the *only* numbers where f *can change sign* are:

- points of discontinuity
- points where the value of the function is 0 (i.e. $x-$intercepts).

Definition: A real number x is a **partition number** for a function f if f is discontinuous at x or $f(x) = 0$.



Steps for constructing a sign chart

Constructing a sign chart for f

1. Find all partition numbers for f
2. Plot the partition numbers on the number line
3. Pick a test number in each interval (between partition numbers) and determine the sign for that interval. Do this for all such intervals.

Example: Construct a sign chart for $f(x) = \dfrac{3x}{x+2}$. ALWAYS check what number makes the function equal 0!

$0 = \frac{3x}{x+2}$ $\quad 0 = 3x \quad x = 0$

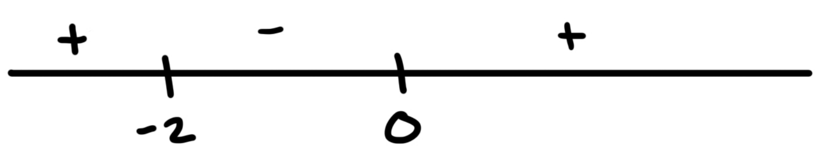
points of discontinuity (whatever makes the denominator 0).

$x = -2$

partition numbers

$x = -2$

$x = 0$



$f(x) = \frac{3x}{x+2}$

$f(-10) = \frac{-30}{-8}$

two negatives make a positive = positive interval

$f(-1) = \frac{-3}{1}$

$f(-1) = -3$

negative interval

$f(10) = \frac{30}{12}$

positive interval

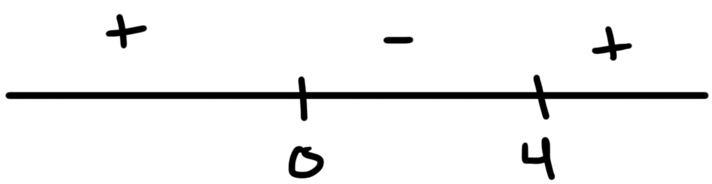
Example: Construct a sign chart for $f(x) = \dfrac{2x}{x-4}$.

$0 = \dfrac{2x}{x-4}$ $\quad x = 0$

partition numbers:

$x = 0$

$x = 4$



$f(x) = \dfrac{2x}{x-4}$

$f(-1) = \dfrac{-2}{-5}$ $\quad f(1) = \dfrac{2}{-3}$ $\quad f(5) = \dfrac{10}{1}$

positive
- partition

negative
- partition

positive
- partition

Example: Construct a sign chart for $f(x) = \dfrac{x-2}{x^2-2x-3}$.

Example: Construct a sign chart for $f(x) = 2x - 4$.

Section 2.4 - The derivative

As mentioned previously, the concept of limits allowed mathematicians to study and answer two major questions:

1. If an object is traveling along a path, what is its rate of change (i.e. velocity) *at a specific moment in time*?
2. How to we find the "slope" of a graph at a given point?

We'll look at both ideas carefully...

Rate of change

$$\text{distance} \ = \ \text{rate} \ \times \ \text{time}$$

$$\text{rate} \ = \frac{\text{distance}}{\text{time}}$$

Example: If you pass mile marker 120 at 8am and mile marker 300 at 11am, then you travelled 180 miles in 3 hours or 180 miles/3 hr = 60 miles per hour.

Of course, this is only your *average* rate of change over the 3 hours of travel.

$\Longrightarrow$ It's likely the case that you went faster than 60 mph at times and slower than 60 mph at times. $\Longleftarrow$

The more useful question: *How do we calculate your rate of change at any given moment of time?*

Formula for the average rate of change

For $y = f(x)$, the **average rate of change from** $x = a$ **to** $x = a + h$ is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}, h \neq 0$$

Example: A small steel ball dropped from a tower will fall a distance of y feet in x seconds where

$$y = f(x) = 16x^2$$

(a) Find the average velocity from $x = 1$ seconds to $x = 2$ seconds.

(b) Find the average velocity from $x = 2$ seconds to $x = 3$ seconds.

(c) Find the average velocity from $x = 2$ seconds to $x = 2 + h$ seconds.

(d) Find the limit of your expression in (c) as $h \to 0$.

(e) What does that limit represent? How can you interpret that limit?

For $y = f(x)$, the **<u>instantaneous</u> rate of change at** $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example: A small steel ball dropped from a tower will fall a distance of y feet in x seconds where

$$y = f(x) = 16x^2$$

(a) Find the instantaneous rate of change at $x = 1$ seconds.

(b) Find the instantaneous rate of change at $x = 2$ seconds.

(c) Find the instantaneous rate of change at $x = a$ seconds.

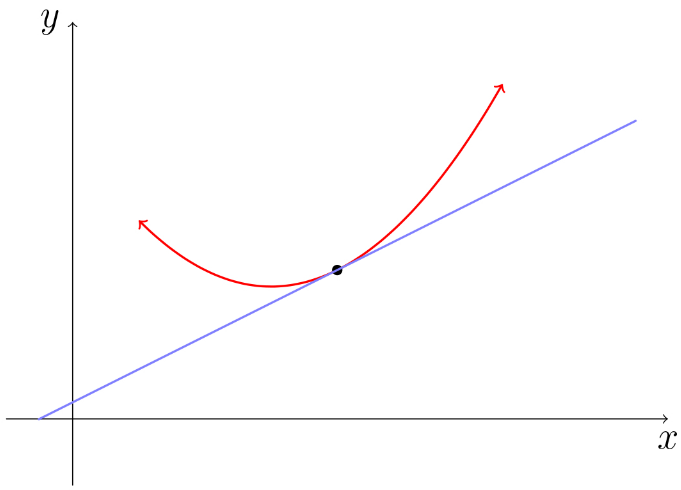
Slope of a graph

The graph of a linear function is a straight line and the slope of that line gives us a lot of useful information about that line.

Example: Sketch a graph of $f(x) = \frac{1}{2}x + 4$.

Capitalizing on that idea: Tangent lines

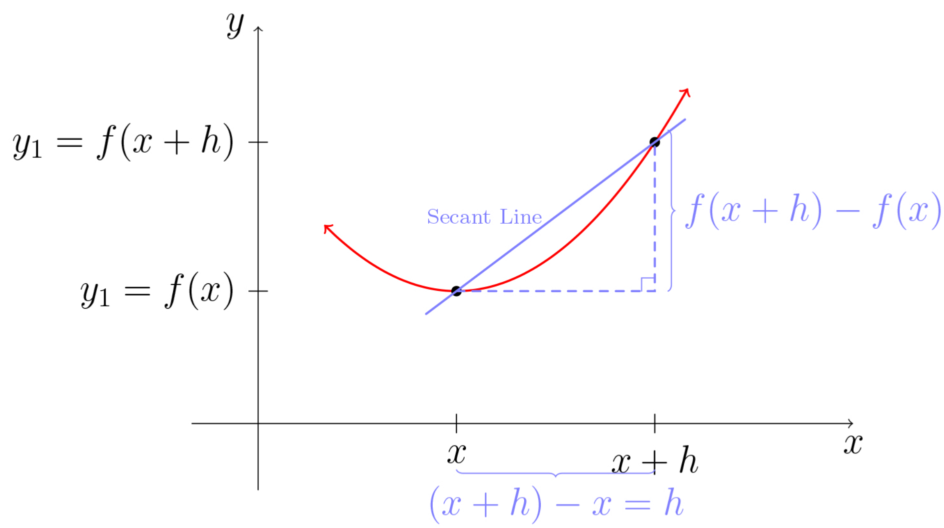
It turns out that if we zoom in on the graph of a given function, the graph will appear to be *close* to being a straight line. Mathematicians capitalized on this idea by focusing on the idea of a *tangent line*; i.e. the line *that "touches" the graph at a given point.* [In other words, the tangent line and the zoomed-in graph are very close.]



Definition: The **slope of the graph at that point** is the slope of the tangent line at that point.

Question: How do we find the slope of the tangent line?

Fact: To calculate the slope of a line we need *two points*.



The slope of the secant line is

$$m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

and the slope of the tangent line is

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: What is the slope of the line tangent to the graph of $f(x) = x^2$ at the point $(2, 4)$?