# 2.1: Introduction to limits

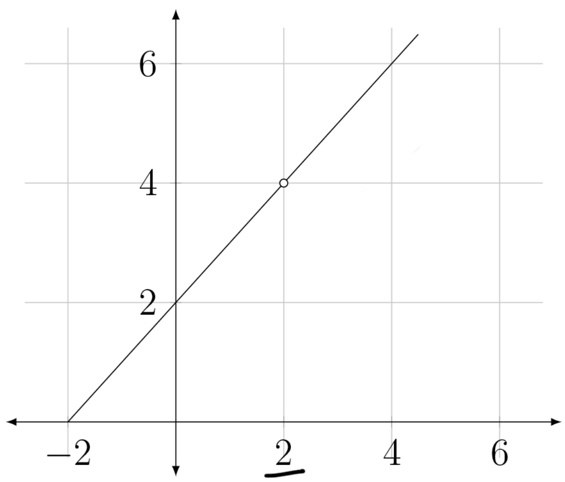
## Why limits?

Continuation of Aug 26 worksheet

*Limits are the cornerstone of Calculus. We won't see limits every day, but they play some role in every topic that we'll cover in this class, so we want to understand the concept.*

**Example:** Consider the following graph of the function $f(x) = \dfrac{x^2 - 4}{x - 2}$.

Note that $x = 2$ is not in the domain of $f$!



X=2 NOT IN DOMAIN

outputs can be very close to 4 but cannot be 4

| $x$ | $f(x) = \dfrac{x^2-4}{x-2}$ |
|---|---|
| 1.9 | |
| 1.99 | |
| 1.999 | |

| $x$ | $f(x) = \dfrac{x^2-4}{x-2}$ |
|---|---|
| 2.1 | |
| 2.01 | |
| 2.001 | |

**Question:** What seems to be true? If we *had to* give a value to $f(2)$, what *should* it be? What makes the most sense?

In the previous example, we would say

$$\lim_{x \to 2} f(x) = 4$$

**Definition of Limit:**
We write

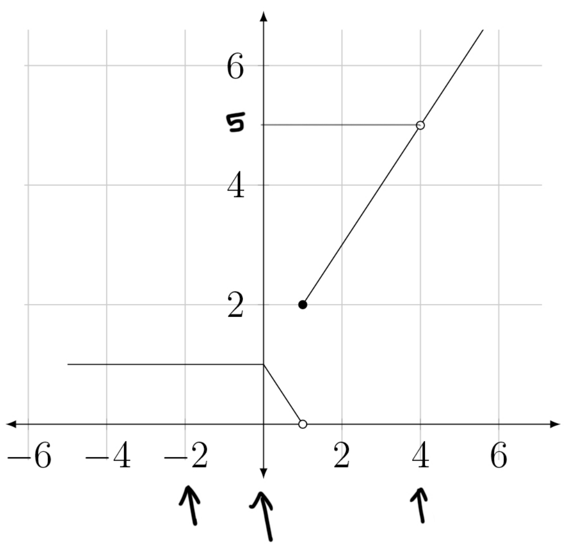
$$\lim_{x \to c} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to c$$

if the functional value $f(x)$ is close to the *single* real number $L$ whenever $x$ is close, but not equal, to $c$ (on either side of $c$).

**Subtle but important point:** The function doesn't have to be defined at $x = c$ but it does have to be defined on both sides of $x = c$. Said another way: $x = c$ doesn't have to be in the domain but all values on both sides of $x = c$ do need to be in the domain.



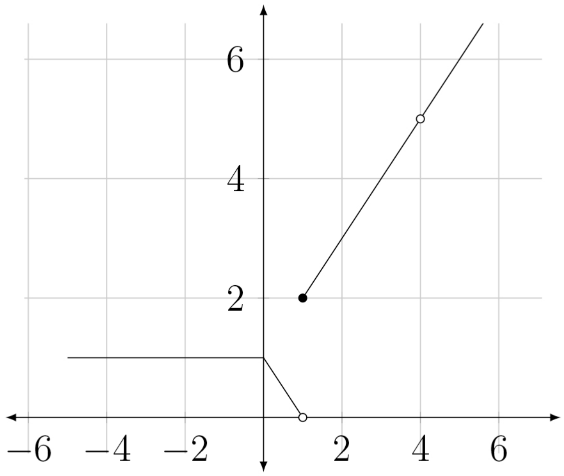

**Examples:** Evaluate (if possible) the limits.

$\lim_{x \to -2} f(x)$ = 1

$\lim_{x \to 0} f(x)$ = 1

$\lim_{x \to 4} f(x)$ = 5

**What can we say about this one?**




$\lim_{x \to 1^-} f(x) = 1 \neq \lim_{x \to 1^+} f(x) = 2$

Does $\lim_{x \to 1} f(x)$ exist?

No because they are not the same

DNE

**Definition of One-sided Limits:**
We write

$$\lim_{x \to c^-} f(x) = K$$

and call $K$ the **left-hand limit** or **limit from the left** if $f(x)$ is close to $K$ whenever $x$ is close, but *to the left* of, $c$ on the real number line.
We write

$$\lim_{x \to c^+} f(x) = L$$

and call $L$ the **right-hand limit** or **limit from the right** if $f(x)$ is close to $L$ whenever $x$ is close, but *to the right* of, $c$ on the real number line.

Theorem:

$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$

**Example** In the example above

$\lim_{x \to 1^-} f(x) = ?$ 0

$\lim_{x \to 1^+} f(x) = ?$ 2

$$\lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

# Properties of limits

Let $f$ and $g$ be two functions, and assume that

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = M$$

where $L$ and $M$ are both real numbers.

1. $\lim_{x\to c} k = k$ for any constant $k$
2. $\lim_{x\to c} x = c$
3. $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$ limit of the sum is the sum of the limit
4. $\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$ limit of the difference is difference of the limit
5. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x) = kL$
6. $\lim_{x\to c}[f(x) \cdot g(x)] = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)] = LM$
7. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$ limit of the quotient is the quotient of the limit
8. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$, where $L > 0$ when $n$ is even

*Note: these same rules apply for one-sided limits too.*

**Example:** Find

$$\lim_{x\to 3}(x^2 - 5x)$$

$\lim(3^2 - 5(3))$

$\lim(9-15)$

$\lim(-6)$

**Example:** Find

$$\lim_{x\to -1}(3x+5)$$

$\lim(3(-1)+5)$

$\lim(-3+5)$

$\lim(2)$

**Example:** Find

$$\lim_{x\to 3}(x^2+3x+5)$$

$\lim(3^2+3(3)+5)$

$\lim(9+9+5)$

$\lim(23)$

$\lim_{x\to 3} f(x)=7 \qquad \lim_{x\to 3} g(x)=2$

(a) Evaluate $\left(\lim_{x\to 3} f(x)-2g(x)\right)$

$7-2(2)$

$7-4$

$=3$

(b) Evaluate $\left(\lim_{x\to 3} f(x)g(x)-1\right)$

$7(2-1)$

$7(1)$

$=7$

**Example:** Find

$$\lim_{x\to 2}\frac{2x-1}{x^3+4}$$

$\lim\left(\frac{2(2)-1}{2^3+4}\right)$

$\lim\left(\frac{3}{20}\right)$

**Theorem 3:**

1. $\lim_{x\to c} f(x)=f(c)$ for any polynomial function $f$

2. $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{f(c)}{g(c)}$ for any rational function $\frac{f}{g}$ with $g(c)\neq 0$.

**Example:** Find

$$\lim_{x\to 3} \sqrt{x+1}$$

$$\lim(\sqrt{3+1})$$

$$\lim(\sqrt{4})$$

$$\lim(2)$$

**Example:** Find

$$\lim_{x\to 3} \frac{1}{\sqrt{x+1}}$$

# Indeterminate form

**Example:** Find

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

If $\lim\limits_{x \to c} f(x) = 0$ and $\lim\limits_{x \to c} g(x) = 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

is said to be **indeterminate**.
If $\lim\limits_{x \to c} f(x) \neq 0$ and $\lim\limits_{x \to c} g(x) = 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

does not exist.

# Evaluate the following limits

**Example:** $\lim_{x\to 0}(3x^4 - 2x^3 + 7)$

**Example:** $\lim_{x\to 1}(2x - 5\sqrt{x+3})$

**Example:** $\lim_{x\to 1}\dfrac{x-1}{x^2+1}$

**Example:** $\lim_{x\to 1}\dfrac{x-1}{x^2-1}$

**Example:** $\lim_{x\to 1}\dfrac{x+1}{x^2-1}$