4.5 - Absolute maxima and minima

Recall (from Section 4.1): throughout a "hill"

We say that $f(c)$ is a **local maximum** if there is an interval (a, b) containing c such that
$$f(x) \leq f(c)$$
for all x in (a, b).

We say that $f(c)$ is a **local minimum** if there is an interval (a, b) containing c such that
$$f(x) \geq f(c)$$
for all x in (a, b).

Now... throughout entire graph

If $f(c) \geq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute maximum** of f.
If $f(c) \leq f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute minimum** of f.

Theorem: A function f that is continuous on a closed interval $[a, b]$ has both an absolute maximum and an absolute minimum on that interval.

Theorem: Absolute extrema (if they exist) *must* occur at critical numbers or at endpoints of the interval.

Procedure to find absolute extrema on a closed interval $[a, b]$:

#1 Check to see if f is continuous on $[a, b]$

#2 Find all critical numbers in (a, b) → where $f'(x) = 0$

#3 Evaluate f for each critical number and each endpoint

#4 The absolute maximum of f on $[a, b]$ is the *largest* value found in Step 3

#5 The absolute minimum of f on $[a, b]$ is the *smallest* value found in Step 3

Example: Find the absolute maximum and absolute minimum of $f(x) = x^3 - 12x$ on $[-5, 5]$.

Find critical numbers

$x = 2$
$x = -2$

$f'(x) = 3x^2 - 12$
$0 = 3x^2 - 12$
$0 = 3(x^2 - 4)$
$0 = x^2 - 4$
$0 = (x-2)(x+2)$
$x = \pm 2$

x	$f(x) = x^3 - 12x$
2	$f(2) = -16$
-2	$f(-2) = 16$
5	$f(5) = 65$ ← Absolute max
-5	$f(-5) = -65$ ← Absolute min

Procedure to find absolute extrema on a closed interval $[a, b]$:

#1 Check to see if f is continuous on $[a, b]$

#2 Find all critical numbers in (a, b)

#3 Evaluate f for each critical number and each endpoint

#4 The absolute maximum of f on $[a, b]$ is the *largest* value found in Step 3

#5 The absolute minimum of f on $[a, b]$ is the *smallest* value found in Step 3

Example: Find the absolute maximum and absolute minimum of $f(x) = x^3 - 12x$ on $[-3, 1]$.

critical numbers

$x = 2$
$x = -2$

$f'(x) = 3x^2 - 12$
$0 = 3(x^2 - 4)$
$0 = (x^2 - 4)$
$0 = (x-2)(x+2)$

x	$f(x) = x^3 - 12x$
~~2~~	~~$(2)^3 - 12(2)$~~ ~~$8 - 24$~~ ~~$= -16$~~ ← Falls outside the interval
-2	$(-2)^3 - 12(-2)$ $-8 + 24$ $= 16$ ← Absolute maximum
-3	$(-3)^3 - 12(-3)$ $-27 + 36$ $= 9$
1	$(1)^3 - 12(1)$ $1 - 12$ $= -11$ ← Absolute minimum

Using the second derivative (i.e. concavity) to find extrema

Let c be a critical number of $f(x)$ such that $f'(c) = 0$.
If $f''(c) > 0$, then $f(c)$ is a local minimum.
If $f''(c) < 0$, then $f(c)$ is a local maximum.

Example: Find the local maxima and minima for $f(x) = x^3 - 9x^2 + 24x - 10$.

$f'(x) = 3x^2 - 18x + 24$

$0 = 3x^2 - 18x + 24$

$0 = 3(x^2 - 6x + 8)$

$0 = x^2 - 6x + 8$

$0 = (x-4)(x-2)$

$x = 4 \quad x = 2$

$f''(x) = 6x - 18$

$f''(4) = 24 - 18 > 0$ Local min

$f''(2) = 12 - 18 < 0$ Local max

Local max $f(2) = 10$

Local min $f(4) = 6$

Let c be a critical number of $f(x)$ such that $f'(c) = 0$.
If $f''(c) > 0$, then $f(c)$ is a local minimum.
If $f''(c) < 0$, then $f(c)$ is a local maximum.

Example: Find the local maxima and minima for $f(x) = xe^{-0.2x}$

$$f'(x) = 1 \cdot e^{-0.2x} + x\left(e^{-0.2x} \cdot (-0.2)\right) = e^{-0.2x}(1 - 0.2x)$$

$$e^{-0.2x}(1 - 0.2x) = 0$$

$$1 - 0.2x = 0$$

$$1 = 0.2x$$

$x = 5$ ← only critical number

$$f''(x) = -0.2e^{-0.2x}(1 - 0.2x) + e^{-0.2x}(-0.2)$$

$$f''(5) = -0.2e^{-0.2(5)}\underbrace{(1 - 0.2(5))}_{=0} + e^{-0.2(5)}(-0.2)$$

$$= 0 + e^{-0.2(5)}(-0.2) < 0$$

$f''(5) < 0$ so $f(5)$ is a local max ☆

$$f(5) = 5 \cdot e^{-1} = \frac{5}{e} \approx 1.8394$$

Theorem: Let f be continuous on the interval (a, b) with only one critical number on (a, b).
If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on (a, b).
If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on (a, b).

Example: Find the absolute extrema of $f(x) = 12 - x - \dfrac{5}{x}$ on $(0, \infty)$.

$f(x) = 12 - x - 5x^{-1}$

$f'(x) = -1 + 5x^{-2} = -1 + \dfrac{5}{x^2}$

$0 = -1 + \dfrac{5}{x^2}$

$0 = -x^2 + 5$

$x^2 = 5$

$x = \pm\sqrt{5}$

$x = \sqrt{5}$ only critical number

$f''(x) = -10x^{-3} = -\dfrac{10}{x^3}$

$f''(\sqrt{5}) = -\dfrac{10}{\sqrt{5}^3} < 0$ Absolute max at $x = \sqrt{5}$

$f(\sqrt{5}) =$

Theorem: Let f be continuous on the interval (a, b) with only one critical number on (a, b).
If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on (a, b).
If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on (a, b).

Example: Find the absolute extrema of $f(x) = 5 \ln x - x$ on $(0, \infty)$.

Example: Find two numbers whose difference is 19 and whose product is a minimum.

Example: Find two *positive* numbers whose product is 19 and whose sum is a minimum.