The quotient rule: If

$$y = f(x) = \frac{T(x)}{B(x)}$$

and if $T'(x)$ and $B'(x)$ exist, then

$$f'(x) = \frac{B(x)T'(x) - T(x)B'(x)}{[B(x)]^2}$$

In words, *the derivative of a quotient is the bottom times the derivative of the top minus the top times derivative of the bottom divided by the bottom squared.*

Example: If $f(x) = \dfrac{2x-1}{5x+3}$, find $f'(x)$.

$T(x) = 2x-1 \quad T'(x) = 2$

$B(x) = 5x+3 \quad B'(x) = 5$

$$\frac{(5x+3)(2) - (2x-1)(5)}{(5x+3)^2}$$

$(5x+3)(5x+3)$

$10x^2 + 15x + 15x + 9$

$10x^2 + 30x + 9$

$$\frac{\cancel{10x} + 6 - \cancel{10x} - 5}{10x^2+30x+9}$$

$$\frac{1}{10x^2+30x+9}$$

Problem: Find $f'(x)$

(a) $f(x) = \dfrac{x-1}{x+4}$

$$f'(x) = \frac{(x+4)(1)-(1)(x-1)}{(x+4)^2}$$

(b) $f(x) = \dfrac{x^2-1}{3x-1}$

$$f'(x) = \frac{(3x-1)(2x)-(x^2-1)(3)}{(3x-1)^2}$$

left off on October 7

Continued on October 9

(c) $f(x) = \dfrac{\ln x}{x+1}$

$$f'(x) = \frac{(x+1)(\frac{1}{x})-(1)(\ln x)}{(x+1)^2}$$

(d) $f(x) = \dfrac{1+e^x}{1-e^x}$

$$f'(x) = \frac{(1-e^x)(e^x)-(-e^x)(1+e^x)}{(1-e^x)^2}$$

(e) $f(x) = \dfrac{2x}{1+\ln x}$

$$f'(x) = \frac{(1+\ln x)(2)-(\frac{1}{x})(2x)}{(1+\ln x)^2}$$

Example: What about $f(x) = \dfrac{1}{x}$? Should I use the quotient rule? Or?

$$f'(x) = \frac{x(0) - 1 \cdot 1}{x^2} = \frac{1}{x^2}$$ quotient rule

product rule - $f'(x) = x^{-1}$

$$f'(x) = -1x^{-2}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f(x) = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$$

Announcements

Test 2 will be given next Wednesday during your recitation meeting.

- 2.5 - Basic differentiation properties
- 2.6 - Differentials
- 2.7 - Marginal analysis ☆
- 3.1 - Quick review of exponentials and logarithms
- 3.2 - Derivatives of Exponential and Logarithmic Functions
- 3.3 - Derivatives of Products and Quotients ☆
- 3.4 - The Chain Rule ☆

A red star represents the sections to review more

3.4 - Chain rule

The general power rule

If $u(x)$ is a differentiable function and n is any real number, and

$$y = f(x) = [u(x)]^n,$$

then

$$f'(x) = n[(u(x)]^{n-1}u'(x)$$

Example: Find the derivative.

(a) $f(x) = (2x-5)^5$

$u'(x) = 2$

$\boxed{f'(x) = 5(2x-5)^4(2)}$

(b) $f(x) = (x^2-x-1)^{3/2}$

$u'(x) = 2x-1$

$\boxed{f'(x) = \frac{3}{2}(x^2-x-1)^{\frac{1}{2}}(2x-1)}$

(c) $f(x) = \sqrt{x^2-x-1}$

$u'(x) = 2x-1$

$\boxed{f'(x) = \frac{1}{2}(x^2-x-1)^{-\frac{1}{2}}(2x-1)}$

(d) $f(x) = \dfrac{1}{x^2+2x-1}$ $(x^2+2x-1)^{-1}$

$\boxed{f'(x) = -1(x^2+2x-1)^{-2}(2x+2)}$

Function composition (quick review)

So far, we've looked at derivative rules that address the different ways that we can *combine* functions; e.g. through the operations $+, -, \times$, and $\div$

We can also combine functions by using one function as *input* to another function; i.e. **function composition**.

$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$

A function m is a **composite** of functions f and g if

$$m(x) = f[g(x)]$$

Example: If $f(x) = \sqrt{x}$ and $g(x) = x + 2$, find $f(g(x))$.

$$\boxed{f(g(x)) = \sqrt{x+2}}$$

$g(f(x)) = \sqrt{x} + 2$

very different functions.
Be careful of the order

Example: If $f(x) = \frac{1}{x}$ and $g(x) = x + 2$, find $f(g(x))$.

$$\boxed{f(g(x)) = \frac{1}{x+2}}$$

$g(f(x)) = \frac{1}{x} + 2$

Breaking down a function...

When taking derivatives it's helpful to be able to take a given function and "break it down into its parts (or stages)."

Example: Let $m(x) = (x^2 - 1)^3$. Can you identify two functions f and g so that

$$m(x) = f(g(x))?$$

$f(g) = x^3$ $\quad g(x) = x^2 - 1$

$\boxed{f(g(x)) = (x^2-1)^3}$

We often refer to $f(x)$ as the *outer* function and $g(x)$ as the *inner* function.

Example: Write each function as a composite of simpler functions.

(a) $f(x) = \sqrt{x^2 + 4}$

$$\boxed{E(x) = \sqrt{x}, \quad I(x) = x^2 + 4}$$

$$f(g(x)) = \sqrt{x^2+4}$$

(b) $g(x) = (x - 1)^2$

$$\boxed{E(x) = x^2, \quad I(x) = x - 1}$$

$$g(f(x)) = (x-1)^2$$

(c) $h(x) = 5e^{2x-1}$

$$\boxed{E(x) = 5e^x, \quad I(x) = 2x - 1}$$

$$h(f(x)) = 5e^{2x-1}$$

(a) $j(x) = \ln(x^2 - x - 1)$

$$\boxed{E(x) = \ln x, \quad I(x) = x^2 - x - 1}$$

$$j(f(x)) = \ln(x^2-x-1)$$

The general chain rule

If $m(x) = E[I(x)]$ is a composite function, then

$$m'(x) = E'[I(x)]I'(x)$$

provided $E'[I(x)]$ and $I'(x)$ exist.

In words, *the derivative of the exterior evaluated at the interior times the derivative of the interior.*

Example: Let $m(x) = (x^2 - 1)^3$. Find $m'(x)$ using the general chain rule.

$E(x) = x^3 \qquad I(x) = x^2 - 1$

$E'(x) = 3x^2 \qquad I'(x) = 2x$

$\boxed{m'(x) = 3(x^2-1)^2(2x)}$

Example: Find the derivative of each.

(a) $f(x) = \sqrt{x^2 + 4}$

$E(x) = \sqrt{x}$ $\quad$ $I(x) = x^2+4$

$E(x) = x^{\frac{1}{2}}$ $\quad$ $I'(x) = 2x$

$E'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$\boxed{m'(x) = \frac{1}{2}(x^2+4)^{-\frac{1}{2}}(2x)}$

(b) $g(x) = (x - 1)^2$

$E(x) = x^2$ $\quad$ $I(x) = x-1$

$E'(x) = 2x$ $\quad$ $I'(x) = 1$

$\boxed{m'(x) = 2(x-1)(1)}$

(c) $h(x) = 5e^{2x-1}$

$E(x) = 5e^x$ $\quad$ $I(x) = 2x-1$

$E'(x) = 5e^x$ $\quad$ $I'(x) = 2$

$\boxed{m'(x) = 5e^{2x-1}(2)}$

(a) $j(x) = \ln(x^2 - x - 1)$

$E(x) = \ln x$ $\quad$ $I(x) = x^2-x-1$

$E'(x) = \frac{1}{x}$ $\quad$ $I'(x) = 2x-1$

$\boxed{m'(x) = \frac{1}{x^2-x-1}(2x-1)}$