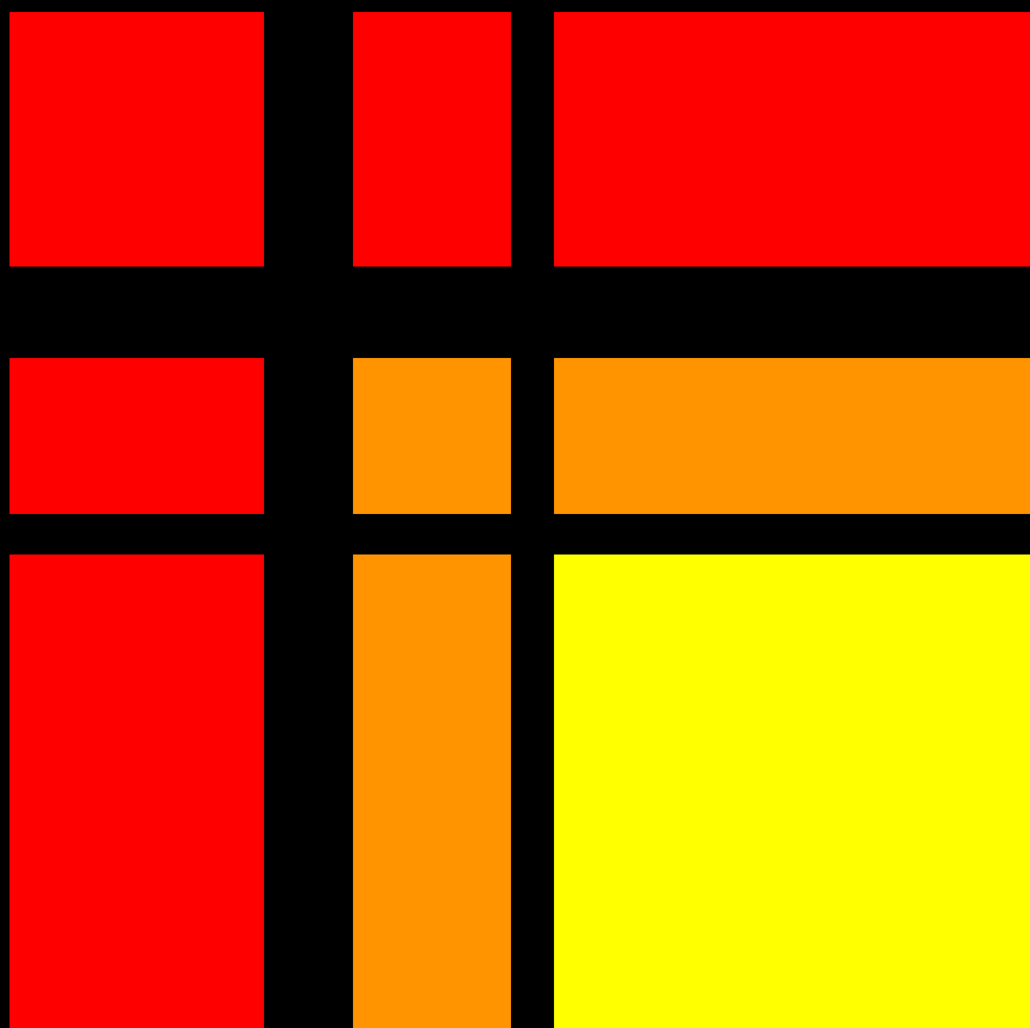
Advanced Linear Algebra

Foundations to Frontiers



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Foundations to Frontiers

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Preface

Robert van de Geijn
Maggie Myers
Austin, 2019

Contents

Week 0

Getting Started

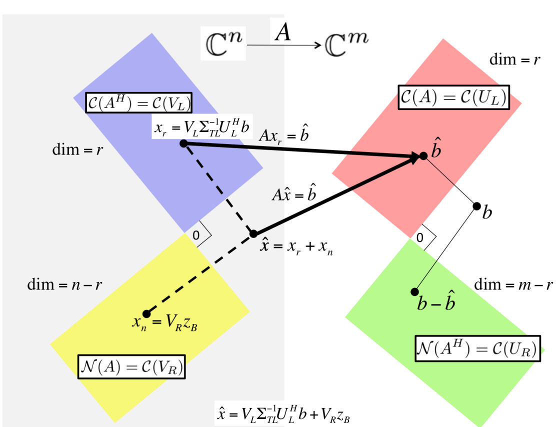
0.1 Opening Remarks

0.1.1 Welcome

YouTube: `https://www.youtube.com/watch?v=KzCTMlvxtQA`

Linear algebra is one of the fundamental tools for computational and data scientists. In Advanced Linear Algebra: Foundations to Frontiers (ALAFF), you build your knowledge, understanding, and skills in linear algebra, practical algorithms for matrix computations, and how floating-point arithmetic, as performed by computers, affects correctness.

The materials are organized into Weeks that correspond to a chunk of information that is covered in a typical on-campus week. These weeks are arranged into three parts:



Part I: Orthogonality

The Singular Value Decomposition (SVD) is possibly the most important result in linear algebra, yet too advanced to cover in an introductory undergraduate course. To be able to get to this topic as quickly as possible, we start by focusing on orthogonality, which is at the heart of image compression, Google's page rank algorithm, and linear least-squares approximation.

Part II: Solving Linear Systems
Solving linear systems, via direct or iterative methods, is at the core of applications in computational science and machine learning. We also leverage these topics to introduce numerical stability of algorithms: the classical study that qualifies and quantifies the "correctness" of an algorithm in the presence of floating point computation and approximation. Along the way, we discuss how to restructure algorithms so that they can attain high performance on modern CPUs.

Algorithm: Compute LU factorization with partial pivoting of A, overwriting A with factors L and U. The pivot vector is returned in p.

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $p \rightarrow \left(\begin{array}{c} p_T \\ \hline p_B \end{array}\right)$.
where A_{TL} is 0×0 and p_T is 0×1
while $n(A_{TL}) < n(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} p_T \\ \hline p_B \end{array}\right) \rightarrow \left(\begin{array}{c} p_0 \\ \hline \pi_1 \\ \hline p_2 \end{array}\right)$$

where α_{11}, λ_{11}, π_1 are 1×1

$\pi_1 = \text{maxi}\left(\begin{array}{c} \alpha_{11} \\ \hline a_{21} \end{array}\right)$

$\left(\begin{array}{c|c|c} a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) := P(\pi_1)\left(\begin{array}{c|c|c} a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$

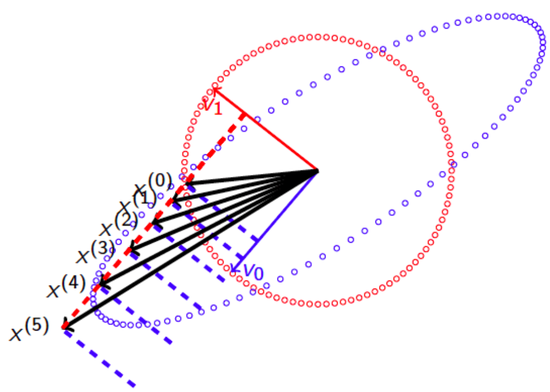
$a_{21} := a_{21}/\alpha_{11}$

$A_{22} := A_{22} - a_{21}a_{12}^T$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} p_T \\ \hline p_B \end{array}\right) \leftarrow \left(\begin{array}{c} p_0 \\ \hline \pi_1 \\ \hline p_2 \end{array}\right)$$

endwhile



Part III: Eigenvalues and Eigenvectors
Many problems in science have the property that if one looks at them in just the right way (in the right basis), they greatly simplify and/ or decouple into simpler subproblems. Eigenvalue and eigenvectors are the key to discovering how to view a linear transformation, represented by a matrix, in that special way. Algorithms for computing them also are the key to practical algorithms for computing the SVD

In this week (Week 0), we walk you through some of the basic course information and help you set up for learning. The week itself is structured like future weeks, so that you become familiar with that structure.

0.1.2 Outline Week 0

Each week is structured so that we give the outline for the week immediately after the "launch:"

- 0.1 Opening Remarks
  - 0.1.1 Welcome
  - 0.1.2 Outline Week 0
  - 0.1.3 What you will learn
- 0.2 Setting Up For ALAFF

0.1.3 What you will learn

The third unit of each week informs you of what you will learn. This describes the knowledge and skills that you can expect to acquire. If you return to this unit after you compete the week, you will be able to use the below to self-assess.

Upon completion of this week, you should be able to

- Navigate the materials.
- Access additional materials from GitHub.
- Track your homework and progress.
- Register for MATLAB online.
- Recognize the structure of a typical week.

0.2 Setting Up For ALAFF

0.2.1 Accessing these notes

For information regarding these and our other materials, visit ulaff.net.

These notes are available in a number of formats:

- As an online book authored with PreTeXt at http://www.cs.utexas.edu/users/flame/laff/alaff/.

- As a PDF at http://www.cs.utexas.edu/users/flame/laff/alaff/ALAFF.pdf.

  If you download this PDF and place it in just the right folder of the materials you will clone from GitHub (see next unit), the links in the PDF to the downloaded material will work.

  During Spring 2020, we will incrementally add weeks (chapters) to that material as the semester progresses. We will be updating the materals frequently as people report typos and we receive feedback from learners. Please consider the environment before you print a copy...

- Eventually, if we perceive there is demand, we may offer a printed copy of these notes from Lulu.com, a self-publishing service. This will not happen until Summer 2020, at the earliest.

Homework 0.2.1.1 If the book has chapters numbered 0 through n and you print a new copy every time a new chapter is added (first you print chapter 0, then you print chapters 0 and 1, and so forth), how many chapters (multiplicity counted) will you print?

If the book has chapters 0 through 12 and each chapter has 50 pages, how many pages do you print?

Answer. Number of chapters printed: $(n+1)(n+2)/2$

Now prove it!

Number of pages printed: $(12+1)(12+2)/2x50 = 4550$

0.2.2 Cloning the ALAFF repository

We have placed all materials on GitHub, a development environment for software projects. In our case, we use it to disseminate the various activities associated with this course.

On the computer on which you have chosen to work, "clone" the GitHub repository for this course:

- Visit `https://github.com/ULAFF/ALAFF`

- Click on

  and copy https://github.com/ULAFF/ALAFF.git.

- On the computer where you intend to work, in a terminal session on the command line in the directory where you would like to place the materials, execute

  ```
  git clone https://github.com/ULAFF/ALAFF.git
  ```

  This will create a local copy (clone) of the materials.

- Sometimes we will update some of the files from the repository. When this happens you will want to execute, in the cloned directory,

```
git stash save
```

which saves any local changes you have made, followed by

```
git pull
```

which updates your local copy of the repository, followed by

```
git stash pop
```

which restores local changes you made. This last step may require you to "merge" files that were changed in the repository that conflict with local changes.

Upon completion of the cloning, you will have a directory structure similar to that given in Figure 0.2.2.1.

```
Users
└─rvdg
  └─ALAFF
    ├─ALAFF.pdf ..................... PDF of notes (with solutions).
    │
    └─Assignments
      ├─Week00 ........... Assignments for Week 01.
      │ │
      │ └─matlab ......... Programming assignments.
      │
      ├─Week01 ........... Assignments for Week 1.
      │ │
      │ └─matlab ......... Programming assignments.
      │
      └─: .5 Week12
        ├─C
        └─matlab
```

Figure 0.2.2.1 Directory structure for your ALAFF materials. In this example, we cloned the repository in Robert's home directory, rvdg.

0.2.3 MATLAB

We will use Matlab to translate algorithms into code and to experiment with linear algebra.

There are a number of ways in which you can use Matlab:

- Via MATLAB that is installed on the same computer as you will execute your performance experiments. This is usually called a "desktop installation of Matlab."

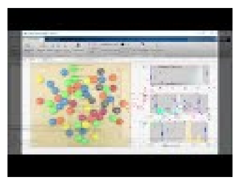
- Via MATLAB Online. You will have to transfer files from the computer where you are performing your experiments to MATLAB Online. You could try to set up MATLAB Drive, which allows you to share files easily between computers and with MATLAB Online. Be warned that there may be a delay in when files show up, and as a result you may be using old data to plot if you aren't careful!

If you are using these materials as part of an offering of the Massive Open Online Course (MOOC) titled "Advanced Linear Algebra: Foundations to Frontiers," you will be given a temporary license to Matlab, courtesy of MathWorks. In this case, there will be additional instructions on how to set up MATLAB Online, in the Unit on edX that corresponds to this section.

You need relatively little familiarity with MATLAB in order to learn what we want you to learn in this course. So, you could just skip these tutorials altogether, and come back to them if you find you want to know more about MATLAB and its programming language (M-script).

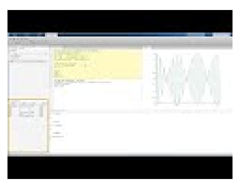
Below you find a few short videos that introduce you to MATLAB. For a more comprehensive tutorial, you may want to visit MATLAB Tutorials at MathWorks and click "Launch Tutorial".

What is MATLAB?



`https://www.youtube.com/watch?v=2sB-NMD9Qhk`

Getting Started with MATLAB Online



`https://www.youtube.com/watch?v=4shp284pGc8`

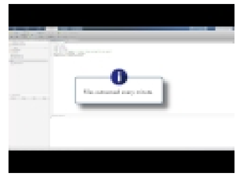
MATLAB Variables

`https://www.youtube.com/watch?v=gPIsIzHJA9I`

MATLAB as a Calculator

`https://www.youtube.com/watch?v=K9xy5kQHDBo`

Managing Files with MATLAB Online



https://www.youtube.com/watch?v=mqYwMnM-x5Q

Remark 0.2.3.1 Some of you may choose to use MATLAB on your personal computer while others may choose to use MATLAB Online. Those who use MATLAB Online will need to transfer some of the downloaded materials to that platform.

0.2.4 Setting up to implement in C (optional)

You may want to return to this unit later in the course. We are still working on adding programming exercises that require C implementation.

In some of the enrichments in these notes and the final week on how to attain performance, we suggest implementing algorithms that are encounted in C. Those who intend to pursue these activities will want to install a Basic Linear Algebra Subprograms (BLAS) library and our libflame library (which not only provides higher level linear algebra functionality, but also allows one to program in a manner that mirrors how we present algorithms.)

0.2.4.1 Installing the BLAS

The Basic Linear Algebra Subprograms (BLAS) are an interface to fundamental linear algebra operations. The idea is that if we write our software in terms of calls to these routines and vendors optimize an implementation of the BLAS, then our software can be easily ported to different computer architectures while achieving reasonable performance.

A popular and high-performing open source implementation of the BLAS is provided by our BLAS-like Library Instantation Software (BLIS). The following steps will install BLIS if you are using the Linux OS (on a Mac, there may be a few more steps, which are discussed later in this unit.)

- Visit the BLIS Github repository.
- Click on

  and copy https://github.com/flame/blis.git.
- In a terminal session, in your home directory, enter

  ```
  git clone https://github.com/flame/blis.git
  ```

  (to make sure you get the address right, you will want to paste the address you copied in the last step.)

- Change directory to blis:

```
cd blis
```

- Indicate a specific version of BLIS so that we all are using the same release:

```
git checkout pfhp
```

- Configure, build, and install with OpenMP turned on.

```
./configure -t openmp -p ~/blis auto
make -j8
make check -j8
make install
```

  The -p ~/blis installs the library in the subdirectory ~/blis of your home directory, which is where the various exercises in the course expect it to reside.

- If you run into a problem while installing BLIS, you may want to consult https://github.com/flame/blis/blob/master/docs/BuildSystem.md.

On Mac OS-X

- You may need to install Homebrew, a program that helps you install various software on you mac. Warning: you may need "root" privileges to do so.

```
$ /usr/bin/ruby -e "$(curl -fsSL https://raw.githubusercontent.com/Homebrew/install/ma
```

  Keep an eye on the output to see if the “Command Line Tools” get installed. This may not be installed if you already have Xcode Command line tools installed. If this happens, post in the "Discussion" for this unit, and see if someone can help you out.

- Use Homebrew to install the gcc compiler:

```
$ brew install gcc
```

  Check if gcc installation overrides clang:

```
$ which gcc
```

  The output should be /usr/local/bin. If it isn’t, you may want to add /usr/local/bin to "the path." I did so by inserting

```
export PATH="/usr/local/bin:$PATH"
```

  into the file .bash_profile in my home directory. (Notice the "period" before "bash_profile"

- Now you can go back to the beginning of this unit, and follow the instructions to install BLIS.

0.2.4.2 Installing libflame

Higher level linear algebra functionality, such as the various decompositions we will discuss in this course, are supported by the LAPACK library [1]. Our libflame library is an implementation of LAPACK that also exports an API for representing algorithms in code in a way that closely reflects the FLAME notation to which you will be introduced in the course.

The libflame library can be cloned from

- https://github.com/flame/libflame.

Instructions on how to install it are at

- https://github.com/flame/libflame/blob/master/INSTALL.

0.3 Enrichments

In each week, we include "enrichments" that allow the participant to go beyond.

0.3.1 Ten surprises from numerical linear algebra

You may find the following list of insights regarding numerical linear algebra, compiled by John D. Cook, interesting:

- John D. Cook. Ten surprises from numerical linear algebra. 2010.

0.3.2 Best algorithms of the 20th century

An article published in SIAM News, a publication of the Society for Industrial and Applied Mathematics, lists the ten most important algorithms of the 20th century [6]:

1. *1946:* John von Neumann, Stan Ulam, and Nick Metropolis, all at the Los Alamos Scientific Laboratory, cook up the *Metropolis algorithm*, also known as the Monte Carlo method.

2. *1947:* George Dantzig, at the RAND Corporation, creates the *simplex method for linear programming*.

3. *1950:* Magnus Hestenes, Eduard Stiefel, and Cornelius Lanczos, all from the Institute for Numerical Analysisat the National Bureau of Standards, initiate the development of *Krylov subspace iteration methods*.

4. *1951:* Alston Householder of Oak Ridge National Laboratory formalizes the *decompositional approach to matrix computations*.

5. *1957:* John Backus leads a team at IBM in developing the *Fortran optimizing compiler*.

6. *1959–61:* J.G.F. Francis of Ferranti Ltd., London, finds a stable method for computing eigenvalues, known as the *QR algorithm*.

7. *1962:* Tony Hoare of Elliott Brothers, Ltd., London, presents *Quicksort*.

8. *1965:* James Cooley of the IBM T.J. Watson Research Center and John Tukey of PrincetonUniversity and AT&T Bell Laboratories unveil the *fast Fourier transform*.

9. *1977:* Helaman Ferguson and Rodney Forcade of Brigham Young University advance an *integer relation detection algorithm*.

10. *1987:* Leslie Greengard and Vladimir Rokhlin of Yale University invent the *fast multipole algorithm*.

Of these, we will explicitly cover three: the decomposition method to matrix computations, Krylov subspace methods, and the QR algorithm. Although not explicitly covered, your understanding of numerical linear algebra will also be a first step towards understanding some of the other numerical algorithms listed.

0.4 Wrap Up

0.4.1 Additional Homework

For a typical week, additional assignments may be given in this unit.

0.4.2 Summary

In a typical week, we provide a quick summary of the highlights in this unit.

Part I

Orthogonality

Week 1

Norms

1.1 Opening

1.1.1 Why norms?

YouTube: `https://www.youtube.com/watch?v=DKX3TdQWQ90`

The following exercises expose some of the issues that we encounter when computing.

We start by computing $b = Ux$, where U is upper triangular.

Homework 1.1.1.1 Compute

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} =$$

Solution.

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$$

Next, let's examine the slightly more difficult problem of finding a vector x that satisfies $Ux = b$.

Homework 1.1.1.2 Solve

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$$

Solution. We can recognize the relation between this problem and Homework 1.1.1.1 and hence deduce the answer without computation:

$$\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

The point of these two homework exercises is that if one creates a (nonsingular) $n \times n$ matrix U and vector x of size n, then computing $b = Ux$ followed by solving $U\widehat{x} = b$ should leave us with a vector $\widehat{x}$ such that $x = \widehat{x}$.

Remark 1.1.1.1 We don't "teach" Matlab in this course. Instead, we think that Matlab is intuitive enough that we can figure out what the various commands mean. We can always investigate them by typing

```
help <command>
```

in the command window. For example, for this unit you may want to execute

```
help format
help rng
help rand
help triu
help *
help \
help diag
help abs
help min
help max
```

If you want to learn more about Matlab, you may want to take some of the tutorials offered by Mathworks at `https://www.mathworks.com/support/learn-with-matlab-tutorials.html`.

Let us see if Matlab can compute the solution of a triangular matrix correctly.

Homework 1.1.1.3 In Matlab's command window, create a random upper triangular matrix U:

```
format long

rng( 0 );
n = 3
U = triu( rand( n,n ) )
x = rand( n,1 )
```

Report results in long format. Seed the random number generator so that we all create the same random matrix U and vector x.

```
b = U * x;
```
Compute right-hand side b from known solution x.

```
xhat = U \ b;
```
Solve $U\widehat{x} = b$.

```
xhat - x
```
Report the difference between $\widehat{x}$ and x.

What do we notice?

Next, check how close $U\widehat{x}$ is to $b = Ux$:

```
b - U * xhat
```
This is known as the residual.

What do we notice?

Solution. A script with the described commands can be found in Assignments/Week01/matlab/Test_Upper_triangular_solve_3.m.

Some things we observe:

- $\widehat{x} - x$ does not equal zero. This is due to the fact that the computer stores floating point numbers and computes with floating point arithmetic, and as a result roundoff error happens.
- The difference is small (notice the 1.0e-15$*$ before the vector, which shows that each entry in $\widehat{x} - x$ is around 10^{-15}.
- The residual $b - U\widehat{x}$ is small.
- Repeating this with a much larger n make things cumbersome since very long vectors are then printed.

To be able to compare more easily, we will compute the Euclidean length of $\widehat{x} - x$ instead using the Matlab command norm(xhat - x). By adding a semicolon at the end of Matlab commands, we suppress output.

Homework 1.1.1.4 Execute

```
format long
```
Report results in long format.

```
rng( 0 );
n = 100;
U = triu( rand( n,n ) );
x = rand( n,1 );
```
Seed the random number generator so that we all create the same random matrix U and vector x.

```
b = U * x;
```
Compute right-hand side b from known solution x.

```
xhat = U \ b;
```
Solve $U\widehat{x} = b$

```
norm( xhat - x )
```
Report the Euclidean length of the difference between $\widehat{x}$ and x.

What do we notice?

Next, check how close $U\widehat{x}$ is to $b = Ux$, again using the Euclidean length:

```
norm( b - U * xhat  )
```

What do we notice?

Solution. A script with the described commands can be found in Assignments/Week01/matlab/Test_Upper_triangular_solve_100.m.

Some things we observe:

- $norm(\widehat{x} - x)$, the Euclidean length of $\widehat{x} - x$, is huge. Matlab computed the wrong answer!

- However, the computed $\widehat{x}$ solves a problem that corresponds to a slightly different right-hand side. Thus, $\widehat{x}$ appears to be the solution to an only slightly changed problem.

The next exercise helps us gain insight into what is going on.

Homework 1.1.1.5 Continuing with the U, x, b, and xhat from Homework 1.1.1.4, consider

- When is an upper triangular matrix singular?
- How large is the smallest element on the diagonal of the U from Homework 1.1.1.4? (min(abs(diag(U))) returns it!)
- If U were singular, how many solutions to $U\widehat{x} = b$ would there be? How can we characterize them?
- What is the relationship between $\widehat{x} - x$ and U?

What have we learned?

Solution.

- When is an upper triangular matrix singular?

  Answer:

  If and only if there is a zero on its diagonal.

- How large is the smallest element on the diagonal of the U from Homework 1.1.1.4? (min(abs(diag(U))) returns it!)

  Answer:

  It is small in magnitude. This is not surprising, since it is a random number and hence as the matrix size increases, the chance of placing a small entry (in magnitude) on the diagonal increases.

- If U were singular, how many solutions to $U\widehat{x} = b$ would there be? How can we characterize them?

  Answer:

  An infinite number. Any vector in the null space can be added to a specific solution to create another solution.

- What is the relationship between $\widehat{x} - x$ and U?

  Answer:

  It maps almost to the zero vector. In other words, it is close to a vector in the null space of the matrix U that has its smallest entry (in magnitude) on the diagonal changed to a zero.

What have we learned? The :"wrong" answer that Matlab computed was due to the fact that matrix U was almost singular.

To mathematically qualify and quantify all this, we need to be able to talk about "small" and "large" vectors, and "small" and "large" matrices. For that, we need to generalize the notion of length. By the end of this week, this will give us some of the tools to more fully understand what we have observed.

YouTube: `https://www.youtube.com/watch?v=2ZEtcnaynnM`

1.1.2 Overview

- 1.1 Opening
  - 1.1.1 Why norms?
  - 1.1.2 Overview
  - 1.1.3 What you will learn
- 1.2 Vector Norms
  - 1.2.1 Absolute value
  - 1.2.2 What is a vector norm?
  - 1.2.3 The vector 2-norm (Euclidean length)
  - 1.2.4 The vector p-norms
  - 1.2.5 Unit ball
  - 1.2.6 Equivalence of vector norms
- 1.3 Matrix Norms
  - 1.3.1 Of linear transformations and matrices
  - 1.3.2 What is a matrix norm?

1.1.3 What you will learn

Numerical analysis is the study of how the perturbation of a problem or data affects the accuracy of computation. This inherently means that you have to be able to measure whether changes are large or small. That, in turn, means we need to be able to quantify whether vectors or matrices are large or small. Norms are a tool for measuring magnitude.

Upon completion of this week, you should be able to

- Prove or disprove that a function is a norm.
- Connect linear transformations to matrices.
- Recognize, compute, and employ different measures of length, which differ and yet are equivalent.
- Exploit the benefits of examining vectors on the unit ball.
- Categorize different matrix norms based on their properties.
- Describe, in words and mathematically, how the condition number of a matrix affects how a relative change in the right-hand side can amplify into relative change in the solution of a linear system.
- Use norms to quantify the conditioning of solving linear systems.

1.2 Vector Norms

1.2.1 Absolute value

Remark 1.2.1.1 Don't Panic!

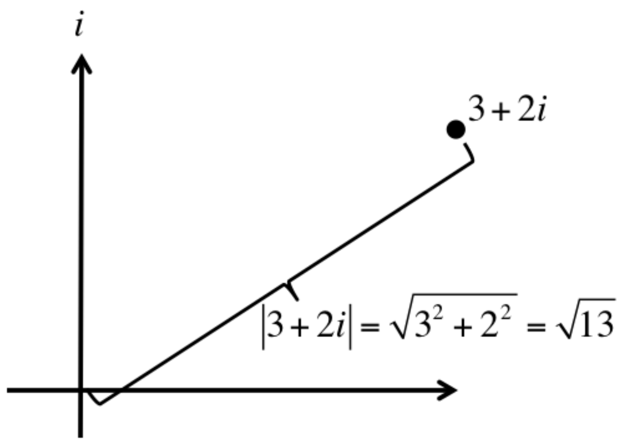
In this course, we mostly allow scalars, vectors, and matrices to be complex-valued. This means we will use terms like "conjugate" and "Hermitian" quite liberally. You will think this is a big deal, but actually, if you just focus on the real case, you will notice that the complex case is just a natural extension of the real case.

YouTube: `https://www.youtube.com/watch?v=V5ZQmR4zTeU`

Recall that $|\cdot| : \mathbb{C} \rightarrow \mathbb{R}$ is the function that returns the absolute value of the input. In other words, if $\alpha = \alpha_r + \alpha_c i$, where α_r and α_c are the real and complex parts of α, respectively, then

$$|\alpha| = \sqrt{\alpha_r^2 + \alpha_c^2}.$$

The absolute value (magnitude) of a complex number can also be thought of as the (Euclidean) distance from the point in the complex plane to the origin of that plane, as illustrated below for the number $3 + 2i$.



Alternatively, we can compute the absolute value as

$$
\begin{array}{l}
|\alpha| \\
\quad = \\
\sqrt{\alpha_r^2 + \alpha_c^2} \\
\quad = \\
\sqrt{\alpha_r^2 - \alpha_c\alpha_r + \alpha_r\alpha_c + \alpha_c^2} \\
\quad = \\
\sqrt{(\alpha_r - \alpha_c i)(\alpha_r + \alpha_c i)} \\
\quad = \\
\sqrt{\overline{\alpha}\alpha} \quad ,
\end{array}
$$

where $\overline{\alpha}$ denotes the complex conjugate of α:

$$\overline{\alpha} = \overline{\alpha_r + \alpha_c i} = \alpha_r - \alpha_c i.$$

Th absolute value function has the following properties:

- $\alpha \neq 0 \Rightarrow |\alpha| > 0$ ($|\cdot|$ is positive definite),
- $|\alpha\beta| = |\alpha||\beta|$ ($|\cdot|$ is homogeneous), and
- $|\alpha + \beta| \leq |\alpha| + |\beta|$ ($|\cdot|$ obeys the triangle inequality).

Norms are functions from a domain to the real numbers that are positive definite, homogeneous, and obey the triangle inequality. This makes the absolute value function an example of a norm.

The below exercises help refresh your fluency with complex arithmetic.

Homework 1.2.1.1

1. $(1+i)(2-i) =$
2. $(2-i)(1+i) =$
3. $\overline{(1-i)}(2-i) =$
4. $\overline{\overline{(1-i)}(2-i)} =$
5. $\overline{(2-i)}(1-i) =$
6. $(1-i)\overline{(2-i)} =$

Solution.

1. $(1+i)(2-i) = 2 + 2i - i - i^2 = 2 + i + 1 = 3 + i$
2. $(2-i)(1+i) = 2 - i + 2i - i^2 = 2 + i + 1 = 3 + i$
3. $\overline{(1-i)}(2-i) = (1+i)(2-i) = 2 - i + 2i - i^2 = 3 + i$

4. $\overline{\overline{(1-i)}(2-i)} = \overline{(1+i)(2-i)} = \overline{2-i+2i-i^2} = \overline{2+i+1} = \overline{3+i} = 3-i$

5. $\overline{(2-i)}(1-i) = (2+i)(1-i) = 2-2i+i-i^2 = 2-i+1 = 3-i$

6. $(1-i)\overline{(2-i)} = (1-i)(2+i) = 2+i-2i-i^2 = 2-i+1 = 3-i$

Homework 1.2.1.2 Let $\alpha, \beta \in \mathbb{C}$.

1. ALWAYS/SOMETIMES/NEVER: $\alpha\beta = \beta\alpha$.

2. ALWAYS/SOMETIMES/NEVER: $\overline{\alpha}\beta = \overline{\beta}\alpha$.

Hint. Let $\alpha = \alpha_r + \alpha_c i$ and $\beta = \beta_r + \beta_c i$, where $\alpha_r, \alpha_c, \beta_r, \beta_c \in \mathbb{R}$.

Answer.

1. ALWAYS: $\alpha\beta = \beta\alpha$.

2. SOMETIMES: $\overline{\alpha}\beta = \overline{\beta}\alpha$.

Solution.

1. ALWAYS: $\alpha\beta = \beta\alpha$.

   Proof:

$$
\begin{array}{l}
\alpha\beta \\
\quad = \qquad < \text{substitute} > \\
(\alpha_r + \alpha_c i)(\beta_r + \beta_c i) \\
\quad = \qquad < \text{multiply out} > \\
\alpha_r\beta_r + \alpha_r\beta_c i + \alpha_c\beta_r i - \alpha_c\beta_c \\
\quad = \qquad < \text{commutativity of real multiplication} > \\
\beta_r\alpha_r + \beta_r\alpha_c i + \beta_c\alpha_r i - \beta_c\alpha_c \\
\quad = \qquad < \text{factor} > \\
(\beta_r + \beta_c i)(\alpha_r + \alpha_c i) \\
\quad = \qquad < \text{substitute} > \\
\beta\alpha.
\end{array}
$$

2. SOMETIMES: $\overline{\alpha}\beta = \overline{\beta}\alpha$.

   An example where it is true: $\alpha = \beta = 0$.

   An example where it is false: $\alpha = 1$ and $\beta = i$. Then $\overline{\alpha}\beta = 1 \times i = i$ and $\overline{\beta}\alpha = -i \times 1 = -i$.

Homework 1.2.1.3 Let $\alpha, \beta \in \mathbb{C}$.

ALWAYS/SOMETIMES/NEVER: $\overline{\overline{\alpha}\beta} = \overline{\beta}\alpha$.

Hint. Let $\alpha = \alpha_r + \alpha_c i$ and $\beta = \beta_r + \beta_c i$, where $\alpha_r, \alpha_c, \beta_r, \beta_c \in \mathbb{R}$.

Answer. ALWAYS

Now prove it!

Solution 1.

$$
\begin{array}{l}
\overline{\overline{\alpha}\beta} \\
\quad = \quad < \alpha = \alpha_r + \alpha_c i; \beta = \beta_r + \beta_c i > \\
\overline{\overline{(\alpha_r + \alpha_c i)}(\beta_r + \beta_c i)} \\
\quad = \quad < \text{ conjugate } \alpha > \\
\overline{(\alpha_r - \alpha_c i)(\beta_r + \beta_c i)} \\
\quad = \quad < \text{ multiply out } > \\
\overline{(\alpha_r \beta_r - \alpha_c \beta_r i + \alpha_r \beta_c i + \alpha_c \beta_c)} \\
\quad = \quad < \text{ conjugate } > \\
\alpha_r \beta_r + \alpha_c \beta_r i - \alpha_r \beta_c i + \alpha_c \beta_c \\
\quad = \quad < \text{ rearrange } > \\
\beta_r \alpha_r + \beta_r \alpha_c i - \beta_c \alpha_r i + \beta_c \alpha_c \\
\quad = \quad < \text{ factor } > \\
(\beta_r - \beta_c i)(\alpha_r + \alpha_c i) \\
\quad = \quad < \text{ definition of conjugation } > \\
\overline{(\beta_r + \beta_c i)}(\alpha_r + \alpha_c i) \\
\quad = \quad < \alpha = \alpha_r + \alpha_c i; \beta = \beta_r + \beta_c i > \\
\overline{\beta}\alpha
\end{array}
$$

Solution 2. Proofs in mathematical textbooks seem to always be wonderfully smooth arguments that lead from the left-hand side of an equivalence to the right-hand side. In practice, you may want to start on the left-hand side, and apply a few rules:

$$
\begin{array}{l}
\overline{\overline{\alpha}\beta} \\
\quad = \quad < \alpha = \alpha_r + \alpha_c i; \beta = \beta_r + \beta_c i > \\
\overline{\overline{(\alpha_r + \alpha_c i)}(\beta_r + \beta_c i)} \\
\quad = \quad < \text{ conjugate } \alpha > \\
\overline{(\alpha_r - \alpha_c i)(\beta_r + \beta_c i)} \\
\quad = \quad < \text{ multiply out } > \\
\overline{(\alpha_r \beta_r - \alpha_c \beta_r i + \alpha_r \beta_c i + \alpha_c \beta_c)} \\
\quad = \quad < \text{ conjugate } > \\
\alpha_r \beta_r + \alpha_c \beta_r i - \alpha_r \beta_c i + \alpha_c \beta_c
\end{array}
$$

and then move on to the right-hand side, applying a few rules:

$$
\begin{array}{l}
\overline{\beta}\alpha \\
\quad = \quad < \alpha = \alpha_r + \alpha_c i; \beta = \beta_r + \beta_c i > \\
\overline{(\beta_r + \beta_c i)}(\alpha_r + \alpha_c i) \\
\quad = \quad < \text{ conjugate } \beta > \\
(\beta_r - \beta_c i)(\alpha_r + \alpha_c i) \\
\quad = \quad < \text{ multiply out } > \\
\beta_r \alpha_r + \beta_r \alpha_c i - \beta_c \alpha_r i + \beta_c \alpha_c .
\end{array}
$$

At that point, you recognize that

$$\alpha_r \beta_r + \alpha_c \beta_r i - \alpha_r \beta_c i + \alpha_c \beta_c \quad = \quad \beta_r \alpha_r + \beta_r \alpha_c i - \beta_c \alpha_r i + \beta_c \alpha_c$$

since the second is a rearrangement of the terms of the first. Optionally, you then go back and presents these insights as a smooth argument that leads from the expression on the left-hand side to the one on the right-hand side:

$$\begin{array}{l}
\overline{\overline{\alpha}\beta} \\
\quad = \quad < \alpha = \alpha_r + \alpha_c i; \beta = \beta_r + \beta_c i > \\
\overline{\overline{(\alpha_r + \alpha_c i)}(\beta_r + \beta_c i)} \\
\quad = \quad < \text{ conjugate } \alpha > \\
\overline{(\alpha_r - \alpha_c i)(\beta_r + \beta_c i)} \\
\quad = \quad < \text{ multiply out } > \\
\overline{(\alpha_r \beta_r - \alpha_c \beta_r i + \alpha_r \beta_c i + \alpha_c \beta_c)} \\
\quad = \quad < \text{ conjugate } > \\
\alpha_r \beta_r + \alpha_c \beta_r i - \alpha_r \beta_c i + \alpha_c \beta_c \\
\quad = \quad < \text{ rearrange } > \\
\beta_r \alpha_r + \beta_r \alpha_c i - \beta_c \alpha_r i + \beta_c \alpha_c \\
\quad = \quad < \text{ factor } > \\
(\beta_r - \beta_c i)(\alpha_r + \alpha_c i) \\
\quad = \quad < \text{ definition of conjugation } > \\
\overline{(\beta_r + \beta_c i)}(\alpha_r + \alpha_c i) \\
\quad = \quad < \alpha = \alpha_r + \alpha_c i; \beta = \beta_r + \beta_c i > \\
\overline{\beta}\alpha.
\end{array}$$

Solution 3. Yet another way of presenting the proof uses an "equivalence style proof." The idea is to start with the equivalence you wish to prove correct:

$$\overline{\overline{\alpha}\beta} = \overline{\beta}\alpha$$

and through a sequence of equivalent statement argue that this evaluates to TRUE:

$$\begin{array}{l}
\overline{\overline{\alpha}\beta} = \overline{\beta}\alpha \\
\quad \Leftrightarrow \quad < \alpha = \alpha_r + \alpha_c i; \beta = \beta_r + \beta_c i > \\
\overline{\overline{(\alpha_r + \alpha_c i)}(\beta_r + \beta_c i)} = \overline{(\beta_r + \beta_c i)}(\alpha_r + \alpha_c i) \\
\quad \Leftrightarrow \quad < \text{ conjugate } \times 2 > \\
\overline{(\alpha_r - \alpha_c i)(\beta_r + \beta_c i)} = (\beta_r - \beta_c i)(\alpha_r + \alpha_c i) \\
\quad \Leftrightarrow \quad < \text{ multiply out } \times 2 > \\
\overline{\alpha_r \beta_r + \alpha_r \beta_c i - \alpha_c \beta_r i + \alpha_c \beta_c} = \beta_r \alpha_r + \beta_r \alpha_c i - \beta_c \alpha_r i + \beta_c \alpha_c \\
\quad \Leftrightarrow \quad < \text{ conjugate } > \\
\alpha_r \beta_r - \alpha_r \beta_c i + \alpha_c \beta_r i + \alpha_c \beta_c = \beta_r \alpha_r + \beta_r \alpha_c i - \beta_c \alpha_r i + \beta_c \alpha_c \\
\quad \Leftrightarrow \quad < \text{ subtract equivalent terms from left-hand side and right-hand side } > \\
0 = 0 \\
\quad \Leftrightarrow \quad < \text{ algebra } > \\
\text{TRUE}.
\end{array}$$

By transitivity of equivalence, we conclude that $\overline{\overline{\alpha}\beta} = \overline{\beta}\alpha$ is TRUE.

Homework 1.2.1.4 Let $\alpha \in \mathbb{C}$.

ALWAYS/SOMETIMES/NEVER: $\overline{\alpha}\alpha \in \mathbb{R}$

Answer. ALWAYS.

Now prove it!

Solution. Let $\alpha = \alpha_r + \alpha_c i$. Then

$$\begin{array}{l}
\overline{\alpha}\alpha \\
\quad = \qquad < \text{ instantiate } > \\
\overline{(\alpha_r + \alpha_c i)}(\alpha_r + \alpha_c i) \\
\quad = \qquad < \text{ conjugate } > \\
(\alpha_r - \alpha_c i)(\alpha_r + \alpha_c i) \\
\quad = \qquad < \text{ multiply out } > \\
\alpha_r^2 + \alpha_c^2,
\end{array}$$

which is a real number.

Homework 1.2.1.5 Prove that the absolute value function is homogeneous: $|\alpha\beta| = |\alpha||\beta|$ for all $\alpha, \beta \in \mathbb{C}$.

Solution.

$$\begin{array}{l}
|\alpha\beta| = |\alpha||\beta| \\
\quad \Leftrightarrow \qquad < \text{ squaring both sides simplifies } > \\
|\alpha\beta|^2 = |\alpha|^2|\beta|^2 \\
\quad \Leftrightarrow \qquad < \text{ instantiate } > \\
|(\alpha_r + \alpha_c i)(\beta_r + \beta_c i)|^2 = |\alpha_r + \alpha_c i|^2|\beta_r + \beta_c i|^2 \\
\quad \Leftrightarrow \qquad < \text{ algebra } > \\
|(\alpha_r\beta_r - \alpha_c\beta_c) + (\alpha_r\beta_c + \alpha_c\beta_r)i|^2 = (\alpha_r^2 + \alpha_c^2)(\beta_r^2 + \beta_c^2) \\
\quad \Leftrightarrow \qquad < \text{ algebra } > \\
(\alpha_r\beta_r - \alpha_c\beta_c)^2 + (\alpha_r\beta_c + \alpha_c\beta_r)^2 = (\alpha_r^2 + \alpha_c^2)(\beta_r^2 + \beta_c^2) \\
\quad \Leftrightarrow \qquad < \text{ algebra } > \\
\alpha_r^2\beta_r^2 - 2\alpha_r\alpha_c\beta_r\beta_c + \alpha_c^2\beta_c^2 + \alpha_r^2\beta_c^2 + 2\alpha_r\alpha_c\beta_r\beta_c + \alpha_c^2\beta_r^2 \\
\qquad\qquad\qquad = \alpha_r^2\beta_r^2 + 2\alpha_c^2\beta_r^2 + \alpha_c^2\beta_c^2 \\
\quad \Leftrightarrow \qquad < \text{ subtract equivalent terms from both sides } > \\
0 = 0 \\
\quad \Leftrightarrow \qquad < \text{ algebra } > \\
T
\end{array}$$

Homework 1.2.1.6 Let $\alpha \in \mathbb{C}$.

ALWAYS/SOMETIMES/NEVER: $|\overline{\alpha}| = |\alpha|$.

Answer. ALWAYS

Now prove it!

Solution. Let $\alpha = \alpha_r + \alpha_c i$.

$$
\begin{array}{l}
|\overline{\alpha}| \\
\quad = \quad < \text{ instantiate } > \\
|\overline{\alpha_r + \alpha_c i}| \\
\quad = \quad < \text{ conjugate } > \\
|\alpha_r - \alpha_c i| \\
\quad = \quad < \text{ definition of } | \cdot | > \\
\sqrt{\alpha_r^2 + \alpha_c^2} \\
\quad = \quad < \text{ definition of } | \cdot | > \\
|\alpha_r + \alpha_c i| \\
\quad = \quad < \text{ instantiate } > \\
|\alpha|
\end{array}
$$

1.2.2 What is a vector norm?

YouTube: `https://www.youtube.com/watch?v=CTrUVfLGcNM`

A vector norm extends the notion of an absolute value to vectors. It allows us to measure the magnitude (or length) of a vector. In different situations, a different measure may be more appropriate.

Definition 1.2.2.1 Vector norm. Let $\nu : \mathbb{C}^m \rightarrow \mathbb{R}$. Then ν is a (vector) norm if for all $x, y \in \mathbb{C}^m$ and all $\alpha \in \mathbb{C}$

- $x \neq 0 \Rightarrow \nu(x) > 0$ (ν is positive definite),
- $\nu(\alpha x) = |\alpha|\nu(x)$ (ν is homogeneous), and
- $\nu(x + y) \leq \nu(x) + \nu(y)$ (ν obeys the triangle inequality).

$\Diamond$

Homework 1.2.2.1 TRUE/FALSE: If $\nu : \mathbb{C}^m \rightarrow \mathbb{R}$ is a norm, then $\nu(0) = 0$.

Hint. From context, you should be able to tell which of these 0's denotes the zero vector of a given size and which is the scalar 0.

$0x = 0$ (multiplying any vector x by the scalar 0 results in a vector of zeroes).

Answer. TRUE.

Now prove it.

Solution. Let $x \in \mathbb{C}^m$ and, just for clarity this first time, $\vec{0}$ be the zero vector of size m so that 0 is the scalar zero. Then

$$\begin{array}{l} \nu(\vec{0}) \\ \quad = \quad < 0 \cdot x = \vec{0} > \\ \nu(0 \cdot x) \\ \quad = \quad < \nu(\cdots) \text{ is homogeneous } > \\ 0\nu(x) \\ \quad = \quad < \text{ algebra } > \\ 0 \end{array}$$

Remark 1.2.2.2 We typically use $\| \cdot \|$ instead of $\nu(\cdot)$ for a function that is a norm.

1.2.3 The vector 2-norm (Euclidean length)

YouTube: `https://www.youtube.com/watch?v=bxDDpUZEqBs`

The length of a vector is most commonly measured by the "square root of the sum of the squares of the elements," also known as the Euclidean norm. It is called the 2-norm because it is a member of a class of norms known as p-norms, discussed in the next unit.

Definition 1.2.3.1 Vector 2-norm. The vector 2-norm $\| \cdot \|_2 : \mathbb{C}^m \to \mathbb{R}$ is defined for $x \in \mathbb{C}^m$ by

$$\|x\|_2 = \sqrt{|\chi_0|^2 + \cdots + |\chi_{m-1}|^2} = \sqrt{\sum_{i=0}^{m-1} |\chi_i|^2}.$$

Equivalently, it can be defined by

$$\|x\|_2 = \sqrt{x^H x}$$

or

$$\|x\|_2 = \sqrt{\overline{\chi}_0 \chi_0 + \cdots + \overline{\chi}_{m-1} \chi_{m-1}} = \sqrt{\sum_{i=0}^{m-1} \overline{\chi}_i \chi_i}.$$

$\Diamond$

Remark 1.2.3.2 The notation x^H requires a bit of explanation. If

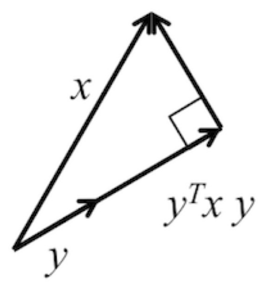
$$x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_m \end{pmatrix}$$

then the row vector

$$x^H = \left(\begin{array}{ccc} \overline{\chi}_0 & \cdots & \overline{\chi}_m \end{array} \right)$$

is the Hermitian transpose of x (or, equivalently, the Hermitian transpose of the vector x that is viewed as a matrix) and $x^H y$ can be thought of as the dot product of x and y or, equivalently, as the matrix-vector multiplication of the matrix x^H times the vector y.

To prove that the 2-norm is a norm (just calling it a norm doesn't mean it is, after all), we need a result known as the Cauchy-Schwartz inequality. This inequality relates the magnitude of the dot product of two vectors to the product of their 2-norms: if $x, y \in \mathbb{R}^m$, then $|x^T y| \leq \|x\|_2 \|y\|_2$. To motivate this result before we rigorously prove it, recall from your undergraduate studies that the component of x in the direction of a vector y of unit length is given by $(y^T x) y$, as illustrated by



The length of the component of x in the direction of y then equals

$$\begin{array}{l} \|(y^T x) y\|_2 \\ \quad = \quad < \text{ definition } > \\ \sqrt{(y^T x)^T y^T (y^T x) y} \\ \quad = \quad < z\alpha = \alpha z > \\ \sqrt{(x^T y)^2 y^T y} \\ \quad = \quad < y \text{ has unit length } > \\ |y^T x| \\ \quad = \quad < \text{ definition } > \\ |x^T y|. \end{array}$$

Thus $|x^T y| \leq \|x\|_2$ (since a component should be shorter than the whole). If y is not of unit length (but a nonzero vector), then $|x^T \frac{y}{\|y\|_2}| \leq \|x\|_2$ or, equivalently, $|x^T y| \leq \|x\|_2 \|y\|_2$.

We now state this result as a theorem, generalized to complex valued vectors:

Theorem 1.2.3.3 Cauchy-Schwartz inequality. *Let* $x, y \in \mathbb{C}^m$. *Then* $|x^H y| \leq \|x\|_2 \|y\|_2$.

Proof. Assume that $x \neq 0$ and $y \neq 0$, since otherwise the inequality is trivially true. We can then choose $\widehat{x} = x/\|x\|_2$ and $\widehat{y} = y/\|y\|_2$. This leaves us to prove that $|\widehat{x}^H \widehat{y}| \leq 1$ since $\|\widehat{x}\|_2 = \|\widehat{y}\|_2 = 1$.

Pick

$$\alpha = \left\{ \begin{array}{ll} 1 & \text{if } x^H x = 0 \\ \widehat{y}^H \widehat{x} / |\widehat{x}^H \widehat{y}| & \text{otherwise.} \end{array} \right.$$

so that $|\alpha| = 1$ and $\alpha \widehat{x}^H \widehat{y}$ is real and nonnegative. Note that since it is real we also know that

$$\begin{array}{l} \alpha\widehat{x}^H\widehat{y} \\ \quad = \qquad < \beta = \overline{\beta} \text{ if } \beta \text{ is real } > \\ \overline{\alpha\widehat{x}^H\widehat{y}} \\ \quad = \qquad < \text{ property of complex conjugation } > \\ \overline{\alpha}\widehat{y}^H\widehat{x} \end{array}.$$

Now,

$$\begin{array}{l} 0 \\ \quad \leq \qquad < \|\cdot\|_2 \text{ is nonnegative definite } > \\ \|\widehat{x} - \alpha\widehat{y}\|_2^2 \\ \quad = \qquad < \|z\|_2^2 = z^H z > \\ (\widehat{x} - \alpha\widehat{y})^H(\widehat{x} - \alpha\widehat{y}) \\ \quad = \qquad < \text{ multiplying out } > \\ \widehat{x}^H\widehat{x} - \overline{\alpha}\widehat{y}^H\widehat{x} - \alpha\widehat{x}^H\widehat{y} + \overline{\alpha}\alpha\widehat{y}^H\widehat{y} \\ \quad = \qquad < \text{ above assumptions and observations } > \\ 1 - 2\alpha\widehat{x}^H\widehat{y} + |\alpha|^2 \\ \quad = \qquad < \alpha\widehat{x}^H\widehat{y} = |\widehat{x}^H\widehat{y}|; |\alpha| = 1 > \\ 2 - 2|\widehat{x}^H\widehat{y}|. \end{array}$$

Thus $|\widehat{x}^H\widehat{y}| \leq 1$ and therefore $|x^H y| \leq \|x\|_2\|y\|_2$. ■

The proof of Theorem 1.2.3.3 does not employ any of the intuition we used to motivate it in the real valued case just before its statement. We leave it to the reader to prove the Cauchy-Schartz inequality for real-valued vectors by modifying (simplifying) the proof of Theorem 1.2.3.3.

Ponder This 1.2.3.1 Let $x, y \in \mathbb{R}^m$. Prove that $|x^T y| \leq \|x\|_2\|y\|_2$ by specializing the proof of Theorem 1.2.3.3.

The following theorem states that the 2-norm is indeed a norm:

Theorem 1.2.3.4 *The vector 2-norm is a norm.*

We leave its proof as an exercise.

Homework 1.2.3.2 Prove Theorem 1.2.3.4.

Solution. To prove this, we merely check whether the three conditions are met:

Let $x, y \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$ be arbitrarily chosen. Then

- $x \neq 0 \Rightarrow \|x\|_2 > 0$ ($\|\cdot\|_2$ is positive definite):

  Notice that $x \neq 0$ means that at least one of its components is nonzero. Let's assume that $\chi_j \neq 0$. Then

  $$\|x\|_2 = \sqrt{|\chi_0|^2 + \cdots + |\chi_{m-1}|^2} \geq \sqrt{|\chi_j|^2} = |\chi_j| > 0.$$

- $\|\alpha x\|_2 = |\alpha| \|x\|_2$ ($\| \cdot \|_2$ is homogeneous):

$$
\begin{array}{l}
\|\alpha x\|_2 \\
\quad = \quad < \text{ scaling a vector scales its components; definition } > \\
\sqrt{|\alpha \chi_0|^2 + \cdots + |\alpha \chi_{m-1}|^2} \\
\quad = \quad < algebra > \\
\sqrt{|\alpha|^2 |\chi_0|^2 + \cdots + |\alpha|^2 |\chi_{m-1}|^2} \\
\quad = \quad < \text{ algebra } > \\
\sqrt{|\alpha|^2 (|\chi_0|^2 + \cdots + |\chi_{m-1}|^2)} \\
\quad = \quad < \text{ algebra } > \\
|\alpha| \sqrt{|\chi_0|^2 + \cdots + |\chi_{m-1}|^2} \\
\quad = \quad < \text{ definition } > \\
|\alpha| \|x\|_2.
\end{array}
$$

- $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ ($\| \cdot \|_2$ obeys the triangle inequality):

$$
\begin{array}{l}
\|x + y\|_2^2 \\
\quad = \quad < \|z\|_2^2 = \sqrt{z^H z} > \\
(x + y)^H (x + y) \\
\quad = \quad < \text{ distribute } > \\
x^H x + y^H x + x^H y + y^H y \\
\quad = \quad < \overline{\beta} + \beta = 2\text{Real}(\beta) > \\
x^H x + 2\text{Real}(x^H y) + y^H y \\
\quad \leq \quad < \text{ algebra } > \\
x^H x + 2|\text{Real}(x^H y)| + y^H y \\
\quad \leq \quad < \text{ algebra } > \\
x^H x + 2|x^H y| + y^H y \\
\quad \leq \quad < \text{ algebra; Cauchy-Schwartz } > \\
\|x\|_2^2 + 2\|x\|_2 \|y\|_2 + \|y\|_2^2 \\
\quad = \quad < \text{ algebra } > \\
(\|x\|_2 + \|y\|_2)^2.
\end{array}
$$

  Taking the square root (an increasing function that hence maintains the inequality) of both sides yields the desired result.

Throughout this course, we will reason about subvectors and submatrices. Let's get some practice:

Homework 1.2.3.3 Partition $x \in \mathbb{C}^m$ into subvectors:

$$
x = \left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{M-1} \end{array} \right).
$$

ALWAYS/SOMETIMES/NEVER: $\|x_i\|_2 \leq \|x\|_2$.

Answer. ALWAYS

Now prove it!

Solution.

$$
\begin{array}{l}
\|x\|_2^2 \\
\quad = \quad < \text{ partition vector } > \\
\left\| \left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{M-1} \end{array} \right) \right\|_2^2 \\
\quad = \quad < \text{ equivalent definition } > \\
\left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{M-1} \end{array} \right)^H \left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{M-1} \end{array} \right) \\
\quad = \quad < \text{ dot product of partitioned vectors } > \\
x_0^H x_0 + x_1^H x_1 + \cdots + x_{M-1}^H x_{M-1} \\
\quad = \quad < \text{ equivalent definition } > \\
\|x_0\|_2^2 + \|x_1\|_2^2 + \cdots + \|x_{M-1}\|_2^2 \\
\quad \geq \quad < \text{ algebra } > \\
\|x_i\|_2^2
\end{array}
$$

so that $\|x_i\|_2^2 \leq \|x\|_2^2$. Taking the square root of both sides shows that $\|x_i\|_2 \leq \|x\|_2$.

1.2.4 The vector p-norms

YouTube: `https://www.youtube.com/watch?v=WGBMnmgJek8`

A vector norm is a measure of the magnitude of a vector. The Euclidean norm (length) is merely the best known such measure. There are others. A simple alternative is the 1-norm.

Definition 1.2.4.1 Vector 1-norm. The vector 1-norm, $\| \cdot \|_1 : \mathbb{C}^m \rightarrow \mathbb{R}$, is defined for $x \in \mathbb{C}^m$ by

$$\|x\|_1 = |\chi_0| + |\chi_1| + \cdots + |\chi_{m-1}| = \sum_{i=0}^{m-1} |\chi_i|.$$

$\Diamond$

Homework 1.2.4.1 Prove that the vector 1-norm is a norm.

Solution. We show that the three conditions are met:

Let $x, y \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$ be arbitrarily chosen. Then

- $x \neq 0 \Rightarrow \|x\|_1 > 0$ ($\| \cdot \|_1$ is positive definite):

  Notice that $x \neq 0$ means that at least one of its components is nonzero. Let's assume that $\chi_j \neq 0$. Then
  $$\|x\|_1 = |\chi_0| + \cdots + |\chi_{m-1}| \geq |\chi_j| > 0.$$

- $\|\alpha x\|_1 = |\alpha| \|x\|_1$ ($\| \cdot \|_1$ is homogeneous):

  $$\begin{array}{l} \|\alpha x\|_1 \quad = \quad < \text{ scaling a vector-scales-its-components; definition } > \\ |\alpha \chi_0| + \cdots + |\alpha \chi_{m-1}| \\ \quad = \quad < \text{ algebra } > \\ |\alpha||\chi_0| + \cdots + |\alpha||\chi_{m-1}| \\ \quad = \quad < \text{ algebra } > \\ |\alpha|(|\chi_0| + \cdots + |\chi_{m-1}|) \\ \quad = \quad < \text{ definition } > \\ |\alpha|\|x\|_1. \end{array}$$

- $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ ($\| \cdot \|_1$ obeys the triangle inequality):

  $$\begin{array}{l} \|x + y\|_1 \\ \quad = \quad < \text{ vector addition; definition of 1-norm } > \\ |\chi_0 + \psi_0| + |\chi_1 + \psi_1| + \cdots + |\chi_{m-1} + \psi_{m-1}| \\ \quad \leq \quad < \text{ algebra } > \\ |\chi_0| + |\psi_0| + |\chi_1| + |\psi_1| + \cdots + |\chi_{m-1}| + |\psi_{m-1}| \\ \quad = \quad < \text{ commutivity } > \\ |\chi_0| + |\chi_1| + \cdots + |\chi_{m-1}| + |\psi_0| + |\psi_1| + \cdots + |\psi_{m-1}| \\ \quad = \quad < \text{ associativity; definition } > \\ \|x\|_1 + \|y\|_1. \end{array}$$

The vector 1-norm is sometimes referred to as the "taxi-cab norm". It is the distance that a taxi travels, from one point on a street to another such point, along the streets of a city that has square city blocks.

Another alternative is the infinity norm.

Definition 1.2.4.2 Vector ∞-norm. The vector ∞-norm, $\| \cdot \|_\infty : \mathbb{C}^m \rightarrow \mathbb{R}$, is defined for $x \in \mathbb{C}^m$ by
$$\|x\|_\infty = \max(|\chi_0|, \ldots, |\chi_{m-1}|) = \max_{i=0}^{m-1} |\chi_i|.$$

$\Diamond$

The infinity norm simply measures how large the vector is by the magnitude of its largest entry.

Homework 1.2.4.2 Prove that the vector ∞-norm is a norm.

Solution. We show that the three conditions are met:

Let $x, y \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$ be arbitrarily chosen. Then

- $x \neq 0 \Rightarrow \|x\|_\infty > 0$ ($\| \cdot \|_\infty$ is positive definite):

  Notice that $x \neq 0$ means that at least one of its components is nonzero. Let's assume that $\chi_j \neq 0$. Then
  $$\|x\|_\infty = \max_{i=0}^{m-1} |\chi_i| \geq |\chi_j| > 0.$$

- $\|\alpha x\|_\infty = |\alpha| \|x\|_\infty$ ($\| \cdot \|_\infty$ is homogeneous):
  $$\begin{array}{rcl} \|\alpha x\|_\infty & = & \max_{i=0}^{m-1} |\alpha \chi_i| \\ & = & \max_{i=0}^{m-1} |\alpha||\chi_i| \\ & = & |\alpha| \max_{i=0}^{m-1} |\chi_i| \\ & = & |\alpha| \|x\|_\infty. \end{array}$$

- $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$ ($\| \cdot \|_\infty$ obeys the triangle inequality):
  $$\begin{array}{rcl} \|x + y\|_\infty & = & \max_{i=0}^{m-1} |\chi_i + \psi_i| \\ & \leq & \max_{i=0}^{m-1} (|\chi_i| + |\psi_i|) \\ & \leq & \max_{i=0}^{m-1} |\chi_i| + \max_{i=0}^{m-1} |\psi_i| \\ & = & \|x\|_\infty + \|y\|_\infty. \end{array}$$

In this course, we will primarily use the vector 1-norm, 2-norm, and ∞-norms. For completeness, we briefly discuss their generalization: the vector p-norm.

Definition 1.2.4.3 Vector p-norm. Given $p \geq 1$, the vector p-norm $\| \cdot \|_p : \mathbb{C}^m \rightarrow \mathbb{R}$ is defined for $x \in \mathbb{C}^m$ by

$$\|x\|_p = \sqrt[p]{|\chi_0|^p + \cdots + |\chi_{m-1}|^p} = \left(\sum_{i=0}^{m-1} |\chi_i|^p \right)^{1/p}.$$

$\Diamond$

Theorem 1.2.4.4 *The vector p-norm is a norm.*

The proof of this result is very similar to the proof of the fact that the 2-norm is a norm. It depends on Hölder's inequality, which is a generalization of the Cauchy-Schwartz inequality:

Theorem 1.2.4.5 Hölder's inequality. *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $x, y \in \mathbb{C}^m$ then $|x^H y| \leq \|x\|_p \|y\|_q$.*

We skip the proof of Hölder's inequality and Theorem 1.2.4.4. You can easily find proofs for these results, should you be interested.

Remark 1.2.4.6 The vector 1-norm and 2-norm are obviously special cases of the vector p-norm. It can be easily shown that the vector ∞-norm is also related:

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty .$$

Ponder This 1.2.4.3 Consider Homework 1.2.3.3. Try to elegantly formulate this question in the most general way you can think of. How do you prove the result?

Ponder This 1.2.4.4 Consider the vector norm $\| \cdot \| : \mathbb{C}^m \to \mathbb{R}$, the matrix $A \in \mathbb{C}^{m \times n}$ and the function $f : \mathbb{C}^n \to \mathbb{R}$ defined by $f(x) = \|Ax\|$. For what matrices A is the function f a norm?

1.2.5 Unit ball

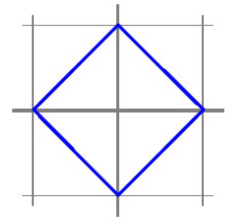
YouTube: `https://www.youtube.com/watch?v=aJgrpp7uscw`

In 3-dimensional space, the notion of the unit ball is intuitive: the set of all points that are a (Euclidean) distance of one from the origin. Vectors have no position and can have more than three components. Still the unit ball for the 2-norm is a straight forward extension to the set of all vectors with length (2-norm) one. More generally, the unit ball for any norm can be defined:

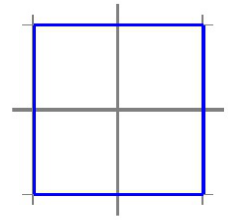
Definition 1.2.5.1 Unit ball. Given norm $\| \cdot \| : \mathbb{C}^m \to \mathbb{R}$, the unit ball with respect to $\| \cdot \|$ is the set $\{ x \mid \|x\| = 1 \}$ (the set of all vectors with norm equal to one). We will use $\|x\| = 1$ as shorthand for $\{ x \mid \|x\| = 1 \}$. $\Diamond$

Homework 1.2.5.1 Although vectors have no position, it is convenient to visualize a vector $x \in \mathbb{R}^2$ by the point in the plane to which it extends when rooted at the origin. For example, the vector $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ can be so visualized with the point $(2, 1)$. With this in mind, match the pictures on the right corresponding to the sets on the left:

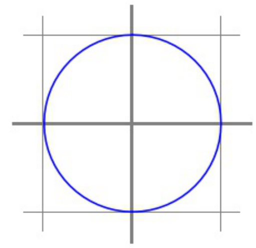
(a) $\|x\|_2 = 1$. (1)



(b) $\|x\|_1 = 1$. (2)

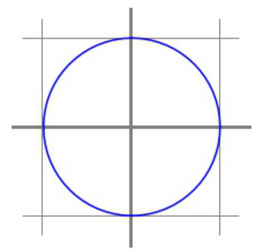


(c) $\|x\|_\infty = 1$. (3)

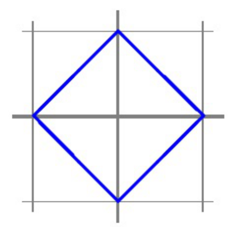


Solution.

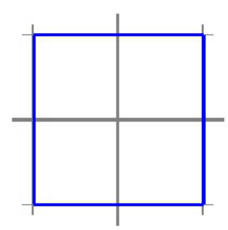
(a) $\|x\|_2 = 1$. (3)



(b) $\|x\|_1 = 1$. (1)



(c) $\|x\|_\infty = 1$. (2)



YouTube: `https://www.youtube.com/watch?v=Ov77sE90P58`

1.2.6 Equivalence of vector norms

YouTube: `https://www.youtube.com/watch?v=qjZyKHvL13E`

Homework 1.2.6.1 Fill out the following table:

x	$\|x\|_1$	$\|x\|_\infty$	$\|x\|_2$
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$			
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$			
$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$			

Solution.

x	$\|x\|_1$	$\|x\|_\infty$	$\|x\|_2$
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	1	1	1
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	3	1	$\sqrt{3}$
$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$	4	2	$\sqrt{1^2 + (-2)^2 + (-1)^2} = \sqrt{6}$

In this course, norms are going to be used to reason that vectors are "small" or "large". It would be unfortunate if a vector were small in one norm yet large in another norm. Fortunately, the following theorem excludes this possibility:

YouTube: `https://www.youtube.com/watch?v=I1W6ErdEyoc`

Theorem 1.2.6.1 Equivalence of vector norms. *Let $\| \cdot \| : \mathbb{C}^m \rightarrow \mathbb{R}$ and $|||\cdot||| : \mathbb{C}^m \rightarrow \mathbb{R}$ both be vector norms. Then there exist positive scalars σ and τ such that for all $x \in \mathbb{C}^m$*

$$\sigma \|x\| \leq |||x||| \leq \tau \|x\|.$$

Proof. The proof depends on a result from real analysis (sometimes called "advanced calculus") that states that $\sup_{x \in S} f(x)$ is attained for some vector $x \in S$ as long as f is continuous and S is a compact (closed and bounded) set. For any norm $\| \cdot \|$, the unit ball $\|x\| = 1$ is a compact set. When a supremum is an element in S, it is called the maximum instead and $\sup_{x \in S} f(x)$ can be restated as $\max_{x \in S} f(x)$.

Those who have not studied real analysis (which is not a prerequisite for this course) have to take this on faith. It is a result that we will use a few times in our discussion.

We prove that there exists a τ such that for all $x \in \mathbb{C}^m$

$$|||x||| \leq \tau \|x\|,$$

leaving the rest of the proof as an exercise.

Let $x \in \mathbb{C}^m$ be an arbitary vector. W.l.o.g. assume that $x \neq 0$. Then

$$\begin{array}{l} |||x||| \\ \quad = \quad < \text{ algebra } > \\ \frac{|||x|||}{\|x\|} \|x\| \\ \quad \leq \quad < \text{ algebra } > \\ \left(\sup_{z \neq 0} \frac{|||z|||}{\|z\|} \right) \|x\| \\ \quad = \quad < \text{ change of variables: } y = z / \|z\| > \\ \left(\sup_{\|y\|=1} |||y||| \right) \|x\| \\ \quad = \quad < \text{ the set } \|y\| = 1 \text{ is compact } > \\ \left(\max_{\|y\|=1} |||y||| \right) \|x\| \end{array}$$

The desired τ can now be chosen to equal $\max_{\|y\|=1} |||y|||$. ■

Homework 1.2.6.2 Complete the proof of Theorem 1.2.6.1.

Solution. We need to prove that

$$\sigma \|x\| \leq |||x|||.$$

From the first part of the proof of Theorem 1.2.6.1, we know that there exists a $\rho > 0$ such that

$$\|x\| \leq \rho |||x|||$$

and hence

$$\frac{1}{\rho}\|x\| \leq |||x|||.$$

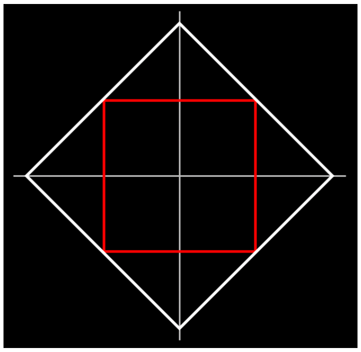
We conclude that

$$\sigma \|x\| \leq |||x|||$$

where $\sigma = 1/\rho$.

Example 1.2.6.2

- Let $x \in \mathbb{R}^2$. Use the picture



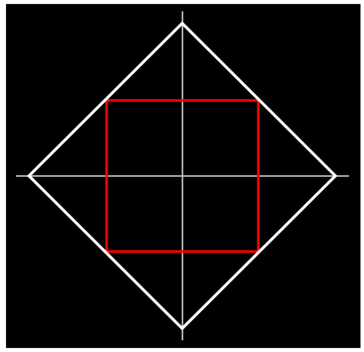
  to determine the constant C such that $\|x\|_1 \leq C\|x\|_\infty$. Give a vector x for which $\|x\|_1 = C\|x\|_\infty$.

- For $x \in \mathbb{R}^2$ and the C you determined in the first part of this problem, prove that $\|x\|_1 \leq C\|x\|_\infty$.

- Let $x \in \mathbb{C}^m$. Extrapolate from the last part the constant C such that $\|x\|_1 \leq C\|x\|_\infty$ and then prove the inequality. Give a vector x for which $\|x\|_1 = C\|x\|_\infty$.

Solution.

- Consider the picture



- ∘ The red square represents all vectors such that $\|x\|_\infty = 1$ and the white square represents all vectors such that $\|x\|_1 = 2$.
- ∘ All points on or outside the red square represent vectors y such that $\|y\|_\infty \geq 1$. Hence *if* $\|y\|_1 = 2$ *then* $\|y\|_\infty \geq 1$.
- ∘ Now, pick any $z \neq 0$. Then $\|2z/\|z\|_1\|_1 = 2$). Hence

$$\|2z/\|z\|_1\|_\infty \geq 1$$

which can be rewritten as

$$\|z\|_1 \leq 2\|z\|_\infty.$$

Thus, $C = 2$ works.

- ∘ Now, from the picture it is clear that $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has the property that $\|x\|_1 = 2\|x\|_\infty$. Thus, the inequality is "tight."

- We now prove that $\|x\|_1 \leq 2\|x\|_\infty$ for $x \in \mathbb{R}^2$:

$$
\begin{array}{l}
\|x\|_1 \\
\quad = \qquad < \text{ definition } > \\
|\chi_0| + |\chi_1| \\
\quad \leq \qquad < \text{ algebra } > \\
\max(|\chi_0|, |\chi_1|) + \max(|\chi_0|, |\chi_1|) \\
\quad = \qquad < \text{ algebra } > \\
2\max(|\chi_0|, |\chi_1|) \\
\quad = \qquad < \text{ definition } > \\
2\|x\|_\infty.
\end{array}
$$

- From the last part we extrapolate that $\|x\|_1 \leq m\|x\|_\infty$.

$\|x\|_1$
$= \quad <$ definition $>$
$\sum_{i=0}^{m-1} |\chi_i|$
$\leq \quad <$ algebra $>$
$\sum_{i=0}^{m-1} \left(\max_{j=0}^{m-1} |\chi_j| \right)$
$= \quad <$ algebra $>$
$m \max_{j=0}^{m-1} |\chi_j|$
$= \quad <$ definition $>$
$m\|x\|_\infty$.

Equality holds (i.e., $\|x\|_1 = m\|x\|_\infty$) for $x = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

Some will be able to go straight for the general result, while others will want to seek inspiration from the picture and/or the specialized case where $x \in \mathbb{R}^2$. □

Homework 1.2.6.3 Let $x \in \mathbb{C}^m$. The following table organizes the various bounds:

	$\|x\|_1 \leq C_{1,2}\|x\|_2$	$\|x\|_1 \leq C_{1,\infty}\|x\|_\infty$
$\|x\|_2 \leq C_{2,1}\|x\|_1$		$\|x\|_2 \leq C_{2,\infty}\|x\|_\infty$
$\|x\|_\infty \leq C_{\infty,1}\|x\|_1$	$\|x\|_\infty \leq C_{\infty,2}\|x\|_2$	

For each, determine the constant $C_{x,y}$ and prove the inequality, including that it is a tight inequality.

Hint: look at the hint!

Hint. $\|x\|_1 \leq \sqrt{m}\|x\|_2$:

This is the hardest one to prove. Do it last and use the following hint:

Consider $y = \begin{pmatrix} \chi_0/|\chi_0| \\ \vdots \\ \chi_{m-1}/|\chi_{m-1}| \end{pmatrix}$ and employ the Cauchy-Schwartz inequality.

Solution 1 ($\|x\|_1 \leq C_{1,2}\|x\|_2$). $\|x\|_1 \leq \sqrt{m}\|x\|_2$:

Consider $y = \begin{pmatrix} \chi_0/|\chi_0| \\ \vdots \\ \chi_{m-1}/|\chi_{m-1}| \end{pmatrix}$. Then

$$|x^H y| = \left| \sum_{i=0}^{m-1} \overline{\chi_i}\chi_i/|\chi_i| \right| = \left| \sum_{i=0}^{m-1} |\chi_i|^2/|\chi_i| \right| = \left| \sum_{i=0}^{m-1} |\chi_i| \right| = \|x\|_1.$$

We also notice that $\|y\|_2 = \sqrt{m}$.

From the Cauchy-Swartz inequality we know that

$$\|x\|_1 = |x^H y| \leq \|x\|_2\|y\|_2 = \sqrt{m}\|x\|_2.$$

If we now choose

$$x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

then $\|x\|_1 = m$ and $\|x\|_2 = \sqrt{m}$ so that $\|x\|_1 = \sqrt{m}\|x\|_2$.

Solution 2 ($\|x\|_1 \leq C_{1,\infty}\|x\|_\infty$). $\|x\|_1 \leq m\|x\|_\infty$:

See Example 1.2.6.2.

Solution 3 ($\|x\|_2 \leq C_{2,1}\|x\|_1$:). $\|x\|_2 \leq \|x\|_1$:

$$\begin{array}{l} \|x\|_2^2 \\ \quad = \qquad < \text{ definition } > \\ \sum_{i=0}^{m-1} |\chi_i|^2 \\ \quad \leq \qquad < \text{ algebra } > \\ \left(\sum_{i=0}^{m-1} |\chi_i|\right)^2 \\ \quad = \qquad < \text{ definition } > \\ \|x\|_1^2. \end{array}$$

Taking the square root of both sides yields $\|x\|_2 \leq \|x\|_1$.

If we now choose

$$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

then $\|x\|_2 = \|x\|_1$.

Solution 4 ($\|x\|_2 \leq C_{2,\infty}\|x\|_\infty$). $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$:

$$\begin{array}{l} \|x\|_2^2 \\ \quad = \qquad < \text{ definition } > \\ \sum_{i=0}^{m-1} |\chi_i|^2 \\ \quad \leq \qquad < \text{ algebra } > \\ \sum_{i=0}^{m-1} \left(\max_{j=0}^{m-1} |\chi_j|\right)^2 \\ \quad = \qquad < \text{ definition } > \\ \sum_{i=0}^{m-1} \|x\|_\infty^2 \\ \quad = \qquad < \text{ algebra } > \\ m\|x\|_\infty^2. \end{array}$$

Taking the square root of both sides yields $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$.

Consider

$$x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

then $\|x\|_2 = \sqrt{m}$ and $\|x\|_\infty = 1$ so that $\|x\|_2 = \sqrt{m}\|x\|_\infty$.

Solution 5 ($\|x\|_\infty \leq C_{\infty,1}\|x\|_1$:). $\|x\|_\infty \leq \|x\|_1$:

$$\begin{array}{l} \|x\|_\infty \\ \quad = \quad < \text{ definition } > \\ \max_{i=0}^{m-1} |\chi_i| \\ \quad \leq \quad < \text{ algebra } > \\ \sum_{i=0}^{m-1} |\chi_i| \\ \quad = \quad < \text{ definition } > \\ \|x\|_1. \end{array}$$

Consider

$$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then $\|x\|_\infty = 1 = \|x\|_1$.

Solution 6 ($\|x\|_\infty \leq C_{\infty,2}\|x\|_2$). $\|x\|_\infty \leq \|x\|_2$:

$$\begin{array}{l} \|x\|_\infty^2 \\ \quad = \quad < \text{ definition } > \\ \left(\max_{i=0}^{m-1} |\chi_i|\right)^2 \\ \quad = \quad < \text{ algebra } > \\ \max_{i=0}^{m-1} |\chi_i|^2 \\ \quad \leq \quad < \text{ algebra } > \\ \sum_{i=0}^{m-1} |\chi_i|^2 \\ \quad = \quad < \text{ definition } > \\ \|x\|_2^2. \end{array}$$

Taking the square root of both sides yields $\|x\|_\infty \leq \|x\|_2$.

Consider

$$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then $\|x\|_\infty = 1 = \|x\|_2$.

Solution 7 (Table of constants).

	$\|x\|_1 \leq \sqrt{m}\|x\|_2$	$\|x\|_1 \leq m\|x\|_\infty$
$\|x\|_2 \leq \|x\|_1$		$\|x\|_2 \leq \sqrt{m}\|x\|_\infty$
$\|x\|_\infty \leq \|x\|_1$	$\|x\|_\infty \leq \|x\|_2$	

Remark 1.2.6.3 The bottom line is that, modulo a constant factor, if a vector is "small" in one norm, it is "small" in all other norms. If it is "large" in one norm, it is "large" in all other norms.

1.3 Matrix Norms

1.3.1 Of linear transformations and matrices

YouTube: `https://www.youtube.com/watch?v=xlkiZEbYh38`

We briefly review the relationship between linear transformations and matrices, which is key to understanding why linear algebra is all about matrices and vectors.

Definition 1.3.1.1 Linear transformations and matrices. Let $L : \mathbb{C}^n \to \mathbb{C}^m$. Then L is said to be a linear transformation if for all $\alpha \in \mathbb{C}$ and $x, y \in \mathbb{C}^n$

- $L(\alpha x) = \alpha L(x)$. That is, scaling first and then transforming yields the same result as transforming first and then scaling.
- $L(x + y) = L(x) + L(y)$. That is, adding first and then transforming yields the same result as transforming first and then adding.

◊

The importance of linear transformations comes in part from the fact that many problems in science boil down to, given a function $F : \mathbb{C}^n \rightarrow \mathbb{C}^m$ and vector $y \in \mathbb{C}^m$, find x such that $F(x) = y$. This is known as an inverse problem. Under mild conditions, F can be locally approximated with a linear transformation L and then, as part of a solution method, one would want to solve $Lx = y$.

The following theorem provides the link between linear transformations and matrices:

Theorem 1.3.1.2 *Let $L : \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a linear transformation, $x \in \mathbb{C}^n$, and $v_0, v_1, \cdots, v_{n-1} \in \mathbb{C}^m$. Then*

$$L(\chi_0 v_0 + \chi_1 v_1 + \cdots + \chi_{n-1} v_{n-1}) = \chi_0 L(v_0) + \chi_1 L(v_1) + \cdots + \chi_{n-1} L(v_{n-1}),$$

where

$$x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix}.$$

Proof. A simple inductive proof yields the result. For details, see Week 2 of Linear Algebra: Foundations to Frontiers (LAFF) [20]. ■

The following set of vectors ends up playing a crucial role throughout this course:

Definition 1.3.1.3 Standard basis vector. In this course, we will use $e_j \in \mathbb{C}^m$ to denote the standard basis vector with a "1" in the position indexed with j. So,

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ \longleftarrow j \\ \\ \\ \\ \end{array}$$

◊

Key is the fact that any vector $x \in \mathbb{C}^n$ can be written as a linear combination of the standard basis vectors of $\mathbb{C}^n$:

$$\begin{aligned} x &= \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \chi_0 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \chi_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \chi_{n-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\ &= \chi_0 e_0 + \chi_1 e_1 + \cdots + \chi_{n-1} e_{n-1}. \end{aligned}$$

Hence, if L is a linear transformation,

$$\begin{aligned} L(x) &= L(\chi_0 e_0 + \chi_1 e_1 + \cdots + \chi_{n-1} e_{n-1}) \\ &= \chi_0 \underbrace{L(e_0)}_{a_0} + \chi_1 \underbrace{L(e_1)}_{a_1} + \cdots + \chi_{n-1} \underbrace{L(e_{n-1})}_{a_{n-1}}. \end{aligned}$$

If we now let $a_j = L(e_j)$ (the vector a_j is the transformation of the standard basis vector e_j and collect these vectors into a two-dimensional array of numbers:

$$A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \tag{1.3.1}$$

then we notice that information for evaluating $L(x)$ can be found in this array, since L can then alternatively computed by

$$L(x) = \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}.$$

The array A in (1.3.1) we call a **matrix** and the operation $Ax = \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}$ we call **matrix-vector multiplication**. Clearly

$$Ax = L(x).$$

Remark 1.3.1.4 Notation. In these notes, as a rule,

- Roman upper case letters are used to denote matrices.
- Roman lower case letters are used to denote vectors.
- Greek lower case letters are used to denote scalars.

Corresponding letters from these three sets are used to refer to a matrix, the row or columns of that matrix, and the elements of that matrix. If $A \in \mathbb{C}^{m\times n}$ then

$$\begin{array}{l} A \\ \quad = \quad < \text{ partition } A \text{ by columns and rows } > \\ \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) = \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \widetilde{a}_1^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array} \right) \\ \quad = \quad < \text{ expose the elements of } A > \\ \left(\begin{array}{c|c|c|c} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \hline \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \hline \vdots & \vdots & & \vdots \\ \vdots & & & \\ \hline \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{array} \right) \end{array}$$

We now notice that the standard basis vector $e_j \in \mathbb{C}^m$ equals the column of the $m \times m$ **identity matrix** indexed with j:

$$I = \left(\begin{array}{c|c|c|c} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{array} \right) = \left(\begin{array}{c|c|c|c} e_0 & e_1 & \cdots & e_{m-1} \end{array} \right) = \left(\begin{array}{c} \widetilde{e}_0^T \\ \hline \widetilde{e}_1^T \\ \hline \vdots \\ \hline \widetilde{e}_{m-1}^T \end{array} \right).$$

Remark 1.3.1.5 The important thing to note is that a matrix is a convenient representation of a linear transformation and matrix-vector multiplication is an alternative way for evaluating that linear transformation.

YouTube: `https://www.youtube.com/watch?v=cCFAnQmwwIw`

Let's investigate matrix-matrix multiplication and its relationship to linear transformations. Consider two linear transformations

$$\begin{array}{lll} L_A : \mathbb{C}^k \to \mathbb{C}^m & \text{represented by} & \text{matrix } A \\ L_B : \mathbb{C}^n \to \mathbb{C}^k & \text{represented by} & \text{matrix } B \end{array}$$

and define

$$L_C(x) = L_A(L_B(x)),$$

as the composition of L_A and L_B. Then it can be easily shown that L_C is also a linear transformation. Let $m \times n$ matrix C represent L_C. How are A, B, and C related? If we let c_j equal the column of C indexed with j, then because of the link between matrices, linear transformations, and standard basis vectors

$$c_j = L_C(e_j) = L_A(L_B(e_j)) = L_A(b_j) = Ab_j,$$

where b_j equals the column of B indexed with j. Now, we say that $C = AB$ is the product of A and B defined by

$$\left(\begin{array}{c|c|c|c} c_0 & c_1 & \cdots & c_{n-1} \end{array} \right) = A \left(\begin{array}{c|c|c|c} b_0 & b_1 & \cdots & b_{n-1} \end{array} \right) = \left(\begin{array}{c|c|c|c} Ab_0 & Ab_1 & \cdots & Ab_{n-1} \end{array} \right)$$

and define the matrix-matrix multiplication as the operation that computes

$$C := AB,$$

which you will want to pronounce "C becomes A times B" to distinguish assignment from equality. If you think carefully how individual elements of C are computed, you will realize that they equal the usual "dot product of rows of A with columns of B."

YouTube: `https://www.youtube.com/watch?v=g_9RbA5EOIc`

As already mentioned, throughout this course, it will be important that you can think about matrices in terms of their columns and rows, and matrix-matrix multiplication (and other operations with matrices and vectors) in terms of columns and rows. It is also important to be able to think about matrix-matrix multiplication in three different ways. If we partition each matrix by rows and by columns:

$$C = \left(\begin{array}{c|c|c} c_0 & \cdots & c_{n-1} \end{array} \right) = \left(\begin{array}{c} \widetilde{c}_0^T \\ \hline \vdots \\ \hline \widetilde{c}_{m-1}^T \end{array} \right), A = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{k-1} \end{array} \right) = \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array} \right),$$

and

$$B = \left(\begin{array}{c|c|c} b_0 & \cdots & b_{n-1} \end{array} \right) = \left(\begin{array}{c} \widetilde{b}_0^T \\ \hline \vdots \\ \hline \widetilde{b}_{k-1}^T \end{array} \right),$$

then $C := AB$ can be computed in the following ways:

1. By columns:

$$\left(\begin{array}{c|c|c} c_0 & \cdots & c_{n-1} \end{array} \right) := A \left(\begin{array}{c|c|c} b_0 & \cdots & b_{n-1} \end{array} \right) = \left(\begin{array}{c|c|c} Ab_0 & \cdots & Ab_{n-1} \end{array} \right).$$

   In other words, $c_j := Ab_j$ for all columns of C.

2. By rows:

$$\left(\begin{array}{c} \widetilde{c}_0^T \\ \hline \vdots \\ \hline \widetilde{c}_{m-1}^T \end{array} \right) := \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array} \right) B = \left(\begin{array}{c} \widetilde{a}_0^T B \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T B \end{array} \right).$$

   In other words, $\widetilde{c}_i^T = \widetilde{a}_i^T B$ for all rows of C.

3. One you may not have thought about much before:

$$C := \left(\begin{array}{c|c|c} a_0 & \cdots & a_{k-1} \end{array} \right) \left(\begin{array}{c} \widetilde{b}_0^T \\ \hline \vdots \\ \hline \widetilde{b}_{k-1}^T \end{array} \right) = a_0 \widetilde{b}_0^T + \cdots + a_{k-1} \widetilde{b}_{k-1}^T,$$

   which should be thought of as a sequence of rank-1 updates, since each term is an outer product and an outer product has rank of at most one.

These three cases are special cases of the more general observation that, if we can partition C, A, and B by blocks (submatrices),

$$C = \left(\begin{array}{c|c|c} C_{0,0} & \cdots & C_{0,N-1} \\ \hline \vdots & & \vdots \\ \hline C_{M-1,0} & \cdots & C_{M-1,N-1} \end{array} \right), \left(\begin{array}{c|c|c} A_{0,0} & \cdots & A_{0,K-1} \\ \hline \vdots & & \vdots \\ \hline A_{M-1,0} & \cdots & A_{M-1,K-1} \end{array} \right),$$

and

$$\left(\begin{array}{c|c|c} B_{0,0} & \cdots & B_{0,N-1} \\ \hline \vdots & & \vdots \\ \hline B_{K-1,0} & \cdots & B_{K-1,N-1} \end{array}\right),$$

where the partitionings are "conformal", then

$$C_{i,j} = \sum_{p=0}^{K-1} A_{i,p} B_{p,j}.$$

Remark 1.3.1.6 If the above review of linear transformations, matrices, matrix-vector multiplication, and matrix-matrix multiplication makes you exclaim "That is all a bit too fast for me!" then it is time for you to take a break and review Weeks 2-5 of our introductory linear algebra course "Linear Algebra: Foundations to Frontiers." Information, including notes [20] (optionally downloadable for free) and a link to the course on edX [21] (which can be audited for free) can be found at `http://ulaff.net`.

1.3.2 What is a matrix norm?

YouTube: `https://www.youtube.com/watch?v=6DsBTz1eU7E`

A matrix norm extends the notions of an absolute value and vector norm to matrices:

Definition 1.3.2.1 Matrix norm. Let $\nu : \mathbb{C}^{m\times n} \to \mathbb{R}$. Then ν is a (matrix) norm if for all $A, B \in \mathbb{C}^{m\times n}$ and all $\alpha \in \mathbb{C}$

- $A \neq 0 \Rightarrow \nu(A) > 0$ (ν is positive definite),
- $\nu(\alpha A) = |\alpha|\nu(A)$ (ν is homogeneous), and
- $\nu(A + B) \leq \nu(A) + \nu(B)$ (ν obeys the triangle inequality).

◊

Homework 1.3.2.1 Let $\nu : \mathbb{C}^{m\times n} \to \mathbb{R}$ be a matrix norm.

ALWAYS/SOMETIMES/NEVER: $\nu(0) = 0$.

Hint. Review the proof on Homework 1.2.2.1.

Answer. ALWAYS.

Now prove it.

Solution. Let $A \in \mathbb{C}^{m \times n}$. Then

$$
\begin{array}{l}
\nu(0) \\
\quad = \quad < 0 \cdot A = 0 > \\
\nu(0 \cdot A) \\
\quad = \quad < \| \cdot \|_\nu \text{ is homogeneous } > \\
0 \nu(A) \\
\quad = \quad < \text{ algebra } > \\
0
\end{array}
$$

Remark 1.3.2.2 As we do with vector norms, we will typically use $\| \cdot \|$ instead of $\nu(\cdot)$ for a function that is a matrix norm.

1.3.3 The Frobenius norm

YouTube: `https://www.youtube.com/watch?v=0ZHnGgrJXa4`

Definition 1.3.3.1 The Frobenius norm. The Frobenius norm $\| \cdot \|_F : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ is defined for $A \in \mathbb{C}^{m \times n}$ by

$$
\|A\|_F = \sqrt{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}|^2} = \sqrt{\begin{array}{ccccccc} |a_{0,0}|^2 & + & \cdots & + & |a_{0,n-1}|^2 & + \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ |a_{m-1,0}|^2 & + & \cdots & + & |a_{m-1,n-1}|^2. & \end{array}}
$$

◊

One can think of the Frobenius norm as taking the columns of the matrix, stacking them on top of each other to create a vector of size $m \times n$, and then taking the vector 2-norm of the result.

Homework 1.3.3.1 Partition $m \times n$ matrix A by columns:

$$
A = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{n-1} \end{array} \right).
$$

Show that

$$
\|A\|_F^2 = \sum_{j=0}^{n-1} \|a_j\|_2^2.
$$

Solution.

$$
\begin{array}{l}
\|A\|_F \\
\quad = \quad < \text{ definition } > \\
\sqrt{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\alpha_{i,j}|^2} \\
\quad = \quad < \text{ commutativity of addition } > \\
\sqrt{\sum_{j=0}^{n-1}\sum_{i=0}^{m-1}|\alpha_{i,j}|^2} \\
\quad = \quad < \text{ definition of vector 2-norm } > \\
\sqrt{\sum_{j=0}^{n-1}\|a_j\|_2^2}
\end{array}
$$

Homework 1.3.3.2 Prove that the Frobenius norm is a norm.

Solution. Establishing that this function is positive definite and homogeneous is straight forward. To show that the triangle inequality holds it helps to realize that if $A = \left(a_0 \mid a_1 \mid \cdots \mid a_{n-1} \right)$ then

$$
\begin{array}{l}
\|A\|_F \\
\quad = \quad < \text{ definition } > \\
\sqrt{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\alpha_{i,j}|^2} \\
\quad = \quad < \text{ commutativity of addition } > \\
\sqrt{\sum_{j=0}^{n-1}\sum_{i=0}^{m-1}|\alpha_{i,j}|^2} \\
\quad = \quad < \text{ definition of vector 2-norm } > \\
\sqrt{\sum_{j=0}^{n-1}\|a_j\|_2^2} \\
\quad = \quad < \text{ definition of vector 2-norm } > \\
\sqrt{\left\| \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \right\|_2^2}.
\end{array}
$$

In other words, it equals the vector 2-norm of the vector that is created by stacking the columns of A on top of each other. One can then exploit the fact that the vector 2-norm obeys the triangle inequality.

Homework 1.3.3.3 Partition $m \times n$ matrix A by rows:

$$
A = \begin{pmatrix} \widetilde{a}_0^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{pmatrix}.
$$

Show that

$$
\|A\|_F^2 = \sum_{i=0}^{m-1} \|\widetilde{a}_i\|_2^2,
$$

where $\widetilde{a}_i = \widetilde{a}_i^{T\,T}$.

Solution.

$\|A\|_F$
$= \quad < \text{definition} >$
$\sqrt{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}|^2}$
$= \quad < \text{definition of vector 2-norm} >$
$\sqrt{\sum_{j=0}^{m-1} \|\widetilde{a}_j\|_2^2}.$

Let us review the definition of the transpose of a matrix (which we have already used when defining the dot product of two real-valued vectors and when identifying a row in a matrix):

Definition 1.3.3.2 Transpose. If $A \in \mathbb{C}^{m \times n}$ and

$$A = \left(\begin{array}{c|c|c|c} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \hline \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \hline \vdots & \vdots & & \vdots \\ \vdots & & & \\ \hline \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{array} \right)$$

then its **transpose** is defined by

$$A^T = \left(\begin{array}{c|c|c|c} \alpha_{0,0} & \alpha_{1,0} & \cdots & \alpha_{n-1,0} \\ \hline \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{n-1,1} \\ \hline \vdots & \vdots & & \vdots \\ \vdots & & & \\ \hline \alpha_{0,m-1} & \alpha_{1,m-1} & \cdots & \alpha_{n-1,m-1} \end{array} \right).$$

◊

For complex-valued matrices, it is important to also define the **Hermitian transpose** of a matrix:

Definition 1.3.3.3 Hermitian transpose. If $A \in \mathbb{C}^{m \times n}$ and

$$A = \left(\begin{array}{c|c|c|c} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \hline \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \hline \vdots & \vdots & & \vdots \\ \vdots & & & \\ \hline \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{array} \right)$$

then its **Hermitian transpose** is defined by

$$
A^H = \overline{A}^T \left(\begin{array}{c|c|c|c}
\overline{\alpha}_{0,0} & \overline{\alpha}_{1,0} & \cdots & \overline{\alpha}_{m-1,0} \\ \hline
\overline{\alpha}_{0,1} & \overline{\alpha}_{1,1} & \cdots & \overline{\alpha}_{m-1,1} \\ \hline
\vdots & \vdots & & \vdots \\
\vdots & & & \\ \hline
\overline{\alpha}_{0,n-1} & \overline{\alpha}_{1,n-1} & \cdots & \overline{\alpha}_{m-1,n-1}
\end{array} \right),
$$

where $\overline{A}$ denotes the **conjugate of a matrix**, in which each element of the matrix is conjugated. ◊

We note that

- $\overline{A}^T = \overline{A^T}$.
- If $A \in \mathbb{R}^{m \times n}$, then $A^H = A^T$.
- If $x \in \mathbb{C}^m$, then x^H is defined consistent with how we have used it before.
- If $\alpha \in \mathbb{R}$, then $\alpha^H = \overline{\alpha}$.

  (If you view the scalar as a matrix and then Hermitian transpose it, you get the matrix with as only element $\overline{\alpha}$.)

Don't Panic!. While working with complex-valued scalars, vectors, and matrices may appear a bit scary at first, you will soon notice that it is not really much more complicated than working with their real-valued counterparts.

Homework 1.3.3.4 Let $A \in \mathbb{C}^{m \times k}$ and $B \in \mathbb{C}^{k \times n}$. Using what you once learned about matrix transposition and matrix-matrix multiplication, reason that $(AB)^H = B^H A^H$.

Solution.

$$
\begin{array}{ll}
(AB)^H & \\
= & < X^H = \overline{X^T} > \\
\overline{(AB)^T} & \\
= & < \text{ you once discovered that } (AB)^T = B^T A^T > \\
\overline{B^T A^T} & \\
= & < \text{ you may check separately that } \overline{XY} = \overline{X}\,\overline{Y} > \\
\overline{B^T}\;\overline{A^T} & \\
= & < \overline{X^T} = \overline{X}^T > \\
B^H A^H &
\end{array}
$$

Definition 1.3.3.4 Hermitian. A matrix $A \in \mathbb{C}^{m \times m}$ is **Hermitian** if and only if $A = A^H$. ◊

Obviously, if $A \in \mathbb{R}^{m \times m}$, then A is a Hermitian matrix if and only if A is a symmetric matrix.

Homework 1.3.3.5 Let $A \in \mathbb{C}^{m \times n}$.

ALWAYS/SOMETIMES/NEVER: $\|A^H\|_F = \|A\|_F$.

Answer. ALWAYS

Solution.

$$
\begin{array}{l}
\|A\|_F \\
\quad = \quad < \text{ definition } > \\
\sqrt{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}|^2} \\
\quad = \quad < \text{ commutativity of addition } > \\
\sqrt{\sum_{j=0}^{n-1} \sum_{i=0}^{m-1} |\alpha_{i,j}|^2} \\
\quad = \quad < \text{ change of variables } > \\
\sqrt{\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\alpha_{j,i}|^2} \\
\quad = \quad < \text{ algebra } > \\
\sqrt{\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\overline{\alpha_{j,i}}|^2} \\
\quad = \quad < \text{ definition } > \\
\|A^H\|_F
\end{array}
$$

Similarly, other matrix norms can be created from vector norms by viewing the matrix as a vector. It turns out that, other than the Frobenius norm, these aren't particularly interesting in practice. An example can be found in Homework 1.6.1.6.

Remark 1.3.3.5 The Frobenius norm of a $m \times n$ matrix is easy to compute (requiring $O(mn)$ computations). The functions $f(A) = \|A\|_F$ and $f(A) = \|A\|_F^2$ are also differentiable. However, you'd be hard-pressed to find a meaningful way of linking the definition of the Frobenius norm to a measure of an underlying linear transformation (other than by first transforming that linear transformation into a matrix).

1.3.4 Induced matrix norms

YouTube: `https://www.youtube.com/watch?v=M6ZVBRFnYcU`

Recall from Subsection 1.3.1 that a matrix, $A \in \mathbb{C}^{m \times n}$, is a 2-dimensional array of numbers that represents a linear transformation, $L : \mathbb{C}^n \to \mathbb{C}^m$, such that for all $x \in \mathbb{C}^n$ the matrix-vector multiplication Ax yields the same result as does $L(x)$.

The question "What is the norm of matrix A?" or, equivalently, "How 'large' is A?" is the same as asking the question "How 'large' is L?" What does this mean? It suggests that what we really want is a measure of how much linear transformation L or, equivalently, matrix A "stretches" (magnifies) the "length" of a vector. This observation motivates a class of matrix norms known as induced matrix norms.

Definition 1.3.4.1 Induced matrix norm. Let $\| \cdot \|_\mu : \mathbb{C}^m \rightarrow \mathbb{R}$ and $\| \cdot \|_\nu : \mathbb{C}^n \rightarrow \mathbb{R}$ be vector norms. Define $\| \cdot \|_{\mu,\nu} : \mathbb{C}^{m\times n} \rightarrow \mathbb{R}$ by

$$\|A\|_{\mu,\nu} = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu}{\|x\|_\nu}.$$

◊

Matrix norms that are defined in this way are said to be **induced** matrix norms.

Remark 1.3.4.2 In context, it is obvious (from the column size of the matrix) what the size of vector x is. For this reason, we will write

$$\|A\|_{\mu,\nu} = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu}{\|x\|_\nu} \quad \text{as} \quad \|A\|_{\mu,\nu} = \sup_{x\neq 0} \frac{\|Ax\|_\mu}{\|x\|_\nu}.$$

Let us start by interpreting this. How "large" A is, as measured by $\|A\|_{\mu,\nu}$, is defined as the most that A magnifies the length of nonzero vectors, where the length of the vector, x, is measured with norm $\| \cdot \|_\nu$ and the length of the transformed vector, Ax, is measured with norm $\| \cdot \|_\mu$.

Two comments are in order. First,

$$\sup_{x\neq 0} \frac{\|Ax\|_\mu}{\|x\|_\nu} = \sup_{\|x\|_\nu = 1} \|Ax\|_\mu.$$

This follows from the following sequence of equivalences:

$$\begin{array}{l} \sup_{x\neq 0} \frac{\|Ax\|_\mu}{\|x\|_\nu} \\ \quad = \quad < \text{ homogeneity } > \\ \sup_{x\neq 0} \|\frac{Ax}{\|x\|_\nu}\|_\mu \\ \quad = \quad < \text{ norms are associative } > \\ \sup_{x\neq 0} \|A\frac{x}{\|x\|_\nu}\|_\mu \\ \quad = \quad < \text{ substitute } y = x/\|x\|_\nu > \\ \sup_{\|y\|_\nu = 1} \|Ay\|_\mu. \end{array}$$

Second, the "sup" (which stands for supremum) is used because we can't claim yet that there is a nonzero vector x for which

$$\sup_{x\neq 0} \frac{\|Ax\|_\mu}{\|x\|_\nu}$$

is attained or, alternatively, a vector, x, with $\|x\|_\nu = 1$ for which

$$\sup_{\|x\|_\nu = 1} \|Ax\|_\mu$$

is attained. In words, it is not immediately obvious that there is a vector for which the supremum is attained. The fact is that there is always such a vector x. The proof again depends on a result from real analysis, also employed in Proof 1.2.6.1, that states that $\sup_{x \in S} f(x)$ is attained for some vector $x \in S$ as long as f is continuous and S is a compact set. For any norm, $\|x\| = 1$ is a compact set. Thus, we can replace sup by max from here on in our discussion.

We conclude that the following two definitions are equivalent definitions to the one we already gave:

Definition 1.3.4.3 Let $\| \cdot \|_\mu : \mathbb{C}^m \rightarrow \mathbb{R}$ and $\| \cdot \|_\nu : \mathbb{C}^n \rightarrow \mathbb{R}$ be vector norms. Define $\| \cdot \|_{\mu,\nu} : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ by

$$\|A\|_{\mu,\nu} = \max_{x \neq 0} \frac{\|Ax\|_\mu}{\|x\|_\nu}.$$

or, equivalently,

$$\|A\|_{\mu,\nu} = \max_{\|x\|_\nu = 1} \|Ax\|_\mu.$$

$\Diamond$

Remark 1.3.4.4 In this course, we will often encounter proofs involving norms. Such proofs are much cleaner if one starts by strategically picking the most convenient of these two definitions. Until you gain the intuition needed to pick which one is better, you may have to start your proof using one of them and then switch to the other one if the proof becomes unwieldy.

Theorem 1.3.4.5 $\| \cdot \|_{\mu,\nu} : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ *is a norm.*

Proof. To prove this, we merely check whether the three conditions are met:

Let $A, B \in \mathbb{C}^{m \times n}$ and $\alpha \in \mathbb{C}$ be arbitrarily chosen. Then

- $A \neq 0 \Rightarrow \|A\|_{\mu,\nu} > 0$ ($\| \cdot \|_{\mu,\nu}$ is positive definite):

  Notice that $A \neq 0$ means that at least one of its columns is not a zero vector (since at least one element is nonzero). Let us assume it is the jth column, a_j, that is nonzero. Let e_j equal the column of I (the identity matrix) indexed with j. Then

  $$\begin{array}{l} \|A\|_{\mu,\nu} \\ \quad = \quad < \text{ definition } > \\ \max_{x \neq 0} \frac{\|Ax\|_\mu}{\|x\|_\nu} \\ \quad \geq \quad < e_j \text{ is a specific vector } > \\ \frac{\|Ae_j\|_\mu}{\|e_j\|_\nu} \\ \quad = \quad < Ae_j = a_j > \\ \frac{\|a_j\|_\mu}{\|e_j\|_\nu} \\ \quad > \quad < \text{ we assumed that } a_j \neq 0 > \\ 0. \end{array}$$

- $\|\alpha A\|_{\mu,\nu} = |\alpha| \|A\|_{\mu,\nu}$ ($\| \cdot \|_{\mu,\nu}$ is homogeneous):

$$
\begin{array}{l}
\|\alpha A\|_{\mu,\nu} \\
\quad = \quad < \text{ definition } > \\
\max_{x \neq 0} \frac{\|\alpha A x\|_\mu}{\|x\|_\nu} \\
\quad = \quad < \text{ homogeneity } > \\
\max_{x \neq 0} |\alpha| \frac{\|A x\|_\mu}{\|x\|_\nu} \\
\quad = \quad < \text{ algebra } > \\
|\alpha| \max_{x \neq 0} \frac{\|A x\|_\mu}{\|x\|_\nu} \\
\quad = \quad < \text{ definition } > \\
|\alpha| \|A\|_{\mu,\nu}.
\end{array}
$$

- $\|A+B\|_{\mu,\nu} \leq \|A\|_{\mu,\nu} + \|B\|_{\mu,\nu}$ ($\| \cdot \|_{\mu,\nu}$ obeys the triangle inequality).

$$
\begin{array}{l}
\|A+B\|_{\mu,\nu} \\
\quad = \quad < \text{ definition } > \\
\max_{x \neq 0} \frac{\|(A+B)x\|_\mu}{\|x\|_\nu} \\
\quad = \quad < \text{ distribute } > \\
\max_{x \neq 0} \frac{\|Ax+Bx\|_\mu}{\|x\|_\nu} \\
\quad \leq \quad < \text{ triangle inequality } > \\
\max_{x \neq 0} \frac{\|Ax\|_\mu + \|Bx\|_\mu}{\|x\|_\nu} \\
\quad \leq \quad < \text{ algebra } > \\
\max_{x \neq 0} \left(\frac{\|Ax\|_\mu}{\|x\|_\nu} + \frac{\|Bx\|_\mu}{\|x\|_\nu} \right) \\
\quad \leq \quad < \text{ algebra } > \\
\max_{x \neq 0} \frac{\|Ax\|_\mu}{\|x\|_\nu} + \max_{x \neq 0} \frac{\|Bx\|_\mu}{\|x\|_\nu} \\
\quad = \quad < \text{ definition } > \\
\|A\|_{\mu,\nu} + \|B\|_{\mu,\nu}.
\end{array}
$$

■

When $\| \cdot \|_\mu$ and $\| \cdot \|_\nu$ are the same norm (but possibly for different sizes of vectors), the induced norm becomes

Definition 1.3.4.6 Define $\| \cdot \|_\mu : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ by

$$\|A\|_\mu = \max_{x \neq 0} \frac{\|Ax\|_\mu}{\|x\|_\mu}$$

or, equivalently,

$$\|A\|_\mu = \max_{\|x\|_\mu = 1} \|Ax\|_\mu.$$

◊

Homework 1.3.4.1 Consider the vector p-norm $\| \cdot \|_p : \mathbb{C}^n \rightarrow \mathbb{R}$ and let us denote the induced matrix norm by $||| \cdot ||| : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ for this exercise: $|||A||| = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$.

ALWAYS/SOMETIMES/NEVER: $|||y||| = \|y\|_p$ for $y \in \mathbb{C}^m$.

Answer. ALWAYS

Solution.

$|||y|||$

$=$ $<$ definition $>$

$\max_{x\neq 0} \frac{\|yx\|_p}{\|x\|_p}$

$=$ $< x$ is a scalar since y is a matrix with one column, and hence $\|x\|_p = |x| >$

$\max_{x\neq 0} |x| \frac{\|y\|_p}{|x|}$

$=$ $<$ algebra $>$

$\max_{x\neq 0} \|y\|_p$

$=$ $<$ algebra $>$

$\|y\|$

This last exercise is important. One can view a vector $x \in \mathbb{C}^m$ as an $m \times 1$ matrix. What this last exercise tells us is that regardless of whether we view x as a matrix or a vector, $\|x\|_p$ is the same.

We already encounted the vector p-norms as an important class of vector norms. The matrix p-norm is induced by the corresponding vector norm, as defined by

Definition 1.3.4.7 Matrix p-norm. For any vector p-norm, define the corresponding matrix p-norm $\|\cdot\|_p : \mathbb{C}^{m\times n} \to \mathbb{R}$ by

$$\|A\|_p = \max_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} \quad \text{or, equivalently,} \quad \|A\|_p = \max_{\|x\|_p = 1} \|Ax\|_p.$$

$\lozenge$

Remark 1.3.4.8 The matrix p-norms with $p \in \{1, 2, \infty\}$ will play an important role in our course, as will the Frobenius norm. As the course unfolds, we will realize that in practice the matrix 2-norm is of great theoretical importance but difficult to evaluate, except for special matrices. The 1-norm, ∞-norm, and Frobenius norms are straightforward and relatively cheap to compute (for an $m \times n$ matrix, computing these costs $O(mn)$ computation).

1.3.5 The matrix 2-norm

YouTube: `https://www.youtube.com/watch?v=wZAlH_K9XeI`

Let us instantiate the definition of the vector p norm for the case where $p = 2$, giving us a matrix norm induced by the vector 2-norm or Euclidean norm:

Definition 1.3.5.1 Matrix 2-norm. Define the matrix 2-norm $\|\cdot\|_2 : \mathbb{C}^{m\times n} \to \mathbb{R}$ by

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2=1} \|Ax\|_2.$$

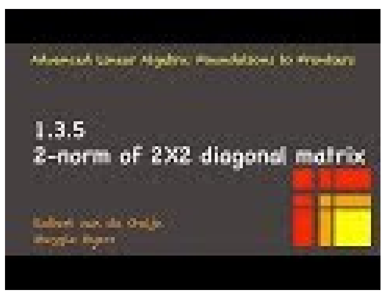
$\Diamond$

Remark 1.3.5.2 The problem with the matrix 2-norm is that it is hard to compute. At some point later in this course, you will find out that if A is a Hermitian matrix ($A = A^H$), then $\|A\|_2 = |\lambda_0|$, where λ_0 equals the eigenvalue of A that is largest in magnitude. You may recall from your prior linear algebra experience that computing eigenvalues involves computing the roots of polynomials, and for polynomials of degree three or greater, this is a nontrivial task. We will see that the matrix 2-norm plays an important role in the theory of linear algebra, but less so in practical computation.

Example 1.3.5.3 Show that

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 = \max(|\delta_0|, |\delta_1|).$$

Solution.



YouTube: `https://www.youtube.com/watch?v=B2rz0i5BB3A`
[slides (PDF)] [LaTeX source] $\square$

Remark 1.3.5.4 The proof of the last example builds on a general principle: Showing that $\max_{x\in D} f(x) = \alpha$ for some function $f : D \to R$ can be broken down into showing that both

$$\max_{x\in D} f(x) \le \alpha$$

and

$$\max_{x\in D} f(x) \ge \alpha.$$

In turn, showing that $\max_{x\in D} f(x) \ge \alpha$ can often be accomplished by showing that there exists a vector $y \in D$ such that $f(y) = \alpha$ since then

$$\alpha = f(y) \le \max_{x\in D} f(x).$$

We will use this technique in future proofs involving matrix norms.

Homework 1.3.5.1 Let $D \in \mathbb{C}^{m \times m}$ be a diagonal matrix with diagonal entries $\delta_0, \ldots, \delta_{m-1}$. Show that

$$\|D\|_2 = \max_{j=0}^{m-1} |\delta_j|.$$

Solution. First, we show that $\|D\|_2 = \max_{\|x\|_2=1} \|Dx\|_2 \leq \max_{i=0}^{m-1} |\delta_i|$:

$$\begin{array}{l}
\|D\|_2^2 \\
\quad = \quad < \text{ definition } > \\
\max_{\|x\|_2=1} \|Dx\|_2^2 \\
\quad = \quad < \text{ diagonal vector multiplication } > \\
\max_{\|x\|_2=1} \left\| \left(\begin{array}{c} \delta_0 \chi_0 \\ \vdots \\ \delta_{m-1} \chi_{m-1} \end{array} \right) \right\|_2^2 \\
\quad = \quad < \text{ definition } > \\
\max_{\|x\|_2=1} \sum_{i=0}^{m-1} |\delta_i \chi_i|^2 \\
\quad = \quad < \text{ homogeneity } > \\
\max_{\|x\|_2=1} \sum_{i=0}^{m-1} |\delta_i|^2 |\chi_i|^2 \\
\quad \leq \quad < \text{ algebra } > \\
\max_{\|x\|_2=1} \sum_{i=0}^{m-1} \left[\max_{j=0}^{m-1} |\delta_j| \right]^2 |\chi_i|^2 \\
\quad = \quad < \text{ algebra } > \\
\left[\max_{j=0}^{m-1} |\delta_j| \right]^2 \max_{\|x\|_2=1} \sum_{i=0}^{m-1} |\chi_i|^2 \\
\quad = \quad < \|x\|_2 = 1 > \\
\left[\max_{j=0}^{m-1} |\delta_j| \right]^2 .
\end{array}$$

Next, we show that there is a vector y with $\|y\|_2 = 1$ such that $\|Dy\|_2 = \max_{i=0}^{m-1} |\delta_i|$: Let j be such that $|\delta_j| = \max_{i=0}^{m-1} |\delta_i|$ and choose $y = e_j$. Then

$$\begin{array}{l}
\|Dy\|_2 \\
\quad = \quad < y = e_j > \\
\|De_j\|_2 \\
\quad = \quad < D = \text{diag}(\delta_0, \ldots, \delta_{m-1}) > \\
\|\delta_j e_j\|_2 \\
\quad = \quad < \text{ homogeneity } > \\
|\delta_j| \|e_j\|_2 \\
\quad = \quad < \|e_j|_2 = 1 > \\
|\delta_j| \\
\quad = \quad < \text{ choice of } j > \\
\max_{i=0}^{m-1} |\delta_i|
\end{array}$$

Hence $\|D\|_2 = \max_{j=0}^{m-1} |\delta_j|$.

Homework 1.3.5.2 Let $y \in \mathbb{C}^m$ and $x \in \mathbb{C}^n$.

ALWAYS/SOMETIMES/NEVER: $\| y x^H \|_2 = \| y \|_2 \| x \|_2$.

Hint. Prove that $\| y x^H \|_2 \geq \| y \|_2 \| x \|_2$ and that there exists a vector z so that $\frac{\| y x^H z \|_2}{\| z \|_2} = \| y \|_2 \| x \|_2$.

Answer. ALWAYS

Now prove it!

Solution. W.l.o.g. assume that $x \neq 0$.

We know by the Cauchy-Schwartz inequality that $| x^H z | \leq \| x \|_2 \| z \|_2$. Hence

$$
\begin{array}{l}
\| y x^H \|_2 \\
\quad = \quad < \text{ definition } > \\
\max_{\| z \|_2 = 1} \| y x^H z \|_2 \\
\quad = \quad < \| \cdot \|_2 \text{ is homogenius } > \\
\max_{\| z \|_2 = 1} | x^H z | \| y \|_2 \\
\quad \leq \quad < \text{ Cauchy-Schwartz inequality } > \\
\max_{\| z \|_2 = 1} \| x \|_2 \| z \|_2 \| y \|_2 \\
\quad = \quad < \| z \|_2 = 1 > \\
\| x \|_2 \| y \|_2 .
\end{array}
$$

But also

$$
\begin{array}{l}
\| y x^H \|_2 \\
\quad = \quad < \text{ definition } > \\
\max_{z \neq 0} \| y x^H z \|_2 / \| z \|_2 \\
\quad \geq \quad < \text{ specific } z > \\
\| y x^H x \|_2 / \| x \|_2 \\
\quad = \quad < x^H x = \| x \|_2^2 ; \text{ homogeneity } > \\
\| x \|_2^2 \| y \|_2 / \| x \|_2 \\
\quad = \quad < \text{ algebra } > \\
\| y \|_2 \| x \|_2 .
\end{array}
$$

Hence

$$
\| y x^H \|_2 = \| y \|_2 \| x \|_2 .
$$

Homework 1.3.5.3 Let $A \in \mathbb{C}^{m \times n}$ and a_j its column indexed with j. ALWAYS/SOMETIMES/NEVER: $\| a_j \|_2 \leq \| A \|_2$.

Hint. What vector has the property that $a_j = A x$?

Answer. ALWAYS.

Now prove it!

Solution.

$$\begin{array}{l} \|a_j\|_2 \\ \quad = \\ \|Ae_j\|_2 \\ \quad \leq \\ \max_{\|x\|_2=1} \|Ax\|_2 \\ \quad = \\ \|A\|_2. \end{array}$$

Homework 1.3.5.4 Let $A \in \mathbb{C}^{m\times n}$. Prove that

- $\|A\|_2 = \max_{\|x\|_2=\|y\|_2=1} |y^H Ax|$.
- $\|A^H\|_2 = \|A\|_2$.
- $\|A^H A\|_2 = \|A\|_2^2$.

Hint. Proving $\|A\|_2 = \max_{\|x\|_2=\|y\|_2=1} |y^H Ax|$ requires you to invoke the Cauchy-Schwartz inequality from Theorem 1.2.3.3.

Solution.

- $\|A\|_2 = \max_{\|x\|_2=\|y\|_2=1} |y^H Ax|$:

$$\begin{array}{l} \max_{\|x\|_2=\|y\|_2=1} |y^H Ax| \\ \quad \leq \qquad < \text{ Cauchy-Schwartz } > \\ \max_{\|x\|_2=\|y\|_2=1} \|y\|_2 \|Ax\|_2 \\ \quad = \qquad < \|y\|_2 = 1 > \\ \max_{\|x\|_2=1} \|Ax\|_2 \\ \quad = \qquad < \text{ definition } > \\ \|A\|_2. \end{array}$$

Also, we know there exists x with $\|x\|_2 = 1$ such that $\|A\|_2 = \|Ax\|_2$. Let $y = Ax/\|Ax\|_2$. Then

$$\begin{array}{l} |y^H Ax| \\ \quad = \qquad < \text{ instantiate } > \\ \left| \frac{(Ax)^H(Ax)}{\|Ax\|_2} \right| \\ \quad = \qquad < z^H z = \|z\|_2^2 > \\ \left| \frac{\|Ax\|_2^2}{\|Ax\|_2} \right| \\ \quad = \qquad < \text{ algebra } > \\ \|Ax\|_2 \\ \quad = \qquad < x \text{ was chosen so that } \|Ax\|_2 = \|A\|_2 > \\ \|A\|_2 \end{array}$$

Hence the bound is attained. We conclude that $\|A\|_2 = \max_{\|x\|_2=\|y\|_2=1} |y^H Ax|$.

- $\|A^H\|_2 = \|A\|_2$:

$$\begin{array}{l} \|A^H\|_2 \\ \quad = \quad < \text{ first part of homework } > \\ \max_{\|x\|_2=\|y\|_2=1} |y^H A^H x| \\ \quad = \quad < |\overline{\alpha}| = |\alpha| > \\ \max_{\|x\|_2=\|y\|_2=1} |x^H A y| \\ \quad = \quad < \text{ first part of homework } > \\ \|A\|_2. \end{array}$$

- $\|A^H A\|_2 = \|A\|_2^2$:

$$\begin{array}{l} \|A^H A\|_2 \\ \quad = \quad < \text{ first part of homework } > \\ \max_{\|x\|_2=\|y\|_2=1} |y^H A^H A x| \\ \quad \geq \quad < \text{ restricts choices of } y > \\ \max_{\|x\|_2=1} |x^H A^H A x| \\ \quad = \quad < z^H z = \|z\|_2^2 > \\ \max_{\|x\|_2=1} \|Ax\|_2^2 \\ \quad = \quad < \text{ algebra } > \\ \left(\max_{\|x\|_2=1} \|Ax\|_2\right)^2 \\ \quad = \quad < \text{ definition } > \\ \|A\|_2^2. \end{array}$$

  So, $\|A^H A\|_2 \geq \|A\|_2^2$.

  Now, let's show that the maximum is attained. Let $x = y$ equal the unit length vector such that $\|Ax\|_2 = \|A\|_2$. Then

$$|y^H A^H A x| = |x^H A^H A x| = \|Ax\|_2^2 = \|A\|_2^2.$$

  Thus, the maximum is attained and hence $\|A^H A\|_2 = \max_{\|x\|_2=\|y\|_2=1} |y^H A^H A x| = \|A\|_2^2$.

Homework 1.3.5.5 Partition $A = \left(\begin{array}{c|c|c} A_{0,0} & \cdots & A_{0,N-1} \\ \hline \vdots & & \vdots \\ \hline A_{M-1,0} & \cdots & A_{M-1,N-1} \end{array}\right)$.

ALWAYS/SOMETIMES/NEVER: $\|A_{i,j}\|_2 \leq \|A\|_2$.

Hint. Using Homework 1.3.5.4 choose v_j and w_i such that $\|A_{i,j}\|_2 = |w_i^H A_{i,j} v_j|$.

Solution. Choose v and w such that

$$v = \left(\begin{array}{c} 0 \\ \hline \vdots \\ \hline 0 \\ \hline v_j \\ \hline 0 \\ \hline \vdots \\ \hline 0 \end{array} \right), \quad w = \left(\begin{array}{c} 0 \\ \hline \vdots \\ \hline 0 \\ \hline w_i \\ \hline 0 \\ \hline \vdots \\ \hline 0 \end{array} \right), \quad \text{and} \quad w^H A v = w_i^H A_{i,j} v_j.$$

Then, by Homework 1.3.5.4

$$\|A\|_2 = \max_{\|x\|_2 = \|y\|_2 = 1} |y^H A x| \geq |w^H A v| = |w_i^H A_{i,j} v_j| = \|A_{i,j}\|_2.$$

1.3.6 Computing the matrix 1-norm and ∞-norm

YouTube: `https://www.youtube.com/watch?v=QTKZdGQ2C6w`

The matrix 1-norm and matrix ∞-norm are of great importance because, unlike the matrix 2-norm, they are easy and relatively cheap to compute.. The following exercises show how to practically compute the matrix 1-norm and ∞-norm.

Homework 1.3.6.1 Let $A \in \mathbb{C}^{m \times n}$ and partition $A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right)$. ALWAYS/ SOMETIMES/NEVER: $\|A\|_1 = \max_{0 \leq j < n} \|a_j\|_1$.

Hint. Prove it for the real valued case first.

Answer. ALWAYS

Solution. Let J be chosen so that $\max_{0 \leq j < n} \|a_j\|_1 = \|a_J\|_1$. Then

$$
\begin{array}{l}
\|A\|_1 \\
\quad = \quad < \text{ definition } > \\
\max_{\|x\|_1=1} \|Ax\|_1 \\
\quad = \quad < \text{ expose the columns of } A \text{ and elements of } x > \\
\max_{\|x\|_1=1} \left\| \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{array} \right) \right\|_1 \\
\quad = \quad < \text{ definition of matrix-vector multiplication } > \\
\max_{\|x\|_1=1} \|\chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}\|_1 \\
\quad \leq \quad < \text{ triangle inequality } > \\
\max_{\|x\|_1=1} (\|\chi_0 a_0\|_1 + \|\chi_1 a_1\|_1 + \cdots + \|\chi_{n-1} a_{n-1}\|_1) \\
\quad = \quad < \text{ homogeneity } > \\
\max_{\|x\|_1=1} (|\chi_0|\|a_0\|_1 + |\chi_1|\|a_1\|_1 + \cdots + |\chi_{n-1}|\|a_{n-1}\|_1) \\
\quad \leq \quad < \text{ choice of } a_J > \\
\max_{\|x\|_1=1} (|\chi_0|\|a_J\|_1 + |\chi_1|\|a_J\|_1 + \cdots + |\chi_{n-1}|\|a_J\|_1) \\
\quad = \quad < \text{ factor out } \|a_J\|_1 > \\
\max_{\|x\|_1=1} (|\chi_0| + |\chi_1| + \cdots + |\chi_{n-1}|) \|a_J\|_1 \\
\quad = \quad < \text{ algebra } > \\
\|a_J\|_1.
\end{array}
$$

Also,

$$
\begin{array}{l}
\|a_J\|_1 \\
\quad = \quad < e_J \text{ picks out column } J > \\
\|Ae_J\|_1 \\
\quad \leq \quad < e_J \text{ is a specific choice of } x > \\
\max_{\|x\|_1=1} \|Ax\|_1.
\end{array}
$$

Hence

$$\|a_J\|_1 \leq \max_{\|x\|_1=1} \|Ax\|_1 \leq \|a_J\|_1$$

which implies that

$$\max_{\|x\|_1=1} \|Ax\|_1 = \|a_J\|_1 = \max_{0 \leq j < n} \|a_j\|_1.$$

Homework 1.3.6.2 Let $A \in \mathbb{C}^{m \times n}$ and partition $A = \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \widetilde{a}_1^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array} \right)$.

ALWAYS/SOMETIMES/NEVER:

$$\|A\|_\infty = \max_{0 \leq i < m} \|\widetilde{a}_i\|_1 (= \max_{0 \leq i < m} (|\alpha_{i,0}| + |\alpha_{i,1}| + \cdots + |\alpha_{i,n-1}|))$$

Notice that in this exercise $\widetilde{a}_i$ is really $(\widetilde{a}_i^T)^T$ since $\widetilde{a}_i^T$ is the label for the ith row of matrix

A.

Hint. Prove it for the real valued case first.

Answer. ALWAYS

Solution. Partition $A = \left(\begin{array}{c} \widetilde a_0^T \\ \hline \vdots \\ \hline \widetilde a_{m-1}^T \end{array} \right)$. Then

$$
\begin{array}{l}
\|A\|_\infty \\
\quad = \quad < \text{ definition } > \\
\max_{\|x\|_\infty=1} \|Ax\|_\infty \\
\quad = \quad < \text{ expose rows } > \\
\max_{\|x\|_\infty=1} \left\| \left(\begin{array}{c} \widetilde a_0^T \\ \hline \vdots \\ \hline \widetilde a_{m-1}^T \end{array} \right) x \right\|_\infty \\
\quad = \quad < \text{ matrix-vector multiplication } > \\
\max_{\|x\|_\infty=1} \left\| \left(\begin{array}{c} \widetilde a_0^T x \\ \hline \vdots \\ \hline \widetilde a_{m-1}^T x \end{array} \right) \right\|_\infty \\
\quad = \quad < \text{ definition of } \| \cdots \|_\infty > \\
\max_{\|x\|_\infty=1} \left(\max_{0 \le i < m} |a_i^T x| \right) \\
\quad = \quad < \text{ expose } a_i^T x > \\
\max_{\|x\|_\infty=1} \max_{0 \le i < m} | \sum_{p=0}^{n-1} \alpha_{i,p} \chi_p | \\
\quad \le \quad < \text{ triangle inequality } > \\
\max_{\|x\|_\infty=1} \max_{0 \le i < m} \sum_{p=0}^{n-1} |\alpha_{i,p} \chi_p| \\
\quad = \quad < \|x\|_\infty = 1 > \\
\max_{\|x\|_\infty=1} \max_{0 \le i < m} \sum_{p=0}^{n-1} (|\alpha_{i,p}||\chi_p|) \\
\quad \le \quad < \text{ algebra } > \\
\max_{\|x\|_\infty=1} \max_{0 \le i < m} \sum_{p=0}^{n-1} (|\alpha_{i,p}|(\max_k |\chi_k|)) \\
\quad = \quad < \text{ definition of } \| \cdot \|_\infty > \\
\max_{\|x\|_\infty=1} \max_{0 \le i < m} \sum_{p=0}^{n-1} (|\alpha_{i,p}| \|x\|_\infty) \\
\quad = \quad < \|x\|_\infty = 1 > \\
\max_{0 \le i < m} \sum_{p=0}^{n-1} |\alpha_{i,p}| \\
\quad = \quad < \text{ definition of } \| \cdot \|_1 > \\
\|\widetilde a_i\|_1
\end{array}
$$

so that $\|A\|_\infty \le \max_{0 \le i < m} \|\widetilde a_i\|_1$.

We also want to show that $\|A\|_\infty \ge \max_{0 \le i < m} \|\widetilde a_i\|_1$. Let k be such that $\max_{0 \le i < m} \|\widetilde a_i\|_1 = \|\widetilde a_k\|_1$ and pick $y = \left(\begin{array}{c} \psi_0 \\ \vdots \\ \psi_{n-1} \end{array} \right)$ so that $\widetilde a_k^T y = |\alpha_{k,0}| + |\alpha_{k,1}| + \cdots + |\alpha_{k,n-1}| = \|\widetilde a_k\|_1$. (This is a matter of picking $\psi_j = |\alpha_{k,j}|/\alpha_{k,j}$ if $\alpha_{k,j} \neq 0$ and $\psi_j = 1$ otherwise. Then $|\psi_j| = 1$, and hence $\|y\|_\infty = 1$ and $\psi_j \alpha_{k,j} = |\alpha_{k,j}|$.) Then

$$
\begin{array}{l}
\|A\|_\infty \\
\quad = \quad < \text{ definition } > \\
\max_{\|x\|_1=1} \|Ax\|_\infty \\
\quad = \quad < \text{ expose rows } > \\
\max_{\|x\|_1=1} \left\| \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array} \right) x \right\|_\infty \\
\quad \geq \quad < y \text{ is a specific } x > \\
\left\| \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array} \right) y \right\|_\infty \\
\quad = \quad < \text{ matrix-vector multiplication } > \\
\left\| \left(\begin{array}{c} \widetilde{a}_0^T y \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T y \end{array} \right) \right\|_\infty \\
\quad \geq \quad < \text{ algebra } > \\
|\widetilde{a}_k^T y| \\
\quad = \quad < \text{ choice of } y > \\
\|\widetilde{a}_k\|_1 . \\
\quad = \quad < \text{ choice of } k > \\
\max_{0 \leq i < m} \|\widetilde{a}_i\|_1
\end{array}
$$

Remark 1.3.6.1 The last homework provides a hint as to how to remember how to compute the matrix 1-norm and ∞-norm: Since $\|x\|_1$ must result in the same value whether x is considered as a vector or as a matrix, we can remember that the matrix 1-norm equals the maximum of the 1-norms of the columns of the matrix: Similarly, considering $\|x\|_\infty$ as a vector norm or as matrix norm reminds us that the matrix ∞-norm equals the maximum of the 1-norms of vectors that become the rows of the matrix.

1.3.7 Equivalence of matrix norms

YouTube: `https://www.youtube.com/watch?v=Csqd4AnH7ws`

Homework 1.3.7.1 Fill out the following table:

A	$\|A\|_1$	$\|A\|_\infty$	$\|A\|_F$	$\|A\|_2$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$				
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$				
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$				

Hint. For the second and third, you may want to use Homework 1.3.5.2 when computing the 2-norm.

Solution.

A	$\|A\|_1$	$\|A\|_\infty$	$\|A\|_F$	$\|A\|_2$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1	1	$\sqrt{3}$	1
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	4	3	$2\sqrt{3}$	$2\sqrt{3}$
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	3	1	$\sqrt{3}$	$\sqrt{3}$

To compute the 2-norm of I, notice that

$$\|I\|_2 = \max_{\|x\|_2=1} \|Ix\|_2 = \max_{\|x\|_2=1} \|x\|_2 = 1.$$

Next, notice that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

and

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.$$

which allows us to invoke the result from Homework 1.3.5.2.

We saw that vector norms are equivalent in the sense that if a vector is "small" in one norm, it is "small" in all other norms, and if it is "large" in one norm, it is "large" in all other norms. The same is true for matrix norms.

Theorem 1.3.7.1 Equivalence of matrix norms. *Let* $\| \cdot \| : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ *and* $||| \cdot ||| : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ *both be matrix norms. Then there exist positive scalars* σ *and* τ *such that for all* $A \in \mathbb{C}^{m \times n}$

$$\sigma \|A\| \leq |||A||| \leq \tau \|A\|.$$

Proof. The proof again builds on the fact that the supremum over a compact set is achieved and can be replaced by the maximum.

We will prove that there exists a τ such that for all $A \in \mathbb{C}^{m \times n}$

$$|||A||| \leq \tau \|A\|$$

leaving the rest of the proof to the reader.

Let $A \in \mathbb{C}^{m \times n}$ be an arbitary matrix. W.l.o.g. assume that $A \neq 0$ (the zero matrix). Then

$$\begin{array}{l}
|||A||| \\
\quad = \quad < \text{ algebra } > \\
\frac{|||A|||}{\|A\|} \|A\| \\
\quad \leq \quad < \text{ algebra } > \\
\left(\sup_{Z \neq 0} \frac{|||Z|||}{\|Z\|} \right) \|A\| \\
\quad = \quad < \text{ homogeneity } > \\
\left(\sup_{Z \neq 0} |||\frac{Z}{\|Z\|}||| \right) \|A\| \\
\quad = \quad < \text{ change of variables } B = Z/\|Z\| > \\
\left(\sup_{\|B\|=1} |||B||| \right) \|A\| \\
\quad = \quad < \text{ the set } \|B\| = 1 \text{ is compact } > \\
\left(\max_{\|B\|=1} |||B||| \right) \|A\|
\end{array}$$

The desired τ can now be chosen to equal $\max_{\|B\|=1} |||B|||$. ■

Remark 1.3.7.2 The bottom line is that, modulo a constant factor, if a matrix is "small" in one norm, it is "small" in any other norm.

Homework 1.3.7.2 Given $A \in \mathbb{C}^{m \times n}$show that $\|A\|_2 \leq \|A\|_F$. For what matrix is equality attained?

Solution. Let x with $\|x\|_2 = 1$ be such that $\|Ax\|_2 = \|A\|_2$ Then

$$
\begin{array}{l}
\|A\|_2^2 \\
\quad = \quad < \text{ how } x \text{ is chosen } > \\
\|Ax\|_2^2 \\
\quad = \quad < \text{ partition } A \text{ by columns and expose elements of } x > \\
\left\| \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \left(\begin{array}{c} \chi_0 \\ \vdots \\ \chi_{n-1} \end{array} \right) \right\|_2^2 \\
\quad = \quad < \text{ matrix-vector multiplication } > \\
\|\chi_0 a_0 + \cdots + \chi_{n-1} a_{n-1}\|_2^2 \\
\quad \leq \quad < \text{ many times: vector norm are homogeneous} \\
\qquad\quad\;\; \text{and the triangle inequality } > \\
\left(|\chi_0| \|a_0\|_2 + \cdots + |\chi_{n-1}| \|a_{n-1}\|_2 \right)^2 \\
\quad \leq \quad < |\chi_j| \leq 1 \text{ since } \|x\|_2 = 1 > \\
\left(\|a_0\|_2 + \cdots + \|a_{n-1}\|_2 \right)^2 \\
\quad = \quad < \text{ Homework 1.3.3.1 } > \\
\|A\|_F^2
\end{array}
$$

Hence $\|A\|_2^2 \leq \|A\|_F^2$. Taking the square root of both side yields $\|A\|_2 \leq \|A\|_F$.

Equality is attained for

$$
A = \left(\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right).
$$

(Indeed it equals for any $A = x$ where x is a nonzero vector.)

Homework 1.3.7.3 Let $A \in \mathbb{C}^{m \times n}$. The following table summarizes the equivalence of various matrix norms:

	$\|A\|_1 \leq \sqrt{m}\|A\|_2$	$\|A\|_1 \leq m\|A\|_\infty$	$\|A\|_1 \leq \sqrt{m}\|A\|_F$
$\|A\|_2 \leq \sqrt{n}\|A\|_1$		$\|A\|_2 \leq \sqrt{m}\|A\|_\infty$	$\|A\|_2 \leq \|A\|_F$
$\|A\|_\infty \leq m\|A\|_1$	$\|A\|_\infty \leq \sqrt{n}\|A\|_2$		$\|A\|_\infty \leq \sqrt{m}\|A\|_F$
$\|A\|_F \leq \sqrt{n}\|A\|_1$	$\|A\|_F \leq ?\|A\|_2$	$\|A\|_F \leq \sqrt{m}\|A\|_\infty$	

For each, prove the inequality, including that it is a tight inequality for some nonzero A.

(Skip $\|A\|_F \leq ?\|A\|_2$: we will revisit it in Week 2.)

Solution.

- $\|A\|_1 \leq \sqrt{m}\|A\|_2$:

$\|A\|_1$
$= \quad < \text{ definition } >$
$\max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1}$
$\leq \quad < \|z\|_1 \leq \sqrt{m}\|z\|_2 >$
$\max_{x\neq 0} \frac{\sqrt{m}\|Ax\|_2}{\|x\|_1}$
$\leq \quad < \|z\|_1 \geq \|z\|_2 >$
$\max_{x\neq 0} \frac{\sqrt{m}\|Ax\|_2}{\|x\|_2}$
$= \quad < \text{ algebra; definition } >$
$\sqrt{m}\|A\|_2$

Equality is attained for $A = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

- $\|A\|_1 \leq m\|A\|_\infty$:

$\|A\|_1$
$= \quad < \text{ definition } >$
$\max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1}$
$\leq \quad < \|z\|_1 \leq m\|z\|_\infty >$
$\max_{x\neq 0} \frac{m\|Ax\|_\infty}{\|x\|_1}$
$\leq \quad < \|z\|_1 \geq \|z\|_\infty >$
$\max_{x\neq 0} \frac{m\|Ax\|_\infty}{\|x\|_\infty}$
$= \quad < \text{ algebra; definition } >$
$m\|A\|_\infty$

Equality is attained for $A = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

- $\|A\|_1 \leq \sqrt{m}\|A\|_F$:

  It pays to show that $\|A\|_2 \leq \|A\|_F$ first. Then

$\|A\|_1$
$\leq \quad < \text{ last part } >$
$\sqrt{m}\|A\|_2$
$\leq \quad < \text{ some other part:} \|A\|_2 \leq \|A\|_F >$
$\sqrt{m}\|A\|_F.$

Equality is attained for $A = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

- $\|A\|_2 \leq \sqrt{m}\|A\|_1$:

$$
\begin{array}{l}
\|A\|_2 \\
\quad = \quad \langle \text{ definition } \rangle \\
\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \\
\quad \leq \quad \langle \|z\|_2 \leq \|z\|_1 \rangle \\
\max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_2} \\
\quad \leq \quad \langle \sqrt{m}\|z\|_2 \geq \|z\|_1 \rangle \\
\max_{x \neq 0} \frac{\sqrt{m}\|Ax\|_1}{\|x\|_1} \\
\quad = \quad \langle \text{ algebra; definition } \rangle \\
\sqrt{m}\|A\|_1.
\end{array}
$$

Equality is attained for $A = \left(1 \mid 1 \mid \cdots \mid 1 \right)$.

- $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$:

$$
\begin{array}{l}
\|A\|_2 \\
\quad = \quad \langle \text{ definition } \rangle \\
\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \\
\quad \leq \quad \langle \|z\|_2 \leq \sqrt{m}\|z\|_\infty \rangle \\
\max_{x \neq 0} \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_2} \\
\quad \leq \quad \langle \|z\|_2 \geq \|z\|_\infty \rangle \\
\max_{x \neq 0} \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_\infty} \\
\quad = \quad \langle \text{ algebra; definition } \rangle \\
\sqrt{m}\|A\|_\infty.
\end{array}
$$

Equality is attained for $A = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

- $\|A\|_2 \leq \|A\|_F$:

Let x with $\|x\|_2 = 1$ be such that $\|Ax\|_2 = \|A\|_2$ Then

$$
\begin{array}{l}
\|A\|_2^2 \\
\quad = \quad < \text{ how } x \text{ is chosen } > \\
\|Ax\|_2^2 \\
\quad = \quad < \text{ partition } A \text{ by columns and expose elements of } x > \\
\left\| \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \left(\begin{array}{c} \chi_0 \\ \vdots \\ \chi_{n-1} \end{array} \right) \right\|_2^2 \\
\quad = \quad < \text{ matrix-vector multiplication } > \\
\|\chi_0 a_0 + \cdots + \chi_{n-1} a_{n-1}\|_2^2 \\
\quad \leq \quad < \text{ many times: vector norms are homogeneous} \\
\qquad\qquad \text{and the triangle inequality } > \\
(|\chi_0| \|a_0\|_2 + \cdots + |\chi_{n-1}| \|a_{n-1}\|_2)^2 \\
\quad \leq \quad < |\chi_j| \leq 1 \text{ since } \|x\|_2 = 1 > \\
(\|a_0\|_2 + \cdots + \|a_{n-1}\|_2)^2 \\
\quad = \quad < \text{ Homework 1.3.3.1 } > \\
\|A\|_F^2
\end{array}
$$

Hence $\|A\|_2^2 \leq \|A\|_F^2$. Taking the square root of both side yields $\|A\|_2 \leq \|A\|_F$.

Equality is attained for

$$A = \left(\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right).$$

(Indeed it equals for any $A = x$ where x is a nonzero vector.)

- Please share more solutions!

1.3.8 Submultiplicative norms

YouTube: `https://www.youtube.com/watch?v=TvthvYGt9x8`

There are a number of properties that we would like for a matrix norm to have (but not all norms do have). Recalling that we would like for a matrix norm to measure by how much a vector is "stretched," it would be good if for a given matrix norm, $\| \cdots \| : \mathbb{C}^{m \times n} \to \mathbb{R}$, there are vector norms $\| \cdot \|_\mu : \mathbb{C}^m \rightarrow \mathbb{R}$ and $\| \cdot \|_\nu : \mathbb{C}^n \rightarrow \mathbb{R}$ such that, for arbitrary nonzero $x \in \mathbb{C}^n$, the matrix norm bounds by how much the vector is stretched:

$$\frac{\| A x \|_\mu}{\| x \|_\nu} \leq \| A \|$$

or, equivalently,

$$\| A x \|_\mu \leq \| A \| \| x \|_\nu$$

where this second formulation has the benefit that it also holds if $x = 0$. When this relationship between the involved norms holds, the matrix norm is said to be subordinate to the vector norms:

Definition 1.3.8.1 Subordinate matrix norm. A matrix norm $\| \cdot \| : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ is said to be subordinate to vector norms $\| \cdot \|_\mu : \mathbb{C}^m \rightarrow \mathbb{R}$ and $\| \cdot \|_\nu : \mathbb{C}^n \rightarrow \mathbb{R}$ if, for all $x \in \mathbb{C}^n$,

$$\| A x \|_\mu \leq \| A \| \| x \|_\nu .$$

If $\| \cdot \|_\mu$ and $\| \cdot \|_\nu$ are the same norm (but perhaps for different m and n), then $\| \cdot \|$ is said to be subordinate to the given vector norm. $\Diamond$

Fortunately, all the norms that we will employ in this course are subordinate matrix norms.

Homework 1.3.8.1 ALWAYS/SOMETIMES/NEVER: The Frobenius norm is subordinate to the vector 2-norm.

Answer. TRUE

Now prove it.

Solution. W.l.o.g., assume $x \neq 0$.

$$\| A x \|_2 = \frac{\| A x \|_2}{\| x \|_2} \| x \|_2 \leq \max_{y \neq 0} \frac{\| A y \|_2}{\| y \|_2} \| x \|_2 = \max_{\| y \|_2 = 1} \| A y \|_2 \| x \|_2 = \| A \|_2 \| x \|_2 .$$

So, it suffices to show that $\| A \|_2 \leq \| A \|_F$. But we showed that in Homework 1.3.7.2.

Theorem 1.3.8.2 *Induced matrix norms, $\| \cdot \|_{\mu,\nu} : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$, are subordinate to the norms, $\| \cdot \|_\mu$ and $\| \cdot \|_\nu$, that induce them.*

Proof. W.l.o.g. assume $x \neq 0$. Then

$$\| A x \|_\mu = \frac{\| A x \|_\mu}{\| x \|_\nu} \| x \|_\nu \leq \max_{y \neq 0} \frac{\| A y \|_\mu}{\| y \|_\nu} \| x \|_\nu = \| A \|_{\mu,\nu} \| x \|_\nu .$$

$\blacksquare$

Corollary 1.3.8.3 *Any matrix p-norm is subordinate to the corresponding vector p-norm.*

Another desirable property that not all norms have is that

$$\| A B \| \leq \| A \| \| B \| .$$

This requires the given norm to be defined for all matrix sizes..

Definition 1.3.8.4 Consistent matrix norm. A matrix norm $\| \cdot \| : \mathbb{C}^{m \times n}$ is said to be a consistent matrix norm if it is defined for all m and n, using the same formula for all m and n. ◊

Obviously, this definition is a bit vague. Fortunately, it is pretty clear that all the matrix norms we will use in this course, the Frobenius norm and the p-norms, are all consistently defined for all matrix sizes.

Definition 1.3.8.5 Submultiplicative matrix norm. A consistent matrix norm $\| \cdot \| : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ is said to be submultiplicative if it satisfies

$$\|AB\| \leq \|A\|\|B\|.$$

◊

Theorem 1.3.8.6 *Let $\| \cdot \| : \mathbb{C}^n \rightarrow \mathbb{R}$ be a vector norm defined for all n. Define the corresponding induced matrix norm as*

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|.$$

Then for any $A \in \mathbb{C}^{m \times k}$ and $B \in \mathbb{C}^{k \times n}$ the inequality $\|AB\| \leq \|A\|\|B\|$ holds.

In other words, induced matrix norms are submultiplicative. To prove this theorem, it helps to first prove a simpler result:

Lemma 1.3.8.7 *Let $\|\cdot\| : \mathbb{C}^n \rightarrow \mathbb{R}$ be a vector norm defined for all n and let $\|\cdot\| : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ be the matrix norm it induces. Then $\| \cdot \|$ is a submultiplicative norm.*

Proof. If $x = 0$, the result obviously holds since then $\|Ax\| = 0$ and $\|x\| = 0$. Let $x \neq 0$. Then

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \geq \frac{\|Ax\|}{\|x\|}.$$

Rearranging this yields $\|Ax\| \leq \|A\|\|x\|$. ∎

We can now prove the theorem:

Proof.

$$
\begin{array}{l}
\|AB\| \\
\quad = \quad < \text{ definition of induced matrix norm } > \\
\max_{\|x\|=1} \|ABx\| \\
\quad = \quad < \text{ associativity } > \\
\max_{\|x\|=1} \|A(Bx)\| \\
\quad \leq \quad < \text{ lemma } > \\
\max_{\|x\|=1} (\|A\|\|Bx\|) \\
\quad \leq \quad < \text{ lemma } > \\
\max_{\|x\|=1} (\|A\|\|B\|\|x\|) \\
\quad = \quad < \|x\| = 1 > \\
\|A\|\|B\|.
\end{array}
$$

∎

Homework 1.3.8.2 Show that $\|Ax\|_\mu \leq \|A\|_{\mu,\nu}\|x\|_\nu$.

Solution. W.l.o.g. assume that $x \neq 0$.

$$\|A\|_{\mu,\nu} = \max_{y \neq 0} \frac{\|Ay\|_\mu}{\|y\|_\nu} \geq \frac{\|Ax\|_\mu}{\|x\|_\nu}.$$

Rearranging this establishes the result.

Homework 1.3.8.3 Show that $\|AB\|_\mu \leq \|A\|_{\mu,\nu}\|B\|_\nu$.

Solution.

$$\begin{array}{l}
\|AB\|_\mu \\
\quad = \qquad < \text{ definition } > \\
\max_{\|x\|_\nu=1} \|ABx\|_\mu \\
\quad \leq \qquad < \text{ last homework } > \\
\max_{\|x\|_\nu=1} \|A\|_{\mu,\nu}\|Bx\|_\nu \\
\quad = \qquad < \text{ algebra } > \\
\|A\|_\nu \max_{\|x\|_\nu=1} \|Bx\|_\nu \\
\quad = \qquad < \text{ definition } > \\
\|A\|_{\mu,\nu}\|B\|_\nu
\end{array}$$

Homework 1.3.8.4 Show that the Frobenius norm, $\|\cdot\|_F$, is submultiplicative.

Solution.

$$\begin{array}{l}
\|AB\|_F^2 \\
\quad = \qquad < \text{ partition } > \\
\left\| \left(\begin{array}{c} \widetilde{a}_0^H \\ \widetilde{a}_1^H \\ \vdots \\ \widetilde{a}_{m-1}^H \end{array} \right) \left(\begin{array}{c|c|c|c} b_0 & b_1 & \cdots & b_{n-1} \end{array} \right) \right\|_F^2 \\
\quad = \qquad < \text{ partitioned matrix-matrix multiplication } > \\
\left\| \left(\begin{array}{c|c|c|c} \widetilde{a}_0^H b_0 & \widetilde{a}_0^H b_1 & \cdots & \widetilde{a}_0^H b_{n-1} \\ \hline \widetilde{a}_0^H b_0 & \widetilde{a}_0^H b_1 & \cdots & \widetilde{a}_0^H b_{n-1} \\ \hline \vdots & \vdots & & \vdots \\ \hline \widetilde{a}_{m-1}^H b_0 & \widetilde{a}_{m-1}^H b_1 & \cdots & \widetilde{a}_{m-1}^H b_{n-1} \end{array} \right) \right\|_F^2 \\
\quad = \qquad < \text{ definition of Frobenius norm } > \\
\sum_i \sum_j |\widetilde{a}_i^H b_j|^2 \\
\quad \leq \qquad < \text{ Cauchy-Schwartz inequality } > \\
\sum_i \sum_j \|\widetilde{a}_i\|_2^2 \|b_j\|_2^2 \\
\quad = \qquad < \text{ algebra } > \\
\left(\sum_i \|\widetilde{a}_i\|_2^2\right) \left(\sum_j \|b_j\|_2^2\right) \\
\quad = \qquad < \text{ previous observations about the Frobenius norm } > \\
\|A\|_F^2 \|B\|_F^2
\end{array}$$

Hence $\|AB\|_F^2 \leq \|A\|_F^2\|B\|_F^2$. Taking the square root of both sides leaves us with $\|AB\|_F \leq \|A\|_F\|B\|_F$.

Homework 1.3.8.5 For $A \in \mathbb{C}^{m\times n}$ define

$$\|A\| = \max_{i=0}^{m-1} \max_{j=0}^{n-1} |\alpha_{i,j}|.$$

1. TRUE/FALSE: This is a norm.

2. TRUE/FALSE: This is a consistent norm.

3. TRUE/FALSE: This is a submultiplicative norm.

Answer.

1. TRUE

2. TRUE

3. FALSE

Solution.

1. This is a norm. You can prove this by checking the three conditions.

2. It is a consistent norm since it is defined for all m and n.

3. It is submultiplicative if $m = n = 1$

   To show it is not always submultiplicative, you need to find matrices A and B such that $\|AB\| > \|A\|\|B\|$.

   Pick $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\|AB\| = 2$ and $\|A\|\|B\| = 1$.

Remark 1.3.8.8 The important take-away: The norms we tend to use in this course, the p-norms and the Frobenius norm, are all submultiplicative.

Homework 1.3.8.6 Let $A \in \mathbb{C}^{m\times n}$.
ALWAYS/SOMETIMES/NEVER: There exists a vector

$$x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{m-1} \end{pmatrix} \text{ with } |\chi_i| = 1 \text{ for } i = 0, \ldots, n-1$$

such that $\|A\|_\infty = \|Ax\|_\infty$.

Answer. ALWAYS
Now prove it!

Solution. Partition A by rows:

$$A = \begin{pmatrix} \widetilde{a}_0^T \\ \vdots \\ \widetilde{a}_{m-1}^T \end{pmatrix}.$$

We know that there exists k such that $\|\widetilde{a}_k\|_1 = \|A\|_\infty$. Now

$$\begin{array}{l} \|\widetilde{a}_k\|_1 \\ \quad = \quad < \text{ definition of 1-norm } > \\ |\alpha_{k,0}| + \cdots + |\alpha_{k,n-1}| \\ \quad = \quad < \text{ algebra } > \\ \frac{|\alpha_{k,0}|}{\alpha_{k,0}}\alpha_{k,0} + \cdots + \frac{|\alpha_{k,n-1}|}{\alpha_{k,n-1}}\alpha_{k,n-1}. \end{array}$$

where we take $\frac{|\alpha_{k,j}|}{\alpha_{k,j}} = 1$ whenever $\alpha_{k,j} = 0$. Vector

$$x = \begin{pmatrix} \frac{|\alpha_{k,0}|}{\alpha_{k,0}} \\ \vdots \\ \frac{|\alpha_{k,n-1}|}{\alpha_{k,n-1}} \end{pmatrix}$$

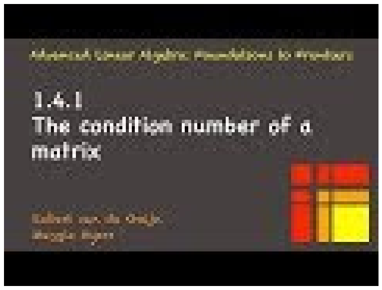
has the desired property.

1.3.9 Summary

YouTube: https://www.youtube.com/watch?v=DyoT2tJhxIs

1.4 Condition Number of a Matrix

1.4.1 Conditioning of a linear system



YouTube: `https://www.youtube.com/watch?v=QwFQNAPKIwk`

A question we will run into later in the course asks how accurate we can expect the solution of a linear system to be if the right-hand side of the system has error in it.

Formally, this can be stated as follows: We wish to solve $Ax = b$, where $A \in \mathbb{C}^{m \times m}$ but the right-hand side has been perturbed by a small vector so that it becomes $b + \delta\!b$.

Remark 1.4.1.1 Notice how the δ touches the b. This is meant to convey that this is a symbol that represents a vector rather than the vector b that is multiplied by a scalar δ.

The question now is how a relative error in b is amplified into a relative error in the solution x.

This is summarized as follows:

$$\begin{array}{rcll} Ax & = & b & \quad\quad \text{exact equation} \\ A(x + \delta\!x) & = & b + \delta\!b & \quad\quad \text{perturbed equation} \end{array}$$

We would like to determine a formula, $\kappa(A, b, \delta\!b)$, that gives us a bound on how much a relative error in b is potentially amplified into a relative error in the solution x:

$$\frac{\|\delta\!x\|}{\|x\|} \leq \kappa(A, b, \delta\!b) \frac{\|\delta\!b\|}{\|b\|}.$$

We assume that A has an inverse since otherwise there may be no solution or there may be an infinite number of solutions. To find an expression for $\kappa(A, b, \delta\!b)$, we notice that

$$\begin{array}{rcl} Ax + A\delta\!x & = & b + \delta\!b \\ Ax \quad\quad & = & b \quad\quad - \\ \hline A\delta\!x & = & \quad\delta\!b \end{array}$$

and from this we deduce that

$$\begin{array}{rcl} Ax & = & b \\ \delta\!x & = & A^{-1}\delta\!b. \end{array}$$

If we now use a vector norm $\| \cdot \|$ and its induced matrix norm $\| \cdot \|$, then

$$\begin{array}{rcl} \|b\| & = & \|Ax\| \leq \|A\| \|x\| \\ \|\delta\!x\| & = & \|A^{-1}\delta\!b\| \leq \|A^{-1}\| \|\delta\!b\| \end{array}$$

since induced matrix norms are subordinate.

From this we conclude that

$$\frac{1}{\|x\|} \leq \|A\| \frac{1}{\|b\|}$$

and

$$\|\delta x\| \leq \|A^{-1}\| \|\delta b\|$$

so that

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|\delta b\|}{\|b\|}.$$

Thus, the desired expression $\kappa(A, b, \delta b)$ doesn't depend on anything but the matrix A:

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\| \|A^{-1}\|}_{\kappa(A)} \frac{\|\delta b\|}{\|b\|}.$$

Definition 1.4.1.2 Condition number of a nonsingular matrix. The value $\kappa(A) = \|A\| \|A^{-1}\|$ is called the condition number of a nonsingular matrix A. $\Diamond$

A question becomes whether this is a pessimistic result or whether there are examples of b and δb for which the relative error in b is amplified by exactly $\kappa(A)$. The answer is that, unfortunately, the bound is tight.

- There is an $\widehat{x}$ for which

  $$\|A\| = \max_{\|x\|=1} \|Ax\| = \|A\widehat{x}\|,$$

  namely the x for which the maximum is attained. This is the direction of maximal magnification. Pick $\widehat{b} = A\widehat{x}$.

- There is an $\widehat{\delta b}$ for which

  $$\|A^{-1}\| = \max_{\|x\| \neq 0} \frac{\|A^{-1}x\|}{\|x\|} = \frac{\|A^{-1}\widehat{\delta b}\|}{\|\widehat{\delta b}\|},$$

  again, the x for which the maximum is attained.

It is when solving the perturbed system

$$A(x + \delta x) = \widehat{b} + \widehat{\delta b}$$

that the maximal magnification by $\kappa(A)$ is observed.

Homework 1.4.1.1 Let $\| \cdot \|$ be a vector norm and corresponding induced matrix norm.

TRUE/FALSE: $\|I\| = 1$.

Answer. TRUE

Solution.

$$\|I\| = \max_{\|x\|=1} \|Ix\| = \max_{\|x\|=1} \|x\| = 1$$

Homework 1.4.1.2 Let $\|\cdot\|$ be a vector norm and corresponding induced matrix norm, and A a nonsingular matrix.

TRUE/FALSE: $\kappa(A) = \|A\|\|A^{-1}\| \geq 1$.

Answer. TRUE

Solution.

$$\begin{array}{l} 1 \\ \quad = \quad < \text{ last homework } > \\ \|I\| \\ \quad = \quad < A \text{ is invertible } > \\ \|AA^{-1}\| \\ \quad \leq \quad < \|\cdot\| \text{ is submultiplicative } > \\ \|A\|\|A^{-1}\|. \end{array}$$

Remark 1.4.1.3 This last exercise shows that there will always be choices for b and δb for which the relative error is at best directly translated into an equal relative error in the solution (if $\kappa(A) = 1$).

1.4.2 Loss of digits of accuracy

YouTube: `https://www.youtube.com/watch?v=-5l9Ov5RXYo`

Homework 1.4.2.1 Let $\alpha = -14.24123$ and $\widehat{\alpha} = -14.24723$. Compute

- $|\alpha| =$
- $|\alpha - \widehat{\alpha}| =$
- $\frac{|\alpha - \widehat{\alpha}|}{|\alpha|} =$
- $\log_{10}\left(\frac{|\alpha - \widehat{\alpha}|}{|\alpha|}\right) =$

Solution. Let $\alpha = -14.24123$ and $\widehat{\alpha} = -14.24723$. Compute

- $|\alpha| = 14.24123$
- $|\alpha - \widehat{\alpha}| = 0.006$
- $\frac{|\alpha - \widehat{\alpha}|}{|\alpha|} \approx 0.00042$

- $\log_{10}\left(\frac{|\alpha-\widehat{\alpha}|}{|\alpha|}\right) \approx -3.4$

The point of this exercise is as follows:

- If you compare $\alpha = -14.24123$
  $\hat{\alpha} = -14.24723$ and you consider $\hat{\alpha}$ to be an approximation of α, then $\hat{\alpha}$ is accurate to four digits: -14.24 is accurate.

- Computing $\log_{10}\left(\frac{|\alpha-\hat{\alpha}|}{|\alpha|}\right)$ tells you approximately how many decimal digits are accurate: 3.4 digits.

Be sure to read the solution to the last homework!

1.4.3 The conditioning of an upper triangular matrix

YouTube: `https://www.youtube.com/watch?v=LGBFyjhjt6U`

We now revisit the material from the launch for the semester. We understand that when solving $Lx = b$, even a small relative change to the right-hand side b can amplify into a large relative change in the solution $\hat{x}$ if the condition number of the matrix is large.

Homework 1.4.3.1 Change the script `Assignments/Week01/matlab/Test_Upper_triangular_solve_100.m` to also compute the condition number of matrix U, $\kappa(U)$. Investigate what happens to the condition number as you change the problem size n.

Since in the example the upper triangular matrix is generated to have random values as its entries, chances are that at least one element on its diagonal is small. If that element were zero, then the triangular matrix would be singular. Even if it is not exactly zero, the condition number of U becomes very large if the element is small.

1.5 Enrichments

1.5.1 Condition number estimation

It has been observed that high-quality numerical software should not only provide routines for solving a given problem, but, when possible, should also (optionally) provide the user with feedback on the conditioning (sensitivity to changes in the input) of the problem. In this enrichment, we relate this to what you have learned this week.

Given a vector norm $\|\cdot\|$ and induced matrix norm $\|\cdot\|$, the condition number of matrix

A using that norm is given by $\kappa(A) = \|A\|\|A^{-1}\|$. When trying to practically compute the condition number, this leads to two issues:

- Which norm should we use? A case has been made in this week that the 1-norm and ∞-norm are canditates since they are easy and cheap to compute.
- It appears that A^{-1} needs to be computed. We will see in future weeks that this is costly: $O(m^3)$ computation when A is $m \times m$. This is generally considered to be expensive.

This leads to the question "Can a reliable estimate of the condition number be cheaply computed?" In this unit, we give a glimpse of how this can be achieved and then point the interested learner to related papers.

Partition $m \times m$ matrix A:

$$A = \left(\begin{array}{c} \widetilde{a}_0^T \\ \vdots \\ \widetilde{a}_{m-1}^T \end{array} \right).$$

We recall that

- The ∞-norm is defined by
$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty.$$
- From Homework 1.3.6.2, we know that the ∞-norm can be practically computed as
$$\|A\|_\infty = \max_{0 \le i < m} \|\widetilde{a}_i\|_1,$$
where $\widetilde{a}_i = (\widetilde{a}_i^T)^T$. This means that $\|A\|_\infty$ can be computed in $O(m^2)$ operations.
- From the solution to Homework 1.3.6.2, we know that there is a vector x with $|\chi_i| = 1$, $0 \le i < m$, such that $\|A\|_\infty = \|Ax\|_1$. This x satisfies $\|x\|_\infty = 1$.

From this we conclude that

$$\|A\|_\infty = \max_{x \in \mathcal{S}} \|Ax\|_\infty,$$

where $\mathcal{S}$ is the set of all vectors x with $|\chi_i| = 1$, $0 \le i < m$.

We will illustrate the techniques that underly efficient condition number estimation by looking at the simpler case where we wish to estimate the condition number of a *real-valued* nonsingular upper triangular $m \times m$ matrix U, using the ∞-norm. Since U is real-valued, $|\chi_i| = 1$ means $\chi_i = \pm 1$. The problem is that it appears we must compute $\|U^{-1}\|_\infty$. Computing U^{-1} when U is dense requires $O(m^3)$ operations (a topic we won't touch on until much later in the course).

Our observations tell us that

$$\|U^{-1}\|_\infty = \max_{x \in \mathcal{S}} \|U^{-1}x\|_\infty,$$

where $\mathcal{S}$ is the set of all vectors x with elements $\chi_i \in \{-1, 1\}$. This is equivalent to

$$\|U^{-1}\|_\infty = \max_{z \in \mathcal{T}} \|z\|_\infty,$$

where $\mathcal{T}$ is the set of all vectors z that satisfy $Uz = y$ for some y with elements $\psi_i \in \{-1, 1\}$. So, we could solve $Uz = y$ for all vectors $y \in \mathcal{S}$, compute the ∞-norm for all those vectors z, and pick the maximum of those values. But that is not practical.

One simple solution is to try to construct a vector y that results in a large amplification (in the ∞-norm) when solving $Uz = y$, and to then use that amplification as an estimate for $\|U^{-1}\|_\infty$. So how do we do this? Consider

$$\underbrace{\begin{pmatrix} \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \upsilon_{m-2,m-2} & \upsilon_{m-2,m-1} \\ 0 & \cdots & 0 & \upsilon_{m-1,m-1} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \vdots \\ \zeta_{m-2} \\ \zeta_{m-1} \end{pmatrix}}_{z} = \underbrace{\begin{pmatrix} \vdots \\ \psi_{m-2} \\ \psi_{m-1} \end{pmatrix}}_{y}.$$

Here is a *heuristic* for picking $y \in \mathcal{S}$:

- We want to pick $\psi_{m-1} \in \{0, 1\}$ in order to construct a vector $y \in \mathcal{S}$. We can pick $\psi_{m-1} = 1$ since picking it equal to -1 will simply carry through negation in the appropriate way in the scheme we are describing.

  From this ψ_{m-1} we can compute ζ_{m-1}.

- Now,

  $$\upsilon_{m-2,m-2}\zeta_{m-2} + \upsilon_{1,2}\zeta_{m-1} = \psi_{m-2}$$

  where ζ_{m-1} is known and ψ_{m-2} can be strategically chosen. We want to z to have a large ∞-norm and hence a *heuristic* is to now pick $\psi_{m-2} \in \{-1, 1\}$ in such a way that ζ_{m-2} is as large as possible in magnitude.

  With this χ_{m-2} we can compute ζ_{m-2}.

- And so forth!

When done, the magnification equals $\|z\|_\infty = |\zeta_k|$, where ζ_k is the element of z with largest magnitude. This approach provides an estimate for $\|U^{-1}\|_\infty$ with $O(m^2)$ operations.

The described method underlies the condition number estimator for LINPACK, developed in the 1970s [11], as described in [7]:

- A.K. Cline, C.B. Moler, G.W. Stewart, and J.H. Wilkinson, An estimate for the condition number of a matrix, SIAM J. Numer. Anal., 16 (1979).

The method discussed in that paper yields a lower bound on $\|A^{-1}\|_\infty$ and with that on $\kappa_\infty(A)$.

Remark 1.5.1.1 Alan Cline has his office on our floor at UT-Austin. G.W. (Pete) Stewart was Robert's Ph.D. advisor. Cleve Moler is the inventor of Matlab. John Wilkinson received the Turing Award for his contributions to numerical linear algebra.

More sophisticated methods are discussed in [15]:

- N. Higham, A Survey of Condition Number Estimates for Triangular Matrices, SIAM Review, 1987.

His methods underlie the LAPACK [1] condition number estimator and are remarkably accurate: most of the time they provides an almost exact estimate of the actual condition number.

1.6 Wrap Up

1.6.1 Additional homework

Homework 1.6.1.1 For $e_j \in \mathbb{R}^n$ (a standard basis vector), compute

- $\|e_j\|_2 =$
- $\|e_j\|_1 =$
- $\|e_j\|_\infty =$
- $\|e_j\|_p =$

Answer. $\|e_j\|_2 = \|e_j\|_1 = \|e_j\|_\infty = \|e_j\|_p = 1$

Homework 1.6.1.2 For $I \in \mathbb{R}^{n\times n}$ (the identity matrix), compute

- $\|I\|_1 =$
- $\|I\|_\infty =$
- $\|I\|_2 =$
- $\|I\|_p =$
- $\|I\|_F =$

Solution.

- $\|I\|_F = \sqrt{n}$
- $\|I\|_1 = 1$
- $\|I\|_\infty = 1$
- $\|I\|_2 = 1$
- $\|I\|_p = 1$

Homework 1.6.1.3 Let $D = \begin{pmatrix} \delta_0 & 0 & \cdots & 0 \\ 0 & \delta_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \delta_{n-1} \end{pmatrix}$ (a diagonal matrix). Compute

- $\|D\|_1 =$
- $\|D\|_\infty =$
- $\|D\|_p =$
- $\|D\|_F =$

Answer.

- $\|D\|_1 = \max_{0 \leq i < m} |\delta_i|$
- $\|D\|_\infty = \max_{0 \leq i < m} |\delta_i|$
- $\|D\|_p = \max_{0 \leq i < m} |\delta_i|$.
- $\|D\|_F = \sqrt{|\delta_0|^2 + \cdots + |\delta_{n-1}|^2}$.

Solution.

$$
\begin{array}{rcl}
\|D\|_p & = & \max_{\|x\|_p=1} \|Dx\|_p \\
& = & \max_{\|x\|_p=1} \left\| \begin{pmatrix} \delta_0 & 0 & \cdots & 0 \\ 0 & \delta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \delta_{n-1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} \right\|_p \\
& = & \max_{\|x\|_p=1} \left\| \begin{pmatrix} \delta_0 \chi_0 \\ \delta_1 \chi_1 \\ \vdots \\ \delta_{n-1} \chi_{n-1} \end{pmatrix} \right\|_p \\
& = & \max_{\|x\|_p=1} \sqrt[p]{|\delta_0 \chi_0|^p + \cdots + |\delta_{n-1} \chi_{n-1}|^p} \\
& = & \max_{\|x\|_p=1} \sqrt[p]{|\delta_0|^p |\chi_0|^p + \cdots + |\delta_{n-1}|^p |\chi_{n-1}|^p} \\
& \leq & \max_{\|x\|_p=1} \sqrt[p]{\max_k |\delta_k|^p |\chi_0|^p + \cdots + \max_k |\delta_k|^p |\chi_{n-1}|^p} \\
& = & \max_k |\delta_k| \max_{\|x\|_p=1} \sqrt[p]{|\chi_0|^p + \cdots + |\chi_{n-1}|^p} \\
& = & \max_k |\delta_k| \max_{\|x\|_p=1} \|x\|_p = \max_k |\delta_k|.
\end{array}
$$

Also,

$$
\|D\|_p \quad = \quad \max_{\|x\|_p=1} \|Dx\|_p \geq \|De_J\|_p = \|\delta_J e_J\|_p = |\delta_J| \|e_J\|_p = |\delta_J| = \max_k |\delta_k|
$$

where J is chosen to be the index so that $= |\delta_J| = \max_k |\delta_k|$.

Thus

$$\max_k |\delta_k| \leq \|D\|_p \leq \max_k |\delta_k|$$

from which we conclude that $\|D\|_p = \max_k |\delta_k|$.

$\|D\|_F = \sqrt{|\delta_0|^2 + \cdots + |\delta_{n-1}|^2}$.

Homework 1.6.1.4 Let $x = \left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{N-1} \end{array} \right)$ and $1 \leq p < \infty$ or $p = \infty$.

ALWAYS/SOMETIMES/NEVER: $\|x_i\|_p \leq \|x\|_p$.

Answer. ALWAYS

Now prove it!

Solution. If $1 \leq p < \infty$, then

$$\|x\|_p^p = \left\| \left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{N-1} \end{array} \right) \right\|_p^p = \|x_0\|_p^p + \|x_1\|_p^p + \cdots + \|x_{N-1}\|_p^p$$

$$\geq \|x_i\|_p^p.$$

Hence $\|x_i\|_p \leq \|x\|_p$.

For $p = \infty$,

$$\|x\|_\infty = \max_{i=0}^{n-1} |\chi_i| = \max(\|x_0\|_\infty, \|x_1\|_\infty, \ldots, \|x_{N-1}\|_\infty) \geq \|x_i\|_\infty.$$

Homework 1.6.1.5 For

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

compute

- $\|A\|_1 =$
- $\|A\|_\infty =$
- $\|A\|_F =$

Solution.

- $\|A\|_1 = 3$
- $\|A\|_\infty = 4$
- $\|A\|_F = \sqrt{1^2 + 2^2 + (-1)^2 + (-1)^2 + 1^2 + 0^2} = \sqrt{8} = 2\sqrt{2}$

Homework 1.6.1.6 For $A \in \mathbb{C}^{m \times n}$ define

$$\|A\| = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\alpha_{i,j}| = \sum \left(\begin{array}{ccc} |\alpha_{0,0}|, & \cdots, & |\alpha_{0,n-1}|, \\ \vdots & & \vdots \\ |\alpha_{m-1,0}|, & \cdots, & |\alpha_{m-1,n-1}| \end{array} \right).$$

- TRUE/FALSE: This function is a matrix norm.
- How can you relate this norm to the vector 1-norm?
- TRUE/FALSE: For this norm, $\|A\| = \|A^H\|$.
- TRUE/FALSE: This norm is submultiplicative.

Answer.

1. This function is a matrix norm.

   TRUE

   Now prove it!

2. How can you relate this norm to the vector 1-norm?

   Short answer: Partition matrix A by columns. This norm equals the 1-norm of the vector created by stacking the columns.

   Now give a detailed answer!

3. For this norm, $\|A\| = \|A^H\|$

   TRUE

   Now prove it!

4. FALSE: This norm is not submultiplicative.

   Now prove it!

Solution.

1. This function is a matrix norm:

   Like in the solution for Homework 1.3.8.5, one way to answer this is to realize that if

$A = \left(a_0 \mid a_1 \mid \cdots \mid a_{n-1} \right)$ then

$$\begin{array}{l} \|A\| \\ \quad = \qquad < \text{ definition } > \\ \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}| \\ \quad = \qquad < \text{ commutativity of sum } > \\ \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} |\alpha_{i,j}| \\ \quad = \qquad < \text{ definition of vector 1-norm } > \\ \sum_{j=0}^{n-1} \|a_j\|_1 \\ \quad = \qquad < \text{ definition of vector 1-norm } > \\ \left\| \left(\begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{array} \right) \right\|_1 . \end{array}$$

In other words, it equals the vector 1-norm of the vector that is created by stacking the columns of A on top of each other. The fact that this is a norm then comes from realizing this connection and exploiting it.

Alternatively, just grind through the three conditions!

2. How can you relate this norm to the vector 1-norm?

   See the answer to the last part.

3. For this norm, $\|A\| = \|A^H\|$.

$$\|A\| = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}| = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} |\alpha_{i,j}| = \|A^H\|.$$

4. That is a very good question... I thought the answer is "no", but I am having trouble finding an example to show this..

Homework 1.6.1.7 Let $A \in \mathbb{C}^{m \times n}$. Partition

$$A = \left(a_0 \mid a_1 \mid \cdots \mid a_{n-1} \right) = \left(\begin{array}{c} \widetilde{a}_0^T \\ \widetilde{a}_1^T \\ \vdots \\ \widetilde{a}_{m-1}^T \end{array} \right).$$

Prove that

- $\|A\|_F = \|A^T\|_F$.
- $\|A\|_F = \sqrt{\|a_0\|_2^2 + \|a_1\|_2^2 + \cdots + \|a_{n-1}\|_2^2}$.
- $\|A\|_F = \sqrt{\|\widetilde{a}_0\|_2^2 + \|\widetilde{a}_1\|_2^2 + \cdots + \|\widetilde{a}_{m-1}\|_2^2}$.

Note that here $\widetilde{a}_i = (\widetilde{a}_i^T)^T$.

Solution.

- $\|A\|_F = \|A^T\|_F$:

$$\begin{array}{rcl} \|A\|_F^2 & = & \sum_{i=0}^{m-1} \left(\sum_{j=0}^{n-1} |\alpha_{i,j}|^2 \right) \\ & = & \sum_{j=0}^{n-1} \left(\sum_{i=0}^{m-1} |\alpha_{i,j}|^2 \right) \\ & = & \|A^T\|_F^2 \end{array}$$

- $\|A\|_F = \sqrt{\|a_0\|_2^2 + \|a_1\|_2^2 + \cdots + \|a_{n-1}\|_2^2}$.

$$\begin{array}{rcl} \|A\|_F^2 & = & \sum_{i=0}^{m-1} \left(\sum_{j=0}^{n-1} |\alpha_{i,j}|^2 \right) \\ & = & \sum_{j=0}^{n-1} \left(\sum_{i=0}^{m-1} |\alpha_{i,j}|^2 \right) \\ & = & \sum_{j=0}^{n-1} \|a_j\|_2^2 \end{array}$$

- $\|A\|_F = \sqrt{\|\widetilde{a}_0\|_2^2 + \|\widetilde{a}_1\|_2^2 + \cdots + \|\widetilde{a}_{m-1}\|_2^2}$.

$$\begin{array}{rcl} \|A\|_F^2 & = & \|A^T\|_F^2 \\ & = & \left\| \left(\begin{array}{c} \widetilde{a}_0^T \\ \widetilde{a}_1^T \\ \vdots \\ \widetilde{a}_{m-1}^T \end{array} \right)^T \right\|_F^2 \\ & = & \left\| \left(\begin{array}{c|c|c|c} \widetilde{a}_0 & \widetilde{a}_1 & \cdots & \widetilde{a}_{n-1} \end{array} \right) \right\|_F^2 \\ & = & \|\widetilde{a}_0\|_2^2 + \|\widetilde{a}_1\|_2^2 + \cdots + \|\widetilde{a}_{m-1}\|_2^2 \end{array}$$

Homework 1.6.1.8 Let $x \in \mathbb{R}^m$ with $\|x\|_1 = 1$.

TRUE/FALSE: $\|x\|_2 = 1$ if and only if $x = \pm e_j$ for some j.

Solution. Obviously, if $x = e_j$ then $\|x\|_1 = \|x|_2 = 1$.

Assume $x \neq e_j$. Then $|\chi_i| < 1$ for all i. But then $\|x\|_2 = \sqrt{|\chi_0|^2 + \cdots + |\chi_{m-1}|^2} < \sqrt{|\chi_0| + \cdots + |\chi_{m-1}|} = \sqrt{1} = 1$.

Homework 1.6.1.9 Prove that if $\|x\|_\nu \leq \beta \|x\|_\mu$ is true for all x, then $\|A\|_\nu \leq \beta \|A\|_{\mu,\nu}$.

Solution.

$$\begin{array}{l} \|A\|_\nu \\ \quad = \quad < \text{ definition } > \\ \max_{\|x\|_\nu = 1} \|Ax\|_\nu \\ \quad \leq \quad < \text{ assumption } > \\ \max_{\|x\|_\nu = 1} \beta \|Ax\|_\mu \\ \quad = \quad < \text{ algebra } > \\ \beta \max_{\|x\|_\nu = 1} \|Ax\|_\mu \\ \quad = \quad < \text{ definition } > \\ \beta \|A\|_{\mu,\nu}. \end{array}$$

1.6.2 Summary

If $\alpha, \beta \in \mathbb{C}$ with $\alpha = \alpha_r + \alpha_c i$ and $\beta = \beta_r + i\beta_c$, where $\alpha_r, \alpha_c, \beta_r, \beta_c \in \mathbb{R}$, then

- Conjugate: $\overline{\alpha} = \alpha_r - \alpha_c i$.
- Product: $\alpha\beta = (\alpha_r\beta_r - \alpha_c\beta_c) + (\alpha_r\beta_c + \alpha_c\beta_r)i$.
- Absolute value: $|\alpha| = \sqrt{\alpha_r^2 + \alpha_c^2} = \sqrt{\overline{\alpha}\alpha}$.

Let $x, y \in \mathbb{C}^m$ with $x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{m-1} \end{pmatrix}$ and $y = \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{m-1} \end{pmatrix}$. Then

- Conjugate:
$$\overline{x} = \begin{pmatrix} \overline{\chi}_0 \\ \vdots \\ \overline{\chi}_{m-1} \end{pmatrix}.$$
- Transpose of vector:
$$x^T = \begin{pmatrix} \chi_0 & \cdots & \chi_{m-1} \end{pmatrix}$$
- Hermitian transpose (conjugate transpose) of vector:
$$x^H = \overline{x}^T = \overline{x^T} = \begin{pmatrix} \overline{\chi}_0 & \cdots & \overline{\chi}_{m-1} \end{pmatrix}.$$
- Dot product (inner product): $x^H y = \overline{x}^T y = \overline{x^T} y = \overline{\chi}_0\psi_0 + \cdots + \overline{\chi}_{m-1}\psi_{m-1} = \sum_{i=0}^{m-1} \overline{\chi}_i\psi_i$.

Definition 1.6.2.1 Vector norm. Let $\| \cdot \| : \mathbb{C}^m \rightarrow \mathbb{R}$. Then $\| \cdot \|$ is a (vector) norm if for all $x, y \in \mathbb{C}^m$ and all $\alpha \in \mathbb{C}$

- $x \neq 0 \Rightarrow \|x\| > 0$ ($\| \cdot \|$ is positive definite),
- $\|\alpha x\| = |\alpha|\|x\|$ ($\| \cdot \|$ is homogeneous), and
- $\|x + y\| \leq \|x\| + \|y\|$ ($\| \cdot \|$ obeys the triangle inequality).

$\Diamond$

- 2-norm (Euclidean length): $\|x\|_2 = \sqrt{x^H x} = \sqrt{|\chi_0|^2 + \cdots + |\chi_{m-1}|^2} = \sqrt{\overline{\chi}_0\chi_0 + \cdots + \overline{\chi}_{m-1}\chi_{m-1}}$ $= \sqrt{\sum_{i=0}^{m-1} |\chi_i|^2}$.
- p-norm: $\|x\|_p = \sqrt[p]{|\chi_0|^p + \cdots + |\chi_{m-1}|^p} = \sqrt[p]{\sum_{i=0}^{m-1} |\chi_i|^p}$.
- 1-norm: $\|x\|_1 = |\chi_0| + \cdots + |\chi_{m-1}| = \sum_{i=0}^{m-1} |\chi_i|$.
- ∞-norm: $\|x\|_\infty = \max(|\chi_0|, \ldots, \chi_{m-1}|) = \max_{i=0}^{m-1} |\chi_i| = \lim_{p\rightarrow\infty} \|x\|_p$.
- Unit ball: Set of all vectors with norm equal to one. Notation: $\|x\| = 1$.

Theorem 1.6.2.2 Equivalence of vector norms. *Let $\|\cdot\| : \mathbb{C}^m \to \mathbb{R}$ and $|||\cdot||| : \mathbb{C}^m \to \mathbb{R}$ both be vector norms. Then there exist positive scalars σ and τ such that for all $x \in \mathbb{C}^m$*

$$\sigma\|x\| \leq |||x||| \leq \tau\|x\|.$$

	$\|x\|_1 \leq \sqrt{m}\|x\|_2$	$\|x\|_1 \leq m\|x\|_\infty$
$\|x\|_2 \leq \|x\|_1$		$\|x\|_2 \leq \sqrt{m}\|x\|_\infty$
$\|x\|_\infty \leq \|x\|_1$	$\|x\|_\infty \leq \|x\|_2$	

Definition 1.6.2.3 Linear transformations and matrices. Let $L : \mathbb{C}^n \to \mathbb{C}^m$. Then L is said to be a linear transformation if for all $\alpha \in \mathbb{C}$ and $x, y \in \mathbb{C}^n$

- $L(\alpha x) = \alpha L(x)$. That is, scaling first and then transforming yields the same result as transforming first and then scaling.
- $L(x + y) = L(x) + L(y)$. That is, adding first and then transforming yields the same result as transforming first and then adding.

◊

Definition 1.6.2.4 Standard basis vector. In this course, we will use $e_j \in \mathbb{C}^m$ to denote the standard basis vector with a "1" in the position indexed with j. So,

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ \longleftarrow j \\ \\ \\ \\ \end{array}$$

◊

If L is a linear transformation and we let $a_j = L(e_j)$ then

$$A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right)$$

is the matrix that represents L in the sense that $Ax = L(x)$.

Partition C, A, and B by rows and columns

$$C = \left(\begin{array}{c|c|c} c_0 & \cdots & c_{n-1} \end{array} \right) = \left(\begin{array}{c} \widetilde{c}_0^T \\ \hline \vdots \\ \hline \widetilde{c}_{m-1}^T \end{array} \right), A = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{k-1} \end{array} \right) = \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array} \right),$$

and

$$B = \left(\begin{array}{c|c|c} b_0 & \cdots & b_{n-1} \end{array} \right) = \left(\begin{array}{c} \widetilde{b}_0^T \\ \hline \vdots \\ \hline \widetilde{b}_{k-1}^T \end{array} \right),$$

then $C := AB$ can be computed in the following ways:

1. By columns:
$$\left(\begin{array}{c|c|c} c_0 & \cdots & c_{n-1} \end{array} \right) := A \left(\begin{array}{c|c|c} b_0 & \cdots & b_{n-1} \end{array} \right) = \left(\begin{array}{c|c|c} Ab_0 & \cdots & Ab_{n-1} \end{array} \right).$$
In other words, $c_j := Ab_j$ for all columns of C.

2. By rows:
$$\left(\begin{array}{c} \widetilde{c}_0^T \\ \hline \vdots \\ \hline \widetilde{c}_{m-1}^T \end{array} \right) := \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array} \right) B = \left(\begin{array}{c} \widetilde{a}_0^T B \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T B \end{array} \right).$$
In other words, $\widetilde{c}_i^T = \widetilde{a}_i^T B$ for all rows of C.

3. As the sum of outer products:
$$C := \left(\begin{array}{c|c|c} a_0 & \cdots & a_{k-1} \end{array} \right) \left(\begin{array}{c} \widetilde{b}_0^T \\ \hline \vdots \\ \hline \widetilde{b}_{k-1}^T \end{array} \right) = a_0 \widetilde{b}_0^T + \cdots + a_{k-1} \widetilde{b}_{k-1}^T,$$
which should be thought of as a sequence of rank-1 updates, since each term is an outer product and an outer product has rank of at most one.

Partition C, A, and B by blocks (submatrices),
$$C = \left(\begin{array}{c|c|c} C_{0,0} & \cdots & C_{0,N-1} \\ \hline \vdots & & \vdots \\ \hline C_{M-1,0} & \cdots & C_{M-1,N-1} \end{array} \right), \left(\begin{array}{c|c|c} A_{0,0} & \cdots & A_{0,K-1} \\ \hline \vdots & & \vdots \\ \hline A_{M-1,0} & \cdots & A_{M-1,K-1} \end{array} \right),$$
and
$$\left(\begin{array}{c|c|c} B_{0,0} & \cdots & B_{0,N-1} \\ \hline \vdots & & \vdots \\ \hline B_{K-1,0} & \cdots & B_{K-1,N-1} \end{array} \right),$$
where the partitionings are "conformal." Then
$$C_{i,j} = \sum_{p=0}^{K-1} A_{i,p} B_{p,j}.$$

Definition 1.6.2.5 Matrix norm. Let $\| \cdot \| : \mathbb{C}^{m \times n} \to \mathbb{R}$. Then $\| \cdot \|$ is a (matrix) norm if for all $A, B \in \mathbb{C}^{m \times n}$ and all $\alpha \in \mathbb{C}$

- $A \neq 0 \Rightarrow \|A\| > 0$ ($\| \cdot \|$ is positive definite),
- $\|\alpha A\| = |\alpha| \|A\|$ ($\| \cdot \|$ is homogeneous), and
- $\|A + B\| \leq \|A\| + \|B\|$ ($\| \cdot \|$ obeys the triangle inequality).

$\Diamond$

Let $A \in \mathbb{C}^{m\times n}$ and

$$A = \begin{pmatrix} \alpha_{0,0} & \cdots \alpha_{0,n-1} \\ \vdots & \vdots \\ \alpha_{m-1,0} & \cdots \alpha_{m-1,n-1} \end{pmatrix} = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{n-1} \end{array} \right) = \left(\begin{array}{c} \widetilde{a}_0^T \\ \hline \vdots \\ \hline \widetilde{a}_{m-1}^T \end{array} \right).$$

Then

- Conjugate of matrix:
$$\overline{A} = \begin{pmatrix} \overline{\alpha}_{0,0} & \cdots & \overline{\alpha}_{0,n-1} \\ \vdots & \vdots & \\ \overline{\alpha}_{m-1,0} & \cdots & \overline{\alpha}_{m-1,n-1} \end{pmatrix}.$$

- Transpose of matrix:
$$A^T = \begin{pmatrix} \alpha_{0,0} & \cdots & \alpha_{m-1,0} \\ \vdots & \vdots & \\ \alpha_{0,n-1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix}.$$

- Conjugate transpose (Hermitian transpose) of matrix:
$$A^H = \overline{A}^T = \overline{A^T} = \begin{pmatrix} \overline{\alpha}_{0,0} & \cdots & \overline{\alpha}_{m-1,0} \\ \vdots & \vdots & \\ \overline{\alpha}_{0,n-1} & \cdots & \overline{\alpha}_{m-1,n-1} \end{pmatrix}.$$

- Frobenius norm: $\|A\|_F = \sqrt{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\alpha_{i,j}|^2} = \sqrt{\sum_{j=0}^{n-1} \|a_j\|_2^2} = \sqrt{\sum_{i=0}^{m-1} \|\widetilde{a}_i\|_2^2}$

- matrix p-norm: $\|A\|_p = \max_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p$.

- matrix 2-norm: $\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2=1} \|Ax\|_2 = \|A^H\|_2$.

- matrix 1-norm: $\|A\|_1 = \max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{\|x\|_1=1} \|Ax\|_1 = \max_{0\leq j<n} \|a_j\|_1 = \|A^H\|_\infty$.

- matrix ∞-norm: $\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{0\leq j<n} \|a_j\|_\infty = \|A^H\|_1$.

Theorem 1.6.2.6 Equivalence of matrix norms. *Let $\|\cdot\| : \mathbb{C}^{m\times n} \to \mathbb{R}$ and $|||\cdot||| : \mathbb{C}^{m\times n} \to \mathbb{R}$ both be matrix norms. Then there exist positive scalars σ and τ such that for all $A \in \mathbb{C}^{m\times n}$*

$$\sigma\|A\| \leq |||A||| \leq \tau\|A\|.$$

	$\|A\|_1 \leq \sqrt{m}\|A\|_2$	$\|A\|_1 \leq m\|A\|_\infty$	$\|A\|_1 \leq \sqrt{m}\|A\|_F$
$\|A\|_2 \leq \sqrt{m}\|A\|_1$		$\|A\|_2 \leq \sqrt{m}\|A\|_\infty$	$\|A\|_2 \leq \|A\|_F$
$\|A\|_\infty \leq m\|A\|_1$	$\|A\|_\infty \leq \sqrt{m}\|A\|_2$		$\|A\|_\infty \leq \sqrt{m}\|A\|_F$
$\|A\|_F \leq \sqrt{m}\|A\|_1$	$\|A\|_F \leq \sqrt{m}\|A\|_2$	$\|A\|_F \leq \sqrt{m}\|A\|_\infty$	

Definition 1.6.2.7 Subordinate matrix norm. A matrix norm $\|\cdot\| : \mathbb{C}^{m\times n} \rightarrow \mathbb{R}$ is said to be subordinate to vector norms $\|\cdot\|_\mu : \mathbb{C}^m \rightarrow \mathbb{R}$ and $\|\cdot\|_\nu : \mathbb{C}^n \rightarrow \mathbb{R}$ if, for all $x \in \mathbb{C}^n$,

$$\|Ax\|_\mu \leq \|A\|\|x\|_\nu.$$

If $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ are the same norm (but perhaps for different m and n), then $\|\cdot\|$ is said to be subordinate to the given vector norm. ♢

Definition 1.6.2.8 Consistent matrix norm. A matrix norm $\|\cdot\| : \mathbb{C}^{m\times n}$ is said to be a consistent matrix norm if it is defined for all m and n, using the same formula for all m and n. ♢

Definition 1.6.2.9 Submultiplicative matrix norm. A consistent matrix norm $\|\cdot\| : \mathbb{C}^{m\times n} \rightarrow \mathbb{R}$ is said to be submultiplicative if it satisfies

$$\|AB\| \leq \|A\|\|B\|.$$

♢

Let $A, \Delta A \in \mathbb{C}^{m\times m}$, $x, \delta x, b, \delta b \in \mathbb{C}^m$, A be nonsingular, and $\|\cdot\|$ be a vector norm and corresponding subordinate matrix norm. Then

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\|\|A^{-1}\|}_{\kappa(A)} \frac{\|\delta b\|}{\|b\|}.$$

Definition 1.6.2.10 Condition number of a nonsingular matrix. The value $\kappa(A) = \|A\|\|A^{-1}\|$ is called the condition number of a nonsingular matrix A. ♢

Week 2

The Singular Value Decomposition

To be released at a future date.

Week 3

The QR Decomposition

To be released at a future date.

Week 4

Linear Least Squares

To be released at a future date.

Part II

Solving Linear Systems

Week 5

The LU and Cholesky Factorizations

To be released at a future date.

Week 6

Numerical Stability

To be released at a future date.

Week 7

Solving Sparse Linear Systems

To be released at a future date.

Week 8

Descent Methods

Part III

The Algebraic Eigenvalue Problem

Week 9

Eigenvalues and Eigenvectors

To be released at a future date.

Week 10

Practical Solution of the Hermitian Eigenvalue Problem

To be released at a future date.

Week 11

The QR Algorithm: Computing the SVD

Week 12

Attaining High Performance

Appendix A

Notation

A.0.1 Householder notation

Alston Householder introduced the convention of labeling matrices with upper case Roman letters (A, B, etc.), vectors with lower case Roman letters (a, b, etc.), and scalars with lower case Greek letters (α, β, etc.). When exposing columns or rows of a matrix, the columns of that matrix are usually labeled with the corresponding Roman lower case letter, and the the individual elements of a matrix or vector are usually labeled with "the corresponding Greek lower case letter," which we can capture with the triplets $\{A, a, \alpha\}$, $\{B, b, \beta\}$, etc.

$$A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) = \left(\begin{array}{c|c|c|c} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{array} \right)$$

and

$$x = \left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{m-1} \end{array} \right),$$

where α and χ is the lower case Greek letters "alpha" and "chi," respectively. You will also notice that in this course we start indexing at zero. We mostly adopt this convention (exceptions include i, j, p, m, n, and k, which usually denote integer scalars.)

Appendix B

Knowledge from Numerical Analysis

Typically, an undergraduate numerical analysis course is considered a prerequisite for a graduate level course on numerical linear algebra. There is, however, relatively few concepts from such a course that is needed to be successful in such a course. In this appendix, we very briefly discuss some of these concepts.

B.0.1 Cost of basic linear algebra operations

B.0.2 Catastrophic cancellation

Typically, an undergraduate course on numerical analysis or numerical methods, in addition to undergraduate linear algebra, is a prerequisite for this course. It is our experience that many learners only have undergraduate linear algebra as background. For this reason, we examine how to compute the roots of a quadratic equation to illustrate **catastrophic cancellation**, a key concept in numerical analysis.

Recall that if

$$\chi^2 + \beta\chi + \gamma = 0$$

then the quadratic formula gives the largest root of this quadratic equation:

$$\chi = \frac{-\beta + \sqrt{\beta^2 - 4\gamma}}{2}.$$

Example B.0.2.1 We use the quadratic equation in the exact order indicated by the parentheses in

$$\chi = \left[\frac{\left[-\beta + \left[\sqrt{[[\beta^2] - [4\gamma]]} \right] \right]}{2} \right],$$

truncating every expression within square brackets to three significant digits, to solve

$$\chi^2 + 25\chi + \gamma = 0$$

$$\begin{aligned}
\chi &= \left[\frac{\left[-25+\left[\sqrt{[[25^2]-[4]]}\right]\right]}{2}\right] = \left[\frac{\left[-25+\left[\sqrt{[625-4]}\right]\right]}{2}\right] \\
&= \left[\frac{\left[-25+\left[\sqrt{621}\right]\right]}{2}\right] = \left[\frac{[-25+24.9]}{2}\right] = \left[\frac{-0.1}{2}\right] = -0.05.
\end{aligned}$$

Now, if you do this to the full precision of a typical calculator, the answer is instead approximately -0.040064. The relative error we incurred is, approximately, $0.01/0.04 = 0.25$.

What is going on here? The problem comes from the fact that there is error in the 24.9 that is encountered after the square root is taken. Since that number is close to in magnitude, but of opposite sign to the -25 to which it is added, the result of $-25 + 24.9$ is mostly error.

This is known as catastrophic cancelation: adding two nearly equal numbers of opposite sign, at least one of which has some error in it related to roundoff, yields a result with large relative error.

Now, one can use an alternative formula to compute the root:

$$\chi = \frac{-\beta + \sqrt{\beta^2 - 4\gamma}}{2} = \frac{-\beta + \sqrt{\beta^2 - 4\gamma}}{2} \times \frac{-\beta - \sqrt{\beta^2 - 4\gamma}}{-\beta - \sqrt{\beta^2 - 4\gamma}},$$

which yields

$$\chi = \frac{2\gamma}{-\beta - \sqrt{\beta^2 - 4\gamma}}.$$

Carrying out the computations, rounding intermediate results, yields $-.0401$. The relative error is now $0.00004/0.040064 \approx .001$. It avoids catastrophic cancellation because now the two numbers of nearly equal magnitude are added instead. $\square$

Remark B.0.2.2 The point is: if possible, avoid creating small intermediate results that amplify into a large relative error in the final result.

Notice that in this example it is not inherently the case that a small relative change in the input is amplified into a large relative change in the output (as is the case when solving a linear system with a poorly conditioned matrix). The problem is with the standard formula that was used. Later we will see that this is an example of an unstable algorithm.

Appendix C

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