Spectral Properties of a Non-compact Operator in Ecology

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Introduction

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- IPMs generalize Leslie matrices by allowing for a continuous structure variable.

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  - and $f(x)$ the fecundity function.

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Theorem

Suppose A is an IPM operator whose kernel $k(y, x)$ is positive and continuous on $[L, U]^2$. Then $\lambda = r(A)$ is an eigenvalue of A, and its eigenvector ψ can be scaled to be positive. Additionally, λ is the asymptotic growth rate of the population, and ψ is the stable stage distribution, in the sense that for any nonzero initial population φ_0,

$$\lim_{n\to\infty} \frac{A^n \varphi_0}{\lambda^n} = \langle \psi, \varphi_0 \rangle \psi.$$

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- Fish have bony skeletons, and hence cannot shrink in length.

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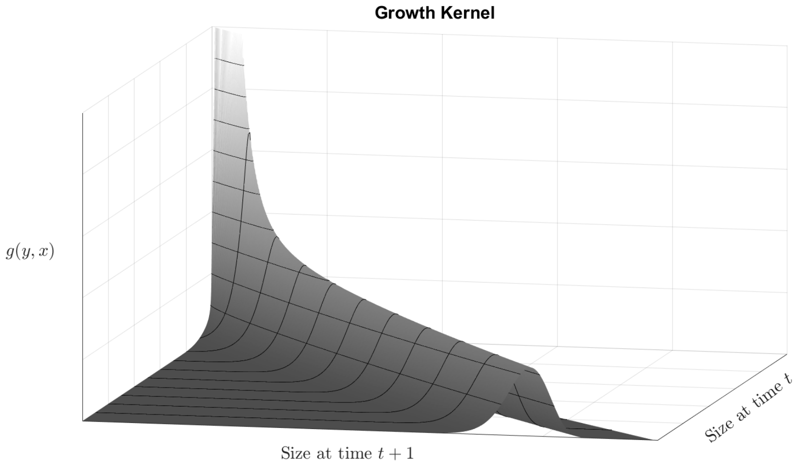
- The assumption that individuals cannot shrink is that

$$g(y,x) = 0, \text{whenever } y < x.$$


Growth Kernel
$g(y,x)$
Size at time $t+1$
Size at time t

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  - Is the operator T still compact?
  - Is $\lambda = r(T)$ still an eigenvalue of T?
  - Are λ and its eigenvector ψ still the asymptotic growth rate and stable stage distribution, respectively, of the population?

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- Let $F : L^1 \to \mathbb{R}$ be the fecundity functional $F\varphi := \int_L^U f(x)\varphi(x)\, dx$.
- Then the IPM operator T can be written $T = GS + bF$, where $b = b(y)$ is the offspring distribution.

T is Not Compact

Theorem (Reichenbach, 2018)

If $g(y, x)$ is the growth subkernel for an IPM which satisfies $g(y, x) = 0$ for $y < x$, then its associated integral operator G is not compact.

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Corollary

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Corollary

The IPM operator $T := GS + bF$ is not compact.

Proof Sketch of Corollary.

An assumption of $s(x)$ is that $0 < s_0 \leq s(x)$ for all $x \in [L, U]$, so we can write

$$G = \frac{T - bF}{s(x)}.$$

Hence, G and T must be compact/non-compact together. □

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- Let $\mathscr{U} \subset L^1$ be the unit ball. A theorem in Dunford & Schwartz [2] says that the set $G(\mathscr{U})$ is weakly compact on $L^1(\Omega)$ iff

$$\lim_{\mu(E)\to 0} \int_E (G\varphi)(t)\, dt = 0 \tag{1}$$

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- Put $\delta_n := \frac{1}{n}(U - L)$, and define $E_n := [U - \delta_n, U]$; then $\mu(E_n) \to 0$.

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  uniformly for $\varphi \in \mathscr{U}$, where μ is the Lebesgue measure.
- Put $\delta_n := \frac{1}{n}(U - L)$, and define $E_n := [U - \delta_n, U]$; then $\mu(E_n) \to 0$.
- Define $h_n(t) := \frac{1}{\delta_n}\chi_{E_n}(t)$; then $h_n \in \mathscr{U}$. The limit (1) is not uniform on the collection $\{h_n\}$. Therefore, G is not weakly compact.

$\square$

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- However, the proof in the written dissertation shows that T^k fails to be compact for all $k \in \mathbb{N}$ when $g(y, x) = 0$ for $y < x$.
- Thus, we will require a wholly different method in order to prove a similar theorem.

Helpful Definitions

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For example, L^∞ is the Banach dual of L^1.

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Given an operator $T : X \to X$, its *resolvent* $R(z, T)$ is defined as $(zI - T)^{-1}$, which is holomorphic in the *resolvent set* $\rho(T) := \mathbb{C} \setminus \sigma(T)$.

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An operator $T : X \to X$ is called *positive* with respect to the cone $K \subset X$ if $T(K) \subset K$.

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its Banach adjoint is given by "transposing" the kernel function:

$$(T^*\varphi^*)(t) = \int_L^U k(y,t)\varphi^*(y)\,dy.$$

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Definition

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For the same cone above, the strictly positive elements of $K^* \subset L^\infty$ are those represented by positive almost-everywhere, essentially bounded functions.

Main Theorem

Theorem (R., 2020)

Suppose that $T : L^1 \to L^1$ *is an IPM operator such that* $g(y, x) = 0$ *for* $y < x$. *Then under biologically reasonable assumptions,* T *has the following properties (among others):*

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- *The spectral radius* $\lambda = r(T)$ *is a positive eigenvalue for* T *and* T^*. *Moreover, the respective eigenvectors* ψ *and* ψ^* *span one-dimensional eigenspaces,* ψ *is quasi-interior,* ψ^* *represents a strictly positive linear functional, and both* ψ, ψ^* *are the only eigenvectors of* T, T^* *which can be scaled to be nonnegative almost-everywhere.*

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- *Suppose ψ is scaled so that $||\psi||_1 = 1$, and ψ^* is scaled so that $\langle \psi, \psi^* \rangle = 1$. Then for any nonzero $\psi_0 \in K$, we have*

$$\lim_{n\to\infty} \frac{T^n \varphi_0}{\lambda^n} = \langle \varphi_0, \psi^* \rangle \psi.$$

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- To apply Marek's theorem, we need to show two things:
  1. T is a *nonsupporting* operator, and
  2. $r(T)$ is a pole of the resolvent $R(z, T)$.

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Suppose $T : X \to X$ is a positive operator with respect to the cone K, and usppose that $\varphi \in K$, $\varphi^* \in K^*$ are both nonzero.

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- T is called *strictly nonsupporting* if for every pair φ, φ^* there is a positive integer $p = p(\varphi)$ such that $\langle T^n \varphi, \varphi^* \rangle > 0$ for $n \geq p$.

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Our non-compact IPM operator $T : L^1 \to L^1$ is in fact strictly non-supporting, and the integer p actually doesn't depend on the choice of $\varphi \in L^1$ either.

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Theorem

The IPM operator T is strictly nonsupporting (hence, nonsupporting) under biologically reasonable assumptions. Additionally, the integer p does not depend on the choice of $\varphi \in L^1$.

$r(T)$ is a Pole of the Resolvent

- Recall that the *resolvent* of an operator T is defined to be $R(z, T) := (zI - T)^{-1}$, which is well-defined in the resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$.

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Theorem (Clement, 1987 [1])

Suppose that $z \in \sigma(T) \setminus \sigma_e(T)$, where $\sigma_e(T)$ denotes the essential spectrum. Then z is a pole of $R(z, T)$.

- With this theorem, you can intuitively think of the essential spectrum as "the points in the spectrum that are not poles".

Definition

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The *essential spectral radius* is the value

$$r_e(T) := \sup\{|z| \mid z \in \sigma_e(T)\}.$$

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Theorem (Schaefer (1960), [7])

Let $K \subset X$ be a normal cone. If $A : X \to X$ is a positive operator with respect to K, then $\lambda = r(A)$ is an element of $\sigma(A)$.

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Theorem (Schaefer (1960), [7])

Let $K \subset X$ be a normal cone. If $A : X \to X$ is a positive operator with respect to K, then $\lambda = r(A)$ is an element of $\sigma(A)$.

- The cone $K \subset L^1$ of nonnegative a.e. functions is normal, and the IPM operator $T : L^1 \to L^1$ is positive w.r.t. K, so $\lambda \in \sigma(T)$.

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- There is a formula, due to Nussbaum [5] for $r_e(T)$:

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where $\beta(T^n) := \beta(T^n(\mathscr{U}))$, and β is the *ball measure of noncompactness* (or ball-MNC).

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The *ball-MNC*, also known as the *Hausdorff-MNC*, of a subset V of the vector space X is given by

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- for $V \subset L^p$, there is a “nice” formula for $\beta(V)$:

$$\beta(V) = \frac{1}{2} \lim_{\delta \to 0} \sup_{\varphi \in V} \sup_{0 < \tau \leq \delta} ||\varphi - \varphi_\tau||_p,$$

where $\varphi_\tau(t) := \varphi(t + \tau)$.

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$$\beta(T^k) = \beta((GS + bF)^k) = \beta((GS)^k).$$

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- This simplifies the problem a lot, and allows us to compute

$$r_e(T) = \lim_{n\to\infty} \beta(T^n)^{1/n} = \lim_{n\to\infty} \beta((GS)^n)^{1/n} = r_e((GS)^n)^{1/n}.$$

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Theorem (R., 2018)

Put $s_1 := \sup\{s(x)\} = s(U)$. *Then*

$$r_e(GS) \le r(GS) \le s_1. \tag{2}$$

If $g(y,x) = 0$ *for* $y < x$, *then equality holds in* (2).

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Proof sketch.

The first inequality follows from the fact that $\sigma_e(GS) \subset \sigma(GS)$, and the second follows from Gelfand's formula for the spectral radius:

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Assuming that $g(y,x) = 0$ for $y < x$, one can use the formula for β, with a properly chosen subsequence of functions in L^1, to show that $s_1 \leq r_e(GS)$. □

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Theorem

Suppose $\mu \in \rho(GS)$*, the resolvent set of* GS*, and define* $\psi := (\mu I - GS)^{-1} b$*. If*

$$F\psi = F(\mu I - GS)^{-1} b = 1,$$

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Suppose $\mu \in \rho(GS)$, the resolvent set of GS, and define $\psi := (\mu I - GS)^{-1}b$. If

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- This characterizes what eigenvectors look like, and tells us exactly when T has an eigenvalue μ (so long as $\mu \in \rho(GS)$).

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- This characterizes what eigenvectors look like, and tells us exactly when T has an eigenvalue μ (so long as $\mu \in \rho(GS)$).
- Recall that we want some μ such that

$$r(T) \geq \mu > r_e(T) = s_1.$$

$r(T)$ is a Pole of the Resolvent

Theorem (R., 2019)

Put $E := (s_1, \infty)$, *and define the function* $P : E \to \mathbb{R}$ *by*

$$P(t) := F(tI - GS)^{-1} b,$$

where GS *satisfies biologically reasonable properties. Then:*

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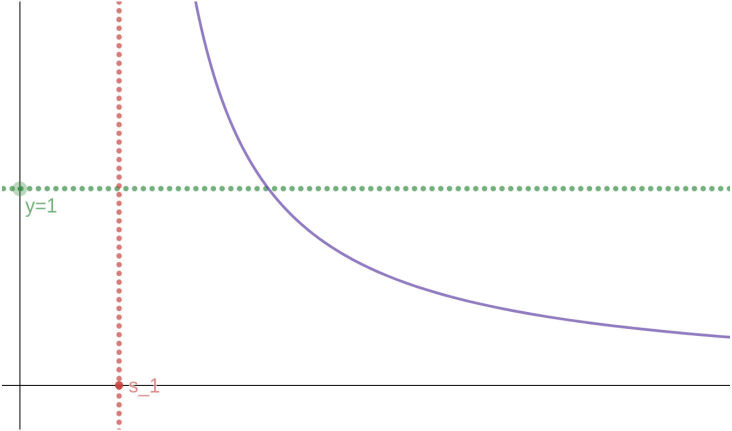
where GS *satisfies biologically reasonable properties. Then:*

1. P *is continuous,*
2. P *is strictly decreasing,*
3. $\lim_{t\to\infty} P(t) = 0$,
4. *and if in addition there is an* $\varepsilon > 0$ *such that* $s(x) \equiv s_1$ *for* $x \in [U - \varepsilon, U]$, *then*
$$\lim_{t\to s_1} P(t) = \infty.$$

$r(T)$ is a Pole of the Resolvent

Here's what $P(t)$ might look like because of this theorem:



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- This theorem guarantees the existence of a (unique) $\mu > s_1$ such that

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- Additionally, for $\lambda = r(T)$,

$$\lambda \geq \mu > s_1 = r_e(GS).$$

  Thus, we can conclude that $\lambda \notin \sigma_e(T)$. This is the last ingredient we needed to prove...

Theorem (R., 2020)

Suppose that $T : L^1 \to L^1$ *is an IPM operator such that* $g(y, x) = 0$ *for* $y < x$. *Then under biologically reasonable assumptions,* T *has the following properties (among others):*

Theorem (R., 2020)

Suppose that $T : L^1 \to L^1$ is an IPM operator such that $g(y,x) = 0$ for $y < x$. Then under biologically reasonable assumptions, T has the following properties (among others):

- *The spectral radius $\lambda = r(T)$ is a positive eigenvalue for T and T^*. Moreover, the respective eigenvectors ψ and ψ^* span one-dimensional eigenspaces, ψ is quasi-interior, ψ^* represents a strictly positive linear functional, and both ψ, ψ^* are the only eigenvectors of T, T^* which can be scaled to be nonnegative almost-everywhere.*

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- *Suppose ψ is scaled so that $||\psi||_1 = 1$, and ψ^* is scaled so that $\langle \psi, \psi^* \rangle = 1$. Then for any nonzero $\psi_0 \in K$, we have*

$$\lim_{n\to\infty} \frac{T^n \varphi_0}{\lambda^n} = \langle \varphi_0, \psi^* \rangle \psi.$$

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Estimating the Spectral Radius

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- An important question is whether one can estimate $\lambda = r(T)$ on a computer when T is not compact.
- The standard methods of doing this when the operator is compact rely on approximating it uniformly with matrices.
- But these methods do not work when T is not compact.
- I was not able to fully resolve this question, but I did prove that the spectral radii of certain compact operators approach the spectral radius of T.

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$$Q(t) := -1 + F(tI - GS)^{-1}b = -1 + \sum_{k=0}^{\infty} \frac{F((GS)^k b)}{t^{k+1}},$$

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where G_δ is the integral operator with kernel equal to $g(y, x)$ on $[L, U - \delta] \times [L, U]$, and 0 otherwise.

Estimating the Spectral Radius

Theorem (R., 2020)

For every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ and a $\delta(N) > 0$ such that for any $n \geq N$ and $\delta < \delta(N)$, we have

$$|z_{n,\delta} - \lambda| < \varepsilon,$$

where $z_{n,\delta}$ is the unique zero of $Q_{n,\delta}$, and λ is the unique zero of Q (i.e., the spectral radius of T).