Spectral Properties of a Non-compact Operator in Ecology

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Introduction

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- IPMs generalize Leslie matrices by allowing for a continuous structure variable.

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  - and $f(x)$ the fecundity function.

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Theorem

Suppose A is an IPM operator whose kernel $k(y,x)$ is positive and continuous on $[L,U]^2$. Then $\lambda = r(A)$ is an eigenvalue of A, and its eigenvector ψ can be scaled to be positive. Additionally, λ is the asymptotic growth rate of the population, and ψ is the stable stage distribution, in the sense that for any nonzero initial population φ_0,

$$\lim_{n\to\infty} \frac{A^n \varphi_0}{\lambda^n} = \langle \psi, \varphi_0 \rangle \psi.$$

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- Fish have bony skeletons, and hence cannot shrink in length.

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$$\int_L^U g(y,x)\,dy = 1, \quad \text{for all } x \in [L, U].$$

- The assumption that individuals cannot shrink is that

$$g(y,x) = 0, \text{whenever } y < x.$$

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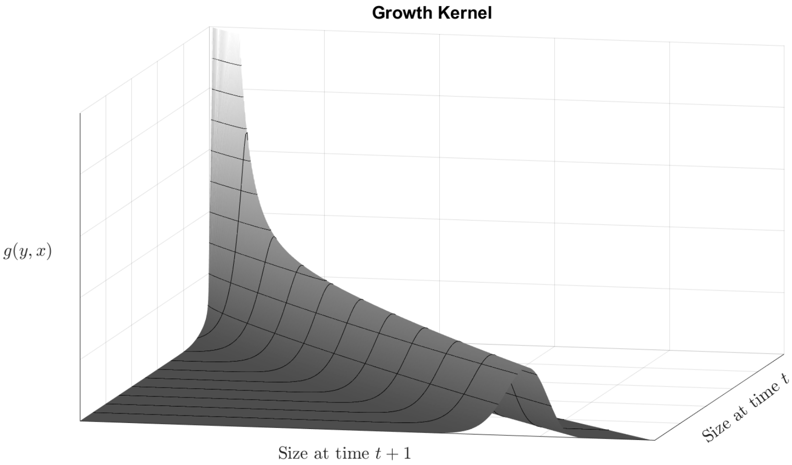

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  - Is the operator T still compact?
  - Is $\lambda = r(T)$ still an eigenvalue of T?
  - Are λ and its eigenvector ψ still the asymptotic growth rate and stable stage distribution, respectively, of the population?

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- Let $S : L^1 \to L^1$ be multiplication by $s(x)$.
- Let $F : L^1 \to \mathbb{R}$ be the fecundity functional $F\varphi := \int_L^U f(x)\varphi(x)\, dx$.
- Then the IPM operator T can be written $T = GS + bF$, where $b = b(y)$ is the offspring distribution.

T is Not Compact

Theorem (Reichenbach, 2018)

If $g(y,x)$ is the growth subkernel for an IPM which satisfies $g(y,x) = 0$ for $y < x$, then its associated integral operator G is not compact.

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Corollary

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Corollary

The IPM operator $T := GS + bF$ is not compact.

Proof Sketch of Corollary.

An assumption of $s(x)$ is that $0 < s_0 \leq s(x)$ for all $x \in [L, U]$, so we can write

$$G = \frac{T - bF}{s(x)}.$$

Hence, G and T must be compact/non-compact together. □

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- Let $\mathscr{U} \subset L^1$ be the unit ball. A theorem in Dunford & Schwartz [1] says that the set $G(\mathscr{U})$ is weakly compact on $L^1(\Omega)$ iff

$$\lim_{\mu(E)\to 0} \int_E (G\varphi)(t)\, dt = 0 \tag{1}$$

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- Put $\delta_n := \frac{1}{n}(U - L)$, and define $E_n := [U - \delta_n, U]$; then $\mu(E_n) \to 0$.

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  uniformly for $\varphi \in \mathscr{U}$, where μ is the Lebesgue measure.
- Put $\delta_n := \frac{1}{n}(U - L)$, and define $E_n := [U - \delta_n, U]$; then $\mu(E_n) \to 0$.
- Define $h_n(t) := \frac{1}{\delta_n}\chi_{E_n}(t)$; then $h_n \in \mathscr{U}$. The limit (1) is not uniform on the collection $\{h_n\}$. Therefore, G is not weakly compact.

$\square$

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- However, the proof in the written dissertation shows that T^k fails to be compact for all $k \in \mathbb{N}$ when $g(y,x) = 0$ for $y < x$.
- Thus, we will require a wholly different method in order to prove a similar theorem.

Helpful Definitions

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For example, L^∞ is the Banach dual of L^1.

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Given an operator $T : X \to X$, its *resolvent* $R(z, T)$ is defined as $(zI - T)^{-1}$, which is holomorphic in the *resolvent set* $\rho(T) := \mathbb{C} \setminus \sigma(T)$.

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An operator $T : X \to X$ is called *positive* with respect to the cone $K \subset X$ if $T(K) \subset K$.

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its Banach adjoint is given by "transposing" the kernel function:

$$(T^*\varphi^*)(t) = \int_L^U k(y,t)\varphi^*(y)\,dy.$$

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Definition

Given a cone K, an element $\varphi^* \in K^*$ is called *strictly positive* if $\langle \varphi, \varphi^* \rangle > 0$ for all nonzero $\varphi \in K$.

For the same cone above, the strictly positive elements of $K^* \subset L^\infty$ are those represented by positive almost-everywhere, essentially bounded functions.

Main Theorem

Theorem (R., 2020)

Suppose that $T : L^1 \to L^1$ *is an IPM operator such that* $g(y, x) = 0$ *for* $y < x$*. Then under biologically reasonable assumptions,* T *has the following properties (among others):*

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- *The spectral radius* $\lambda = r(T)$ *is a positive eigenvalue for* T *and* T^*. *Moreover, the respective eigenvectors* ψ *and* ψ^* *span one-dimensional eigenspaces,* ψ *is quasi-interior,* ψ^* *represents a strictly positive linear functional, and both* ψ, ψ^* *are the only eigenvectors of* T, T^* *which can be scaled to be nonnegative almost-everywhere.*

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- *Suppose* ψ *is scaled so that* $||\psi||_1 = 1$, *and* ψ^* *is scaled so that* $\langle \psi, \psi^* \rangle = 1$. *Then for any nonzero* $\psi_0 \in K$, *we have*

$$\lim_{n\to\infty} \frac{T^n \varphi_0}{\lambda^n} = \langle \varphi_0, \psi^* \rangle \psi.$$

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- To apply Marek's theorem, we need to show two things:
  1. T is a *nonsupporting* operator, and
  2. $r(T)$ is a pole of the resolvent $R(z, T)$.

References I

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