Spectral Properties of a Non-compact Operator in Ecology

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Introduction

- Integral projection models (IPMs) are stage-structured population models of the form

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- IPMs generalize Leslie matrices by allowing for a continuous structure variable.

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  - and $f(x)$ the fecundity function.

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Theorem

Suppose A is an IPM operator whose kernel $k(y,x)$ is positive and continuous on $[L,U]^2$. Then $\lambda = r(A)$ is an eigenvalue of A, and its eigenvector ψ can be scaled to be positive. Additionally, λ is the asymptotic growth rate of the population, and ψ is the stable stage distribution, in the sense that for any nonzero initial population φ_0,

$$\lim_{n\to\infty} \frac{A^n \varphi_0}{\lambda^n} = \langle \psi, \varphi_0 \rangle \psi.$$

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- Fish have bony skeletons, and hence cannot shrink in length.

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- The assumption that individuals cannot shrink is that

$$g(y,x) = 0, \text{whenever } y < x.$$

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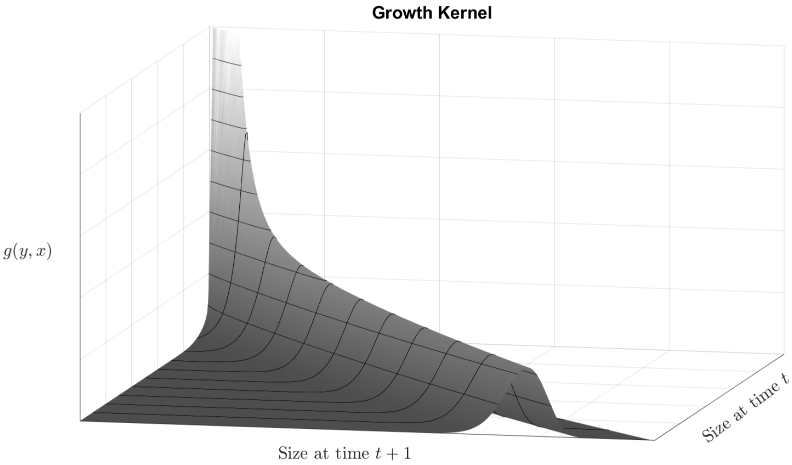

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- With an IPM which models a population whose individuals cannot shrink, we have some questions to answer:
  - Is the operator T still compact?
  - Is $\lambda = r(T)$ still an eigenvalue of T?
  - Are λ and its eigenvector ψ still the asymptotic growth rate and stable stage distribution, respectively, of the population?

T is Not Compact

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References I

S.P. Ellner and M. Rees. "Integral Projection Models for Species with Complex Demography". In: *The American Naturalist* 167.3 (2006), pp. 410–428.

Y. Vindenes et al. "Effects of Climate Change on Trait-Based Dynamics of a Top Predator in Freshwater Ecosystems". In: *The American Naturalist* 183.2 (2014), pp. 243–256.