

QUANTIFYING THE DEGREE OF FINE-TUNING IN  
INFLATIONARY POTENTIALS

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Morgan Presley

# Abstract

To be written later.

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I would like to acknowledge Mathematica's dedicated efforts at making my thesis as difficult to complete as possible.

I would like to dedicate my thesis to my family – both in Illinois and here at Princeton – for their support, without which I could not have finished this thesis.

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# Chapter 1

## General Outline of Thesis

- Introduction & Background Theory
  - Story of Inflation
  - Motivation (Why we need inflation)
  - Original failure of Guth's inflation
  - Slow roll inflation
  - Problems with current theory (domination of young bubbles; infinite multiverse)
  - Current status: trying to impose a measure to make bubbles like us more common
  - But let's ignore the current problems and I'll show that even the simplest models in the current theory must be very fine-tuned to reproduce the current data
- Discussion of Current Models
  - Replication of Planck Plot
  - Large Field vs Hilltop

- How to quantify complexity / fine tuning
  - Discuss Latham's method
  - My replication of Latham's Plot
- Examine Fine-tuning of Large Field Models
  - General Idea
  - Drift Plots for tanh, Rsq, Exp models
- Discussion of Mukhanov's Model
  - Motivation of Mukhanov's potential
  - Discuss how he's doing exactly what we said was a problem
  - My drift plot
- Conclusion



# Chapter 2

## Introduction

### 2.1 How to describe an inflationary theory

The Standard Model that describes all of fundamental physics consists of many different fields (e.g. scalar, vector, Fermion, Boson) and interactions between the given fields. These are put together into a giant universe-encompassing Lagrangian, which then determines all possible interactions. An inflationary theory adds a scalar field  $\phi$  and accompanying potential  $V(\phi)$  to the Standard Model Lagrangian in order to facilitate the drastic expansion required in the early universe. For values of  $\phi$  where the potential is large, the inflationary field encourages the expansion of the universe; when the potential is zero, then the inflationary field does not contribute to expansion.

As an example, consider the potential in Figure 2.1, which shows a potential colloquially called the “Mexican hat” potential:  $V(\phi) = \lambda(\phi^2 - \sigma^2)^2$ . For this potential, the extremum at  $\phi = 0$  is referred to as the false vacuum. At this value of the field, the universe is inflating. However, since the false vacuum is unstable, the field will “roll down the hill” of the potential and end up at the true vacuum,  $\phi = \sigma$ . When the inflationary field settles in the true vacuum (which is stable), the inflationary period ends. Potentials like this which have a false vacuum at  $\phi = 0$  are called hilltop

models. For hilltop models, the inflationary period is dominated by inflation at small values of  $\phi$ . Hence, in a polynomial expansion of the field the lowest order terms will dominate. Thus, the potential will have only a handful of parameters (corresponding to the coefficients of the lowest order terms) requiring constraint. However, as will be discussed in Section ??, such models do not reproduce observational results.

Another type of inflationary potential, which happens to be the form most in line with current observations, is the plateau model, shown in Figure 2.2. In this inflationary scenario the field begins at a large value of  $\phi$  where the potential is nearly flat. The field then rolls down the hill towards smaller values of  $\phi$ , with inflation ending when the potential levels off to zero. Plateau models are examples of large-field models, in which the inflationary period is dominated by inflation at large values of  $\phi$ . Unlike in the hilltop case, inflation in the large-field models depends heavily on the higher order terms of the potential's expansion. Since small perturbations in the higher order terms will result in drastically different behavior, the potential requires an infinite number of constraining parameters. We will explore the sensitivity of such models to higher-order perturbation in Section ??, arguing that this sensitivity reveals an undesirable amount of fine-tuning in such models of inflation.

The main purpose of an inflationary theory is to provide a mechanism to explain the homogeneity, isotropy, and flatness of the universe. As such, it is useful to look at the production of density perturbations during the inflationary period. During inflation, the magnitude of the density perturbations are inversely proportional to  $\dot{\phi}$ . Consequently, in regions where the potential is approximately constant, the perturbations would be unacceptably large. However, steeply-sloped potentials would cause the field to roll off the hill and end inflation before enough expansion took place. Therefore, there is a balancing game to be played in picking potentials.

As we will see in the sections to come, this game of picking potentials to predict observations stumbles across many obstacles. Since the introduction of inflationary

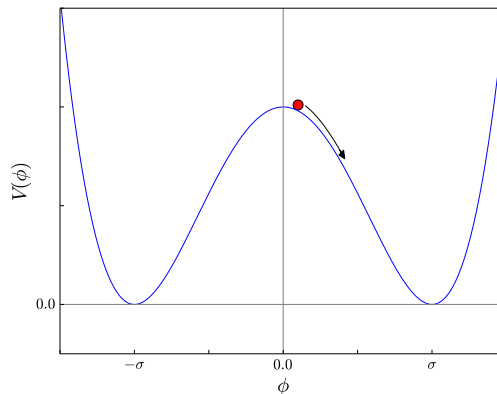


Figure 2.1: A schematic illustration of a hilltop potential, using the example of a Mexican Hat potential:  $V(\phi) = \lambda(\phi^2 - \sigma^2)^2$ . The red ball symbolizes the inflationary field at a given time. The field rolls down the hill of the potential and settles in the vacuum state where the potential is zero, causing inflation to end. Note that in this scenario, the most inflation occurs at small values of  $\phi$  and hence any hilltop model can be reasonably approximated by the first few lowest order terms.

theory 34 years ago, many inflationary potentials have been proposed. However, once they satisfy observations, we have no natural criteria for choosing which among them is the “true” potential describing the inflationary field.

In this paper we explore in the manner of Boyle et al. (2006) a method of quantifying the amount of fine-tuning in an inflationary potential by counting the number of zeros in the equation of state and its derivatives. We find that ... We also explore a new method of quantifying the amount of fine-tuning in large field models that takes into account the sensitivity of the model to perturbations in higher-order terms. We find that ...

## 2.2 History of Inflationary Theory

Inflationary theory was first proposed by Alan Guth as a possible solution to the horizon and flatness problems (Guth, 1981). At the time, the standard cosmological model of the early universe was an adiabatically expanding, radiation-dominated universe. This model had two fundamental problems: the horizon problem and the

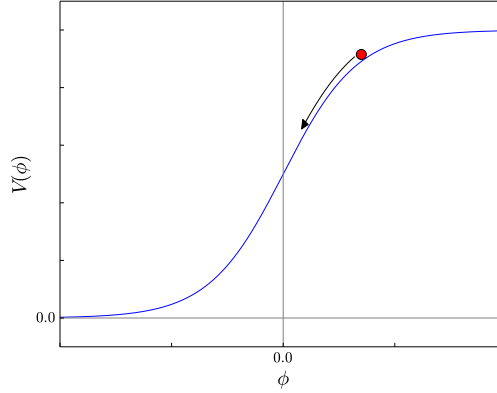


Figure 2.2: A schematic illustration of a plateau potential. The red ball symbolizes the inflationary field at a given time. The field rolls down the hill of the potential and settles in the vacuum state where the potential is zero, causing inflation to end. Note that in this scenario, the most inflation occurs at large values of  $\phi$ . Therefore, unlike in the hilltop case, the highest order terms will dominate.

flatness problem, discussed in more detail in Sections 2.3 and 2.4. Guth’s inflationary model, which proposed a period of exponential expansion in a false vacuum due to supercooling during cosmological phase transitions, solved both the horizon and flatness problems. However, later work by Guth and Weinberg (Guth & Weinberg, 1983) showed that the transition could not be smoothly completed in such a way that would result in a space homogeneously filled with the new phase. Further, a random region within the inhomogeneous space would be unlikely to resemble our own universe.

Andre Linde addressed these problems with a new inflationary theory in which inflation doesn’t occur at the false vacuum but rather as the inflationary field rolled down the hill of the potential towards the true vacuum. [\[MEP: Find Citation to presentation in 1981\]](#) A similar theory was simultaneously developed by Albrecht & Steinhardt (1982). This new inflation resolved the problem of the inhomogeneities because, as mentioned in Section 2.1, density fluctuations are inversely proportional to  $\dot{\phi}$ . Guth’s previous model resulted in deSitter expansion, which corresponds to  $\phi = \text{const}$ , and hence results in unphysically large density fluctuations. Since in this

new model inflation occurs as the field rolls down the hill,  $\dot{\phi} \neq 0$  and perturbations can be made small enough to be physical.

Although the new inflationary model solved some of the problems with Guth's original proposal, new inflation still came with issues of its own. For instance, the inflation field in the new theory often could not begin in thermal equilibrium with other fields, invalidating the use of cosmological phase transitions, which formed the basis of new inflation theory (Linde, 2000). Also, in the new inflationary scenario, inflation can only begin after at a time at least six orders of magnitude later than the Planck time, which is the order of a typical lifetime for a hot, closed universe (Linde, 2005). These problems, together with others, suggested that new inflation needed to be replaced with an even newer theory.

Linde (1983) proposed such a theory, called chaotic inflation, which forms the basis for most modern theories of inflation. Chaotic inflation solves the problems of its predecessors by initiating inflation for any ? initial conditions – even conditions without thermal equilibrium or at the Planck density (Linde, 2005). In chaotic inflation, the inflaton field begins at a point far from the minimum of the potential. The field equation for chaotic inflation includes a viscosity term  $\propto H\dot{\phi}$ , so when the universe is expanding rapidly, the inflaton field rolls slowly down the potential, with the energy density of the field approximately constant. This “slow roll” inflation causes the universe to expand  $\propto e^{Ht}$ . This exponential inflation solves most problems of previous theories because all inhomogeneities in the original universe get stretched  $\sim 10^{10^{12}}$  times after only  $10^{-35}$ s of inflation. (For comparison, the radius of the observable universe is on the order of  $10^{28}$ cm.) Additionally, this solves the flatness problem because even if the universe originally had a small radius of curvature, after inflation that radius is effectively infinite compared to the radius of the observable universe. (Linde, 2000)

However, chaotic inflation still retains a particularly insidious problem, that of

eternal inflation and an inflationary multiverse. Upon closer examination, it has been found that models with large scalar fields during the inflationary period produce large quantum fluctuations in the field. These fluctuations cause the local rate of inflation to change, possibly increasing the rate of inflation (equivalently, delaying the end of inflation) in a certain region. As that region expands, quantum fluctuations in that region can also cause another local section to inflate faster, and this process continues ad infinitum. Thus, once inflation begins, quantum fluctuations preclude it from ending in all regions of space. Further, due to the overwhelming expansion that occurs during inflation, the volume of space that remains inflating completely dominates the volume in which inflation ends.

To make this a little more precise, consider a region of space of radius  $r$  that is undergoing exponential expansion:  $a(t) \propto e^{Ht}$ . The furthest distance a point in that region could influence is  $H^{-1}$ , so if  $r > H^{-1}$ , then that region can be considered its own universe.<sup>1</sup> Following the calculation in ?, we find how much the volume of a region of radius  $H^{-1}$  increases during a time interval  $H^{-1}$ . During that time, the field will decrease due to its classical motion by  $\Delta\phi = \frac{1}{4\pi\phi}$ . However, the average amplitude of quantum fluctuations in the field are  $|\delta\phi| = \frac{H}{2\pi}$ . Therefore, for large enough values of  $\phi$ , the quantum fluctuations will dominate over the classical decrease in the field. In fact, further calculation shows that in the time  $H^{-1}$ , the region grows into  $e^3 \sim 20$  disconnected domains and that in half of those domains the inflationary field will grow. Thus, the growth of the number of disconnected domains with increasing  $\phi$  grows as  $\propto e^{(3-\ln 2)Ht}$ . Also, it turns out that, regardless of the initial value of

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<sup>1</sup>The event horizon is defined as  $d_e(t) = a(t) \int_t^\infty \frac{dt}{a}$ .

## 2.3 Horizon Problem

Simply put, the horizon problem is the realization that the universe is homogenous and isotropic on scales far larger than the particle horizon (the farthest distance from which a point could ever receive information, given the age and expansion rate of the universe). The particle horizon is defined as

$$d_p(t) = a(t) \int_0^t \frac{dt}{a(t)}, \quad (2.1)$$

where  $a(t)$  is the scale factor of the universe, normalized to 1 at the present time. Consider a radiation-dominated universe, which is governed by the Friedmann equation:

$$H^2 = \left( \frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{8\pi G}{3} \frac{\rho_r^0}{a^4}. \quad (2.2)$$

Integrating this equation finds  $a(t) \propto t^{1/2}$ . Putting this into equation (2.1) finds that  $d_p(t) = 2t \propto a^2$ . Now, the size of any homogenous region is proportional to  $a$ . Therefore, at very small times (e.g. the Planck time), the size of the particle horizon will necessarily be smaller than the size of any homogenous region. This is a problem because, in order for a homogenous region to make sense, there must be a time at which the size of the homogenous region was smaller than the particle horizon.

Observations of the CMB show the universe to be homogenous to [Insert ridiculously small scale here], giving the homogenous region a size  $l \sim ct_*$ , where  $t_*$  is the age of the universe at decoupling. The size of the homogenous region it must have originated from must have a size greater than  $l_i \sim ct_* \frac{a_i}{a_*}$ . However, the size of the particle horizon at that time was  $l_h \sim c * t_*$ . Therefore, comparing the two sizes, we find

$$\frac{l_i}{l_h} \sim \frac{a_i}{a_*} \sim \frac{T_*}{T_i}. \quad (2.3)$$

Taking  $T_i$  to be the Planck temperature ( $\sim 10^{32}\text{K}$ ) and  $T_* \sim$

## 2.4 Flatness Problem



# Chapter 3

## Derivation of an Inflationary Theory

### 3.1 Inflationary Friedmann equation

To see how one could implement an inflationary theory, let us first examine a non-inflationary Friedmann equation:

$$H^2 = \left( \frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{8\pi G}{3} \left[ \frac{\rho_m^0}{a^3} + \frac{\rho_r^0}{a^4} + \Lambda + \frac{\sigma^2}{a^6} + \frac{K}{a^2} \right] \quad (3.1)$$

where  $\rho_m^0$  is the energy density of matter in the universe at time  $t = 0$  (set to be the present time),  $\rho_r^0$  is the current energy of radiation,  $\Lambda$  is the cosmological constant,  $\sigma$  is the cosmic anisotropy term, and  $K$  is the curvature of the universe. As we can see by examining the powers of  $a$  in each term, the early universe is dominated by the anisotropy term,  $\propto a^{-6}$ , while the late universe is dominated by the curvature term (and cosmological constant). Therefore, the simplest way to add inflation to the model of the universe is to add an inflationary term to the Friedmann equation:

$$H^2 = \left( \frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{8\pi G}{3} \left[ \frac{\rho_m^0}{a^3} + \frac{\rho_r^0}{a^4} + \Lambda + \frac{\sigma^2}{a^6} + \frac{K}{a^2} + \frac{\rho_\phi^0}{a^{2\epsilon}} \right] \quad (3.2)$$

where  $\rho_\phi^0$  is the energy density of the inflationary field and  $\epsilon$  is a positive exponent. In order to ensure that the universe expands during the inflationary period, we must constrain the value of  $\epsilon$ . To do so, we must examine how an inflation-dominated universe would evolve. In a  $\phi$ -dominated universe, the Friedmann equation becomes:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2 \Omega_\phi}{a^{2\epsilon}}, \quad (3.3)$$

where  $H_0$  is the Hubble constant and  $\Omega_\phi = \frac{8\pi G \rho_\phi}{3H_0^2}$ . Now, solving this equation for  $a(t)$  gives

$$a(t) \propto (\epsilon \sqrt{H_0 \Omega_\phi} t)^{1/\epsilon}, \quad (3.4)$$

and hence

$$\ddot{a}(t) \propto \left(\frac{1-\epsilon}{\epsilon}\right) (H_0 \Omega_\phi)^{1/2} t^{\frac{1-2\epsilon}{\epsilon}}. \quad (3.5)$$

From this, we can see that the condition for an expanding universe is  $\epsilon < 1$ . However, note that at arbitrarily early times (small values of  $a$ ), the anisotropy term will dominate. Thus, in order for inflation to dominate during early times, the value of  $\rho_\phi^0$  compared to  $\sigma^2$  and the other coefficients must be specified such that at the Planck scale the inflationary term will dominate. Also, it is important to note that, in order for inflation to end,  $\epsilon$  must grow to become greater than one, so we know that  $\epsilon$  should be time-dependent.

## 3.2 Inflationary Field & Equation of State

In order to derive a field theory that will produce such an inflationary term consistent with the requirements detailed in Section 3.1, we turn to the equation of state,  $w = \rho/p$ , where  $\rho$  is the energy density and  $p$  is the pressure of the field. The equation of

state is related to  $\epsilon$  by the equation

$$\epsilon = \frac{3}{2}(w + 1). \quad (3.6)$$

Thus, an inflationary field must have an equation of state  $w < -1/3$ . Table 3.1 summarizes the values  $w$  and  $\epsilon$  and the evolution of  $a(t)$  for different types of fields.

Field Type	$w = \rho/p$	$\epsilon = \frac{3}{2}(w + 1)$	$a(t)$	$\ddot{a}(t)$
matter	0	3/2	$\propto t^{2/3}$	$> 0$
radiation	1/3	2	$\propto t^{1/2}$	$< 0$
$\Lambda$	-1	0	$\propto e^t$	$> 0$
inflation	$< -1/3$	$< 1$	$\propto t^{1/\epsilon}$	$> 0$
contraction	$> 1$	$> 3$	$\propto t^{1/\epsilon}$	$< 0$

Table 3.1: The values of  $w$  and  $\epsilon$  and the evolution of  $a(t)$  for different types of fields.

From Table 3.1, we see that an inflationary field must have an equation of state  $w < -1/3$ . From this, we want to put constraints on the Lagrangian density  $\mathcal{L}$  for the inflationary field. Let  $\phi$  be an inflationary field. Its Lagrangian density takes the form:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \quad (3.7)$$

Using the assumptions that  $\phi$  is homogenous (so that  $\partial_j \phi = 0$ ) and isotropic (so that we can use the Minkowski metric  $\eta_{\mu\nu}$ ), we find that the stress-energy tensor of the field is

$$T_{\mu\nu} = -\eta_{\mu 0} \dot{\phi}^2 + \eta_{\mu\nu} (\frac{1}{2} \dot{\phi}^2 + V(\phi)). \quad (3.8)$$

From this, we find that the equation of state is

$$w = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)}. \quad (3.9)$$

Putting in the constraint on  $w$  for inflation, we find that an inflationary field must

satisfy

$$\dot{\phi}^2 < V(\phi). \quad (3.10)$$

[MEP: Is this important? It seems it should be, but we never talked about it before...]

Once we have  $\mathcal{L}$ , we know that the equation of motion for the field is the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (3.11)$$

where  $H$  is the Hubble parameter,  $V(\phi)$  is the inflationary potential, and a dot indicates a time derivative. In our convention, the reduced Planck mass is set to one, i.e.  $M_{pl} = (8\pi G)^{-1/2} \equiv 1$ . Combining this with the Friedmann equation,

$$H = \sqrt{\frac{1}{3} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right)}, \quad (3.12)$$

completely determines the evolution of a universe dominated by an inflationary potential  $V(\phi)$ . It will be useful later on to re-write the Friedmann equation in terms of  $\epsilon$ . Putting together equations (3.3) and (3.4), we find that

$$\epsilon = -\frac{\dot{H}}{H} \quad (3.13)$$

Note that, for general inflationary potentials,  $\epsilon$  will not be constant in time. This is fortuitous since, in order for inflation to end,  $\epsilon$  will have to go from a value greater than 1 to a value less than 1. We can also use this equation to further constrain the inflationary potential with a “graceful exit” condition. [MEP: I think this is true. I need to look up more info & elaborate.]

### 3.3 Calculating Observable Parameters

Of course, the whole point of creating an inflationary theory is to be able to reproduce observational data. When observations from experiments such as WMAP and Planck observe the early universe, they measure the power spectrum of density fluctuations in the primordial plasma. **[MEP: More elaboration on where these come from]**

Instead of working in units of time, where the time at the end of inflation is difficult to pinpoint, we choose instead to work in terms of the number of e-folds until the end of inflation, defined as

$$N = - \int_t^{t_{end}} H dt \quad \text{or equivalently} \quad dN = -H dt. \quad (3.14)$$

Note that  $N$  runs backwards in time, so as time increases the number of e-folds decreases. In these units, the end of inflation is defined to occur at  $N = 0$ , so at this time  $\epsilon = 1$ . Current observations look at a narrow window centered on  $N = 60 \equiv N_*$ .

Figure ?? shows a schematic of the density fluctuations as a function of the wavenumber of the fluctuations ( $\log k \sim \frac{1}{N}$ ). If the fluctuations were purely scale invariant, then the curve would be a horizontal line, the amplitude of the fluctuations independent of their wave number. Since current observations see only a very limited range near  $N_*$ , so they focus on measuring (and we focus on predicting) the deviation from scale invariance at that point. This deviation is called the spectral tilt, denoted  $n_s(k^*)$ , where  $k^*$  is the value of  $k$  at  $N_*$ . Following the work of Wang et al. (1997), we use an approximation of the spectral index to first order in  $\epsilon$ :

$$n_s \approx 1 - 2\epsilon - \frac{d \ln(\epsilon)}{dN} + \mathcal{O}(\epsilon^2), \quad (3.15)$$

where the equation of state is treated as a function of the number of e-foldings instead of a function of time.

In addition to the spectral tilt at  $N_*$ , observations can also constrain the tensor-to-scalar power ratio, typically denoted  $r$ . The tensor-to-scalar ratio is typically computed in the slow-roll approximation (See Section 3.6 for more details), and we follow the conventions of Planck Collaboration et al. (2014) to get

$$r \approx 16\epsilon. \quad (3.16)$$

### 3.4 Fine-tuning in an Inflationary Theory

Now that we have the basic mechanics of an inflationary theory, we see that defining an instance of an inflationary theory reduces to defining the inflationary potential  $V(\phi)$ . In order for inflation to be a predictive theory, we need to find a way to choose which potentials are reasonable and then find whether any of those potentials would generate a universe that conforms with observations.

### 3.5 Eqns for Code

Let  $\phi$  be an inflationary field governed by equations (3.12) and (3.11). For the purposes of our code, we find it easier to work with  $\phi$  not as a function of time but rather as a function of the number of e-folds until the end of inflation, defined in equation (3.14). If we let  $\phi'$  denote the derivative of  $\phi$  with respect to  $N$ , so that  $\dot{\phi} = -H\phi'$  the Friedmann equation becomes

$$H^2 = \frac{V(\phi)}{3 - \frac{1}{2}(\phi')^2}, \quad (3.17)$$

and the Klein-Gordon equation becomes

$$2H^2\phi'' + H^2(\phi')^3 - 6H^2\phi' + 2V'(\phi) = 0. \quad (3.18)$$

## 3.6 Slow Roll Approximation

In the slow roll approximation, we assume that  $\dot{\phi}$  and  $\ddot{\phi}$  are small. In order for this to be true, the inflationary potential must not be varying rapidly, leading us to the slow roll approximations:

$$\left| \frac{V'(\phi)}{V(\phi)} \right| \ll 1 \quad \text{and} \quad \left| \frac{V''(\phi)}{V(\phi)} \right| \ll 1. \quad (3.19)$$

With these approximations, the Friedmann equation becomes

$$3H^2 \approx_{sr} V(\phi), \quad (3.20)$$

and the Klein-Gordon equation becomes

$$3H\dot{\phi} \approx_{sr} -V'(\phi) \quad (3.21)$$

We use these approximations to find the initial conditions for integrating equation (3.18):  $\phi(N = 0)$  and  $\dot{\phi}(N = 0)$ . To do this, we use the fact that, at the end of inflation (i.e. at  $N = 0$ ),  $\epsilon = 1$ . Recall from Section ?? that the condition for accelerated expansion is  $\epsilon < 1$ . Therefore, as the universe transitions out of the inflationary epoch and into a period of decelerating expansion (where  $\epsilon > 1$ ), there must be a point where  $\epsilon = 1$ , marking the end of inflation. Putting together equations (3.6) and (3.9), we find that

$$\epsilon = \frac{3}{2} \left( \frac{\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \right) \approx_{sr} \frac{1}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \quad (3.22)$$

where we have used the slow roll approximations  $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$  and equation (3.21). Setting  $\epsilon = 1$  gives us an equation we can solve for  $\phi(N = 0)$ . Once we have  $\phi(N = 0)$ , we can easily use equation (3.21) to find  $\dot{\phi}(N = 0)$ .

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