Assignment 5. Fourier Transforms.

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Fourier Optics

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Problem Statement

Obtain the Fourier transform for the following functions:

- $ACos(2\pi v_0 t)$
- $ACos^2(2\pi v_0 t)$
- $ACos(2\pi v_0(t - t_0))$
- $ASen(2\pi v_0 t)$
- $ARECT(\frac{t}{T/2})$

Fourier Transform: Cosine

$$f(t) = ACos(2\pi v_0 t) = A\frac{e^{i2\pi v_0 t} + e^{-i2\pi v_0 t}}{2} \tag{1}$$

$$F[f(t)] = \int_{-\infty}^{\infty} ACos(2\pi v_0 t) = A\int_{-\infty}^{\infty} \frac{e^{i2\pi v_0 t} + e^{-i2\pi v_0 t}}{2} e^{-i2\pi vt}\, dt \tag{2}$$

$$= \frac{A}{2}\left(\int_{-\infty}^{\infty} e^{i2\pi v_0 t} e^{-i2\pi vt} + \int_{-\infty}^{\infty} e^{-i2\pi v_0 t} e^{-i2\pi vt}\right) \tag{3}$$

$$= \frac{A}{2}\left(F[e^{i2\pi v_0 t}] + F[e^{-i2\pi v_0 t}]\right) \tag{4}$$

Fourier Transform: Cosine

$$= \frac{A}{2}\left(\delta(v - v_0) + \delta(v + v_0)\right) \tag{5}$$

Fourier Transform: Cosine Squared

$$f(t) = ACos^2(2\pi v_0 t) \tag{6}$$

$$F[f(t)] = \int_{-\infty}^{\infty} ACos^2(2\pi v_0 t)e^{-ikt}\, dt \tag{7}$$

$$F[f(t)] = \int_{-\infty}^{\infty} \frac{A}{4\pi}(1 + Cos(4\pi v_0 t))e^{-ikt}\, dt \tag{8}$$

$$F[f(t)] = \frac{A}{4\pi}\int_{-\infty}^{\infty} 1e^{-ikt}\, dt \; + \frac{A}{4\pi}\int_{-\infty}^{\infty} Cos(4\pi v_0 t)e^{-ikt}\, dt \tag{9}$$

$$F[f(t)] = \frac{A}{4\pi}2\pi\delta(k) + \frac{A}{4\pi}\int_{-\infty}^{\infty} \frac{e^{i4\pi v_0 t} + e^{-i4\pi v_0 t}}{4}e^{-ikt}\, dt \tag{10}$$

This calculation needs to be reviewed.

Fourier Transform: Cosine with phase shift

$$f(t) = ACos(2\pi v_0(t - t_0)) \tag{11}$$

$$F[f(t)] = \int_{-\infty}^{\infty} ACos(2\pi v_0(t - t_0))\, dt \ = \tag{12}$$

$$A \int_{-\infty}^{\infty} \frac{e^{i2\pi v_0(t-t_0)} + e^{-i2\pi v_0(t-t_0)}}{2} e^{-i2\pi vt}\, dt \tag{13}$$

$$\frac{A}{2} \int_{-\infty}^{\infty} \left(e^{i2\pi v_0 t} e^{i2\pi v_0 t_0} + e^{-i2\pi v_0 t} e^{-i2\pi v_0 t_0}\right) e^{-i2\pi vt}\, dt \tag{14}$$

Fourier Transform: Cosine with phase shift

$$
\begin{aligned}
&= \frac{A}{2}\Bigg(e^{i2\pi v_0 t_0}\int_{-\infty}^{\infty} e^{i2\pi v_0 t} e^{-i2\pi v t}\, dt + \\
&\quad e^{-i2\pi v_0 t_0}\int_{-\infty}^{\infty} e^{-i2\pi v_0 t} e^{-i2\pi v t}\, dt
\end{aligned} \tag{15}
$$

$$
= \frac{A}{2}\left(e^{i2\pi v_0 t_0} F[e^{i2\pi v_0 t}] + e^{-i2\pi v_0 t_0} F[e^{-i2\pi v_0 t}]\right) \tag{16}
$$

$$
= \frac{A}{2}\left(e^{i2\pi v_0 t_0}\delta(v - v_0) + e^{-i2\pi v_0 t_0}\delta(v + v_0)\right) \tag{17}
$$

$$
= \frac{A}{2}\left(e^{i2\pi v_0 t_0}\delta(v - v_0) + e^{-i2\pi v_0 t_0}\delta(v + v_0)\right) \tag{18}
$$

Fourier Transform: Sine

$$f(t) = ASin(2\pi v_0 t) \tag{19}$$

$$\begin{gathered} F[f(t)] = \int_{-\infty}^{\infty} ASin(2\pi v_0 t) e^{-i2\pi vt}\, dt \;= \\ A \int_{-\infty}^{\infty} \frac{e^{i2\pi v_0} - e^{-i2\pi v_0}}{2i} e^{-i2\pi vt}\, dt \end{gathered} \tag{20}$$

$$= \frac{A}{2i} \int_{-\infty}^{\infty} e^{i2\pi v_0} e^{-i2\pi vt} - \int_{-\infty}^{\infty} e^{-i2\pi v_0} e^{-i2\pi vt} \tag{21}$$

$$= \frac{A}{2i} \left(F[e^{i2\pi v_0 t}] - F[e^{-i2\pi v_0 t}] \right) \tag{22}$$

$$F[f(t)] = \frac{A}{2i} \left(2\pi\delta(v - v_0) + 2\pi\delta(v + v_0) \right) \tag{23}$$

Fourier Transform: Sine

$$F[f(t)] = \frac{A}{i}\pi\left(\delta(v - v_0) - \delta(v + v_0)\right) \tag{24}$$

Fourier Transform: Rectangular Function

$$f(t) = RECT\left(\frac{t}{T/2}\right) \tag{25}$$

$$F[f(t)] = \int_{-\infty}^{\infty} RECT\left(\frac{t}{T/2}\right) e^{-i\omega t}\, dt \tag{26}$$

It is defined as,

$$RECT\left(\frac{t}{\tau/2}\right) = \prod\left(\frac{t}{\tau/2}\right) = \begin{cases} 1 & for|t| \geq \left(\frac{\tau}{4}\right) \\ 0 & otherwise \end{cases} \tag{27}$$

From the definition of the Fourier transform,

$$F\left[\prod\left(\frac{t}{\tau/2}\right)\right] = \int_{-\infty}^{\infty} \prod\left(\frac{t}{\tau/2}\right) e^{-i\omega t}\, dt \tag{28}$$

Fourier Transform: Rectangular Function

$$F[f(t)] = \int_{-(\tau/4)}^{\tau/4} 1 \cdot e^{-i\omega t}\, dt$$

$$= \left[\frac{e^{-i\omega(\tau/4)} - e^{-i\omega(\tau/2)}}{-i\omega}\right] = \left[\frac{e^{i\omega(\tau/2)} - e^{-i\omega(\tau/4)}}{i\omega}\right] \tag{30}$$

$$= \frac{4\tau\left[e^{i\omega(\tau/4)} - e^{-i\omega(\tau/4)}\right]}{i\omega \cdot 4\tau} = \frac{\tau}{\omega(\tau/4)}\left[\frac{e^{i\omega(\tau/4)} - e^{-i\omega(\tau/2)}}{4i}\right] \tag{31}$$

$$= \frac{\tau}{\omega(\tau/4)} \cdot Sin\omega(\tau/4) = \tau\left[\frac{Sin\omega(\tau/4)}{\omega(\tau/4)}\right] \tag{32}$$

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Fourier Transform: Rectangular Function

$$Sinc(\omega\tau/4)) = \frac{Sin\omega(\tau/4)}{\omega(\tau/4)} \quad (33)$$

$$\therefore F[f(t)] = \tau \cdot Sinc\left(\frac{\omega\tau}{4}\right) \quad (34)$$