# Numerical Modeling of propagation modes in Step-Index Optical Fibers

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**This report will cover building a 2D finite element mode solver using Python and Gmsh. An explanation will be presented to solve the propagation modes of a stepped index optical fiber. A python code was implemented to solve the eigenvalue problem that the proposed Helmholtz equation presents. Thus, graphing these solutions provided us with different light propagation modes for varying core and cladding values. The results have proven to be inconclusive, since it was concluded that smaller core radius sizes have multimodes, and bigger ones have monomodes. These results do not agree with revised references.**



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## 1. INTRODUCTION

Throughout history, scientists have strived to build models that represent phenomena in a clear and manipulable way. The most useful, evolved and accurate models to understand reality are mathematical models.

Modeling consists of translating the situation, element or phenomenon in question into a simpler and more manageable form. This is where mathematics is incorporated in its different fields, to reflect the general properties of modeling. When mathematical models are generated, natural phenomena can be understood, interpreted and even modified.
At present, fiber optics is a very important transmission medium, which is commonly used in data networks due to the great distance that a signal can travel before needing a repeater to recover its intensity. This is a very fine thread of transparent material, glass or plastic materials, through which pulses of light are sent that represent the data to be transmitted.

This report will cover building a 2D finite element mode solver using Python and Gmsh. An explanation will be presented to solve the propagation modes of a stepped index optical fiber. The particularity of these fibers is that there is an abrupt refractive index change at the core-cladding interface [1]. This in order to demonstrate the assertions of [2], that by increasing the radius of the core, each type of fiber has more power confined in its core and less power in its cladding, and in addition to that,that at small values of core radius correspond to monomodes. Next, the physical model that describes the propagation of electromagnetic waves in optical fiber is presented, described by the Helmholtz equation, along with the solution method, the Finite Element Method. Finally, the subsequent implementation of the code in Python, and the graphical interface with the results obtained, their respective analysis and conclusions.

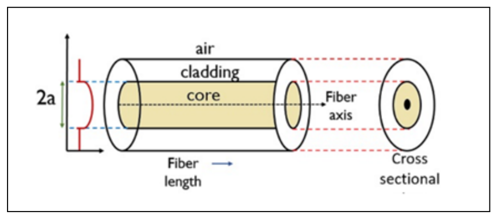


**Fig. 1.** Fibra óptica de índice escalonado

## 2. METHODOLOGY

The methodology for the development of the subject project had four phases. First, the identification of the phenomenon, then the development of the numerical model, the programming, and finally, the analysis. During the analysis and development of the mathematical model, the variables presented in the table 1 were identified.

Also, the following considerations were taken:

1. **There is no magnetization**, since optical fibers do not usually include magnetic constituent materials.
2. Linearity of Maxwell's equations.
3. Time-Harmonic electromagnetic fields.
4. Homogeneous, isotropic and lossless dielectric media.
5. Infinite fiber optic cable length.

### A. Maxwell's Equations

The differential form of Maxwell's equations govern this problem, which govern this problem, was considered. These are presented in the equations 1 to 4 [2] [1].

**Table 1. Variables**

| | Physical significance |
|---|---|
| $E$ | Electrical Field |
| $H$ | Magnetic Field |
| $t$ | Time |
| $\epsilon$ | Permittivity |
| $\mu$ | Permeability |
| $n_{core}$ | Core index of refraction |
| $n_{clad}$ | Cladding index of refraction |
| $\theta$ | Incident angle |
| $\lambda$ | Wavelenght |
| $\beta$ | Propagation constant |
| $\omega$ | Angular frequency |
| $k$ | Wavenumber in vacuum |
| $L$ | Fiber optic lenght |

$$\Delta \cdot D = \rho \tag{1}$$

$$\Delta \cdot B = 0 \tag{2}$$

$$\Delta \times E = -\frac{\partial B}{\partial t} \tag{3}$$

$$\Delta \times H = \frac{\partial D}{\partial t} + J_e \tag{4}$$

Where E and H are the electric and magnetic field, respectively, $D =_\epsilon E$ is the flux density of the electric field and B = μH is the magnetic flux density.

Sometimes it is not necessary to calculate the three-dimensional components of $E$ and $H$. For example, in the waveguide, the wave has no component $z$ of the electric field or has no component $z$ of the magnetic field, so that in the waveguide, as well as in the optical fiber, there is a particular electromagnetic wave ($E$ or $H$) whose radiation plane is always perpendicular to its direction of propagation. These electromagnetic waves are called Transverse Electric (TE) modes or Transverse Magnetic (TM) modes. TE and TM modes are also known as semi-vector modes [2].

### *A.1. Vector Wave Equations*

Before talking about the semi-vector mode, we first examine the vector mode. Time-Harmonic electromagnetic waves are generally noted as:

$$E\,(x,\,y,\,z,\,t) = E\,(x,\,y,\,z)\,e^{iwt} \tag{5}$$

$$H\,(x,\,y,\,z,\,t) = H\,(x,\,y,\,z)\,e^{iwt}$$

In dielectric media, the dielectric permittivity is $\epsilon\,(r) = n^2\,(r)\,\epsilon_0$, the magnetic permeability is denoted as $\mu$, $n(r)$ is the refractive index and $epsilon_0$ is the dielectric permittivity of free space. Since the magnetism in the material is so weak, the permeability is considered to be the same as that of free space, therefore $\mu = \mu_0$ [2].

Rewriting Maxwell's equation from 1 to 9 in phasor form, as:

$$\nabla \cdot \left(\epsilon_0 n^2 E\right) = \rho \tag{6}$$

$$\nabla\Delta \cdot (\mu H) = 0 \tag{7}$$

$$\Delta \times E = -\mu_0 \frac{\partial H}{\partial t} = -iw\mu_0 H \tag{8}$$

$$\Delta \times H = \mu_0 n^2 \frac{\partial E}{\partial t} + J_e = iw\epsilon_0 n^2 E + J_e \tag{9}$$

Now we consider the speed of light in free space as $c$, the wavelength as $\lambda$, the angular frequency as $\omega$, the wave number as $k$, therefore

$$\omega = 2\pi \frac{c}{\lambda} = \frac{k}{\sqrt{\epsilon_0 \mu_0}} \tag{10}$$

Then, the vector mode for the electric field is derived. Take the curl operator on the third equation in the equations 6 and **??** and you get:

$$\nabla(\nabla \times E) = -iw\mu_0 \nabla \times H = -iw\sqrt{\frac{\mu_0}{\epsilon_0}} k \nabla \times H = -i\sqrt{\frac{\mu_0}{\epsilon_0}} k J_e + k^2 n^2 E \tag{11}$$

Where $\nabla^2 = \nabla = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. Thus,

$$\nabla(\nabla \cdot E) - \nabla^2 E = -i\sqrt{\frac{\mu_0}{\epsilon_0}} k J_e + k^2 n^2 E \tag{12}$$

$$(\nabla^2 + k^2 n^2) E = \nabla(\nabla \cdot E) + i\sqrt{\frac{\mu_0}{\epsilon_0}} k J_e \tag{13}$$

Applying divergence to 9, and considering $\nabla(\nabla \times A) = 0$, we obtain

$$\nabla \cdot (\nabla \times H) = 0 \tag{14}$$

$$\nabla J_e + iw\epsilon_0 \nabla \cdot (n^2 E) = \nabla J_e + iw\epsilon_0 E \nabla(n^2) + iw\epsilon_0 n^2 \nabla \cdot E \tag{15}$$

After some lenghty calculations,

$$\nabla E = -\frac{i}{w\epsilon_0} \frac{\nabla \cdot J_e}{n^2} - \frac{\nabla n^2}{n^2} E = -\frac{i}{w\epsilon_0} \frac{\nabla \cdot J_e}{n^2} - E\nabla ln n^2 \tag{16}$$

Substituting equation 16 in equation 13,

$$(\nabla + k^2 n^2) E = -\nabla(E \nabla ln n^2) - \frac{i}{w\epsilon_0} \nabla(\frac{\nabla \cdot J_e}{n^2}) + i\sqrt{\frac{\mu_0}{\epsilon_0}} k J_e \tag{17}$$

This is the vector mode for the electric field cite JIAWEN. Similarly, the vector mode for the magnetic field can be obtained:

$$(\nabla + k^2 n^2) H = (\nabla \cdot H)(\nabla ln n^2) - \nabla \times J_e - J_e \times \nabla ln n^2 \tag{18}$$

### B. Mode Equations

Since the original Maxwell equation requires a three-dimensional differential, the Maxwell equations need to be reduced to two dimensions. To reduce the dimension, consider the scalar form of Maxwell's equations, rewriting the equations and omitting $J_e$:

$$i\omega\epsilon E_x = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) \tag{19}$$

$$i\omega\epsilon E_y = \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)$$

$$i\omega\epsilon E_z = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$i\omega\epsilon E_x = -\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)$$

$$i\omega\epsilon E_y = -\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)$$

$$i\omega\epsilon E_z = -\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

Since the curl operator is expanded to obtain the above equations, a given component can easily be set to zero if that component does not exist in a certain dimension. Considering a detailed process to reduce Maxwell's equation from 3D to 2D, using TE mode. We set $H_x = H_y = 0$, $E_z = 0$ y $\frac{d}{dz} = 0$.

$$\nabla \times E = \epsilon_{ijk}\hat{e}_i\partial_j E_k = \epsilon_{312}\hat{e}_3\partial_1 E_2 - \epsilon_{321}\hat{e}_3\partial_2 E_1 = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{e}_z \tag{20}$$

$$\nabla \times H = \epsilon_{ijk}\hat{e}_i\partial_j H_k = \epsilon_{123}\hat{e}_2\partial_3 H_1 - \epsilon_{213}\hat{e}_1\partial_3 H_2 = \frac{\partial H_z}{\partial y}\hat{e}_x - \frac{\partial H_z}{\partial x}\hat{e}_y \tag{21}$$

Where $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$ are the unit directional vector at $x, y, z$ respectively. Einstein's notation is used here; $1, 2, 3$ represent $x, y, z$ respectively. Then, we substitute equations 20 and 21 in 3 and 4. We obtain

$$i\omega\epsilon E = \frac{\partial H_z}{\partial y}\hat{e}_x - \frac{\partial H_z}{\partial x}\hat{e}_y \tag{22}$$

$$i\omega\mu H = \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)\hat{e}_z$$

In equation 19, $E = (E_x, E_y, 0) = (0, 0, H_z)$. We can rewirte these equations in scalar form

$$i\omega\epsilon E_x = \frac{\partial H_z}{\partial y} \tag{23}$$

$$i\omega\epsilon E_y = -\frac{\partial H_z}{\partial y}$$

$$i\omega\mu H_z = \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

This is the TE mode after dimension reduction. Similarly, you can also get the TM mode after dimension reduction:

$$i\omega\mu H_x = \frac{\partial E_z}{\partial y} \tag{24}$$

$$i\omega\mu H_y = -\frac{\partial E_z}{\partial y}$$

$$i\omega\mu E_z = \left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right)$$

We can do some transformations to avoid calculating the components $H_x$, $H_y$ and use a partial differential equation (PDE) to express $E_z$.

$$i\omega\mu\frac{\partial}{\partial y}H_x = \frac{\partial^2 E_z}{\partial y^2} \tag{25}$$

$$i\omega\mu\frac{\partial}{\partial x}H_x = -\frac{\partial^2 E_z}{\partial x^2}$$

$$-\omega^2\mu E_z = i\mu\omega\left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right)$$

Substituting the first two equations in the equation 24, the wave equation is obtained (Helmholtz equations):

$$\nabla^2 E = -\omega^2\epsilon\mu E = -k^2 E \tag{26}$$

$$\nabla^2 H = -\omega^2\epsilon\mu H = -k^2 H$$

In optical fibers, the refractive index is usually independent of $z$, since light propagates along that direction [2]. We rewrite the envelop part of electromagnetic field in 25 as:

$$E(x, y, z) = E(x, y)e(z) \tag{27}$$

$$H(x, y, z) = H(x, y)h(z)$$

Substituting 28 into equation 26 we obtain

$$e(z)(\nabla^2 + k^2)E_z = E_z\nabla^2 e(z) \tag{28}$$

It is worth to note that $E(x, y)$ is x,y-dependent, $e(z)$ is only z-dependent, therefore the solution of 28 can only be in the form as:

$$e(z) = e^{i\beta z} \tag{29}$$

And the gradient operator can be rewritten as $\nabla = \vec{\nabla}_\perp + \hat{Z}\frac{\partial}{\partial z}$

$$\nabla^2 E_z = \vec{\nabla}_\perp^2 E_z + \frac{\partial E_z}{\partial z^2} = \vec{\nabla}_\perp^2 E_z - \beta^2 E_z \tag{30}$$

and since we neglect the conduction current $J = 0$, we substitute 29 into 17

$$(\nabla^2 + k^2n^2 - \beta^2)E_z = -(\vec{\nabla}_\perp + i\beta\hat{Z})(E_z + \nabla_\perp ln(n^2)) \tag{31}$$

this is the TM mode, and similarly we can get TE mode as:

$$(\nabla^2 + k^2n^2 - \beta^2)H_z = ((\vec{\nabla}_\perp + i\beta\hat{Z}) \times H_z) \times \nabla_\perp ln(n^2) \tag{32}$$

And for most cases, the refractive index are step functions, therefore, 30 and 31 can be written as:

$$\vec{\nabla}_\perp^2 E_z + (k^2n^2 - \beta^2)E_z = 0 \tag{33}$$

$$\vec{\nabla}_\perp^2 H_z + (k^2n^2 - \beta^2)H_z = 0$$

For semi-vector mode, we only need to consider only one of 32 since TE or TM mode must satisfy $Ez = 0$ (TE mode) o $Hz = 0$ (TM mode) [2].

### B.1. Scalar Mode

If we want to obtain a more precise solution with a numerical method without considering any other computational problems, we should always calculate the equations in vector mode. With scalar mode, electromagnetic polarization is omitted and can be expressed by Helmholtz's scalar equation [2]:

$$\nabla_{\Phi} + (k^2{}_0 n^2 - \beta^2)\phi = 0 \tag{34}$$

and $\phi$ can represent an electric field or a magnetic field at all points.

## C. Finite Element Method

The Finite Element Method (FEM) is used to solve Partial Differential Equations (PDE). This method is based on a complete mathematical theory of the weak PDE formulation.

### C.1. Weak Formulation

First, let's start with a fiber mode equation,

$$\nabla_{\perp}^2 E_z + (k^2\beta^2)E_z = f(x), x \in \Omega \tag{35}$$

$$E_z = u(x), x \in \Gamma_D \tag{36}$$

$$\nabla E_z \cdot n = g(x), x \in \Gamma_N \tag{37}$$

$E_z(x)$ is the unknown variable to be solved, $f(x)$, $g(x)$ y $u(x)$ are given functions, $\Gamma \in R$ is the computational domain with the boundary $\partial\Gamma \in \Gamma D + \Gamma N$, $\Gamma D$ is part of the boundary where applying Dirichlet boundary conditions and $\Gamma N$ is part of the boundary where applying Newmann boundary conditions. The weak formulation of 35 is:

$$\int_{\omega} (\nabla_{\perp}^2 E_z + (k^2 - \beta^2)E_z - f).sd\omega = 0 \tag{38}$$

### C.2. Boundary conditions

There are two types of boundary conditions in FEM: essential boundary conditions and natural boundary conditions. With 'essential boundary conditions', we refer to Dirichlet boundary conditions [2]. Since the test function $s$ can have an arbitrary form, we consider a domain $H(\Gamma)$ and a subdomain $H_{\Gamma}$, where

$$H(\Gamma) = h(x) \in \Gamma \tag{39}$$

$$H_0(\Gamma) = \{h(x) \in \Gamma; h(x) = 0, x \in \Gamma_D\} \tag{40}$$

Considering the unkonwn function $T \in H(\Gamma)$ y la función de prueba $s \in H_0(\Gamma)$. Since the unknown function $T$ must be continuous along the gradient in the domain $\Gamma$, and the test function $s$ can have an arbitrary form, we take $s$ as a function that is also continuous along the gradient and $s$ is always zero at the domain limit $\Gamma$ as defined above. In this case, the weak formulation will disappear in the following equation [2]:

$$\begin{aligned}\int_{\Omega} \nabla_{\perp} E_z \cdot \nabla s \, d\Omega = \int_{\Gamma_N} s(\nabla_{\perp} \cdot n) + \int_{\Omega} (k^2 - \beta^2)E_z \cdot s \, d\Omega \\ - \int_{\Omega} f \cdot s \, d\Omega \\ T(x) = u(x), x \in \Gamma\end{aligned} \tag{41}$$

So the integral term of the equation **??** on the surface $\Gamma N$ can be rewritten as $\int g \cdot s$, the weak formulation:

$$\begin{aligned}\int_{\Omega} \nabla_{\perp} E_z \cdot \nabla s \, d\Omega = \int_{\Gamma_N} s \cdot n + \int_{\Omega} (k^2 - \beta^2)E_z \cdot s \, d\Omega \\ - \int_{\Omega} f \cdot s \, d\Omega\end{aligned} \tag{42}$$

We only consider zero surface flux, so the weak formulation of fiber mode vanishes to:

$$\begin{aligned}\int_{\Omega} \nabla_{\perp} E_z \cdot \nabla s \, d\Omega = \int_{\Gamma_N} s \cdot n \, d\omega + \int_{\Omega} (k^2 - \beta^2)E_z \cdot s \, d\Omega \\ - \int_{\Omega} f \cdot s \, d\Omega \\ T(x) = u(x), x \in \Gamma\end{aligned} \tag{43}$$

For an ideal condition, there is no light leakage from the fiber cladding, therefore the Dirichlet boundary condition can be considered as $T(x) = 0, x \in \Gamma$. Then the equation 43 takes the form:

$$\begin{aligned}\int_{\Omega} \nabla_{\perp} E_z \cdot \nabla s \, d\Omega = \int_{\Omega} (k^2 - \beta^2)E_z \cdot s \, d\Omega \\ - \int_{\Omega} f \cdot s \, d\Omega\end{aligned} \tag{44}$$

## D. Meshing

Mesh generation is the key component of the finite element method. A well configured mesh can save a lot of calculation time and guarantee better simulation results [2] [3]. It is known that most of the energy is concentrated in the core of the fiber; there is hardly any energy inside the fiber cladding. So the field inside the fiber cladding is zero almost everywhere [2], and if you put a fine mesh on the cladding part, it will waste a lot of time just to get the same results. Therefore, a "loose" mesh is used for the cladding and a finer one for the area where more energy is concentrated, the core. The result is presented in the figure **??**. For analysis purposes, in this section multiple simulations have been performed varying the radius of the core, based on values given by [2]. Before executing the code, the value of the radius in the python code is modified for each case.

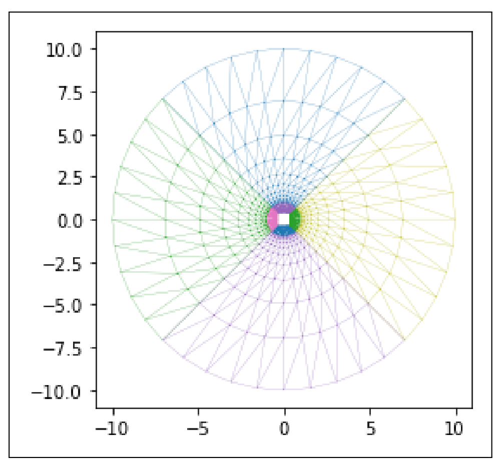


**Fig. 2.** Mesh on the cross section of fiber using Gmsh

It is important to mention that it was chosen to work with triangular elements due to their greater computational simplicity, compared to the analysis that would entail using, for example, quadrilateral elements.

Using linear triangles, the stiffness matrices are

$$K_{\text{local}} = \frac{|J|}{2}(J^{-1}D)^T(J^{-1}D) \quad \textbf{(45)}$$

y

$$M_{\text{local}} = \frac{|J|}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \textbf{(46)}$$

Where $|J|$ is the Jacobian determinant of the transformation and $\mathbf{x}_m$ is the centroid of the triangle.

The code is simple. First we import the libraries that we are going to need: *numpy*, *meshio*, *matplotlib*, and *scipy*. Then we have three main functions: the first assembles the stiffness and mass matrices, the second one calculates the local stiffness and mass matrices, and the third one identifies the restricted and free nodes for the mesh, essentially the boundary of our problem. The second and third will depend on the coordinates of the nodes, the connectivity of the elements, and the index of refraction of the material. This property will later vary according to the region analyzed. After defining these and importing our mesh, we proceed to call the created functions and obtain our results.

## 3. RESULTS AND ANALYSIS

The core and cladding have different refractive index, and the refractive index of the core $n_{core}$ is a little higher than the refractive index of the cladding $n_{clad}$. So when the light hits a certain angle, you will get a total internal refraction between the interface of the core cladding and the cladding; therefore, the light will be transmitted along with the fiber without attenuation [2].

There are two types of widely used step index fibers, one is single mode fiber and the other is multimode fiber. Let's first define the index contrast:

$$\Delta = \frac{n_{core} - n_{clad}}{n_{core}} \quad \textbf{(47)}$$

then the V number can be defined as:

$$V = \frac{2\pi a}{\lambda}\sqrt{n_{core}^2 + n_{clad}^2} \quad \textbf{(48)}$$

where $\lambda$ is the free space wavelength, and a is the radius of fiber core. With V number, we can define the number of mode can transmit along the fiber simultaneously:

$$M \approx \frac{V^2}{2} \quad \textbf{(49)}$$

So we can see that with different core radius, a different number of modes can propagate inside the fiber. We can also see that single-mode fibers have a smaller core and multimode fibers have a larger core. This is the purpose of this project. Carrying out this analysis we will be able to observe the influence that these dimensions have on the energy efficiency of the fibers [2]. With increasing core radius, each type of fiber has more power confined to its core and less power to its cladding [2].

For the simulation, different values were used for the radii of the cladding and the core, in order to demonstrate the relationship between these radii and the modes of propagation.

The eigenvalues obtained were those presented in the figures 3 to 6.

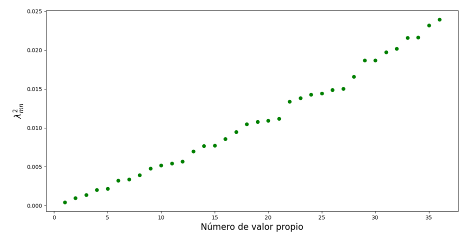


**Fig. 3.** Eigenvalues with $r_{core} = 2.5$ y $r_{clad} = 125$.

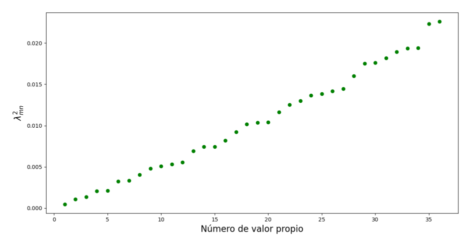


**Fig. 4.** Results with $r_{core} = 12$ y $r_{clad} = 125$.

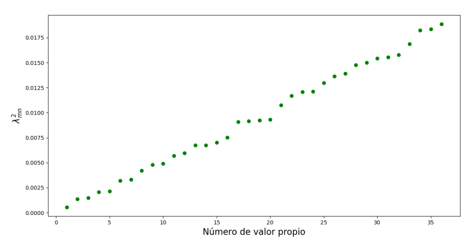


**Fig. 5.** Eigenvalues with $r_{core} = 50$ y $r_{clad} = 125$.

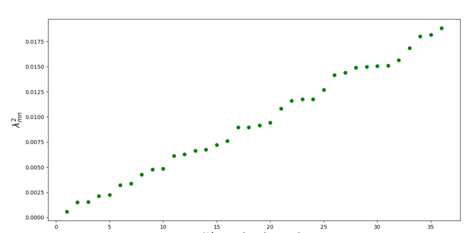


**Fig. 6.** Eigenvalues with $r_{core} = 62$ y $r_{clad} = 125$.

And visualizing our results

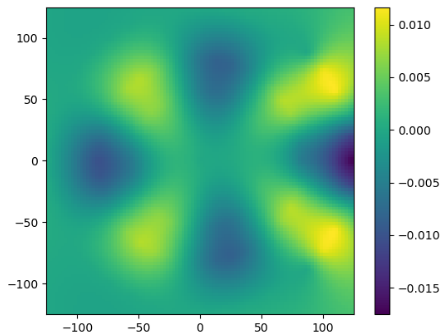


**Fig. 7.** Results with $r_{core} = 2.5$ y $r_{clad} = 125$.

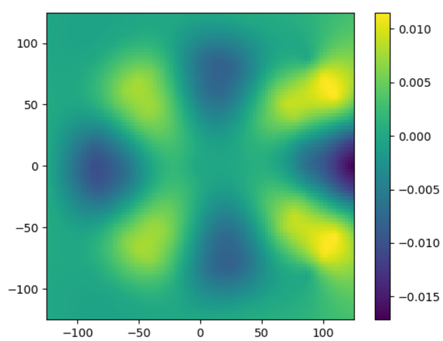


**Fig. 8.** Results with $r_{core} = 12$ y $r_{clad} = 125$.

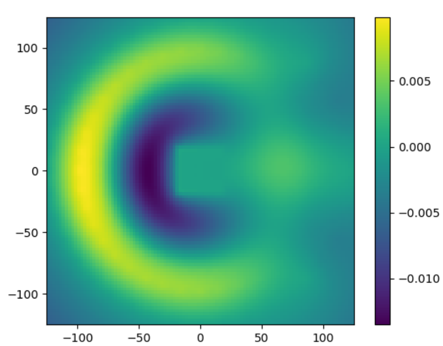


**Fig. 9.** Results with $r_{core} = 50$ y $r_{clad} = 125$.

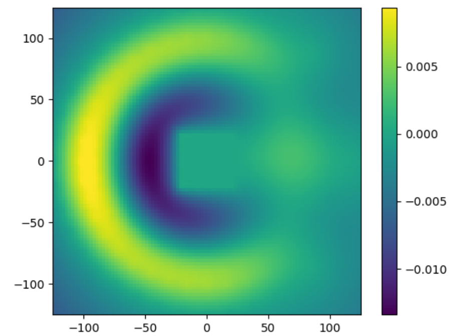


**Fig. 10.** Results with $r_{core} = 62$ y $r_{clad} = 125$.

The form of the solutions should be the one presented in the figure 11.

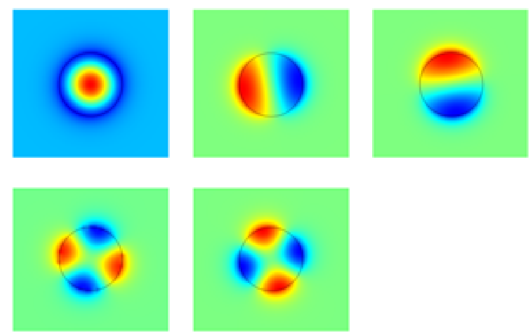

**Fig. 11.** 5 variations of the lowest order step-index fiber mode [2]

Observing these results we can infer that they do not comply with the results obtained by [2], since we can see that multimodes correspond to lower radio values, while monomodes correspond to large radio values.

## 4. CONCLUSIONS

- The proposed eigenvalues problem was successfully solved; obtaining graphical solutions that resemble those that should be observed according to the reviewed literature [2] [1].
- The results obtained do not coincide with those reported in the literature reviewed. It is found that multimodes correspond to lower core radio values and monomodes correspond to higher core radii; which is the opposite relation that should have been found. No further explanations for these results have been found so far.

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