Probabilistic Programming Languages

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Semantics of Probabilistic Programming

Discrete probability (Part II)

Probabilistic PCF - Denotational Semantics

A sound model

Pcoh is a cartesian closed category, CPO-enriched, hence a model of PCF, which is also a model of probabilistic PCF.

Interpretation of terms

$$\left[\!\left[\frac{}{G \vdash \mathtt{bernoulli}(p) : \mathtt{bool\ dist}}\right]\!\right] : \begin{cases}(\gamma, \mathtt{true}) \mapsto p \\ (\gamma, \mathtt{false}) \mapsto 1 - p\end{cases}$$

$$\left[\!\left[\frac{G \vdash d \ : \ t_1 \ \mathtt{dist} \qquad G, x : t_1 \vdash e : t_2}{G \vdash \mathtt{let}\, x = d \,\mathtt{in}\, e \ : \ t_2}\right]\!\right] : (\gamma, a_2) \mapsto$$

$$\sum_a [\![d]\!](\gamma, a)[\![e]\!](\gamma, a, a_2)$$

$$\left[\!\left[\frac{}{G \vdash \mathit{true} : \mathit{bool}}\right]\!\right] : \begin{cases}(\gamma, \mathtt{true}) \mapsto 1 \\ (\gamma, \mathtt{false}) \mapsto 0\end{cases}$$

$$[\![\mathtt{if}\, M \,\mathtt{then}\, N \,\mathtt{else}\, P]\!](\gamma, a) = [\![M]\!](\gamma, \mathtt{true})[\![N]\!](\gamma, a) + [\![M]\!](\gamma, \mathtt{false})[\![P]\!](\gamma, a)$$

Compute the denotational semantics

$$\dfrac{\vdash \text{bern } 0{,}5 : \text{bool dist} \qquad a,b,c : \text{bool} \vdash a+b+c : \text{nat}}{\vdash \underbrace{\text{let } a = \text{sample bern in let } b = \dots \text{ in let } c = \dots \text{ in } a+b+c}_{M} : \text{nat}}$$

bern 0,5	
t	1/2
f	1/2

a	b	c	(a+b+c)	$\mathbb{R}^+$
t	t	t	3	1
t	t	f	2	1
t	t	t	2	0
...				

Bernoulli :

```
let a = sample(bernoulli 0.5) in
let b = sample(bernoulli 0.5) in
let c = sample(bernoulli 0.5) in
a+b+c
```

M : nat	$\mathbb{R}_+$
n	$\sum_{x \in \lvert \text{bool} \rvert} \mathbb{P}(a=x) \sum_{y \in \{t,f\}} \mathbb{P}(a=y) \sum_{z \in \{t,f\}} \mathbb{P}(c=z)\ \mathbb{1}_{a+b+c=n}$
0	$\mathbb{P}(a=f)\,\mathbb{P}(b=f)\,\mathbb{P}(c=f) = \frac{1}{8}$
1	$\mathbb{P}(a=f)\,\mathbb{P}(b=f)\,\mathbb{P}(c=t) + \mathbb{P}(a=f)\,\mathbb{P}(b=t)\,\mathbb{P}(c=f) + \mathbb{P}(a=t)\,\mathbb{P}(b=f)\,\mathbb{P}(c=f) = 3/8$
...	

Examples - Compute the semantics

(2) $a\ b\ c : bool \vdash a=0 \text{ or } b=0 : bool$

(3) $f : nat\ \ a,b,c : bool \vdash a+b+c : nat$

(4) $f : nat, a,b,c : bool \vdash f : nat$

(1) $\vdash$ bern 0.5 : dist bool

(5) $f : nat, a,b,c : bool \vdash$ if $a=0$ or $b=0$ then $a+b+c$ else f

(6)

(7) $f : nat \vdash$ let $a =$ sample bern 0,5 in let $b = \dots$ let $c = \dots$ in if $a=0$ or $b=0$ then $a+b+c$ else f

$\vdash$ fix $\lambda f. \dots$

Funny Bernoulli by rejection sampling :

```
fix lambda f:bool
   let a = sample(bernoulli 0.5) in
   let b = sample(bernoulli 0.5) in
   let c = sample(bernoulli 0.5) in
   if (a==0 or b==0) then a+b+c
                          else f
```

(3) $\cup$ (4) $\Rightarrow$ (5)

(1)

$\lvert bool \rvert$	$\mathbb{R}^+$
t	0,5
f	0,5

Proba Table

Factor

Σ

(2) Relation 0/1.

a b c	a=0 or b=0	$\mathbb{R}^+$
[t][t][ ]	f	1
[t][t][ ]	t	0
[t][f][ ]	t	1
...		...

(3)

a	b	c	(a+b+c)	$\mathbb{R}^+$ (0,1)
[t]	[t]	[t]	3	1
[t]	[t]	[f]	2	1
[t]	[t]	[t]	2	0

(4)

, a b c f	$\mathbb{R}^+$ (0,1)
[n] [ ] [ ] [ ] n	1
otherwise	0

Examples - Compute the semantics

⑥

f	π_6	$\mathbb{R}^+$
$[\,]$	k	$\sum_{x,y,z} \mathbb{P}(a=x)\mathbb{P}(b=x)\mathbb{P}(c=x)\, \mathbb{1}_{a=0 \text{ or } b=0}(x,y)\, \mathbb{1}_{a+b+c=k}(x,y,z)$
$[n]$	k	$\sum_{x,y,z} \mathbb{P}(a=x)\mathbb{P}(b=x)\mathbb{P}(c=x)\, \mathbb{1}_{a=1 \text{ and } b=1}(x,y)\, \mathbb{1}_{k=n} = \frac{1}{4}\mathbb{1}_{k=n}$

Funny Bernoulli by rejection sampling :

```
fix lambda f:bool
  let a = sample(bernoulli 0.5) in
  let b = sample(bernoulli 0.5) in
  let c = sample(bernoulli 0.5) in
  if (a==0 or b==0) then a+b+c
                    else f
```

⑦ F^0: does not use f : $0 \mapsto 1/8$

F^1: uses f at most 1 : $0 \mapsto 1/8 + \frac{1}{4}\cdot F^0 = 1/8\,(1+1/4)$

F_n: uses f at most n times : $0 \mapsto 1/8\,(1+1/4+\dots+1/4^n)$

$\boxed{F = \bigvee F_n : 0 \mapsto \frac{1}{8}\,\frac{1}{3/4} = \frac{2}{6}}$

PCF - Denotational semantics

CPO-enriched : CPO, REL, MREL

$\mathcal{C}(A, B)$ is a complete partial order and $\circ$ is continuous.

$$\left[\!\!\left[\frac{G \vdash e : t \to t}{G \vdash \mathit{fixe} : t} \right]\!\!\right] = \sup F^n \text{ with } \begin{cases} F^0 = \bot \in \mathcal{C}(G, t) \\ F^{n+1} = ev \circ \langle [\![G \vdash e : t \to t]\!], F^n \rangle \end{cases}$$

Probabilistic Language - Type and Semantics

$$[\![G \vdash e \ : \ t]\!] : [\![G]\!] \times [\![t]\!] \to \mathbb{R}^+ \text{ and } [\![G \vdash e \ : \ t \ \texttt{dist}]\!](\gamma, \mathtt{a}) \in \mathrm{P}\,(\mathtt{G} \to \mathtt{t})$$

$$\left[\!\!\left[\frac{}{G \vdash \texttt{bernoulli}(p) : \texttt{bool dist}}\right]\!\!\right] : \begin{cases} (\gamma, \texttt{true}) \mapsto p \\ (\gamma, \texttt{false}) \mapsto 1 - p \end{cases}$$

$$\left[\!\!\left[\frac{G \vdash d \ : \ t_1 \ \texttt{dist} \qquad G, x : t_1 \vdash e : t_2}{G \vdash \texttt{let}\, x = d \,\texttt{in}\, e \ : \ t_2}\right]\!\!\right] : (\gamma, a_2) \mapsto \sum_a [\![e]\!](\gamma, a, a_2)$$

$$\left[\!\!\left[\frac{G \vdash e_1 \ : \ \texttt{bool} \qquad G \vdash e_2 \ : \ t}{G \vdash \texttt{assume}(e_1) \ ; \ e_2 \ : \ t}\right]\!\!\right] (\gamma, a) = \begin{cases} [\![e_2]\!](\gamma, a) & \text{if } [\![e_1]\!](g, true) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\left[\!\!\left[\frac{G \vdash e \ : \ t}{G \vdash \texttt{infer}(e) \ : \ t \ \texttt{dist}}\right]\!\!\right] (\gamma, a) = \frac{[\![G \vdash e \ : \ t]\!](\gamma, a)}{\sum_{a \in |t|} [\![G \vdash e \ : \ t]\!](\gamma, a)}$$

Examples

nat	$\mathbb{R}^+$
$x \in \{1,\dots,6\}$	$1/6$

a	b	(assume $\lvert a-b\rvert=1$)	$\mathbb{R}^+$
x	$x+1$	*	1
x	$x-1$	*	1
otherwise			0

	a	b	Ω	$\mathbb{R}^+$
$x \in \{1 \cdots 5\}$	x	$x+1$	$2x+1$	1
$x \in \{2 \cdots 6\}$	x	$x-1$	$2x-1$	1
	otherwise			0

Dice :

$\vdash \mathrm{Unif}(1,6)$ $a, b \vdash \overbrace{\text{assume } |a-b|=1 \,;\, a+b}^{\Omega} : \text{nat}$

$\vdash \text{Dice} \;;\; \text{nat}$

```
let Dice =
  let a = sample(unif(1,6)) in
  let b = sample(unif(1,6)) in
  assume(|a - b| = 1) ;
  a+b
```

Dice	$\mathbb{R}^+$
k	$\sum_x \mathbb{P}(a=x) \sum_y \mathbb{P}(a=y)\, \mathbb{1}_{\lvert a-b\rvert=1}(x,y)\, \mathbb{1}_{a+b=k}(x,y)$

Cannabis :

```
let Cannabis =
    let b = sample(bernoulli 0.5) in
    if b then let c = sample(bernoulli 0.6) in
              assume c; c
         else true
```

Examples - Compute the semantics

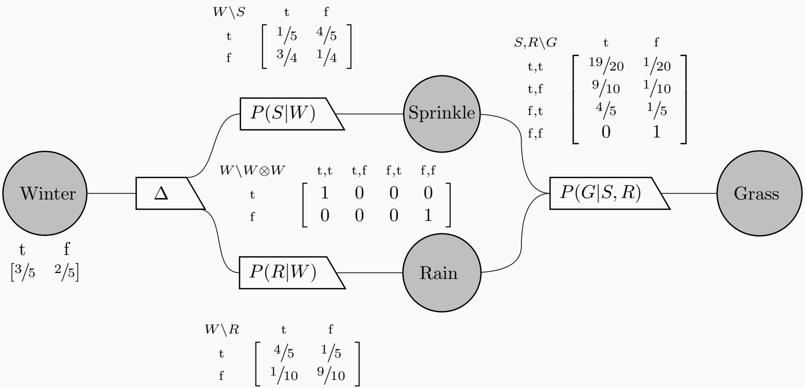
Funny Bernoulli :

```
let FB =
  let a = sample(bernoulli 0.5) in
  let b = sample(bernoulli 0.5) in
  let c = sample(bernoulli 0.5) in
  assume(a==0 or b==0) ;
  a+b+c
```

Semantics of Probabilistic Programming

Graphical Models

Bayesian Network - Example



Compute $\mathbb{P}(G)$ using :

Joint dist. $\mathbb{P}(G,R,S,W) = \mathbb{P}(G|S,R)\ \mathbb{P}(S|W)\ \mathbb{P}(R|W)\ \mathbb{P}(W)$

Marginal $\mathbb{P}(G) = \sum_{(r,s,w)\in|R|\times|S|\times|W|} \mathbb{P}(G,R,S,W)$

Bayesian Network

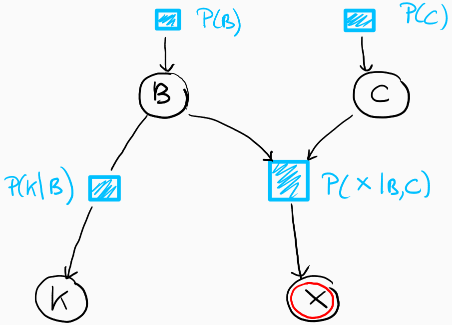
Definition

Directed Acyclic Graph with Conditional Probability Tables
Factors are functions from states to $\mathbb{R}^+$

Computing

Exact inference by variable elimination
Implemented by message passing

Message Passing - Example



Joint distribution

$$P(X, K, B, C) = P(X|B,C)\,P(C)\,P(K|B)\,P(B)$$

Message Passing - Example

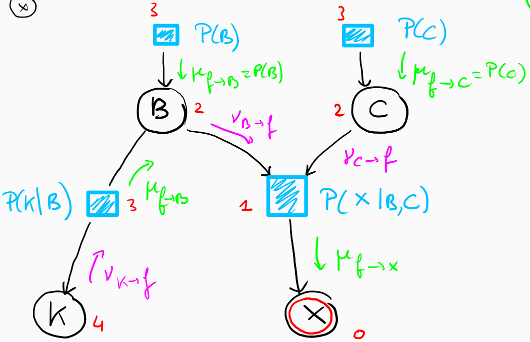
① distance from root X

② Messages from factors

③ Messages from variables

Factors ▨

Variables ⊗



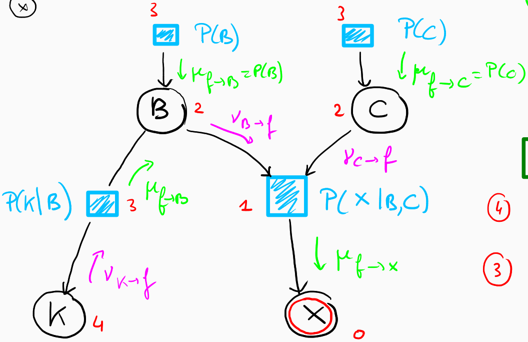
Joint distribution

$$(X, K, B, C) = P(X|B,C)\,P(C)\,P(K|B)\,P(B)$$

Message Passing - Example

Factors $\boxed{}$

Variables $\otimes$



① distance from root X

② Messages from factors

③ Message from variables

Compute From Leaves

④ $\nu_{K\to f} = 1$ (leaf)

③ $\varphi_B = \sum_K P(B)\cdot P(K|B)$

$\varphi_C = P(C)$

② $\varphi = \varphi_B \cdot \varphi_C$

① $\underbrace{\varphi_X}_{\mathbb{P}(X)} = \sum_{B,C} \varphi = \sum_{B,C} \mathbb{P}(C)\cdot \sum_K \mathbb{P}(B)\,\mathbb{P}(K|B)$

Joint distribution & Marginalisation

$$\mathbb{P}(X,K,B,C) = \mathbb{P}(X|B,C)\,\mathbb{P}(C)\,\mathbb{P}(K|B)\,\mathbb{P}(B)$$

$$\mathbb{P}(X) = \sum_{K,B,C} \mathbb{P}(X,K,B,C)$$

Message Passing - Algorithm

Initalisation of leaf nodes :

Messages from leaf node variable are set to 1

Messages from leaf node factors are set to the factor

Propagation

Starting from the further nodes

Variable to factor message : $\nu_{x \to f} = \prod_{g \in \mathtt{ne}(x) \setminus f} \mu_{g \to x}$

Factor to variable message : $\mu_{f \to x} = \sum_{X_f \setminus x} \prod_{y \in \mathtt{ne}(f) \setminus x} \nu_{y \to f}$

Implementation

Dag as an adjacent matrix

Message passing by memoisation

Numerical stability needs logs

Semantics of Probabilistic Programming

Verification of probabilistic Programs

Mc Iver, Morgan, Katoen

Hoare Logic - Principles

Language

$$
\begin{array}{rcl}
e & ::= & \texttt{true} \mid \texttt{false} \mid n \mid x \mid e\ op\ e \qquad \text{states } \Sigma : e \to val \\
op & ::= & + \mid - \mid * \mid = \mid \leq \mid and \mid not \\
s & ::= & \texttt{skip} \mid x := e \mid s; s \mid \texttt{if}\ \ e\ \texttt{then}\ \ s\ \texttt{else}\ \ s \mid \texttt{while}\ \ e\ \texttt{do}\ \ s
\end{array}
$$

Operational semantics

$$
\frac{}{\Sigma, x := e \rightsquigarrow \Sigma\{x \leftarrow \mathrm{ev}_\Sigma(e)\}, \texttt{skip}}
$$

$$
\frac{}{\Sigma, (\texttt{skip}; s) \rightsquigarrow \Sigma, s} \qquad \frac{\Sigma, s_1 \rightsquigarrow \Sigma', s_1'}{\Sigma, (s_1; s_2) \rightsquigarrow \Sigma', (s_1'; s_2)}
$$

$$
\frac{\mathrm{ev}_\Sigma(e) = \texttt{false}}{\Sigma, \texttt{if}\ \ e\ \texttt{then}\ \ s_1 \texttt{else}\ \ s_2 \rightsquigarrow \Sigma, s_1} \qquad \frac{\mathrm{ev}_\Sigma(e) = \texttt{true}}{\Sigma, \texttt{if}\ \ e\ \texttt{then}\ \ s\ \texttt{else}\ \ s \rightsquigarrow \Sigma, s_2}
$$

$$
\frac{\mathrm{ev}_\Sigma(e) = \texttt{false}}{\Sigma, \texttt{while}\ \ e\ \texttt{do}\ \ s \rightsquigarrow \Sigma, (s; \texttt{while}\ \ e\ \texttt{do}\ \ s)} \qquad \frac{\mathrm{ev}_\Sigma(e) = \texttt{true}}{\Sigma, \texttt{while}\ \ e\ \texttt{do}\ \ s \rightsquigarrow \Sigma, \texttt{skip}}
$$

Hoare Logic - Rules

Hoare triple $\{P\}s\{Q\}$ is valid if for any state Σ and Σ' such that $\Sigma, s \rightsquigarrow \Sigma', \texttt{skip}$, $[P]_\Sigma$ valid implies $[Q]_{\Sigma'}$ valid.

Rules

$$\frac{}{\{P[x := E]\}\ x\ :=\ E\ \{P\}} \qquad \frac{\{P\}\ C\ \{Q\} \qquad \{Q\}\ D\ \{R\}}{\{P\}\ C; D\ \{R\}}$$

$$\frac{\{E = \texttt{true}\ \land\ P\}\ C\ \{Q\} \qquad \{E = \texttt{false}\ \land\ P\}\ D\ \{Q\}}{\{P\}\ \texttt{if}\ E\ \texttt{then}\ C\ \texttt{else}\ D\ \{Q\}}$$

$$\frac{\{E = \texttt{true}\ \land\ I\ \}\ C\ \{I\}}{\{I\}\ \texttt{while}\ E\ \texttt{do}\ C\ \texttt{done}\ \{E = \texttt{false}\ \land\ I\}}$$

$$\frac{P' \ \Rightarrow\ P \qquad \{P\}\ C\ \{Q\} \qquad Q\ \Rightarrow\ Q'}{\{P'\}\ C\ \{Q'\}}$$

Soundness

If there is a proof tree with a hoare triple as root, then the program is correct with respect to pre and post conditions.

Weakest Precondition

Definition

Let Q be a predicate on the memory state and C be a program
$\mathrm{WP}(C, \cdot)$ is a predicate transformer, that is
$\mathrm{WP}(C, Q)$ **is the weakset precondition** P **such that** $\langle P \rangle \ \mathtt{C} \ \langle Q \rangle$

Inductive definition

$$\mathrm{WP}(\mathtt{x:=E}, Q(\mathtt{x})) = Q(\mathtt{E})$$
$$\mathrm{WP}(\mathtt{S_1;S_2}, Q) = \mathrm{WP}(\mathtt{S_1}, \mathrm{WP}(\mathtt{S_2}, Q))$$
$$\mathrm{WP}(\mathtt{if\ E\ then\ C\ else\ D}, Q) = (\mathtt{E} \wedge \mathrm{WP}(\mathtt{C}, Q)) \vee (\neg E \wedge \mathrm{WP}(\mathtt{D}, Q))$$
$$\mathrm{WP}(\mathtt{while\ E\ do\ C\ done}\{\mathrm{inv} I, \mathrm{var} V\}, Q) = I$$
$$\text{with} \begin{cases} \mathtt{E{=}true} \wedge I \wedge V = z \Rightarrow \mathrm{WP}(\mathtt{C}, I \wedge V < z) \\ I \Rightarrow V \geq 0 \\ (\mathtt{E{=}false} \wedge I) \Rightarrow Q \end{cases}$$

Examples

$\mathrm{WP}(\texttt{x:=y+3}, x > 3)$ = $x>3\,[x:=y+3]$ = $y+3>3$ = $y>0$.

$\mathrm{WP}(\texttt{n:=n+1}, n > 4)$

$\mathrm{WP}(\texttt{y:=x+2;y:=y-2}, y > 5)$

$\mathrm{WP}(\texttt{if x>2 then y:=1 else y:=-1}, y > 0)$

$\mathrm{WP}(\texttt{while n<>0 do n:=n-1 done } \{I = n \geq 0, V = n\}, n = 0)$

Verifying Probabilistic Programs

Language

$$\begin{array}{rcl} e & ::= & \texttt{true} \mid \texttt{false} \mid n \mid x \mid e\;op\;e \qquad \text{states } \Sigma : e \to val \\ op & ::= & + \mid - \mid * \mid = \mid \leq \mid and \mid not \\ s & ::= & \texttt{skip} \mid x := e \mid s; s \mid \texttt{if } e \texttt{ then } s \texttt{ else } s \mid \texttt{while } e \texttt{ do } s \mid \texttt{bernoul} \end{array}$$

Quantitative predicates : factors associate weights to states

$$\{f | f : |X| \to \mathbb{R}^+ \cup \{\infty\}\}$$

Weakest weighted precondition : $\mathrm{WP}[P]$ transform factors to factors.

Weakest Pre-expectation

Weakest weighted precondition

if f associates scores to states, then $g = \mathrm{WP}[P](f)$ maps a state s to the expected value of f after executing P on s. transform factors to factors.

Inductive definition

$$\mathrm{WP}[\mathtt{x:=E}](f) = f(\mathtt{x:=E}))$$
$$\mathrm{WP}[\mathtt{S}_1;\mathtt{S}_2] = \mathrm{WP}[\mathtt{S}_1](\mathrm{WP}[\mathtt{S}_2](f))$$
$$\mathrm{WP}[\mathtt{if\ E\ then\ C\ else\ D}](f) = [\mathtt{E}]\cdot\mathrm{WP}[\mathtt{C}](f)) + [\neg E]\cdot\mathrm{WP}[\mathtt{D}](f)$$
$$\mathrm{WP}(\mathtt{let\ x\ =\ sample\ bernoulli\ p\ in\ C})(f) = p\cdot\mathrm{WP}[\mathtt{C}](f[x := \mathit{true}])) + (1-p)\cdot\mathrm{WP}[\mathtt{C}](f[x := \mathit{false}])$$
$$\mathrm{WP}[\mathtt{while\ E\ do\ C\ done}](f) = \mathtt{lfp}\ X\cdot\ [\mathtt{E}]\cdot\mathrm{WP}[\mathtt{C}](X)) + [\neg E]\cdot f$$

Weakest Pre-expectation - Example

a=1, b=1

WP[while (a=1 and b=1) do let a = bern 0,5 in let b = bern 0,5 in a+b+c] (res = 0)

WP[while (y=1) { let a = sample bern 0,3 in if a then y := 0 else c := c+1 }] (f = 1)

WP[let a = bern 4/5 in if a then x := 5 else x := 10] (x)

WP[let a = bern 4/5 in if a then x := x+5 else x := 10] (x)

WP[let a = bern 4/5 in if a then x := x+5 else x := 10] (x = 10)

Take home

Semantics for (discrete) probabilistic programs

Probabilistic coherent spaces
Bayesian Networks and message passing
Weakest Predicate Tranformer for probabilistic programs

References

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Probabilistic Weakest Precondition by Katoen & al.