Probabilistic Programming Languages

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Semantics of Probabilistic Programming

Continuous probability

Probabilistic PCF - Discrete Probability

Syntax

$$M, N, P := \underbrace{x \mid \lambda x\, M \mid (M)\, N \mid (M, N)}_{\lambda\text{-calculus}} \mid \underbrace{\mathtt{fix}\, M}_{\text{Recursion}}$$

$$\mid \underbrace{0 \mid \mathtt{succ}\, M}_{\text{Arithmetics}} \mid \underbrace{\mathtt{true} \mid \mathtt{false} \mid \mathtt{if}\, M\, \mathtt{then}\, N\, \mathtt{else}\, P}_{\text{Conditionnal}}$$

$$\mid \underbrace{\mathtt{let\ x = sample(bernoulli}\ p\mathtt{)\ in}\ M}_{\text{Discrete Probability}}\ \forall p \in [0,1]$$

Types

$$\frac{\Gamma, x : \mathtt{bool} \vdash M : B}{\Gamma \vdash \mathtt{let\ x = sample(bernoulli}\ p\mathtt{)\ in}\ M : B}$$

Semantics

Operational : $M \xrightarrow{p} M'$ **Denotational :** $[\![\Gamma \vdash M : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$

Probabilistic Coherent Spaces

Soundness : $[\![\Gamma \vdash M : A]\!] = \sum_{M'} \textbf{Proba}\,(M, M')\, [\![\Gamma \vdash M' : A]\!]$.

Probabilistic PCF - Continuous Probability

Syntax

$$M, N, P := \underbrace{x \mid \lambda x\, M \mid (M)\, N \mid (M, N)}_{\lambda\text{-calculus}} \mid \underbrace{\mathtt{fix}\, M}_{\text{Recursion}}$$

$$\mid \underbrace{\underline{n}, \mid \mathtt{succ}\, M}_{\text{Arithmetics}} \mid \underbrace{\mathtt{true} \mid \mathtt{false} \mid \mathtt{if}\, M\, \mathtt{then}\, N\, \mathtt{else}\, P}_{\text{Conditionnal}}$$

$$\mid \underbrace{\mathtt{Dirac\ M} \mid \mathtt{Uniform\ a:0.\ b:1.} \mid \mathtt{let\ x = sample\ M\ in\ N}}_{\text{Continuous Probability}}$$

$$\mid \underline{r} \mid \underline{f}(M_1, \ldots, M_n)\ \forall r \text{ float } \forall f : \mathbb{R}^n \to \mathbb{R} \text{ measurable}$$

Types

$$\frac{}{\underline{r} : \mathtt{float} \vdash \mathtt{Dirac\ r} : \mathtt{float\ dist}} \qquad \frac{}{\vdash \mathtt{Uniform\ a:0.\ b:1.} : \mathtt{float\ dist}}$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash N : A\ \mathtt{dist}}{\Gamma \vdash \mathtt{let\ x =\ sample}\ N\ \mathtt{in}\ M : B} \qquad \frac{\Gamma \vdash M_i : \mathtt{float\ dist}}{\Gamma, \underline{f} : \mathtt{float}^n \to \mathtt{float} \vdash f(M_1, \ldots, M_n) : \mathtt{float^n\ dist}}$$

Operational Semantics : The evaluation of a program is a markov process described by the probability of reduction from M to N.

Operational Semantics - Discrete Probability

If $\vdash M : \mathtt{nat}$, then $\mathbf{Proba}^\infty(M, \_)$ is the discrete distribution over $\mathbb{N}$ computed by M.

$$\mathbf{Proba}(\mathtt{sample\ Bernoulli\ p}, \underline{0}) = \tfrac{1}{2}$$

Transition Matrix : $\mathbf{Proba}(M, M')$ a stochastic matrix indexed by terms.

$$\mathbf{Proba}(M, M') = \begin{cases} p & \text{if } M \xrightarrow{p} M' \\ 1 & \text{if } M \text{ normal and } M = M' \\ 0 & \text{otherwise.} \end{cases}$$

Iterated Transition Matrix :

$\mathbf{Proba}^k(M, N)$ is the probability that M reduces to N in at most k steps.
$\mathbf{Proba}^\infty(M, N)$ when N is normal is the probability that M reduces to N in any number of steps

Adequation Lemma : If M closed term of type $\mathtt{nat}$,
then for every n, $[\![M]\!]_n = \mathbf{Proba}^\infty(M, \underline{n})$

Operational Semantics - Continuous Probability

If $\vdash M : \texttt{float}$, then $\mathbf{Proba}^{\infty}(M, \_)$ is the continuous distribution over $\mathbb{R}$ computed by M. $\mathbf{Proba}(\texttt{sample Uniform a} : 0.\ \texttt{b} : 1., U) = \int_{x \in U} dx$

The probability to observe U after at most one reduction step applied to M is **Proba**(M , U)

$\mathbf{Proba} : \Lambda^{\Gamma \vdash A} \times \Sigma_{\Lambda^{\Gamma \vdash A}} \to \mathbb{R}^{+}$ is a stochastic **Kernel**, i.e :

for all $M \in \Lambda^{\Gamma \vdash A}$, $\mathbf{Proba}(M, \_)$ is a measure ;

for all $U \in \Sigma_{\Lambda^{\Gamma \vdash A}}$, $\mathbf{Proba}(\_, U)$ is a measurable function.

$\mathbf{Proba}^{\infty}(M, U)$ is the probability to observe U after any steps.

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$\Lambda^{\Gamma \vdash A}$: the set of terms M s.t. $\Gamma \vdash M : A$.

$\textbf{Proba} : \Lambda^{\Gamma \vdash A} \times \Sigma_{\Lambda^{\Gamma \vdash A}} \to \mathbb{R}^+$ is a stochastic **Kernel**, i.e :

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$\Sigma_{\Lambda^{\Gamma\vdash A}}$, i.e. U is measurable : $\forall n, \forall S,\ \{\vec{r}\ s.t.\ S\underline{\vec{r}} \in U\}$ meas. in $\mathbb{R}^n$

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Proba : $\Lambda^{\Gamma \vdash A} \times \Sigma_{\Lambda^{\Gamma \vdash A}} \to \mathbb{R}^+$ is a stochastic **Kernel**, i.e :

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- for all $U \in \Sigma_{\Lambda^{\Gamma \vdash A}}$, $\mathbf{Proba}(\_, U)$ is a measurable function.

Measurable sets and kernels constitute the category Kern.

$\mathbf{Proba}^{\infty}(M, U)$ is the probability to observe U after any steps.

Examples : Distributions

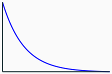
```
sample = sample Uniform a:0. b:1.
```

The Bernoulli distribution takes the value 1 with probability p and the value 0 with probability $1 - p$.

$$p\delta_1 + (1-p)\delta_0$$

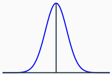
$\texttt{bernoulli}\ p ::= \texttt{let}\, x{=}\texttt{sample}\, \texttt{in}\, x \leq p$
tests if `sample` draws a value within $[0, p]$.

The exponential distribution is specified by its density e^{-x}.



$\texttt{exp} ::= \texttt{let}\, x{=}\texttt{sample}\, \texttt{in} - \log(x)$
by the inversion sampling method.

The standard normal distribution defined by its density $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}$.



$\texttt{gauss} ::= \texttt{let}\, x{=}\texttt{sample}\, \texttt{in}$
$\texttt{let}\, y{=}\texttt{sample}\, \texttt{in}\, \sqrt{-2\log(x)}\cos(2\pi y)$
by the Box Muller method.

Denotational Semantics - First order

Discrete : **PCOH**

For $\vdash M : \texttt{nat}$,
$[\![M]\!]$ a distribution over $\mathbb{N}$

For $\vdash \underline{n} : \texttt{nat}$,
$[\![\underline{n}]\!]_p = \delta_{p,n}$

For $\vdash \texttt{Bernoulli p} : \texttt{bool}$,

$[\![\mathrm{Bern\ p}]\!]_{\mathrm{b}} = \mathrm{p}\delta_{\mathrm{true,b}} + (1-\mathrm{p})\delta_{\mathrm{false,b}}$

For $\vdash N : \texttt{bool}$, $\vdash P : A$, $\vdash Q : A$,

$$[\![\textbf{if } N \textbf{ then } P \textbf{ else } Q]\!]_a = [\![N]\!]_{\texttt{true}}[\![P]\!]_a + [\![N]\!]_{\texttt{false}}[\![Q]\!]_a$$

For $\vdash N : \texttt{nat}$, $\vdash P : A$, $\vdash Q : A$,

$$[\![\texttt{let}\, x{=}N \,\texttt{in}\, P]\!]_a = \sum_{n=0}^{\infty} [\![N]\!]_n \widehat{[\![P]\!]}(n)_a$$

Denotational Semantics - First order

Discrete : **PCOH**

For $\vdash M : \mathtt{nat}$,
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For $\vdash N : \mathtt{bool}$, $\vdash P : A$, $\vdash Q : A$,
$[\![\textbf{if } N \textbf{ then } P \textbf{ else } Q]\!]_a =$
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Continuous : **KERN**

For $\vdash M : \mathtt{float}$,
$[\![M]\!]$ a measure over $\mathbb{R}$

For $\vdash \underline{r} : \mathtt{float}$,
$[\![\underline{r}]\!](U) = \delta_r(U)$

For $\vdash \mathtt{Uniform\ a : 0.\ b : 1. : float}$,

$$[\![\mathtt{Unif\ 0.\ 1.}]\!](\mathtt{U}) = \int_{\mathtt{x}\in\mathbb{R}} \mathbb{1}_{\mathtt{U}}(\mathtt{x})\mathtt{dx}$$

For $\vdash R : \mathtt{bool}$, $\vdash P, Q : A$,
$[\![\textbf{if } R \textbf{ then } P \textbf{ else } Q]\!](U) =$
$[\![R]\!](\{\mathtt{true}\})[\![\mathtt{P}]\!](\mathtt{U}) + [\![\mathtt{R}]\!](\{\mathtt{false}\})[\![\mathtt{Q}]\!](\mathtt{U})$

For $\vdash R : \mathtt{float}$, $\vdash P, Q : A$,
$[\![\mathtt{let}\, x{=}R \,\mathtt{in}\, P]\!](U) =$

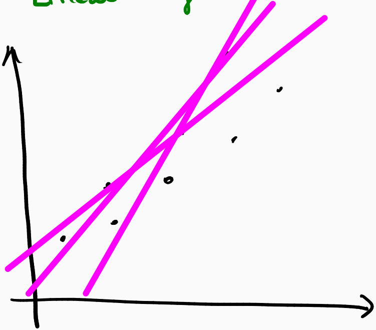
$$\int [\![R]\!](dr)[\![P]\!](\delta_r)(U)$$

Denotational Semantics - Higer-Order

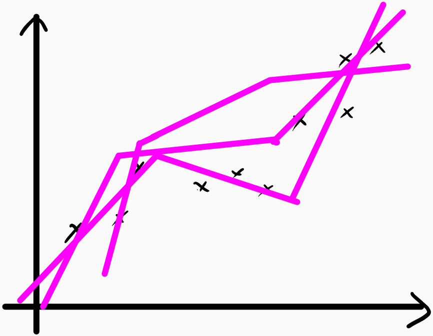
The Program Synthesis problem

Given a data set in R^2 can we build the probabilistic program $\vdash M$: `float -> float` that generated this data set ? We need a distribution over `float -> float`.

Linear regression



Piecewise linear Regression



Denotational Semantics - Higher-Order problem

Théorème (Aumann' 61)

There is no σ-algebra on $\mathbb{R}^{\mathbb{R}}$ such that **eval** : $\mathbb{R}^{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ is measurable.

By contradiction, assume for all Z, Y measurable space, Z^Y measurable and eval measurable:

$$\begin{tikzcd} X \times Y \arrow[d, "\Lambda h \times id"'] \arrow[dr, bend left, "h"] & \\ \mathbb{R}^Y \times Y \arrow[r, "eval"'] & \mathbb{R} \end{tikzcd}$$

$h : x, y \mapsto \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

Assume $X = \mathbb{R}, \mathcal{P}(\mathbb{R})$ and $Y = \mathbb{R}, \mathcal{C}(\mathbb{R})$

all subsets of $\mathbb{R}$

countable or cocountable subsets of $\mathbb{R}$

Then $X \times Y = \mathbb{R} \times \mathbb{R}, \mathcal{P}(\mathbb{R}) \otimes \mathcal{C}(\mathbb{R})$

σ-algebra generated by $U \times V$ by
- complement
- countable unions
- countable intersections

Denotational Semantics - Higher-Order problem

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There is no σ-algebra on $\mathbb{R}^{\mathbb{R}}$ such that **eval** : $\mathbb{R}^{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ is measurable.

$$\begin{tikzcd} X \times Y \arrow[d, "\Lambda h \times id"'] \arrow[dr, bend left, "h"] & \\ \mathbb{R}^Y \times Y \arrow[r, "ev"'] & \mathbb{R} \end{tikzcd}$$

$h : x, y \mapsto \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

. $\Lambda h : X \to \mathbb{R}^Y$ is measurable: $\forall A \in \sigma(\mathbb{R}^Y)$, $\Lambda h^{-1}(A) \in \mathcal{P}(\mathbb{R})$!

is well defined: $\Lambda(h)(x) : Y \to \mathbb{R}$ measurable

$y \mapsto \begin{cases} 1 & x = y \\ 0 & \text{otherwise} \end{cases}$

$\Lambda(h)(x)^{-1}(\{1\}) = \{x\}$ countable $\in \mathcal{C}(\mathbb{R})$

$\Lambda(h)(x)^{-1}(\{0\}) = \mathbb{R} \setminus \{x\}$ co countable $\in \mathcal{C}(\mathbb{R})$

. We assumed that ev is measurable, hence $h = ev \circ (\Lambda h \times id)$ is measurable

Denotational Semantics - Higher-Order problem

Théorème (Aumann' 61)

There is no σ-algebra on $\mathbb{R}^{\mathbb{R}}$ such that **eval** : $\mathbb{R}^{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ is measurable. NO

If h is measurable then so is

$h : x, y \mapsto \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

$h^{-1}(\{1\}) = \{(x,x) \mid x \in \mathbb{R}\} \notin \mathcal{P}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$

$X \times Y = \mathbb{R} \times \mathbb{R}, \quad \mathcal{P}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$

σ-algebra generated by $U \times V$ by
- complement
- countable unions
- countable intersections

(P) If $W \in \mathcal{P}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ then $\exists (b_n) \in \mathbb{R}^{\mathbb{N}}$ st if $\underline{(x,y) \in W}$ and $y \notin b_n$ then $\forall z \notin b_n$, $\underline{(x,z) \in W}$

$\Delta = \{(x,x) \mid x \in \mathbb{R}\}$ does not satisfy P : $\forall (b_n) \in \mathbb{R}^{\mathbb{N}}$, $\forall y \notin b_n$, $(y,y) \in \Delta$ $\overset{\exists z \neq y}{(y,z) \in \Delta}$!

If $U \in \mathcal{P}(\mathbb{R})$, $C \in \mathcal{B}(\mathbb{R})$, then $U \times C$ satisfies P | If W_n satisfy P, then $\cup W_n$ and $\cap W_n$ satisfy P.

Semantics of Probabilistic Programming

Measurable Stable Cones

Measurable Stable Cones - Definition

CSTAB is a **CCC** based on Selinger's **cones** (dcpos with the order induced by addition and a convex structure).

Objects are cones and measurable spaces

Morphisms are stable and measurable functions

1. **Complete cones** (convex dcpos with the order induced by addition) with Scott continuous functions
   However, the category is cartesian but not closed.
2. Complete cones and **Stable functions** (∞-non-decreasing functions) is a CCC.
   However, not every stable function is measurable.
3. **Measurable Cones** (complete cones with **measurable tests**). Measurable paths pass measurable tests and Measurable functions preserve measurable paths.
   CSTAB is a CCC with measurability included !

Results

The category **CSTAB** is a CCC and a model of Real PPCF.

Invariance of the semantics

$$[\![M]\!]_{\Gamma\vdash A} = \int_{\Lambda^{\Gamma\vdash A}} [\![t]\!]_{\Gamma\vdash A}\, \mathbf{Proba}(M, dt)$$

Adequacy

$$[\![M]\!]_{\vdash\mathtt{real}}(U) = \mathbf{Proba}^{\infty}(M, U)$$

Conservative extension of PCOH

If X and Y are probabilistic coherence spaces and $f : X \to Y$ is a stable measurable map, then f is a map of probabilistic coherence spaces. (Crubille 2018).

Step 1 : Complete Cones

A Cone P is analogous to a real normed vector space, except that scalars are $\mathbb{R}^+$ and the norm $\|\_\|_P : P \to \mathbb{R}^+$ satisfies :

$$x + y = 0 \Rightarrow x, y = 0, \qquad \|x + x'\|_P \leq \|x\|_P + \|x'\|_P, \qquad \|\alpha x\|_P = \alpha\|x\|_P$$
$$x + y = x + y' \Rightarrow y = y', \qquad \|x\|_P = 0 \Rightarrow x = 0, \qquad \|x\|_P \leq \|x + x'\|_P$$

The Unit Ball is the set $\mathcal{B}P = \{x \in P \mid \|x\|_P \leq 1\}$.

Order $x \leq_P x'$ if there is a $y \in P$ such that $x' = x + y$. This unique y is denoted as $y = x' - x$.

A Complete Cone is s.t. any non-decreasing $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{B}P$ has a lub and $\|\sup_{n\in\mathbb{N}} x_n\|_P = \sup_{n\in\mathbb{N}} \|x_n\|_P$.

Example of Complete Cones

Meas(X) with X a measurable space.
$\widehat{\mathcal{X}} = \{u \in (\mathbb{R}^+)^{|\mathcal{X}|} \mid \exists\epsilon > 0 \ \epsilon u \in \mathbf{PCOH}\mathcal{X}\}$ if $\mathcal{X} \in \mathbf{PCOH}$.

Step 2 : Stable functions

The category of **complete cones** and **Scott-continuous** functions is not cartesian closed as *currying* fails to be *non-decreasing*.

A function $f : \mathcal{B}P \to Q$ is **n-non-decreasing function** if :

$n = 0$ and f is non-decreasing

$n > 0$ and $\forall u \in \mathcal{B}P$, $\Delta f(x; u) = f(x+u) - f(x)$ is $(n-1)$-non-decreasing in x.

A function is **stable** if it is Scott-continuous and ∞-non-decreasing, i.e. n-non-decreasing for all $n \in \mathbb{N}$.

Complete cones and **stable** functions constitute a **CCC**.

Weak Parallel Or

wpor : $[0,1] \times [0,1] \to [0,1]$ given as $\mathrm{wpor}(s,t) = s + t - st$ is Scott-continuous, but not Stable. Its currying is not Scott-continuous.

Step 3 : The Measurability Problem

Type $\texttt{real}$ is interpreted as $[\![\texttt{real}]\!] = \mathrm{Meas}(\mathbb{R})$,
Closed term $\vdash M : \texttt{real}$ as a measure μ and
Term $x : \texttt{real} \vdash N : \texttt{real}$ as a stable $f : \mathrm{Meas}(\mathbb{R}) \to \mathrm{Meas}(\mathbb{R})$.

Operational semantics

$$\forall r,\ \text{s.t. } M \to r,\ \texttt{let}\, x{=}M \,\texttt{in}\, N \to N\{r/x\}$$

By **Soundness**

$$[\![\texttt{let}\, x{=}M \,\texttt{in}\, N]\!] = \int_{\mathbb{R}} (\ f \ \circ \ \delta\ )(r) \ \ \mu\,(dr)$$

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$[\![N]\!]$

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$[\![N]\!]$ Dirac measure

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Operational semantics

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By **Soundness**

$$[\![\texttt{let}\, x{=}M \,\texttt{in}\, N]\!] = \int_{\mathbb{R}} (\underset{[\![N]\!]}{f} \circ \underset{\text{Dirac measure}}{\delta})(r)\ \underset{[\![M]\!]}{\mu}\,(dr)$$

Step 3 : The Measurability Problem

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Operational semantics

$$\forall r,\ \text{s.t. } M \to r,\ \mathtt{let}\, x{=}M\, \mathtt{in}\, N \to N\{r/x\}$$

By **Soundness**

$$[\![\mathtt{let}\, x{=}M\, \mathtt{in}\, N]\!] = \int_{\mathbb{R}} (\ f \ \circ\ \delta\ )(r)\ \ \mu\ (dr)$$

Thus $f \circ \delta$ needs to be measurable.

There are non measurable stable functions
We need to equip every cone with a notion of measurability

Step 3 : Measurability tests

Measurability tests of $\mathrm{Meas}(\mathbb{R})$ are given by measurable sets of $\mathbb{R}$:

$$\forall U \subseteq \mathbb{R} \text{ measurable}, \ \epsilon_U \in \mathrm{Meas}(\mathbb{R})' : \mu \mapsto \mu(U)$$

For needs of CCC, we parameterized measurable tests of a cone :

Measurable Cone

A cone P with a collection $(\mathrm{M}^n(P))_{n\in\mathbb{N}}$ with $\mathrm{M}^n(P) \subseteq (P')^{\mathbb{R}^n}$ s.t. :

$$0 \in \mathrm{M}^n(P), \quad \ell \in \mathrm{M}^n(P) \text{ and } h : \mathbb{R}^p \to \mathbb{R}^n \Rightarrow \ell \circ h \in \mathrm{M}^p(P)$$

$$\ell \in \mathrm{M}^n(P) \text{ and } x \in P \Rightarrow \left\{ \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^+ \\ \mathrm{Vect}\, r & \mapsto & \ell(\mathrm{Vect}\, r)(x) \end{array} \right. \quad \text{measurable.}$$

Measurable Tests, Paths and Functions

CSTAB is the category of complete and measurable cones with stable and measurable functions.

Let P and Q be measurable and complete cones :

Measurable Test : $\mathrm{M}^n(P) \subseteq (P')^{\mathbb{R}^n}$
Measurable Path : $\mathrm{Path}^n(P) \subseteq P^{\mathbb{R}^n}$ the set of bounded $\gamma : \mathbb{R}^n \to P$
such that $\ell * \gamma : \mathbb{R}^{k+n} \to \mathbb{R}^+$ is measurable with

$$\ell * \gamma : (\mathrm{Vect}\, r, \mathrm{Vect}\, s) \mapsto \ell(\mathrm{Vect}\, r)(\gamma(\mathrm{Vect}\, s))$$

Measurable Functions : Stable functions $f : P \to Q$ such that :

$$\forall n \in \mathbb{N},\ \forall \gamma \in \mathrm{Path}_1^n(P), \quad f \circ \gamma \in \mathrm{Path}^n(Q)$$

If X is a measurable space, then $\mathrm{Meas}(X)$ is equipped with :
$\mathrm{M}^n(X) = \{\epsilon_U : \mathbb{R}^n \to \mathrm{Meas}(X)' \text{ s.t. } \epsilon_U(\mathrm{Vect}\, r)(\mu) = \mu(U),\ U \text{ meas.}\}$
$\mathrm{Path}_1^n(P)$ is the set of stochastic kernels from $\mathbb{R}^n$ to X.

Semantics of Probabilistic Programming

Quasi Borel Spaces

(Ohad Kammar Tutorial)

Measurable Stable Cones - Definition

Quasi Borel Space $X = (|X|, \mathcal{R}(X))$ such that

Random elements : $\mathcal{R}(X) \subset \mathbb{R} \to |X|$
Constants : if $x \in |X|$, then $\lambda r.x \in \mathcal{R}(X)$
Precomposition : if $\alpha \in \mathcal{R}(X)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ measurable, then $\varphi \circ \alpha \in \mathcal{R}(X)$.
Recombination : if $\alpha \in \mathcal{R}(X)^{\mathbb{N}}$ and $\mathbb{R} = \uplus A_n$ and A_n measurable, then $\lambda r.\alpha_n(r)$ (if $r \in A_n$) $\in \mathbb{R}(X)$

Examples

Reals : $(\mathbb{R}, \mathcal{M}eas(\mathbb{R}, \mathbb{R}))$
Discrete QBS : $(|X|, \sigma - \text{simple}(\mathbb{R}, |X|))$
Indiscrete QBS : $(|X|, |X|^{\mathbb{R}})$

Measurable Stable Cones - Definition

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Morphism $f : X \to Y$

Function $f : |X| \to |Y|$ such that
If $\alpha \in \mathcal{R}(X)$, then $f \circ \alpha \in R(Y)$

Measurable Stable Cones - Properties

QBS is a category

Cartesian : $|X \times Y| = |X| \times |Y|$ and
$\mathcal{R}(X \times Y) = \{\lambda r.(\alpha(r), \beta(r)) \mid \alpha \in \mathcal{R}(X), \beta \in \mathcal{R}(Y)$
Closed : $\left|Y^X\right| = \mathbf{QBS}(X, Y)$ and
$\mathcal{R}(Y^X) = \{\alpha \mid \lambda(r, x).\alpha(r)(x) \in \mathbf{QBS}(\mathbb{R} \times X \to Y)\}$
Limits : Coproducts, Quotients, ... as in Sets

QBS is a **conservative extension** of Standard Borel Sets

One uniform distribution is sufficient to generate all probability measures on Borel spaces.
if $\vdash d : \texttt{X dist}$, then there is $\alpha \in \mathcal{R}(X)$ such that
`sample d` $\sim$ `let r = sample uniform a:0. b:1. in` α(`r`)

Measure μ on a QBS is a borel space Σ, a random element $\alpha \in \mathcal{R}(X)$ and a measure on Σ. If $f : X \to \mathbb{R}^+$, then its integral with respect to μ :

$$\int_X \mu f = \int_\Sigma \mu(dr)(f(\alpha(r))$$

Take home

Semantics for (discrete) probabilistic programs

The operational semantics of continuous probability using kernels

The category **Meas** is not a CCC.

The Measurable Cones solution.

The Quasi Borel Spaces solution.

Both are sound models of probabilistic higher order programs

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