Probabilistic Programming Languages

Reactive Probabilistic Programming

Guillaume Baudart

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Uncertainty in embedded systems

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Synchronous languages

- High-level specification language
- Generate correct-by-construction embedded code
- Industrial tool: ANSYS Scade

Challenges

- Noisy environment, perceived through noisy sensors
- Interaction with other autonomous entities

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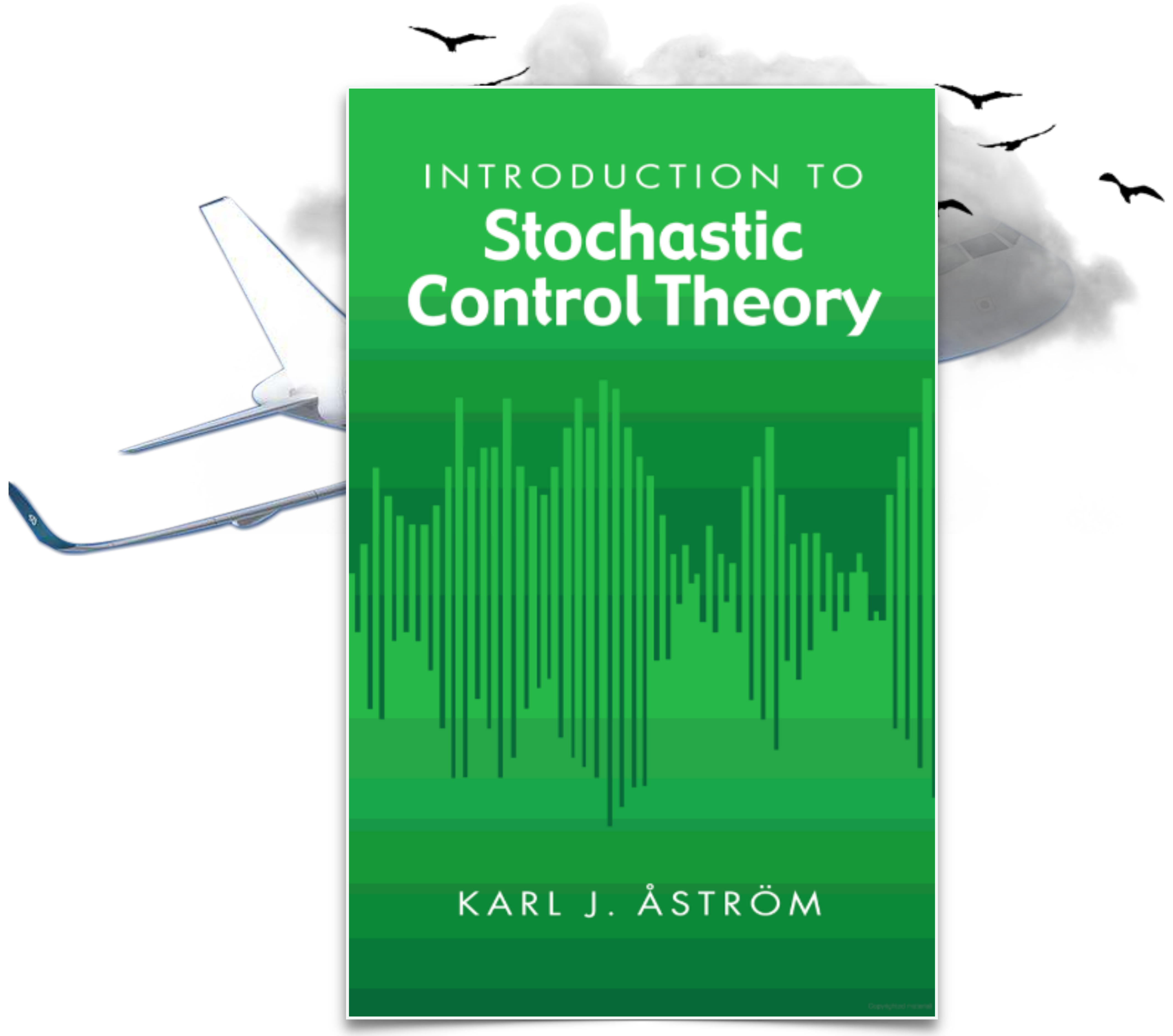
Existing approaches

- Manually implement stochastic controller: *Can be error prone*
- Offline statistical tests: *Requires up-to-date offline data*



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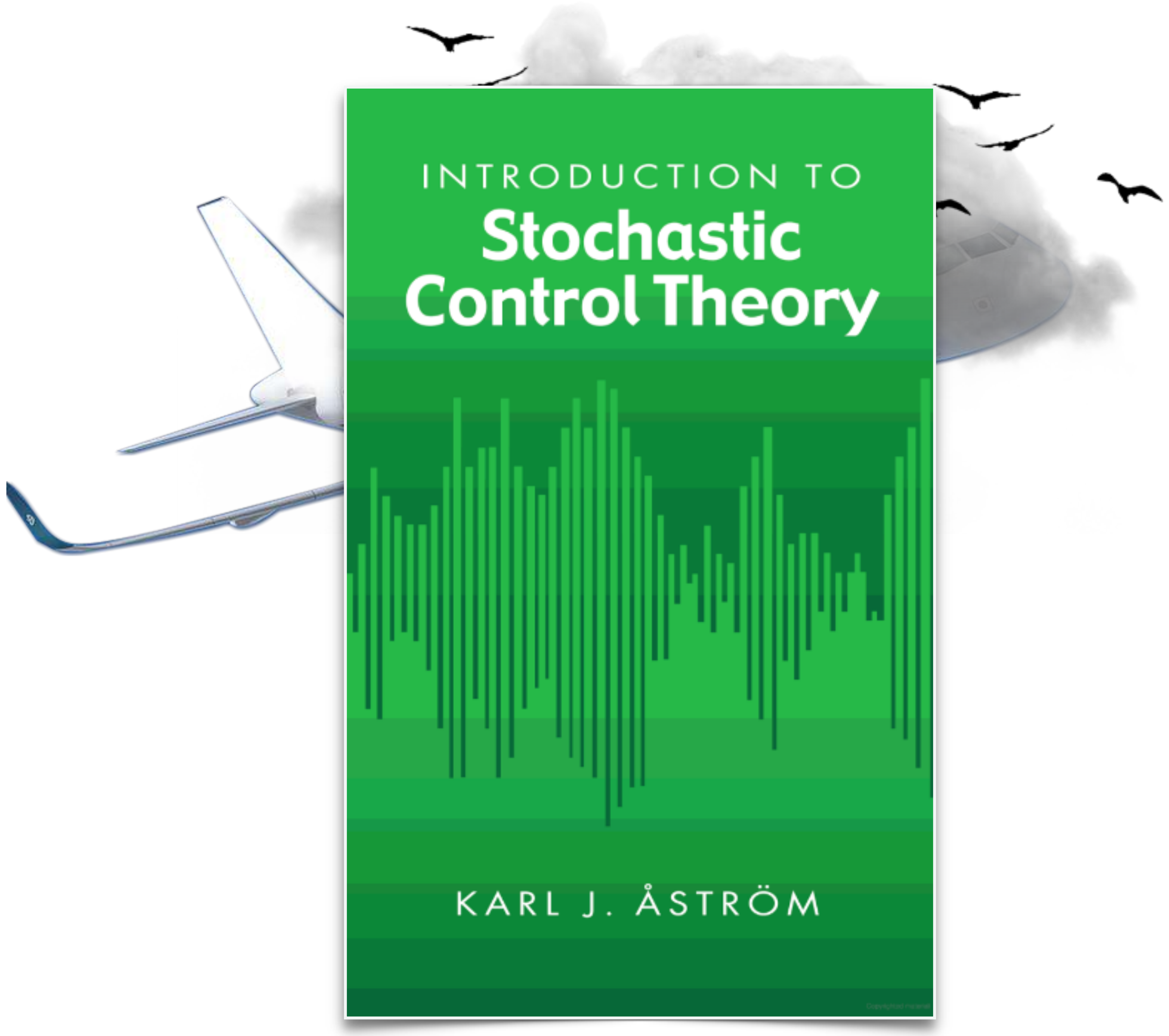
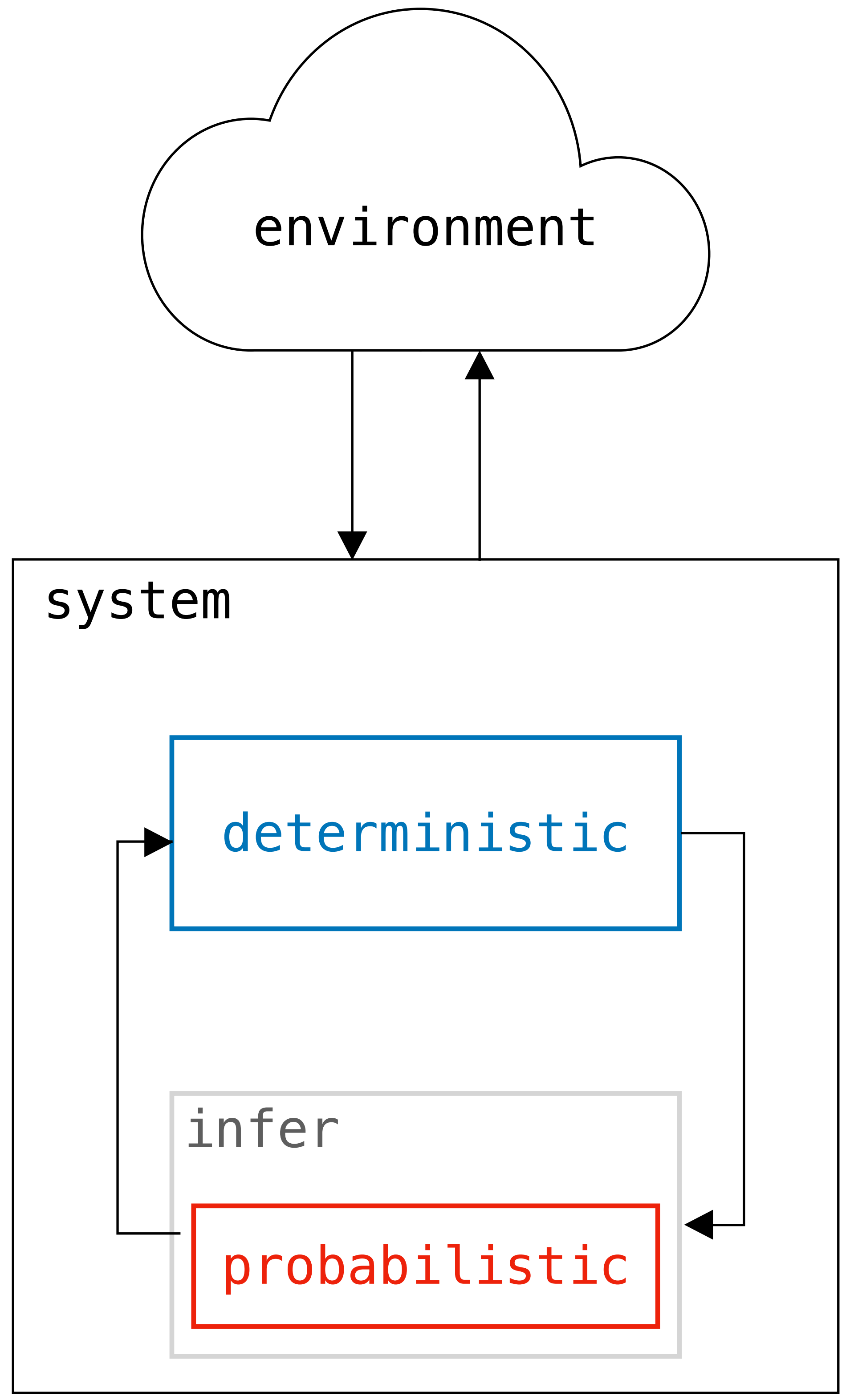
Reactive Probabilistic Programming

- Synchronous languages with probabilistic constructs
- Make the probabilistic model explicit
- Automatically learn posterior distributions from observations

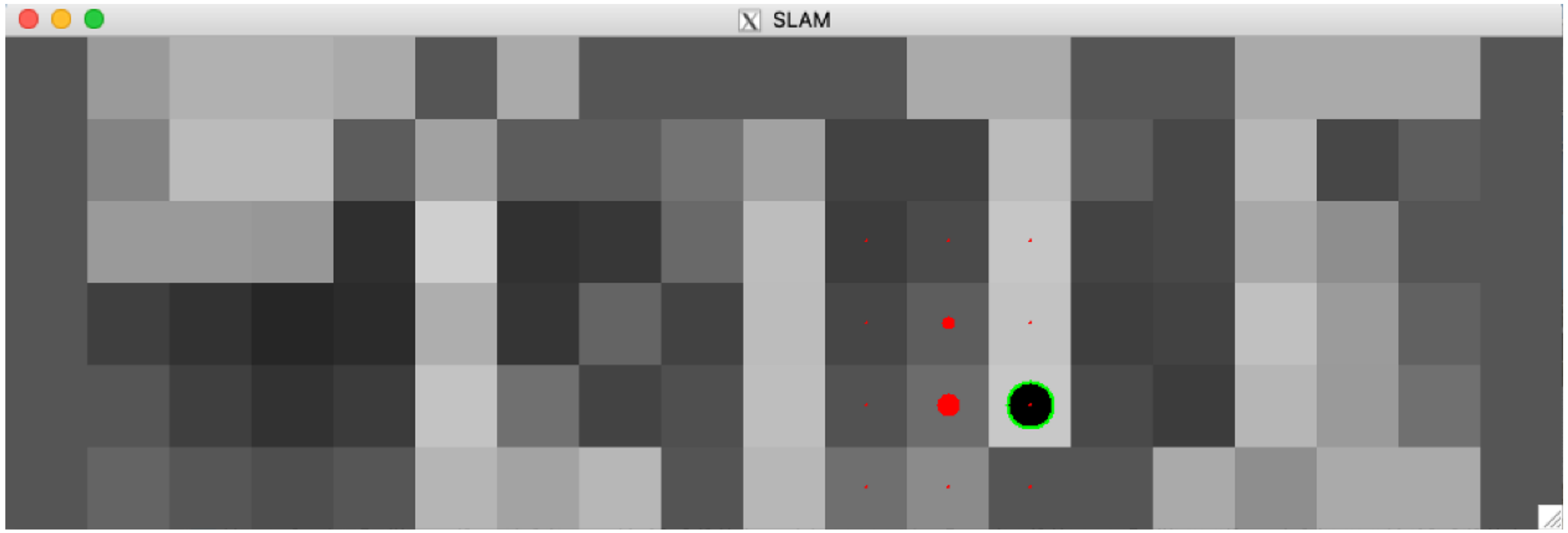


Reactive Probabilistic Programming (Demo)



Simultaneous **L**ocalization **A**nd **M**apping

- Environment: slippery wheels and noisy color sensor
- System: infer current position and map, output command (left/right/up/down)



At each step:

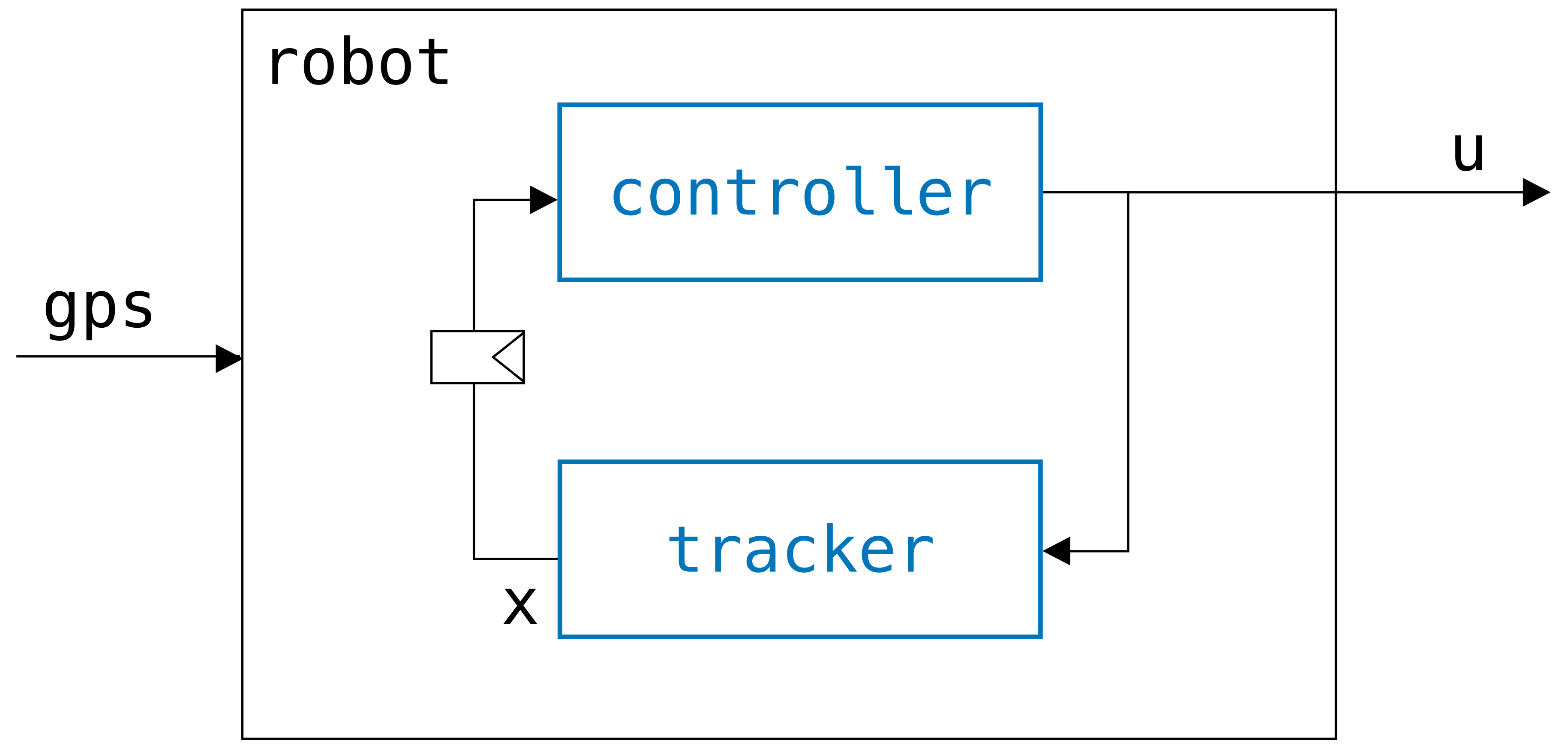
- Move to the next position
- Observe the color of the ground
- Use inferred position to compute next command

●	`exact position + color sensor`
■	`estimated color of a map cell`
🔴	`estimated position`

Reactive systems

Synchronous data-flow languages and block diagrams

- Signal: stream of values
- System: stream processor



```
let node robot (gps) = u where
  rec u = controller (x0 → pre x)
  and x = tracker (u, gps)
```

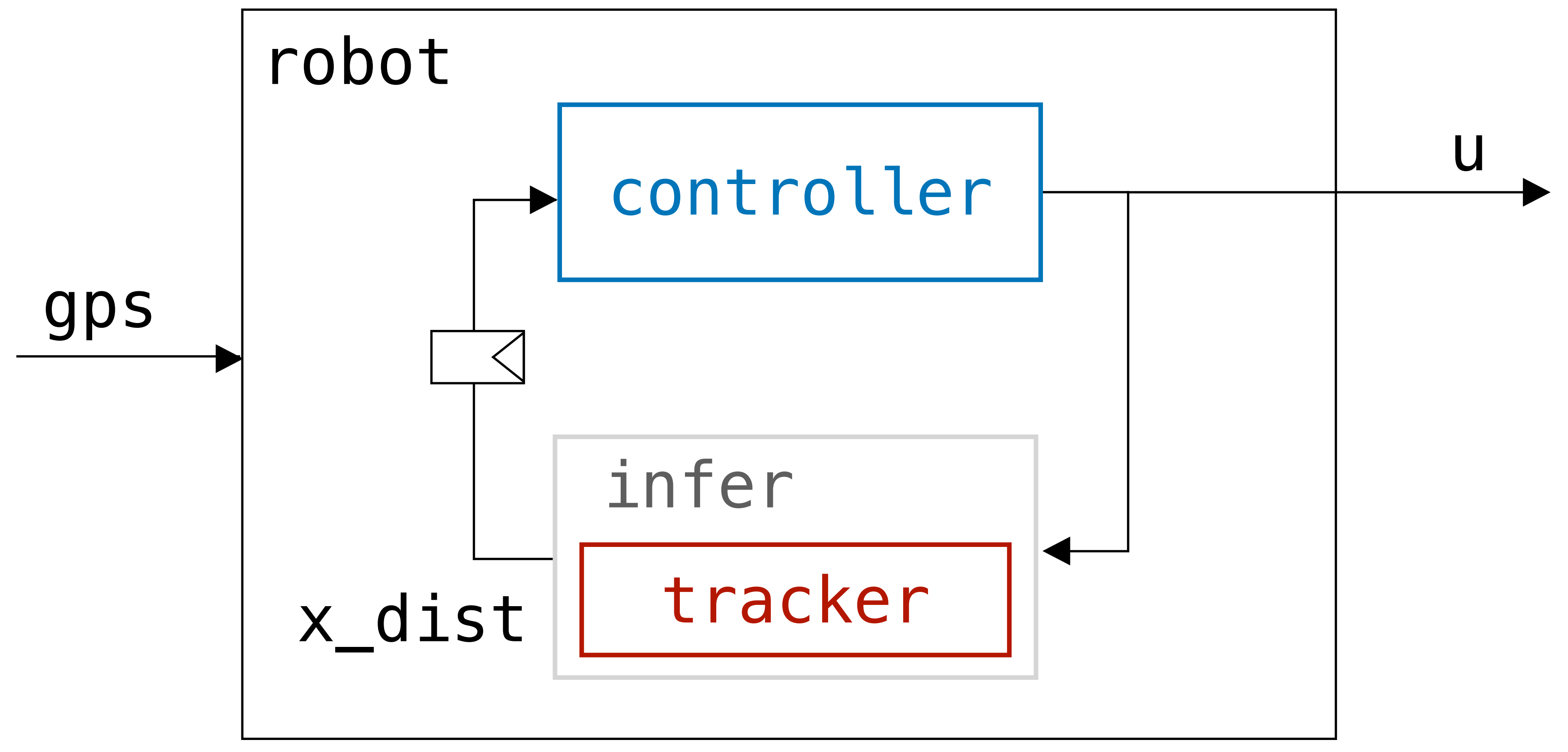
Reactive probabilistic systems

Synchronous data-flow languages and block diagrams

- Signal: stream of values
- System: stream processor

ProbZelus: add support to deal with uncertainty

- Extend a synchronous language
- Parallel composition: deterministic/probabilistic
- Inference-in-the-loop
- Streaming inference



```
let proba robot (gps) = u where
  rec u = controller (x0_dist → pre x_dist)
  and x_dist = infer tracker (u, gps)
```

Synchronous programming

Reactive Probabilistic Programming

`Lustre → Lucid Synchrone → Zelus → ProbZelus`

Dataflow synchronous programming

- Set of stream equations
- Discrete logical time steps
- At each step, compute the current value given inputs and previous values

Stream operations

- Constant are lifted to stream: `1 = 1, 1, 1, ....`
- Temporal operators: `->`, `pre`, `fby`
- Control structures: `reset/every`, `present`, `automaton`

Bourke, Pouzet 2013

Lustre → Lucid Synchrone → Zelus → ProbZelus

Dataflow synchronous programming

- Set of stream equations
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```
node nat v = cpt where
  rec cpt = v → pre cpt + 1
```

$$cpt_n = if\ (n = 0)\ then\ v_0\ else\ cpt_{n-1} + 1$$

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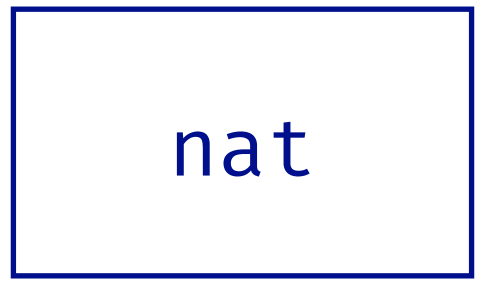

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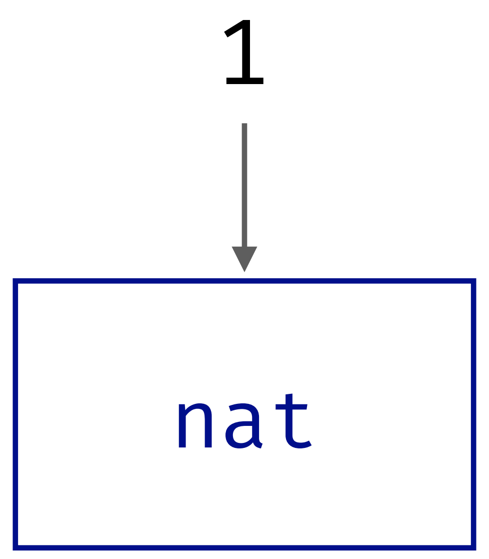

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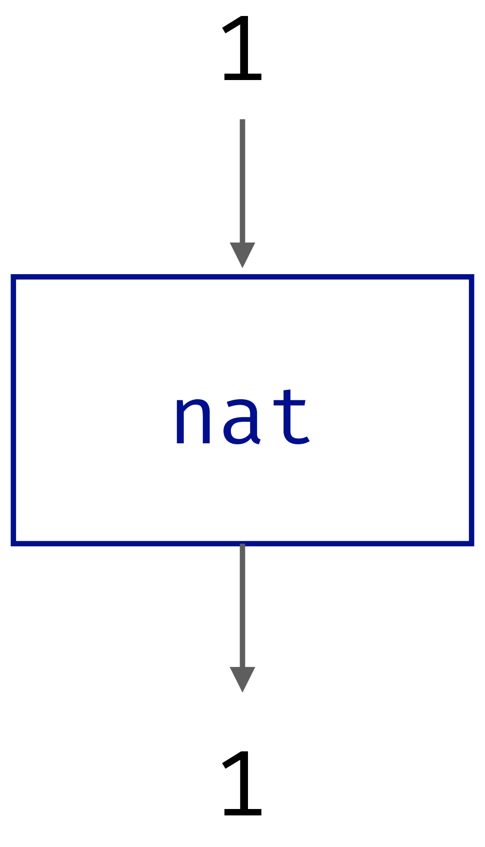

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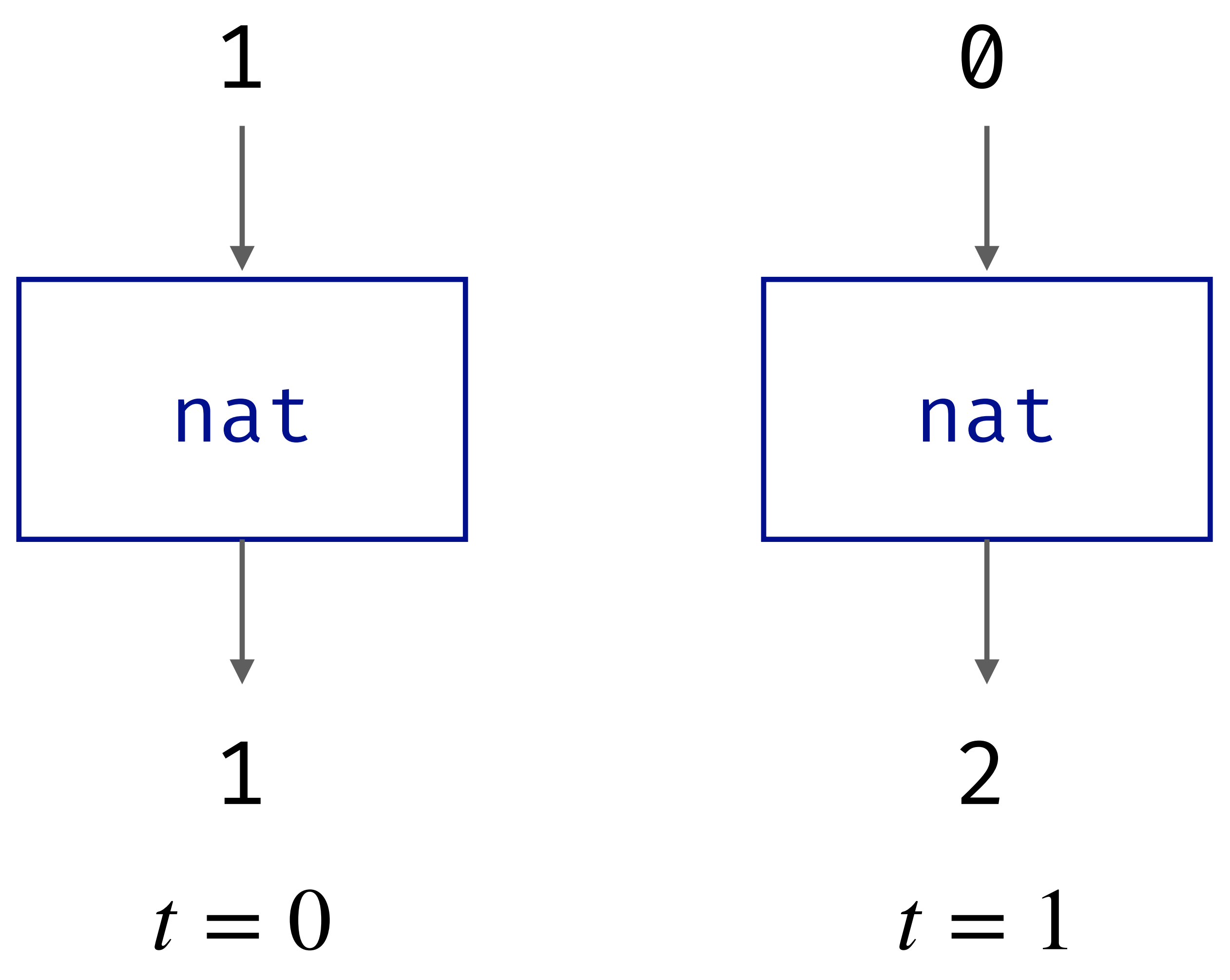

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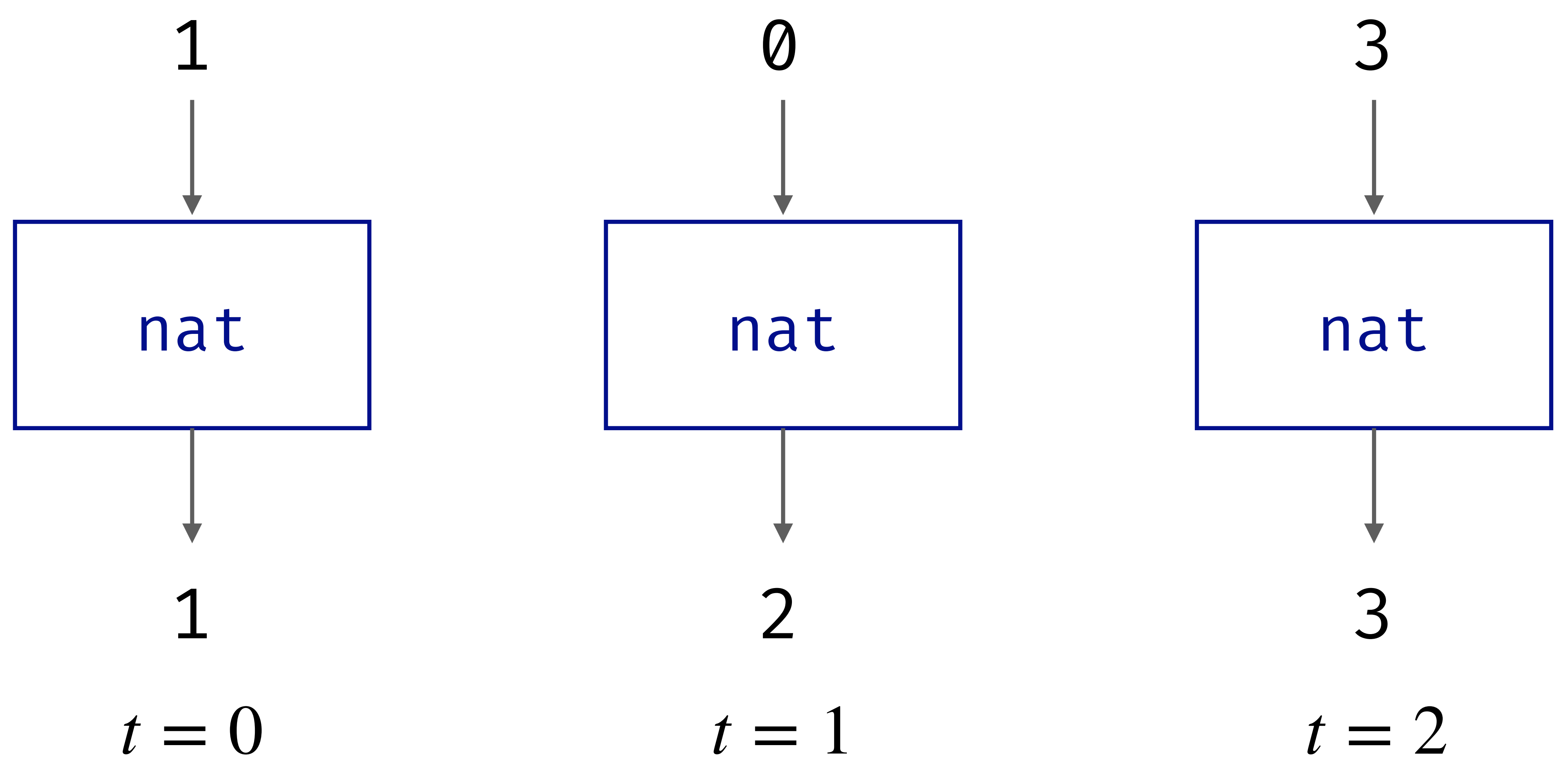

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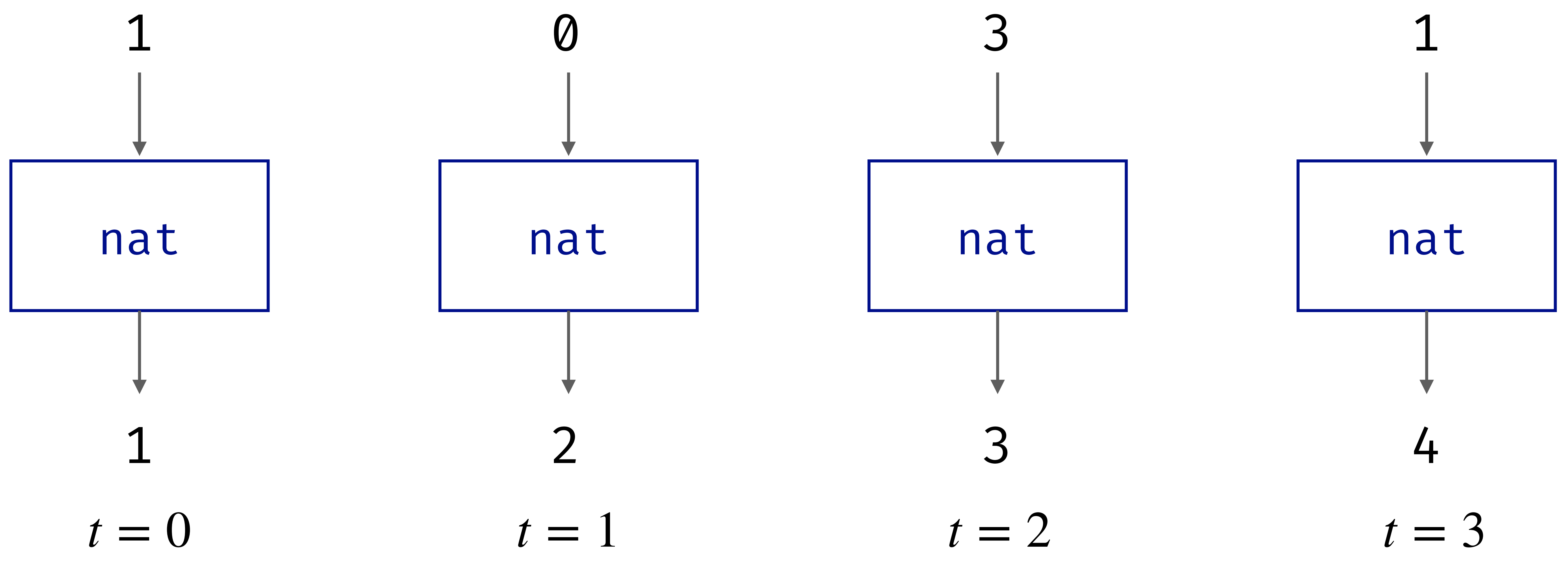

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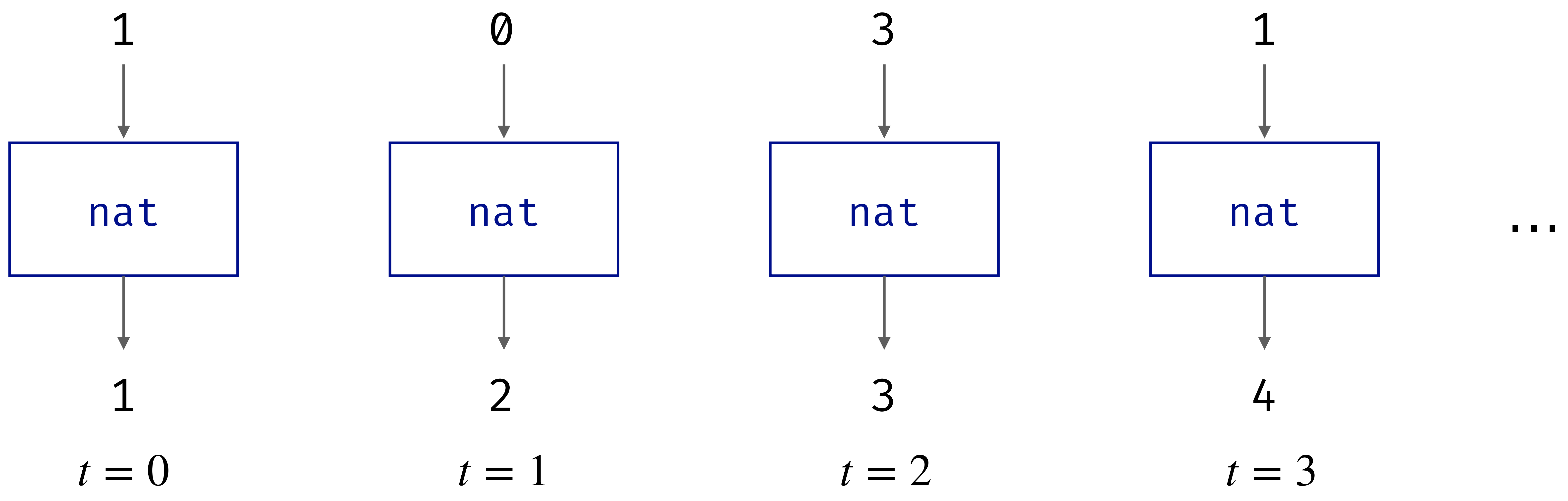

Bourke, Pouzet 2013

Examples

```
let node n1 () = (o1, o2) where
  rec reset o1 = nat 0 every (0 fby o2 = 3)
  and reset o2 = nat 0 every (0 fby o1 = 2)
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```
let node n2 () = o where
  rec o1, o2 = n1 ()
  and o = present (o2 = 0) → o1 else 1
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let node n1 () = (o1, o2) where
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```
let node n3 () = o where
  rec o1, o2 = n1 ()
  and o = present (o2 = 0) → last o + o1 init 1
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let node n1 () = (o1, o2) where
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```

Exercises

pid.zls

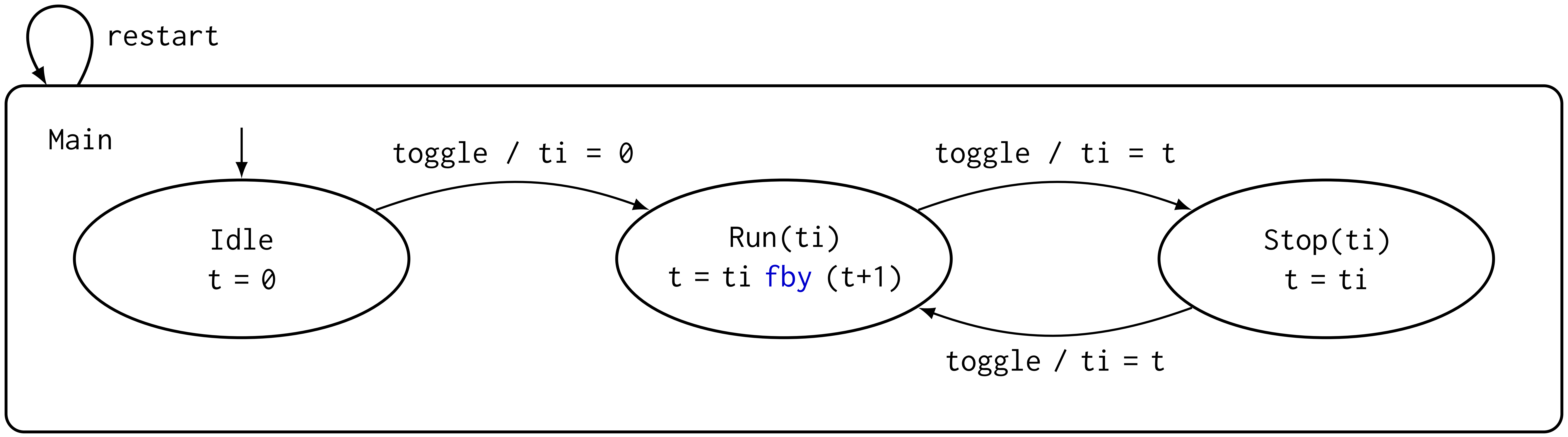
```
let node integr (y, dt) = o where
  rec ???

let node deriv (y, dt) = o where
  rec ???

let node pid ((p, i, d), r, y, dt) = o where
  rec ???
```

Hierarchical automata

```
let node stopwatch (toggle, restart) = t where
  rec automaton
    | Main →
      do automaton
        | Idle     → do t = 0                until toggle then Run(0)
        | Run(ti)  → do t = ti fby (t + 1) until toggle then Stop(t)
        | Stop(ti) → do t = ti               until toggle then Run(t)
      end
      until restart then Main
```



Demo

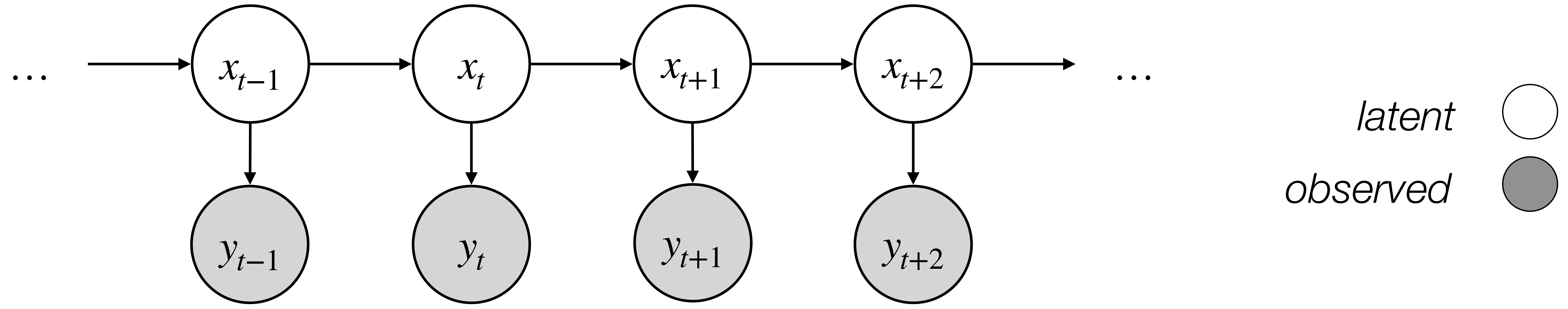
ProbZelus

Reactive Probabilistic Programming

Reactive probabilistic programming

Probabilistic constructs

- `x = sample(d)`: introduce a random variable `x` of distribution `d`
- `observe(d, y)`: condition on the fact that `y` was sampled from `d`
- `infer m y`: compute posterior distribution of `m` given `y`



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let proba tracker (y) = x where
    rec x = x0 → sample(mv_gaussian(f *@ (pre x), q))
    and () = observe(mv_gaussian(h *@ x, r), y)
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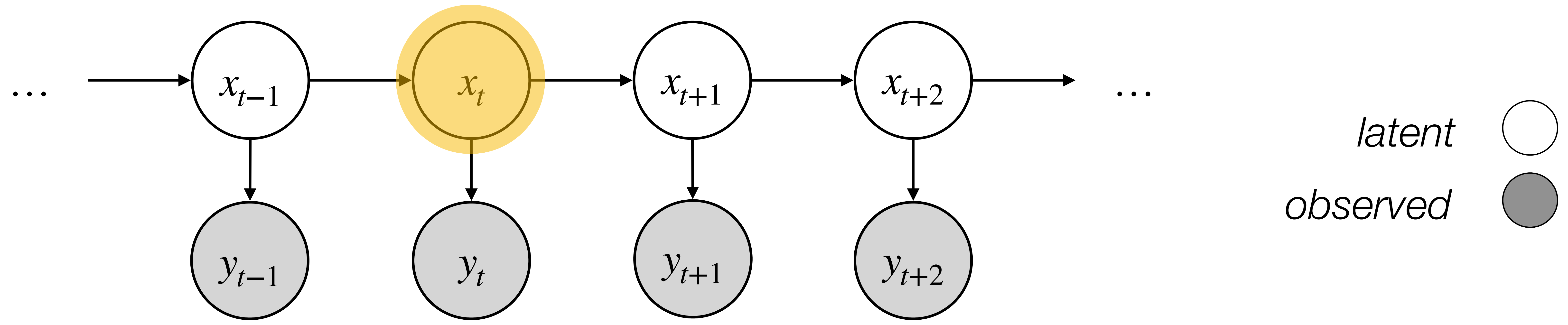

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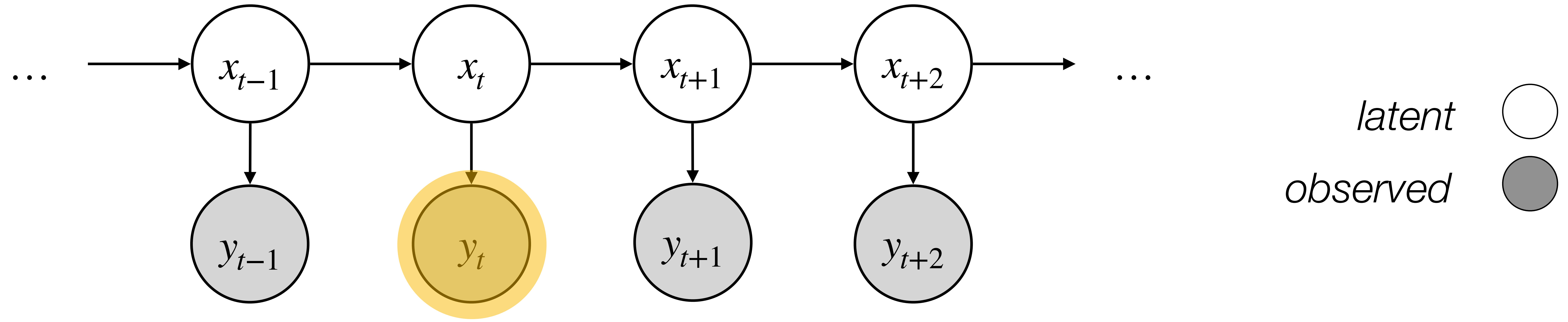

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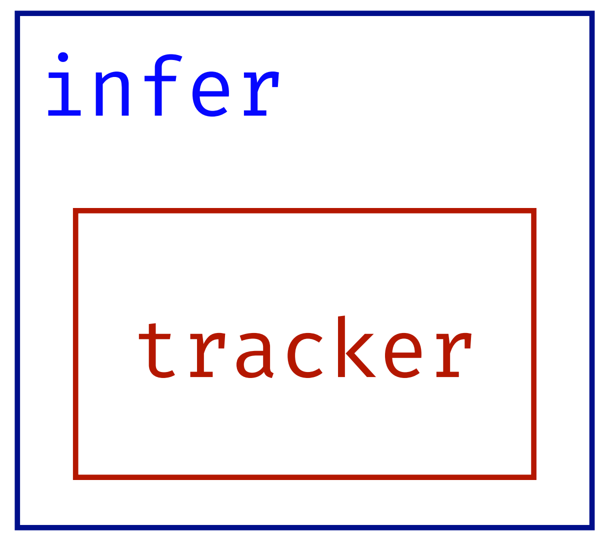

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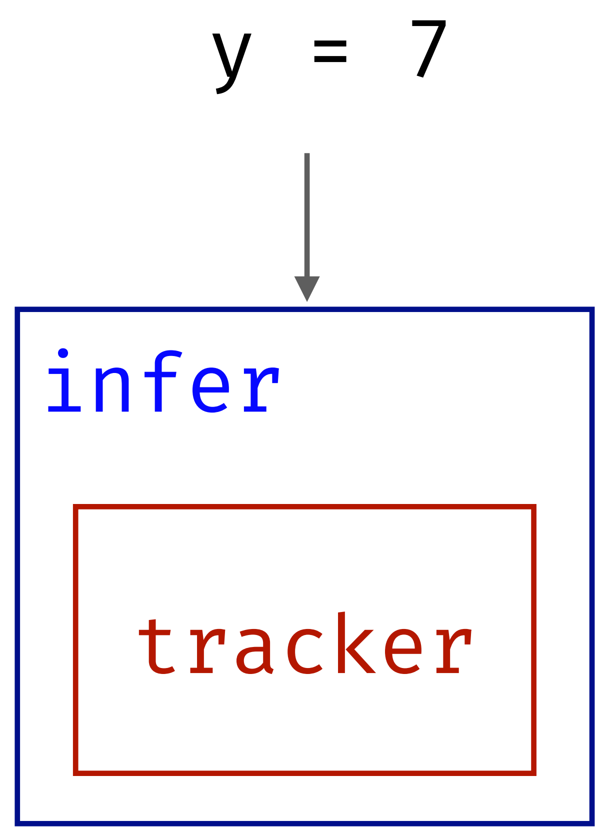

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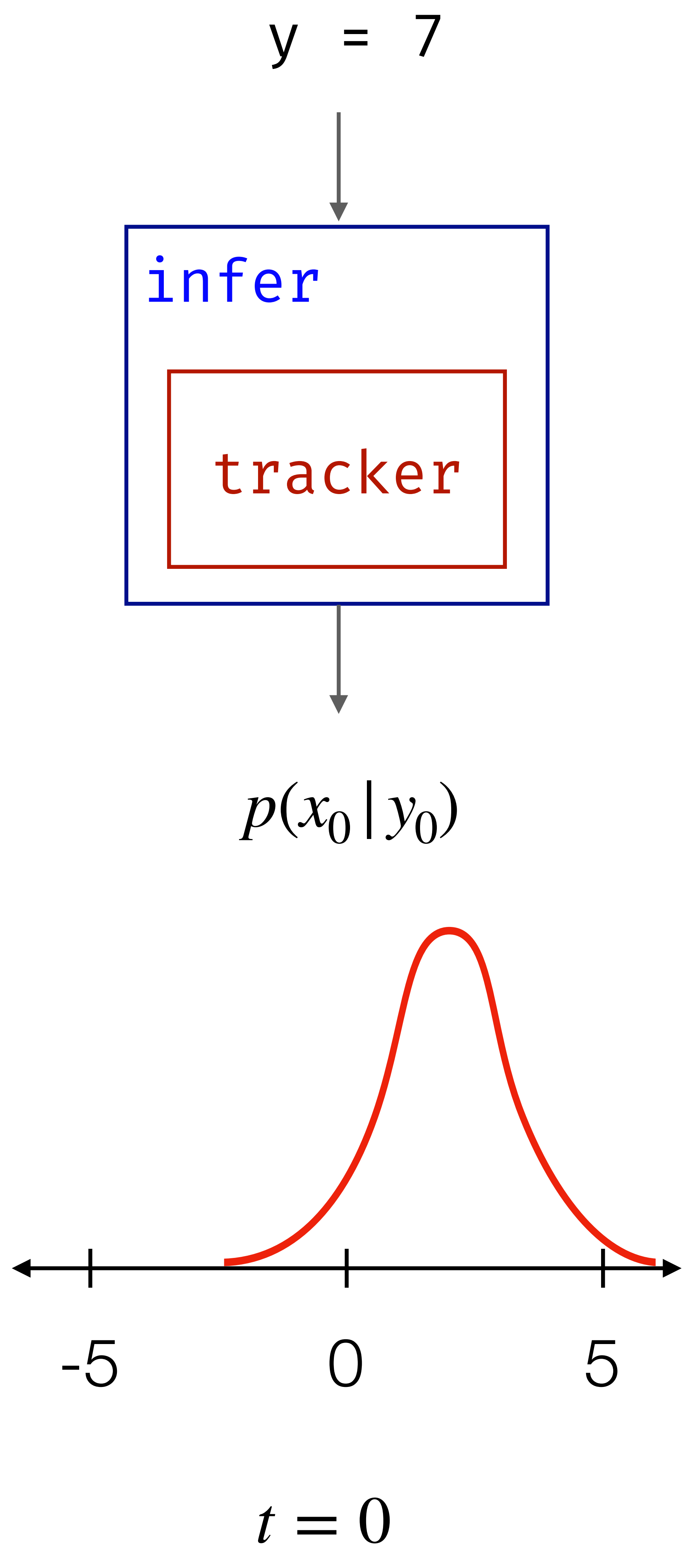

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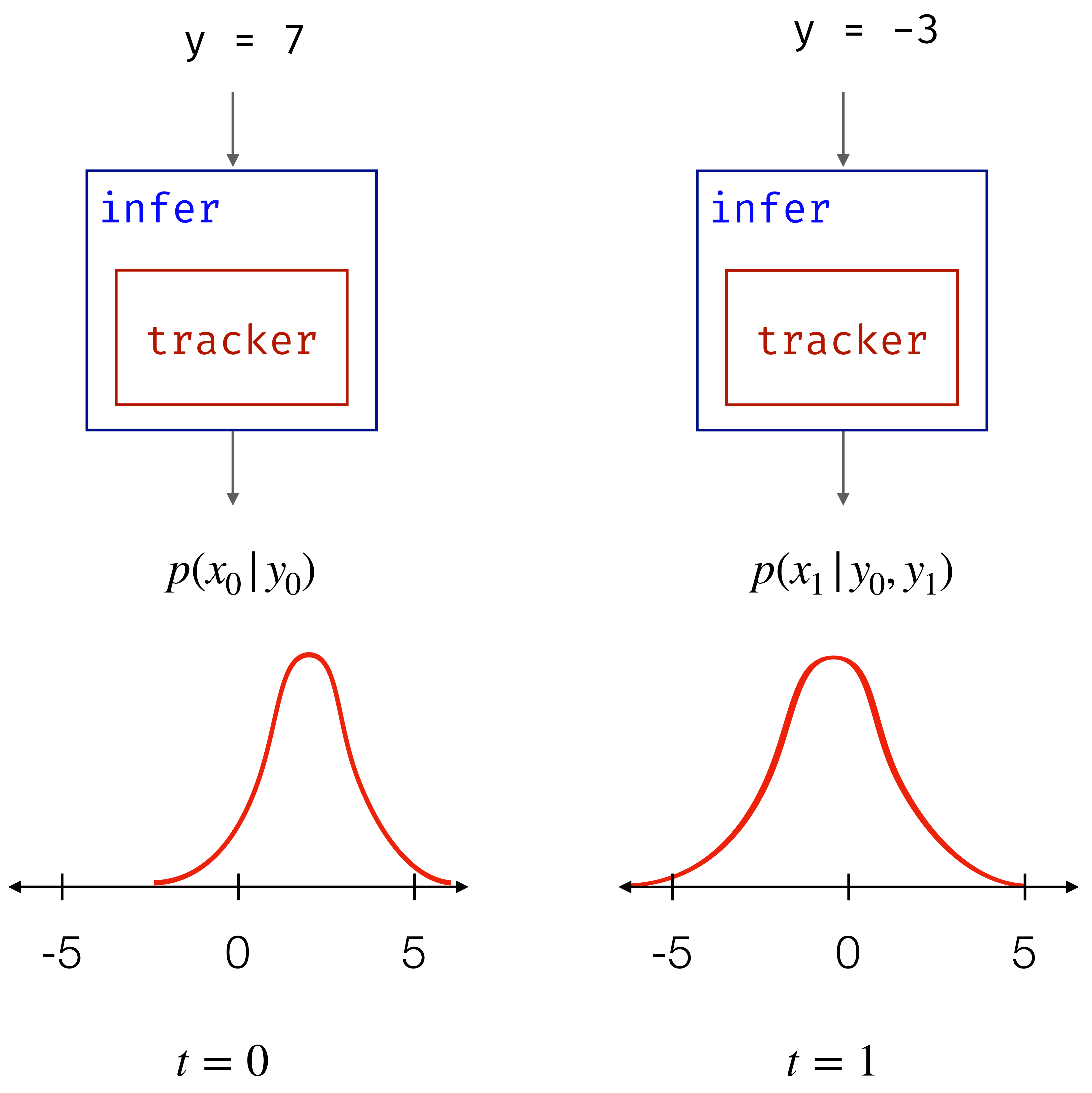

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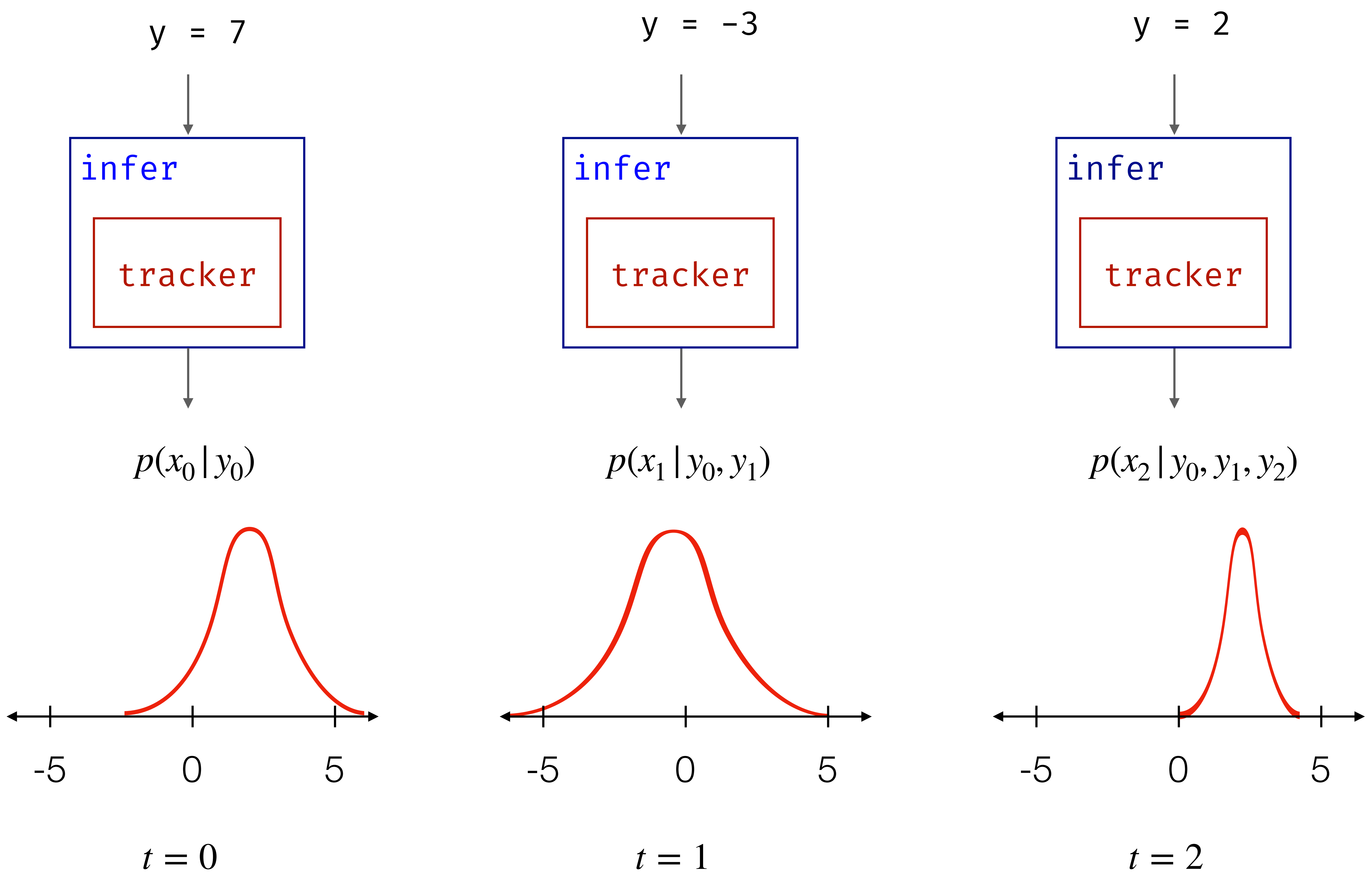

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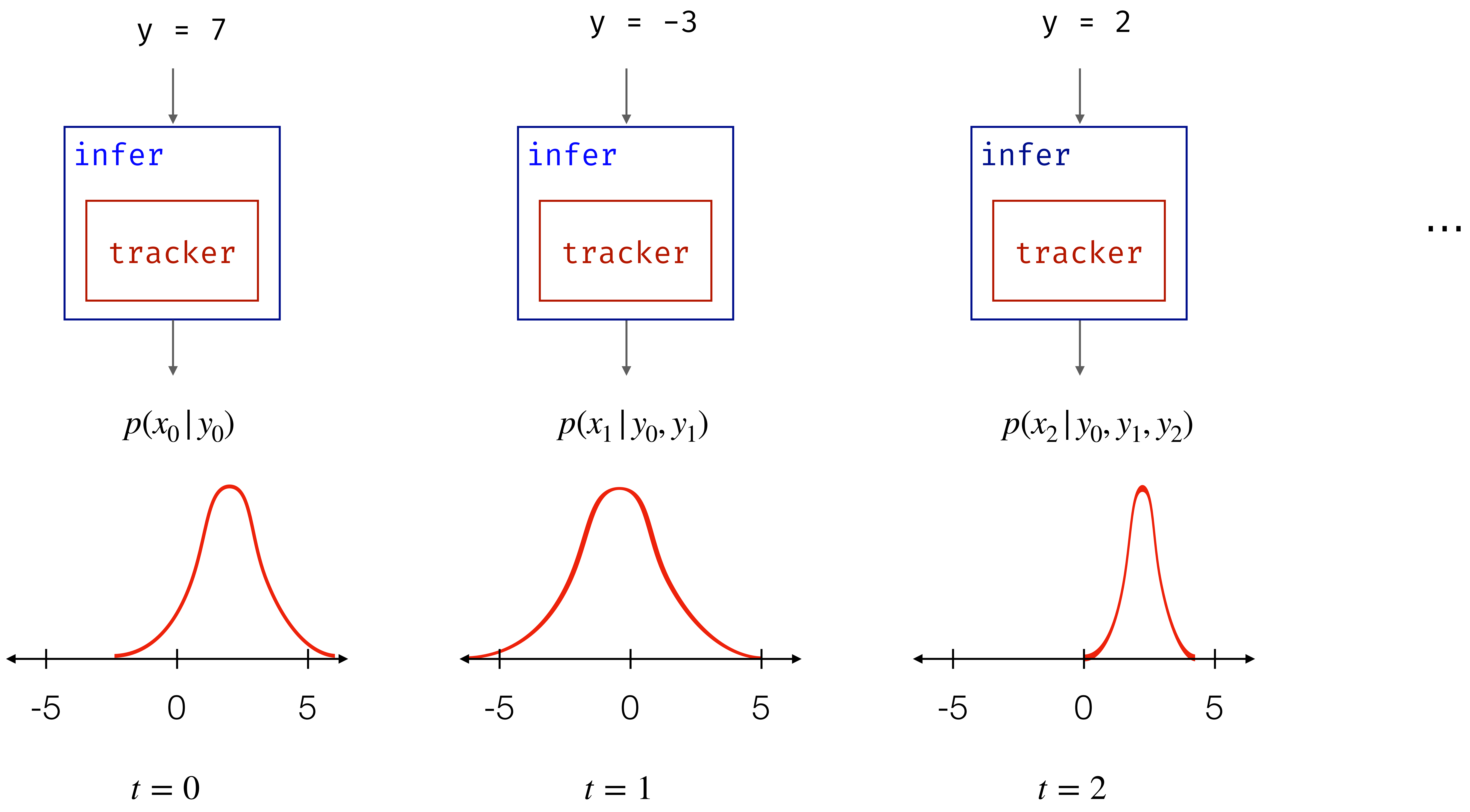


Demo

Co-iterative semantics

Schedule agnostic semantics

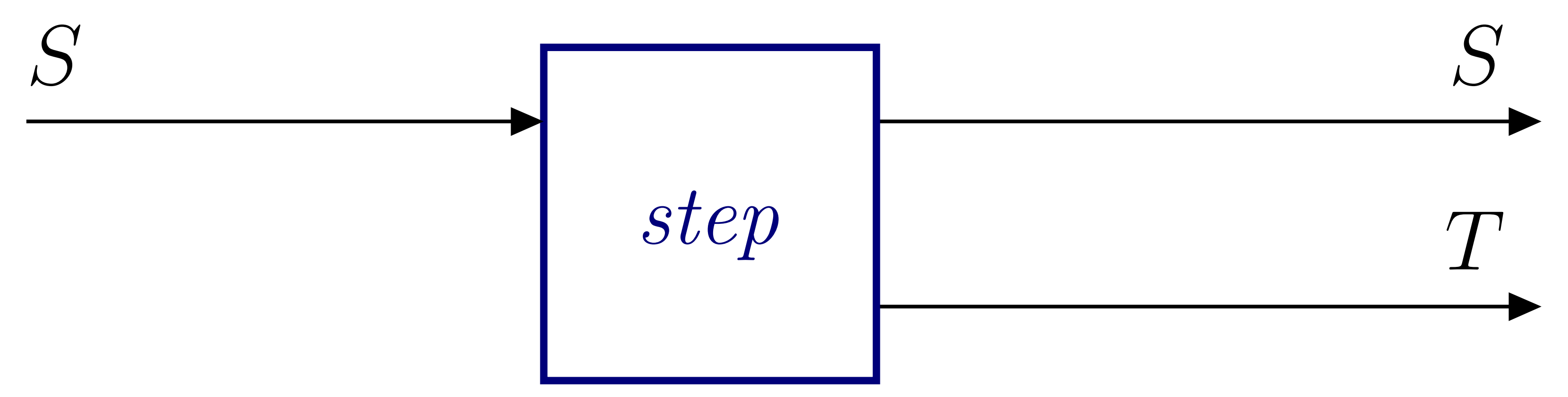
Deterministic semantics

allocation: S

transition: $S \to S \times T$

$$[\![e]\!] = [\![e]\!]^{\text{init}}_{\gamma}\ ,\ [\![e]\!]^{\text{step}}_{\gamma}$$

environment



$$\begin{aligned}
[\![c]\!]^{\text{init}}_{\gamma} &= (\,) \\
[\![c]\!]^{\text{step}}_{\gamma}\,((\,)) &= (\,), c \\
\\
[\![x]\!]^{\text{init}}_{\gamma} &= (\,) \\
[\![x]\!]^{\text{step}}_{\gamma}\,((\,)) &= (\,), \gamma(x) \\
\\
[\![i \to \texttt{pre}\ e]\!]^{\text{init}}_{\gamma} &= (i, [\![e]\!]^{\text{init}}_{\gamma}) \\
[\![i \to \texttt{pre}\ e]\!]^{\text{step}}_{\gamma}\,(p, s) &= \textit{let } s', v = [\![e]\!]^{\text{step}}_{\gamma}\,(s) \textit{ in } (v, s'), p
\end{aligned}$$

Deterministic semantics

allocation: S transition: $S \to S \times T$

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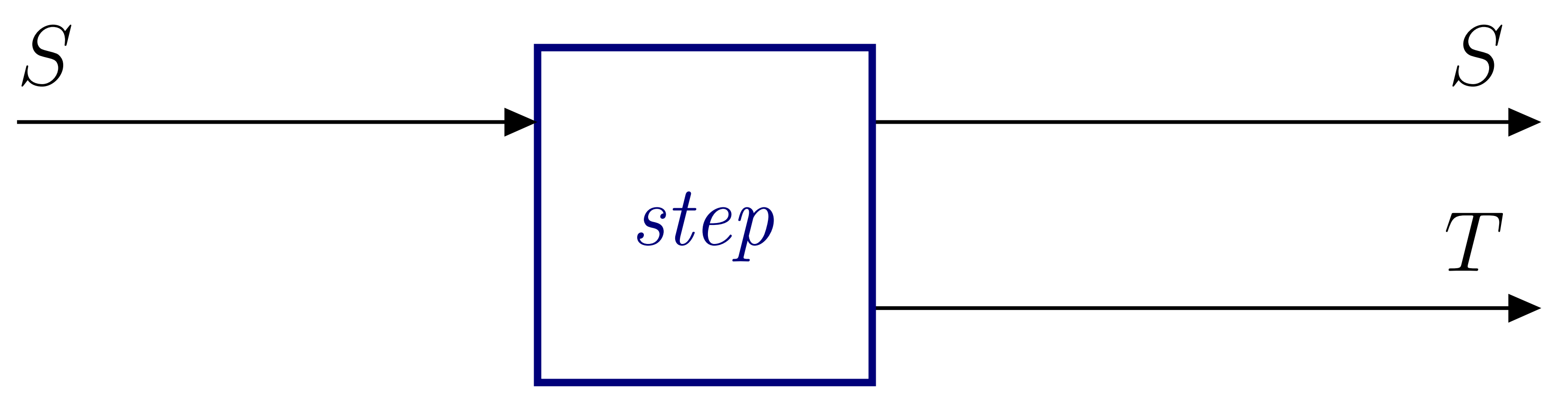

$[\![c]\!]^{\text{init}}_{\gamma}$

$[\![c]\!]^{\text{step}}_{\gamma}(\,(\,)\,)$

$[\![x]\!]^{\text{init}}_{\gamma}$

$[\![x]\!]^{\text{step}}_{\gamma}(\,(\,)\,)$

$[\![i \to$ `pre` $e]$

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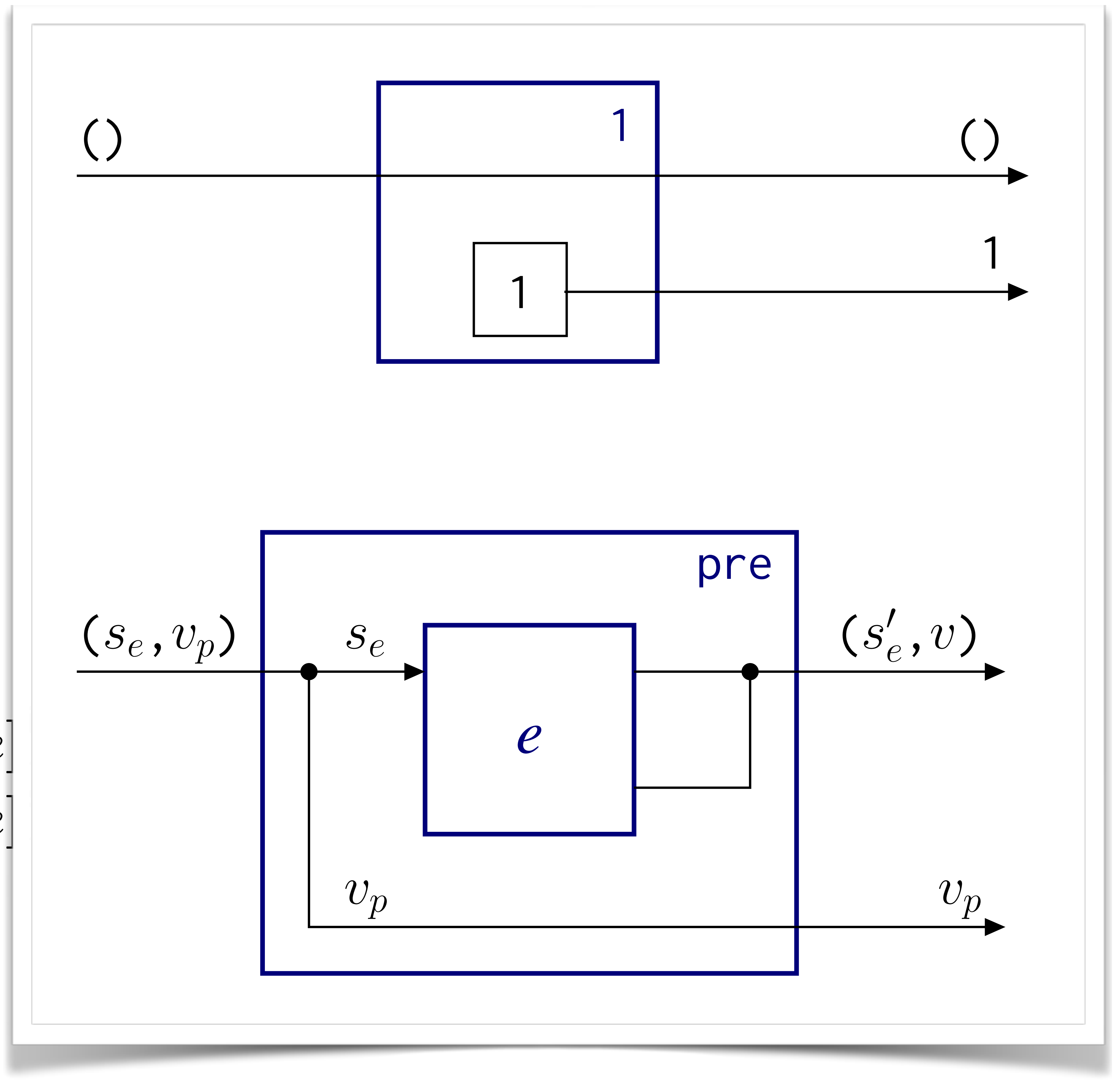

Deterministic equations

$$\left[\!\!\left[\begin{array}{l} e\ \texttt{where rec init}\ x = c \\ \qquad\qquad \texttt{and}\ x = e_x \\ \qquad\qquad \texttt{and}\ y = e_y \end{array}\right]\!\!\right]^{\text{init}}_{\gamma} = c, \left([\![e]\!]^{\text{init}}_{\gamma}, [\![e_x]\!]^{\text{init}}_{\gamma}, [\![e_y]\!]^{\text{init}}_{\gamma}\right)$$

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Deterministic equations

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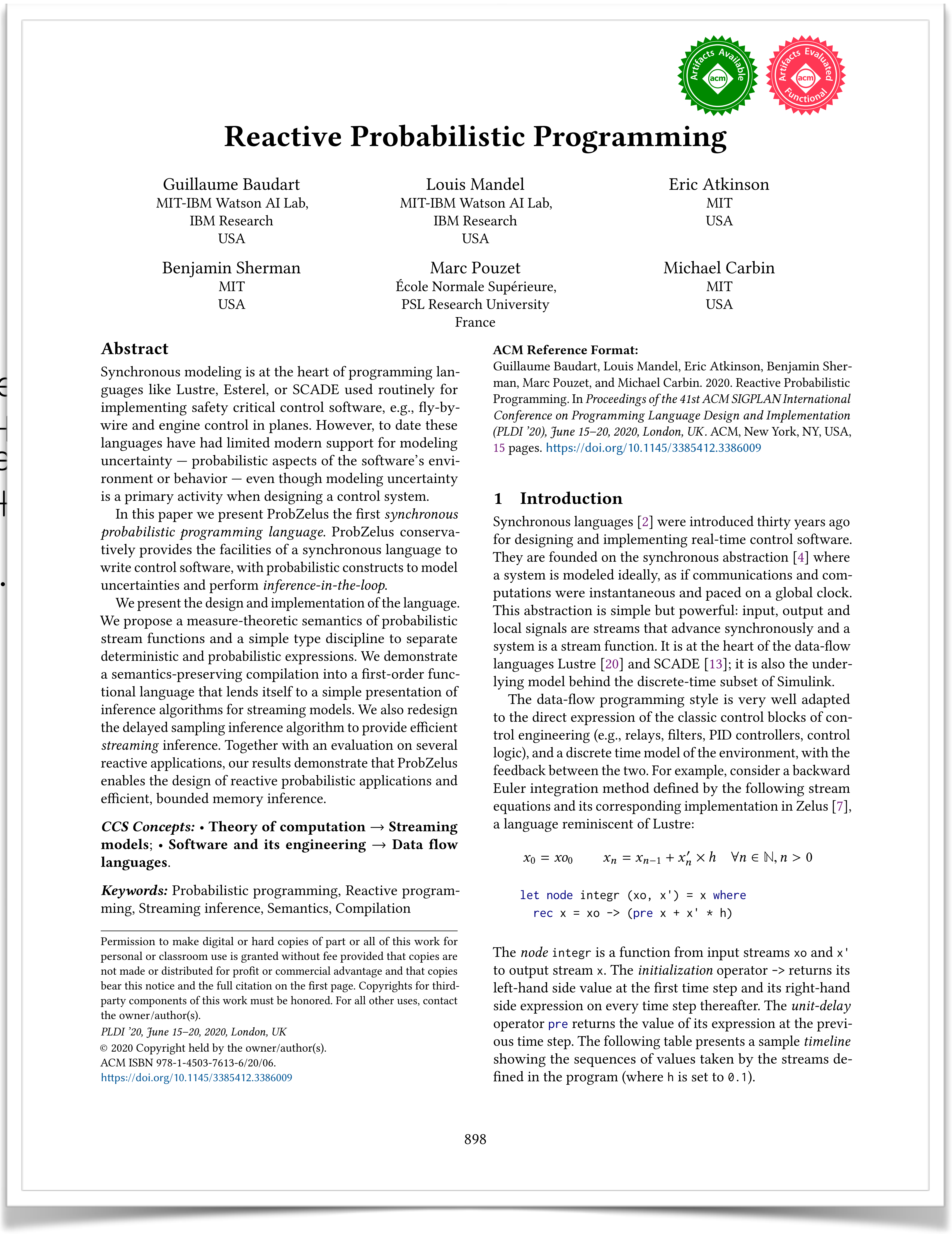

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Abstract

Synchronous modeling is at the heart of programming languages like Lustre, Esterel, or SCADE used routinely for implementing safety critical control software, e.g., fly-by-wire and engine control in planes. However, to date these languages have had limited modern support for modeling uncertainty — probabilistic aspects of the software's environment or behavior — even though modeling uncertainty is a primary activity when designing a control system.

In this paper we present ProbZelus the first *synchronous probabilistic programming language*. ProbZelus conservatively provides the facilities of a synchronous language to write control software, with probabilistic constructs to model uncertainties and perform *inference-in-the-loop*.

We present the design and implementation of the language. We propose a measure-theoretic semantics of probabilistic stream functions and a simple type discipline to separate deterministic and probabilistic expressions. We demonstrate a semantics-preserving compilation into a first-order functional language that lends itself to a simple presentation of inference algorithms for streaming models. We also redesign the delayed sampling inference algorithm to provide efficient *streaming* inference. Together with an evaluation on several reactive applications, our results demonstrate that ProbZelus enables the design of reactive probabilistic applications and efficient, bounded memory inference.

CCS Concepts: • **Theory of computation** → **Streaming models**; • **Software and its engineering** → **Data flow languages**.

Keywords: Probabilistic programming, Reactive programming, Streaming inference, Semantics, Compilation

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1 Introduction

Synchronous languages [2] were introduced thirty years ago for designing and implementing real-time control software. They are founded on the synchronous abstraction [4] where a system is modeled ideally, as if communications and computations were instantaneous and paced on a global clock. This abstraction is simple but powerful: input, output and local signals are streams that advance synchronously and a system is a stream function. It is at the heart of the data-flow languages Lustre [20] and SCADE [13]; it is also the underlying model behind the discrete-time subset of Simulink.

The data-flow programming style is very well adapted to the direct expression of the classic control blocks of control engineering (e.g., relays, filters, PID controllers, control logic), and a discrete time model of the environment, with the feedback between the two. For example, consider a backward Euler integration method defined by the following stream equations and its corresponding implementation in Zelus [7], a language reminiscent of Lustre:

$$x_0 = xo_0 \qquad x_n = x_{n-1} + x'_n \times h \quad \forall n \in \mathbb{N}, n > 0$$

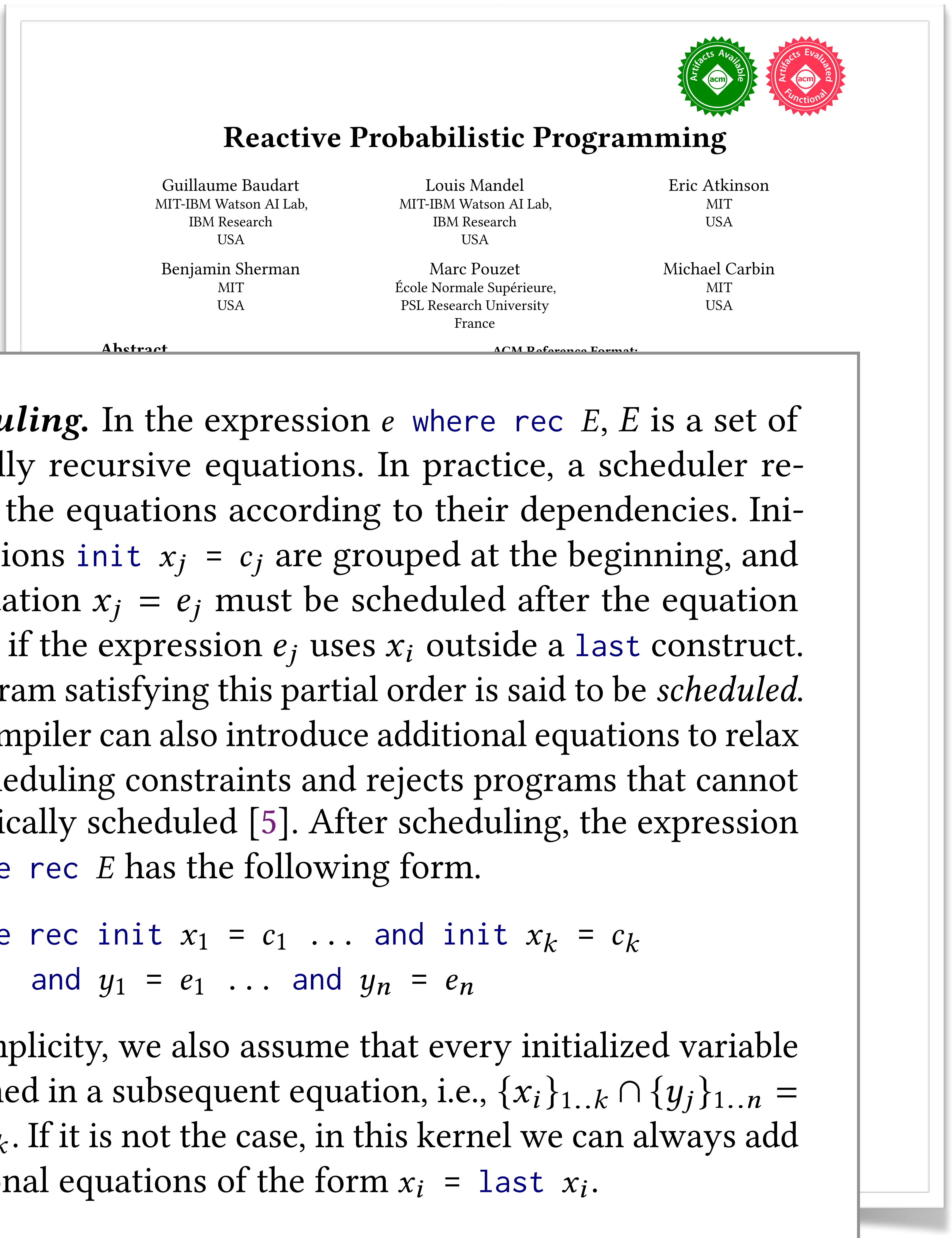
```
let node integr (xo, x') = x where
  rec x = xo -> (pre x + x' * h)
```

The *node* `integr` is a function from input streams `xo` and `x'` to output stream `x`. The *initialization* operator `->` returns its left-hand side value at the first time step and its right-hand side expression on every time step thereafter. The *unit-delay* operator `pre` returns the value of its expression at the previous time step. The following table presents a sample *timeline* showing the sequences of values taken by the streams defined in the program (where `h` is set to `0.1`).

Deterministic equations

$$\left\llbracket \begin{array}{l} e \texttt{ where rec init } x = c \\ \qquad\qquad \texttt{and } x = e_x \\ \qquad\qquad \texttt{and } y = e_y \end{array} \right\rrbracket^{\text{init}}_{\gamma} = c, \Big(\llbracket e \rrbracket^{\text{init}}_{\gamma}, \llbracket e_x \rrbracket^{\text{init}}_{\gamma},$$

$$\left\llbracket \begin{array}{l} e \texttt{ where rec init } x = c \\ \qquad\qquad \texttt{and } x = e_x \\ \qquad\qquad \texttt{and } y = e_y \end{array} \right\rrbracket^{\text{step}}_{\gamma} (p_x, (s, s_x, s_y)) = \begin{array}{l} \textit{let } s'_x, v_x \\ \textit{let } s'_y, v_y \\ \textit{let } s', v = \\ (v_x, (s', s' \end{array}$$



Reactive Probabilistic Programming

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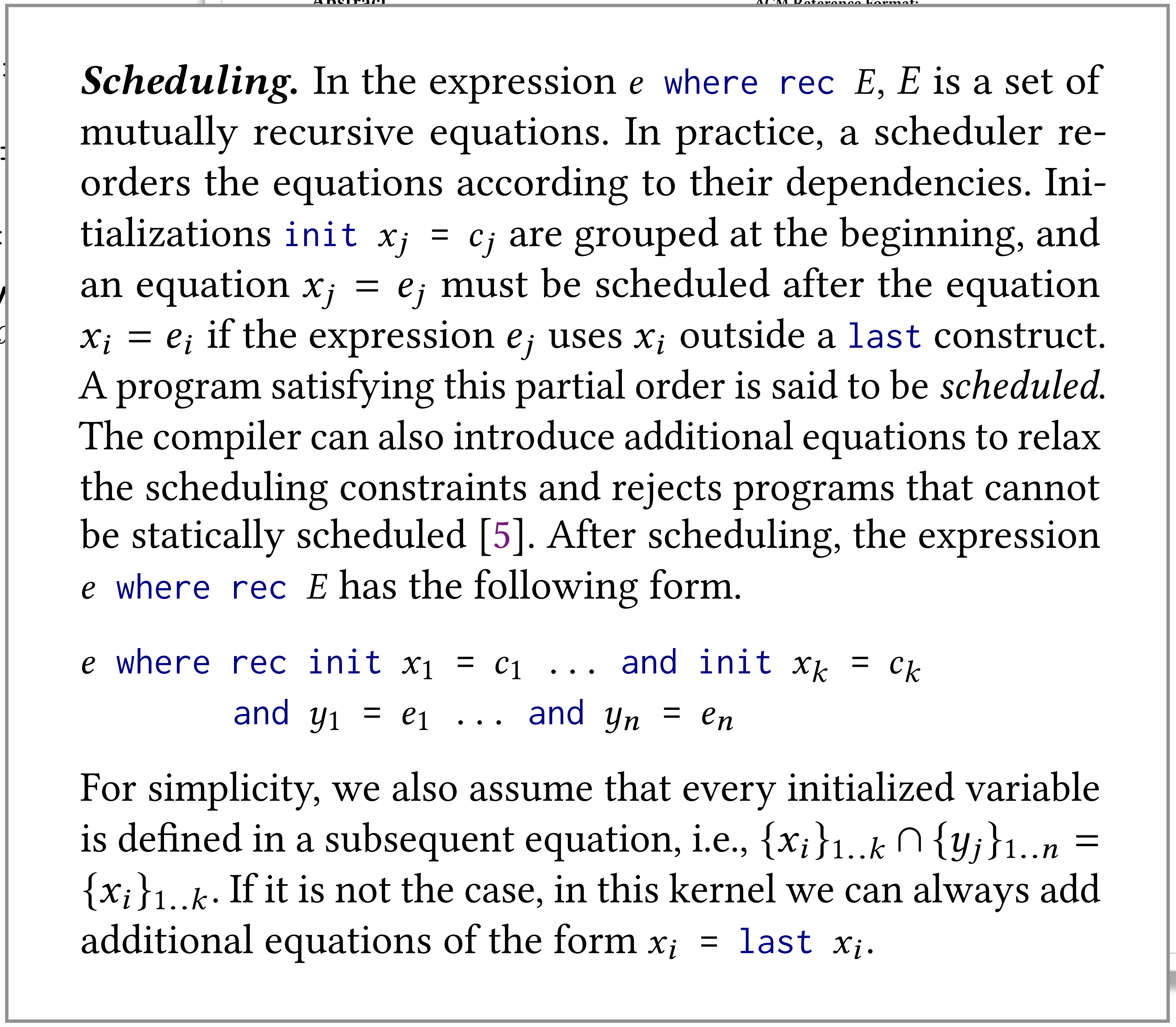


Scheduling. In the expression e `where rec` E, E is a set of mutually recursive equations. In practice, a scheduler reorders the equations according to their dependencies. Initializations `init` x_j = c_j are grouped at the beginning, and an equation $x_j = e_j$ must be scheduled after the equation $x_i = e_i$ if the expression e_j uses x_i outside a `last` construct. A program satisfying this partial order is said to be *scheduled*. The compiler can also introduce additional equations to relax the scheduling constraints and rejects programs that cannot be statically scheduled [5]. After scheduling, the expression e `where rec` E has the following form.

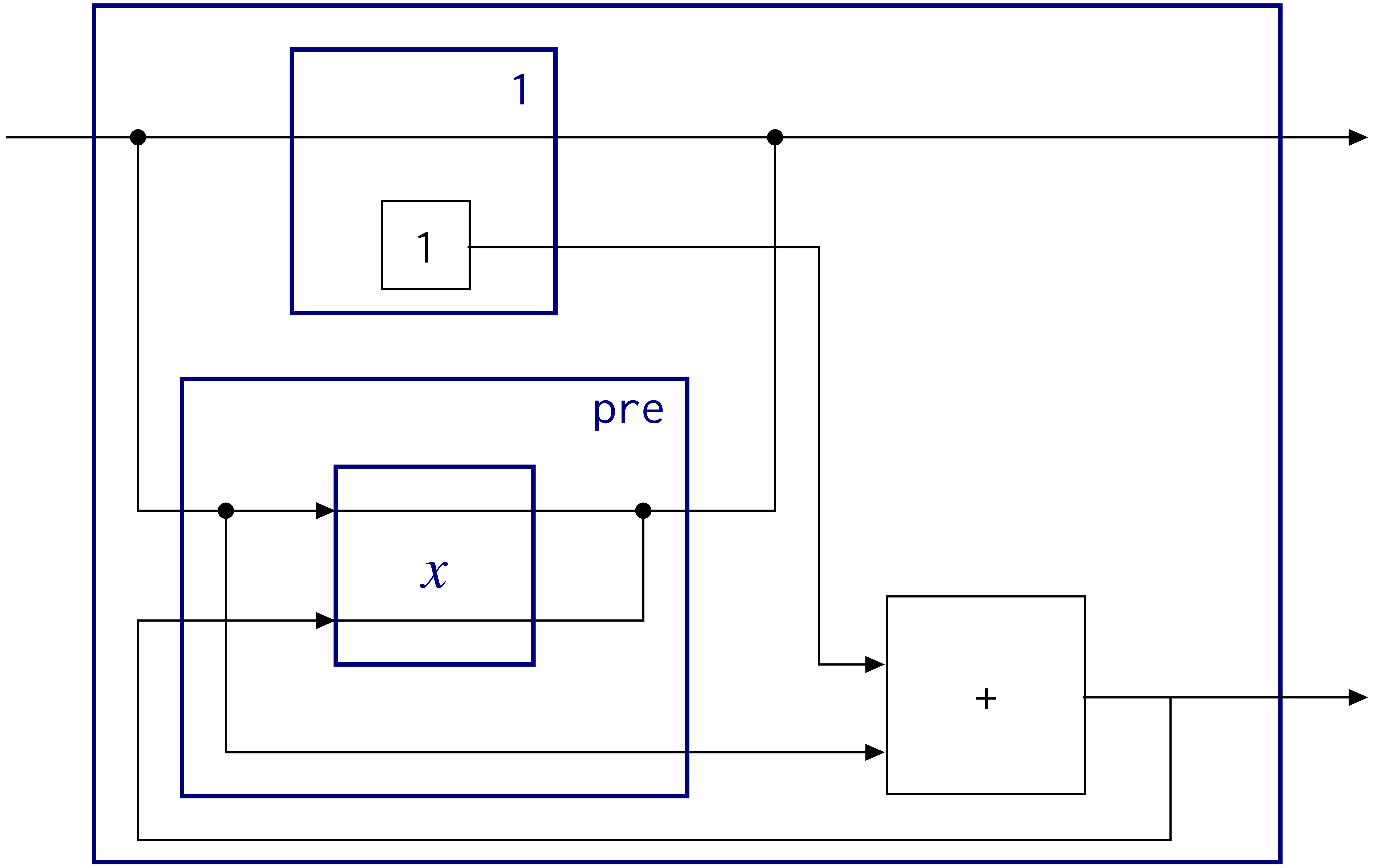
```
e where rec init x₁ = c₁ ... and init xₖ = cₖ
        and y₁ = e₁ ... and yₙ = eₙ
```

For simplicity, we also assume that every initialized variable is defined in a subsequent equation, i.e., $\{x_i\}_{1..k} \cap \{y_j\}_{1..n} = \{x_i\}_{1..k}$. If it is not the case, in this kernel we can always add additional equations of the form x_i = `last` x_i.

Example

`rec x = 1 + pre x`

- Initial state: `(), 0`
- Output: `1, 2, 3, 4, ....`



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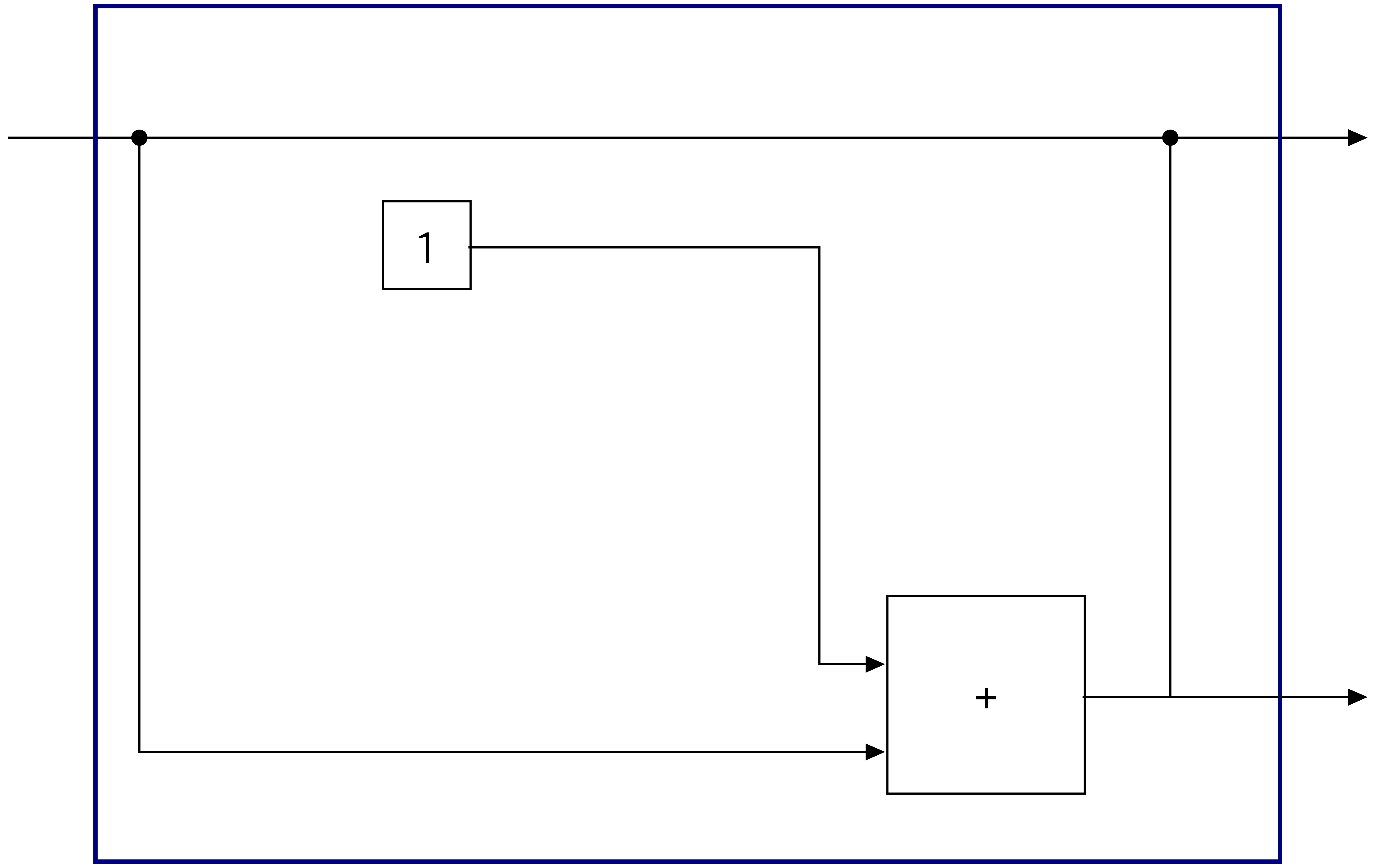

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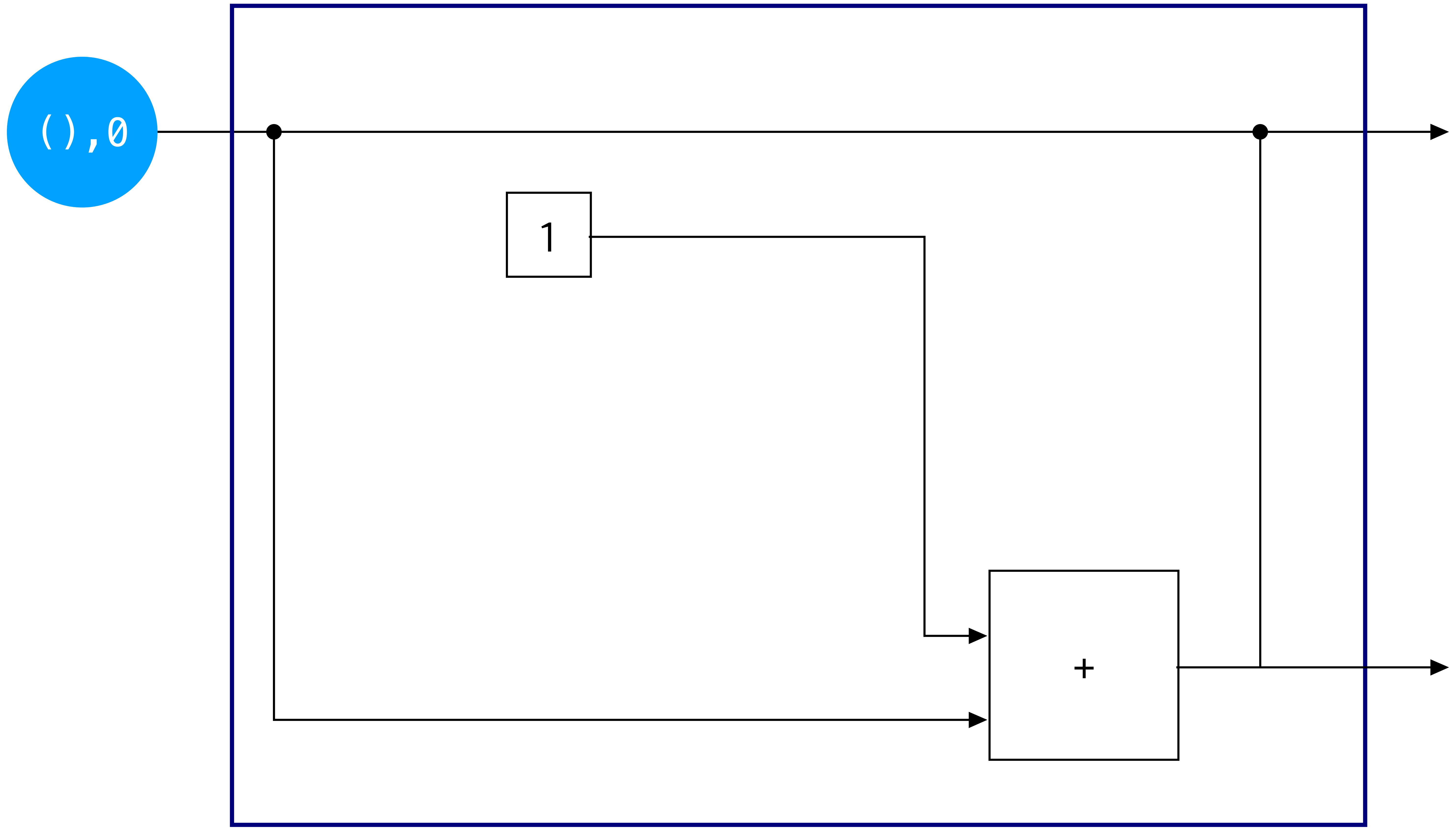

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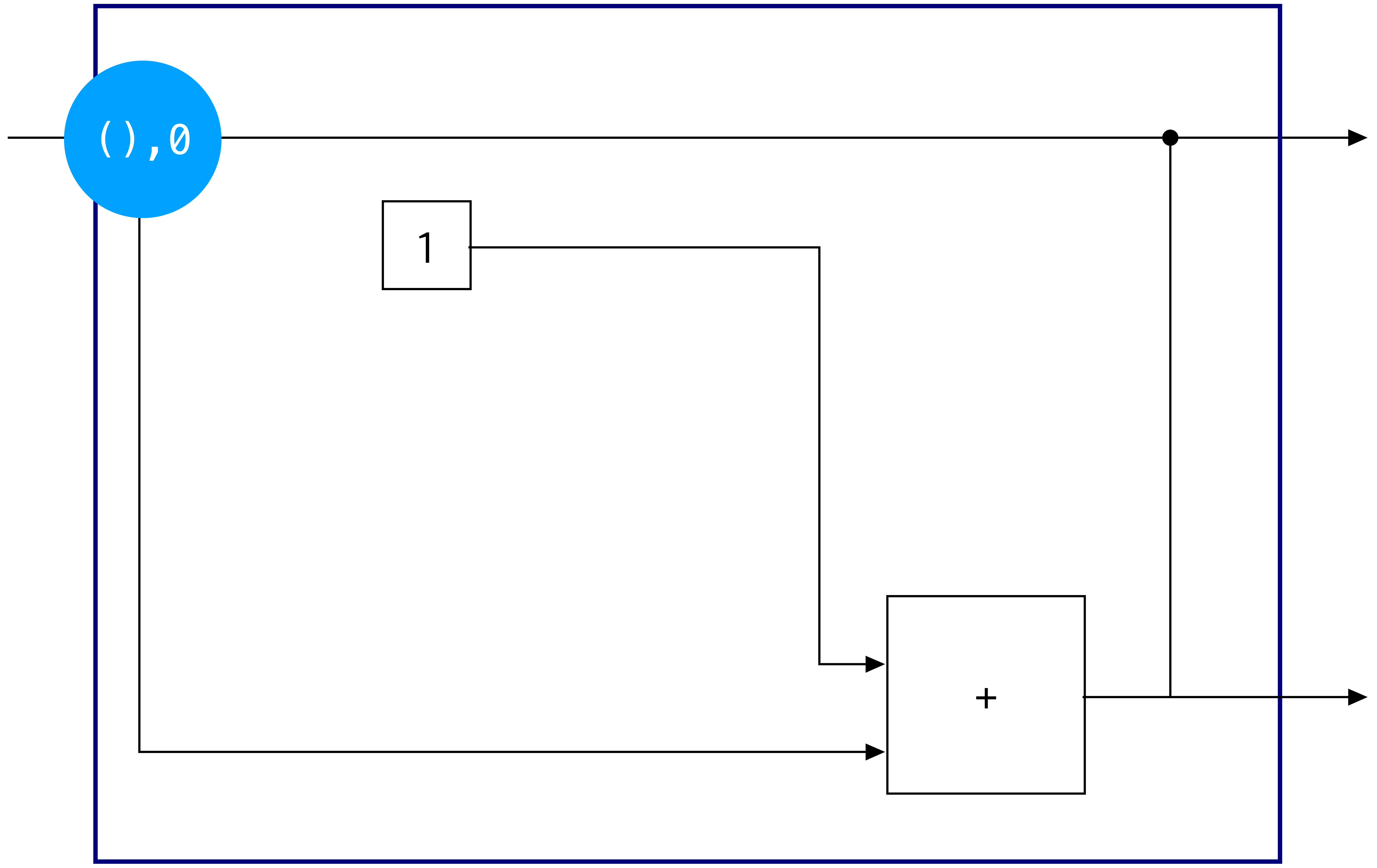

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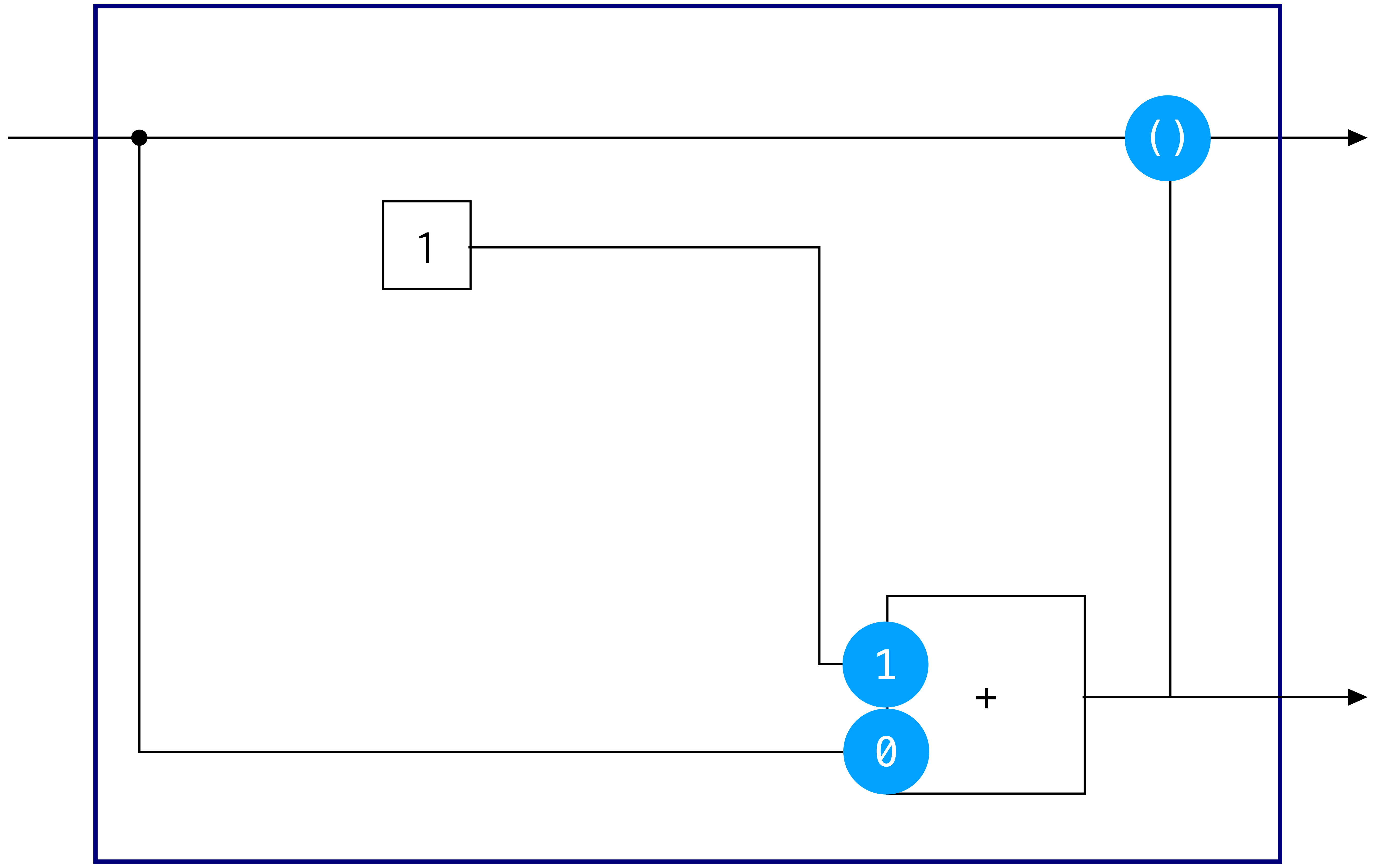

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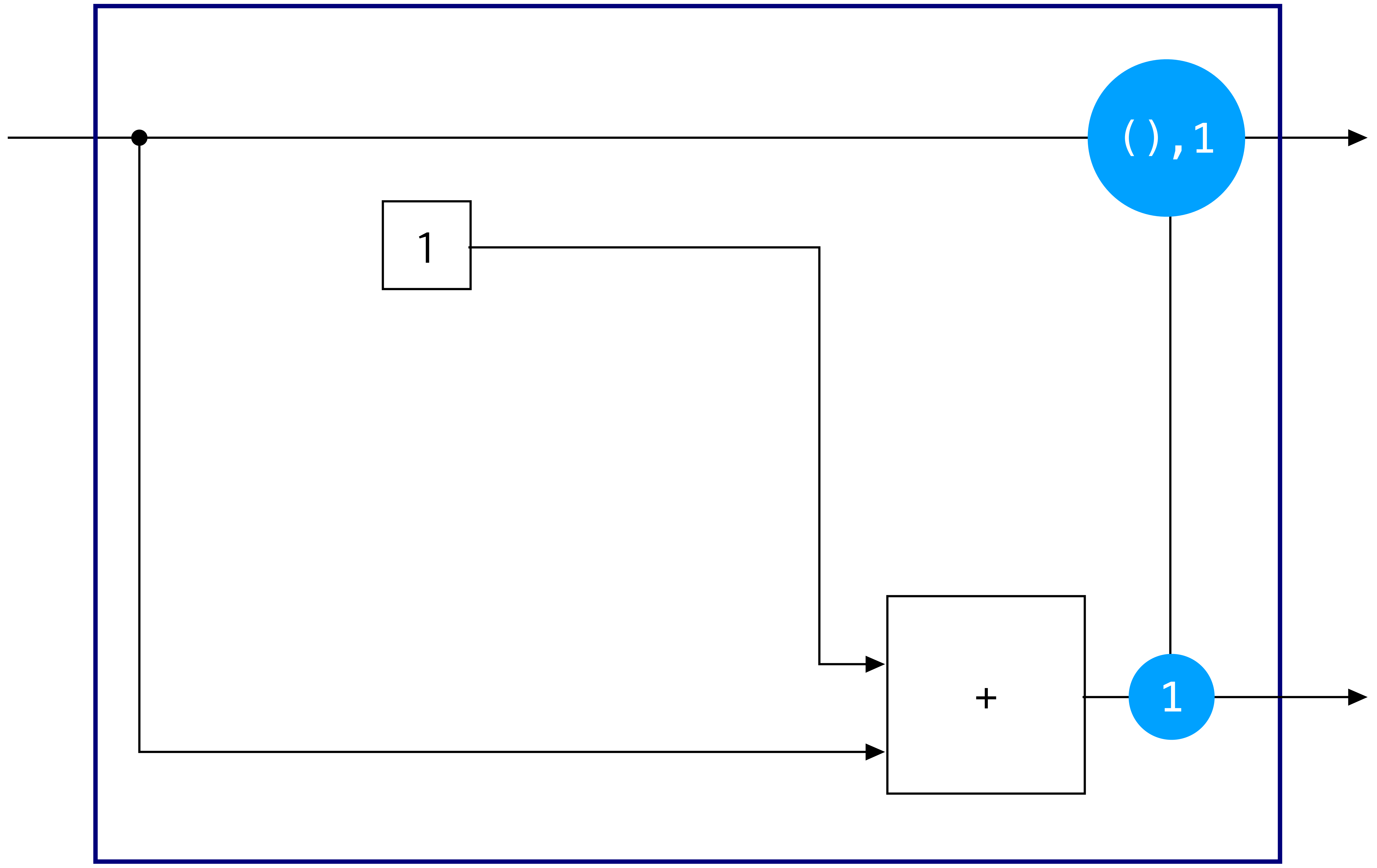

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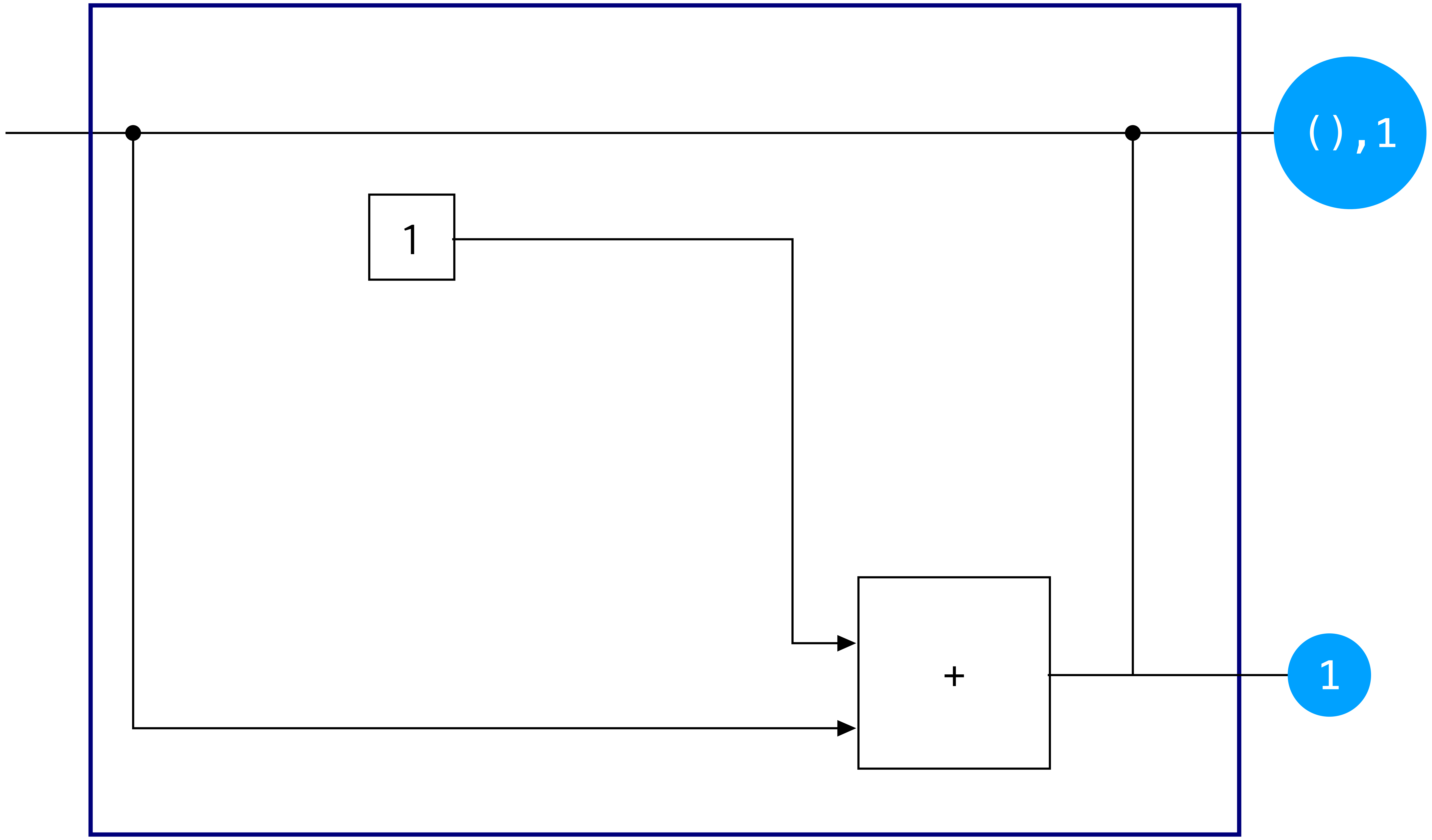

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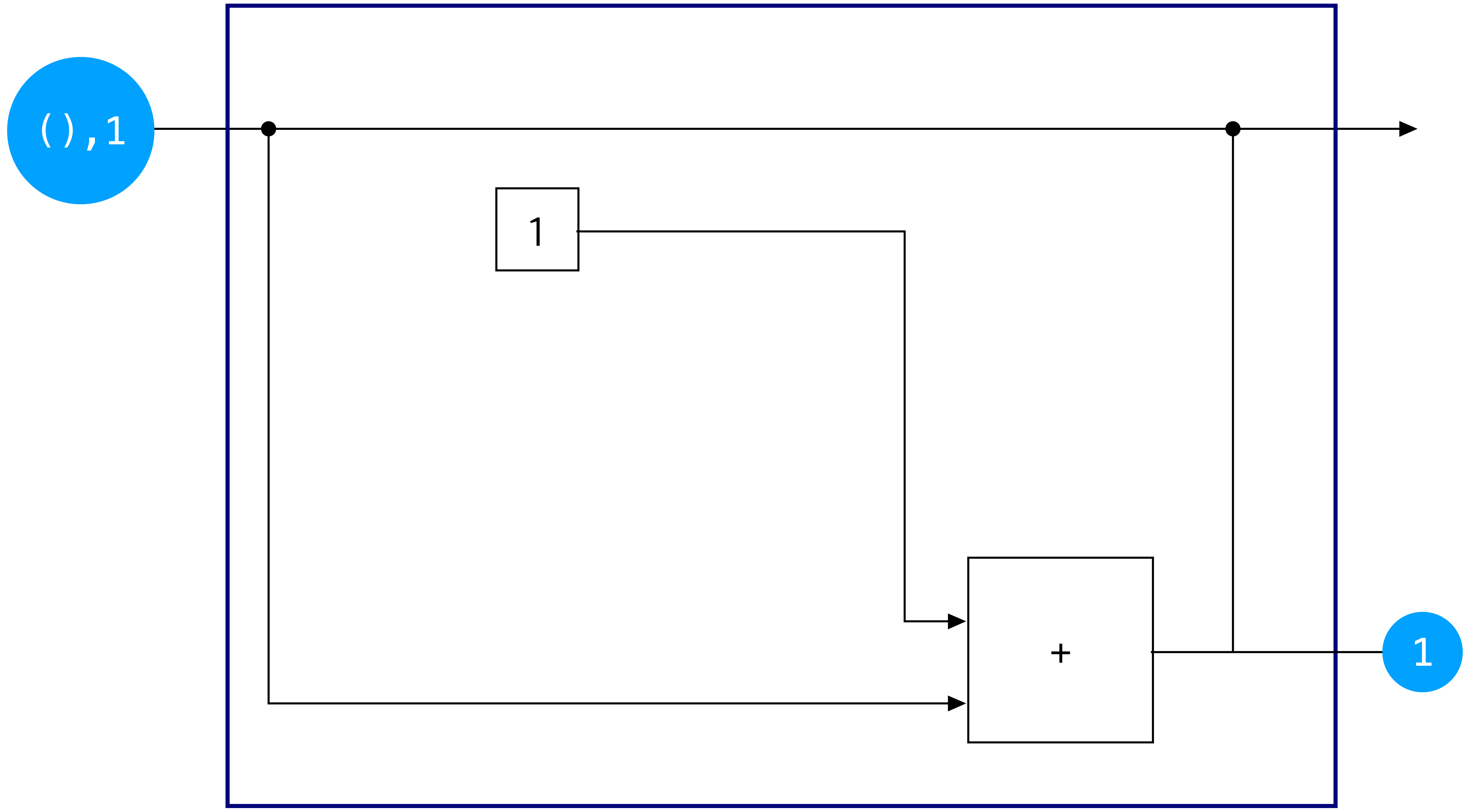

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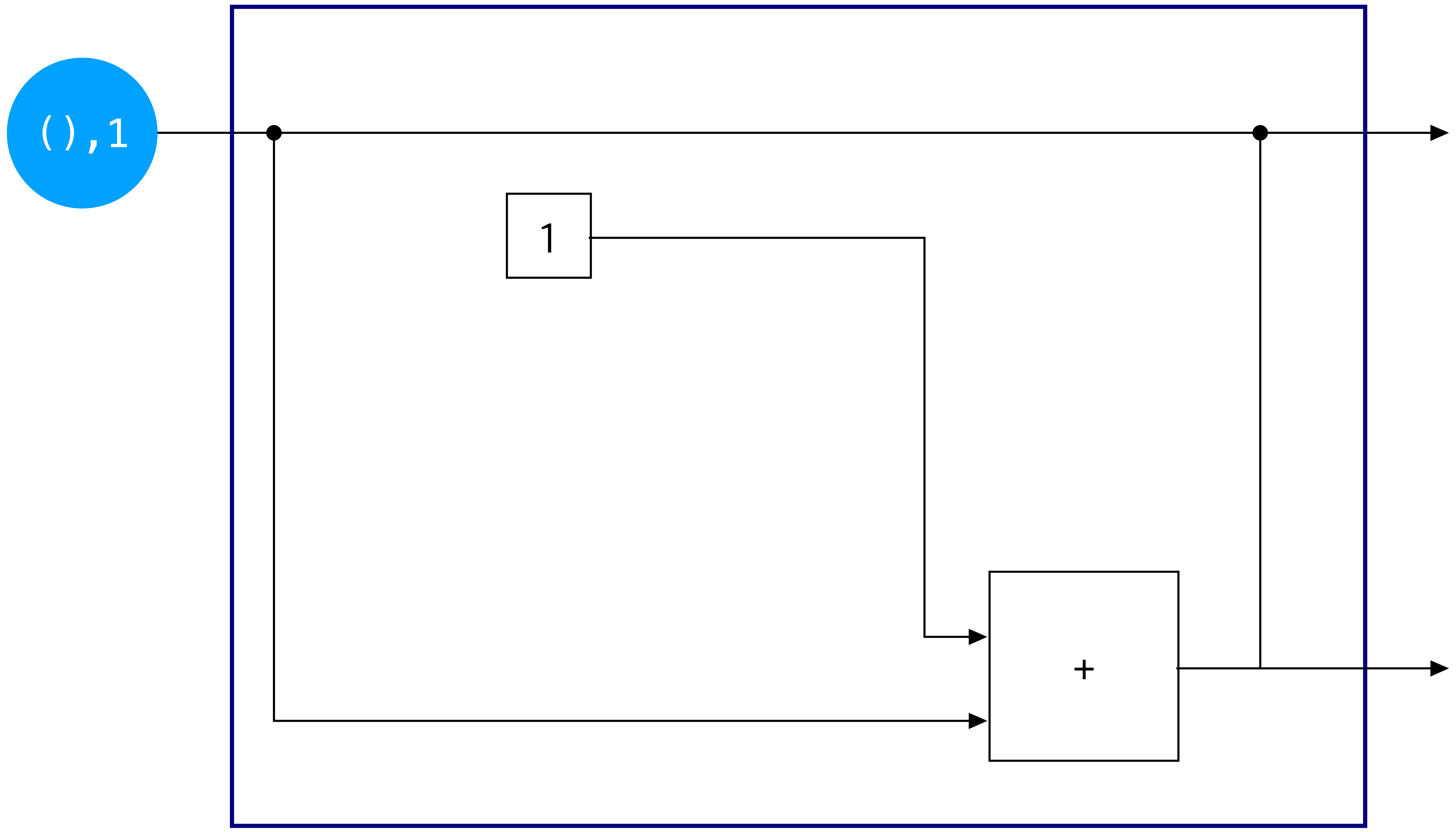

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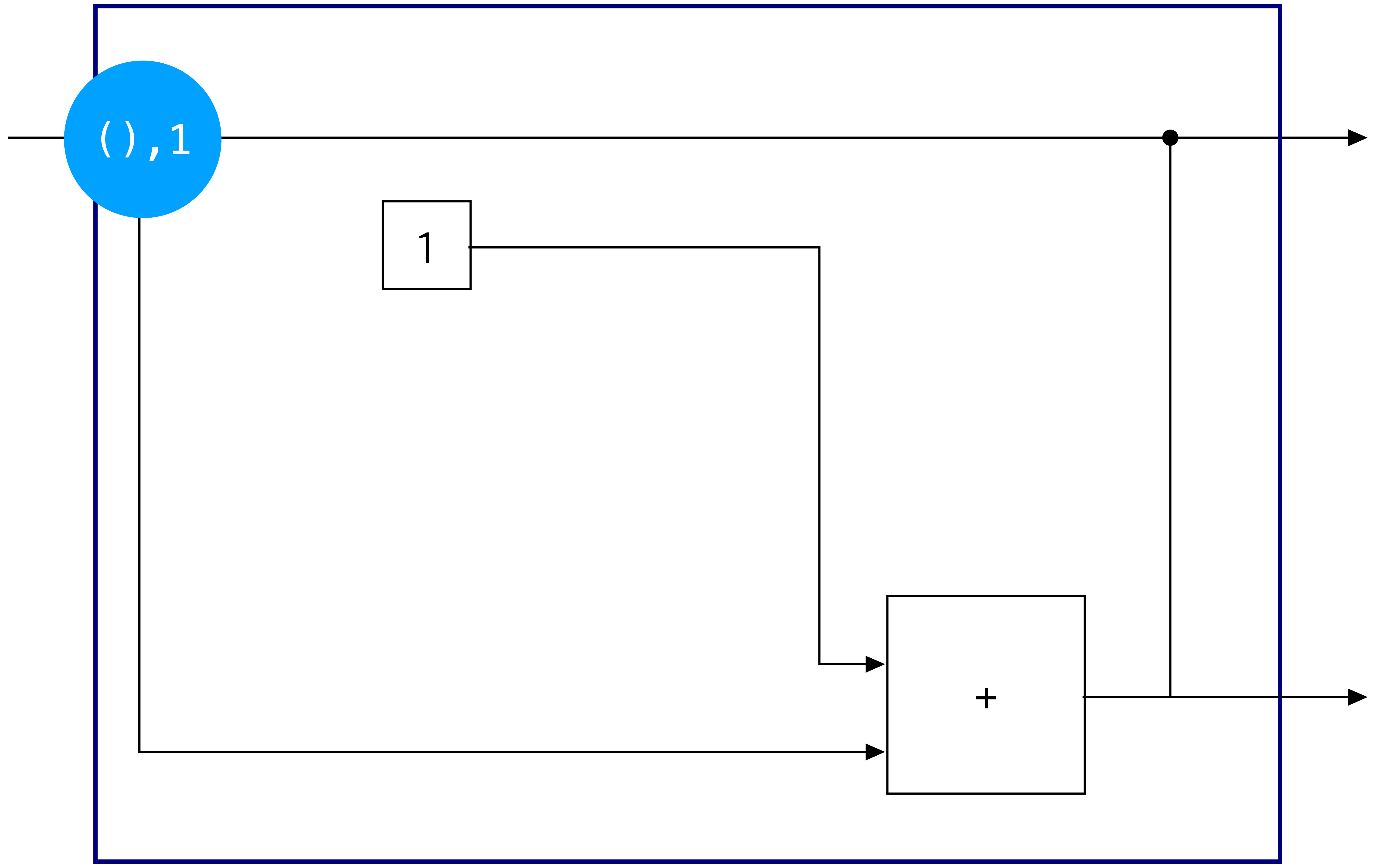

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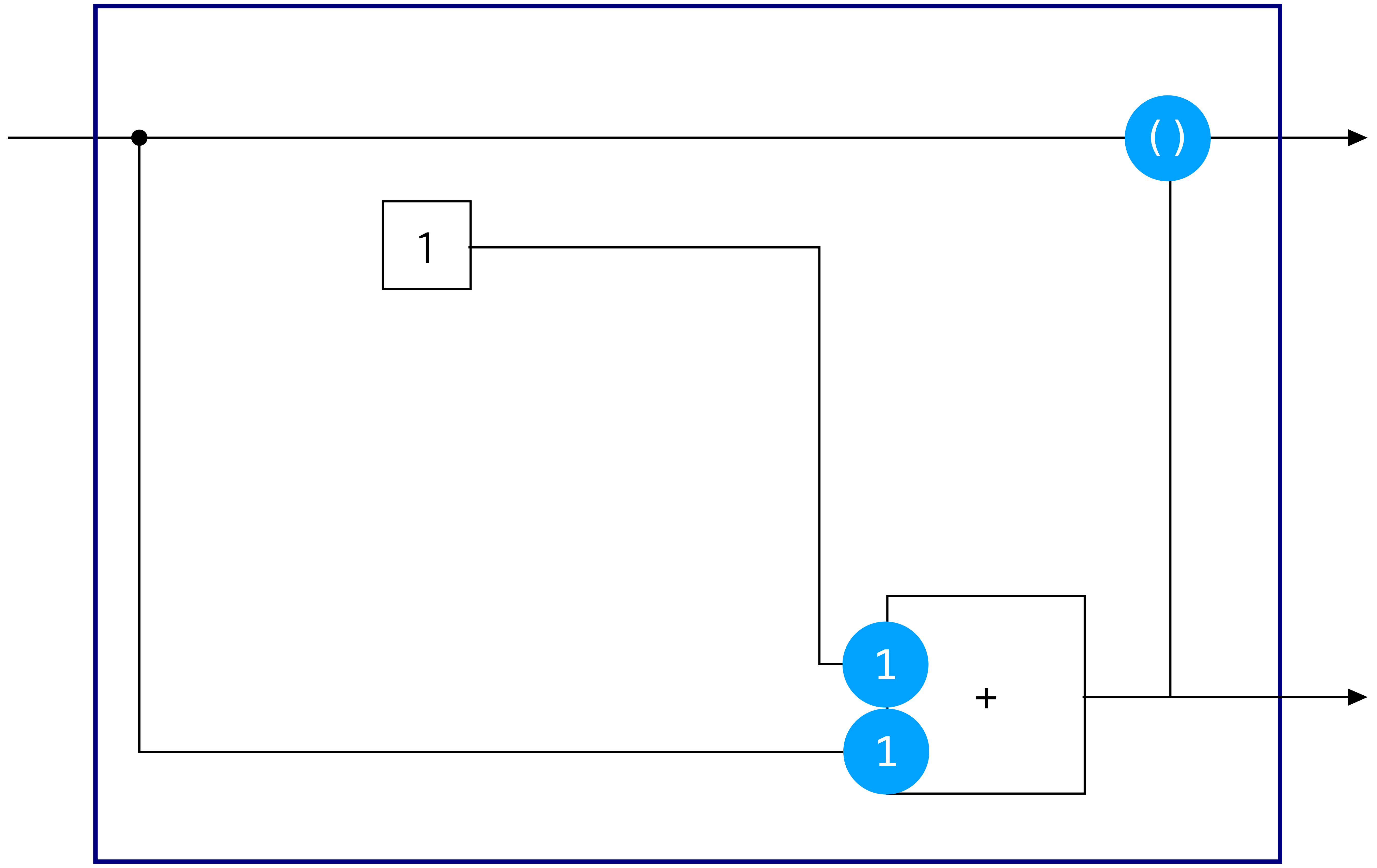

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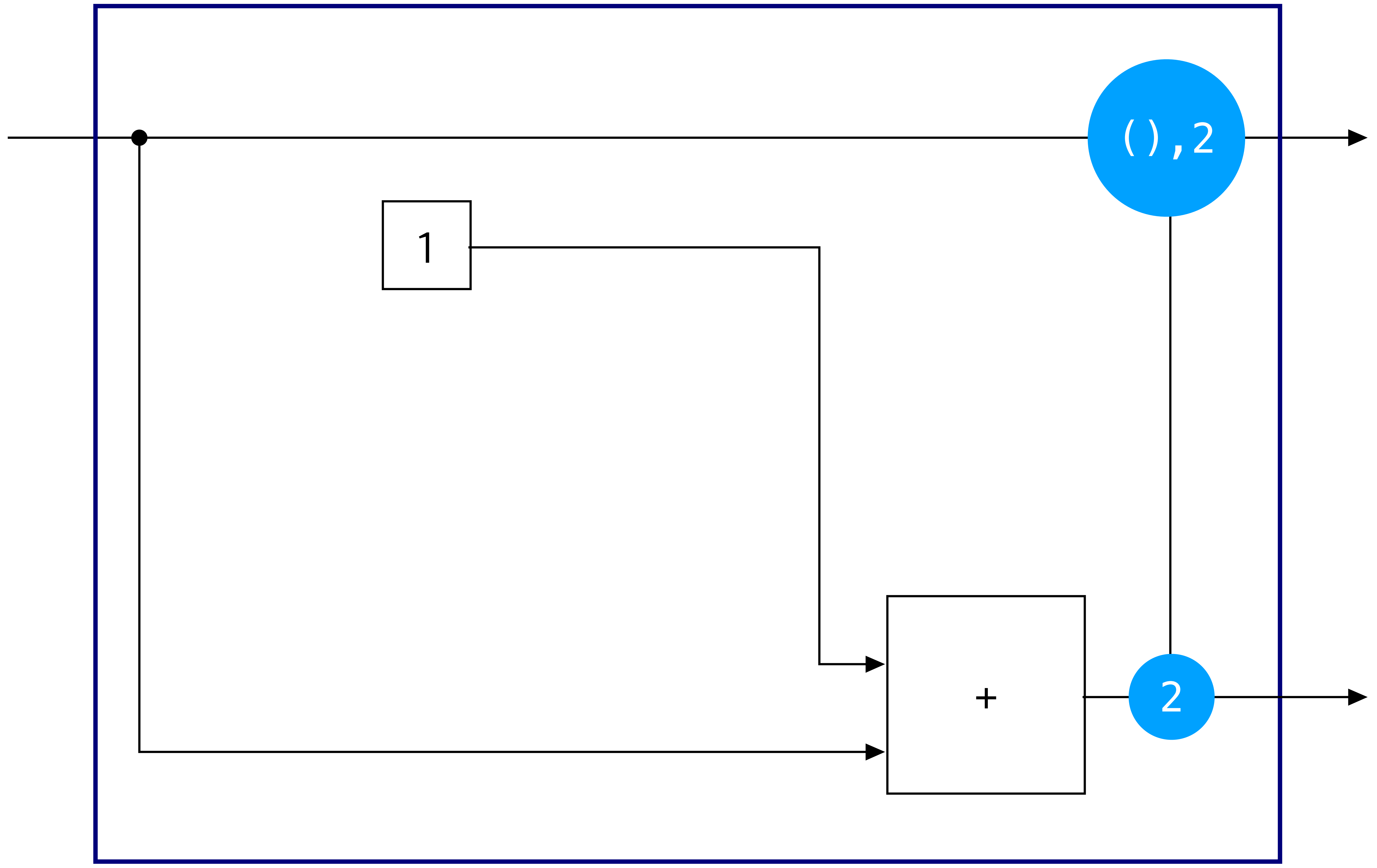

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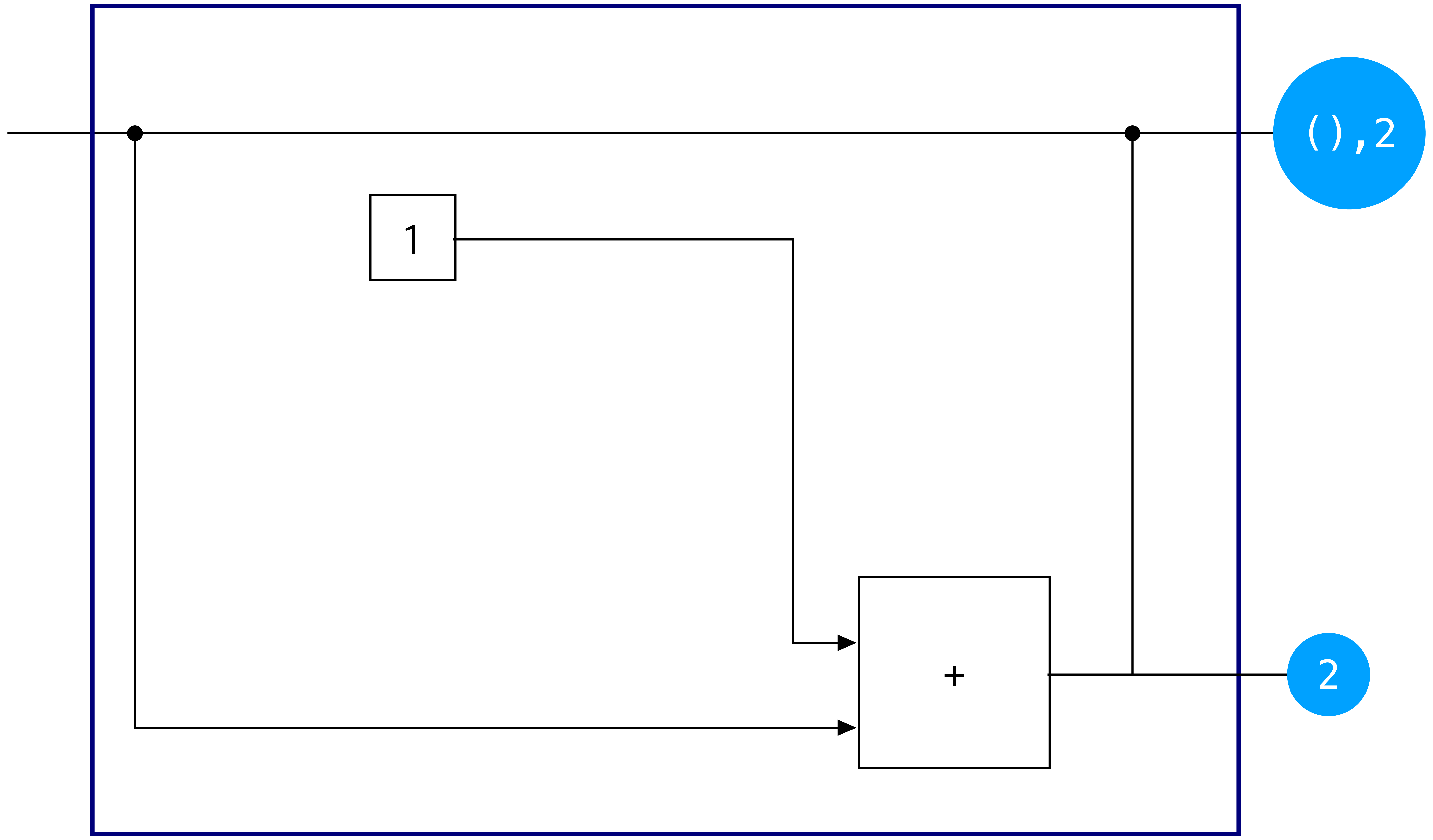

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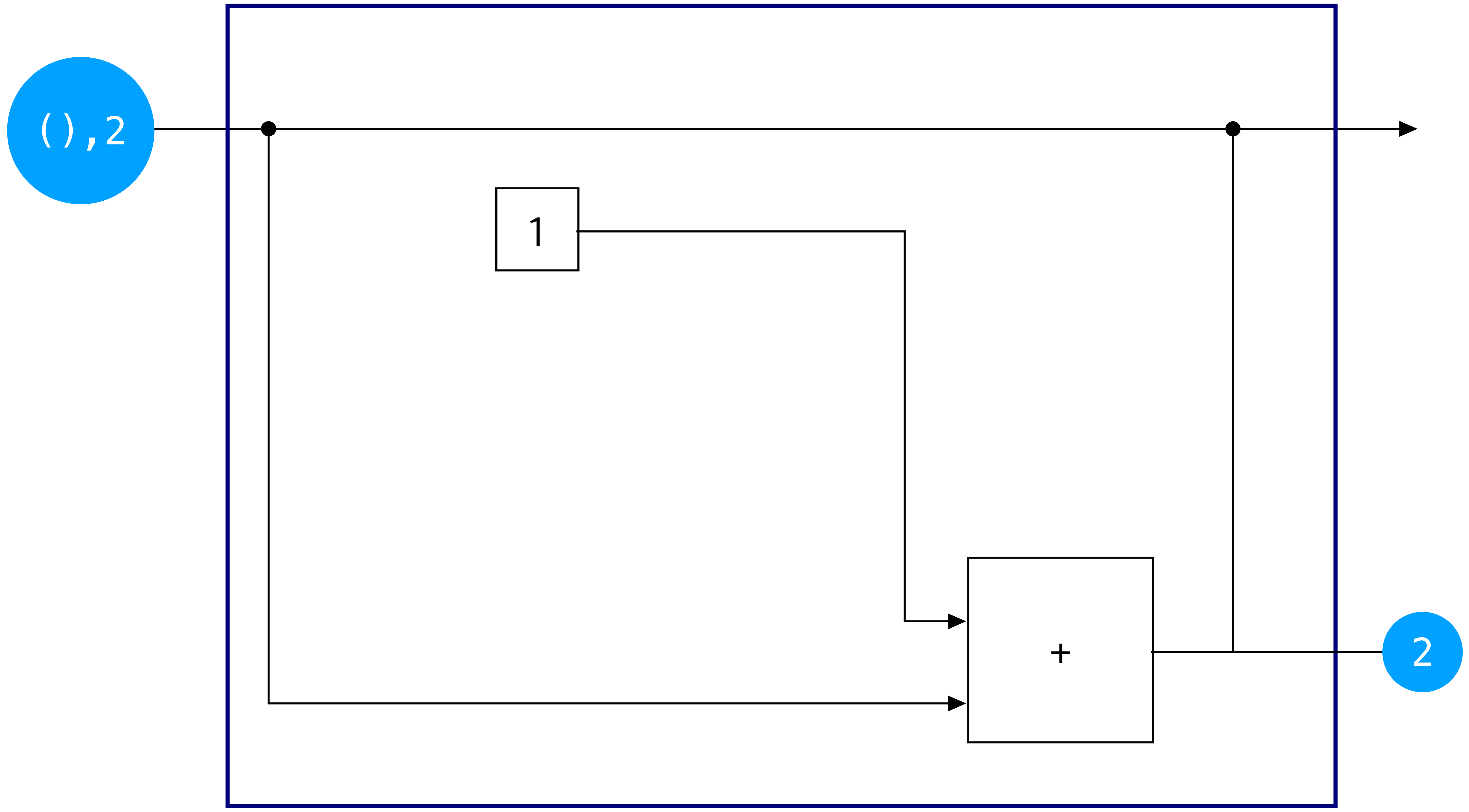
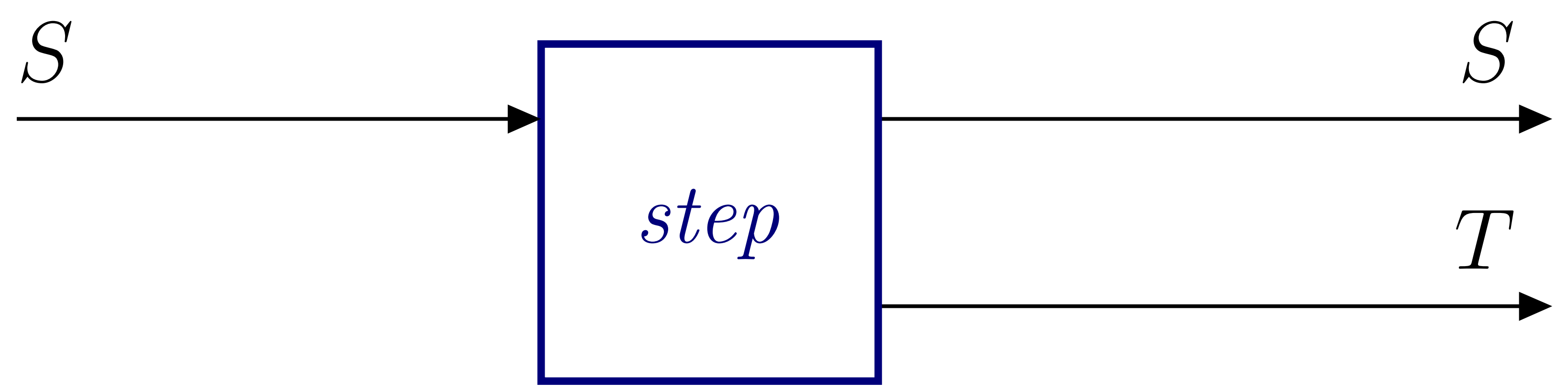


Deterministic vs. probabilistic

Deterministic streams

Transition function returns a pair of state and value

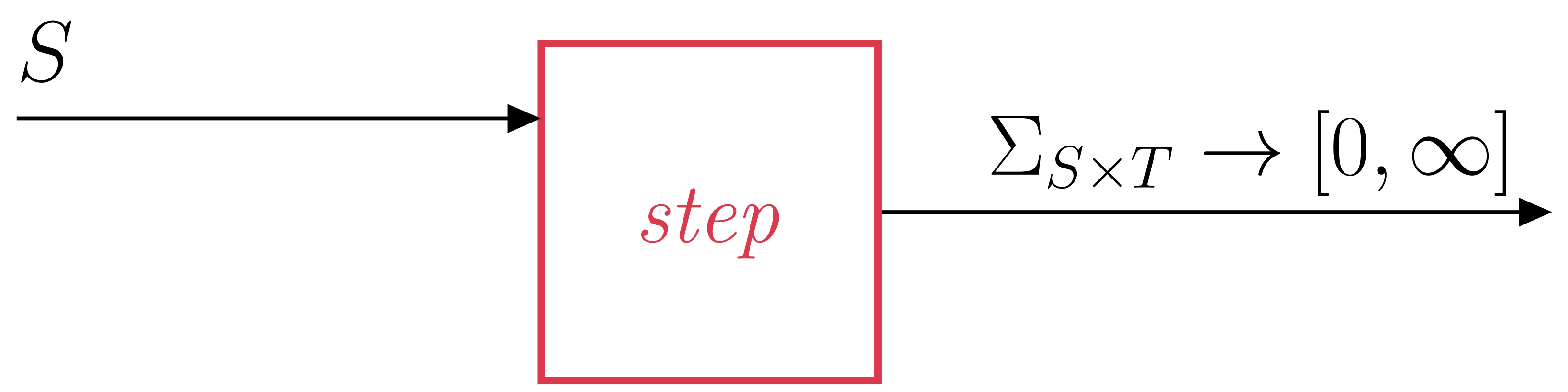
$$\textit{CoStream}(T, S) = S \times (S \to S \times T)$$



Probabilistic streams

Transition function returns a **measure** over (state, value)

$$\textit{CoPStream}(T, S) = S \times (S \to \Sigma_{S \times T} \to [0, \infty])$$



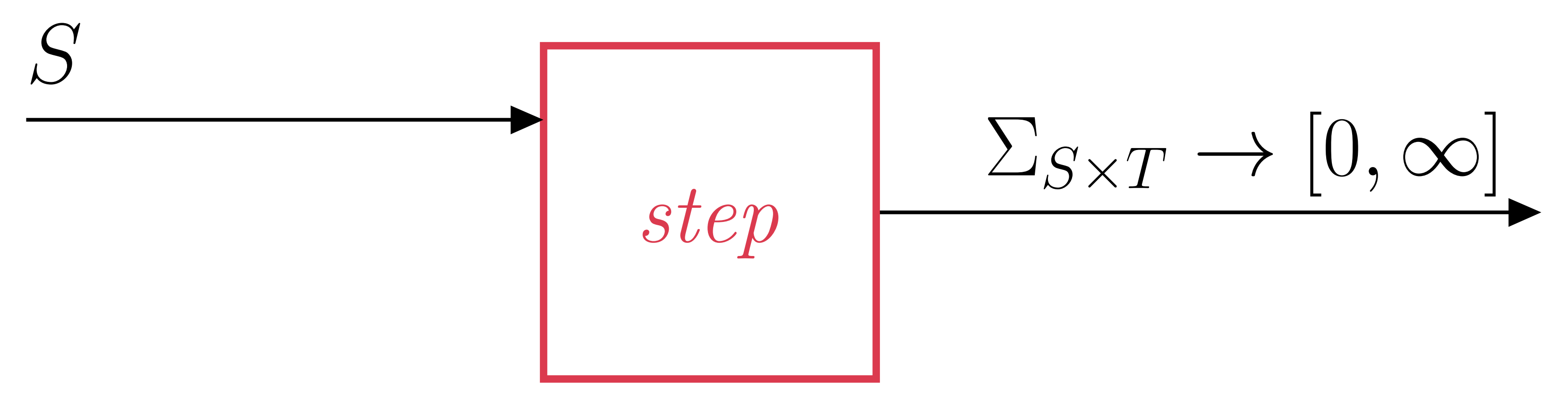
Probabilistic semantics

allocation: S

transition: $S \to \Sigma_{S \times T} \to [0, \infty]$

$$\{\!\![e]\!\!\} = \{\!\![e]\!\!\}^{\text{init}}_{\gamma}\,,\ \{\!\![e]\!\!\}^{\text{step}}_{\gamma}$$

environment

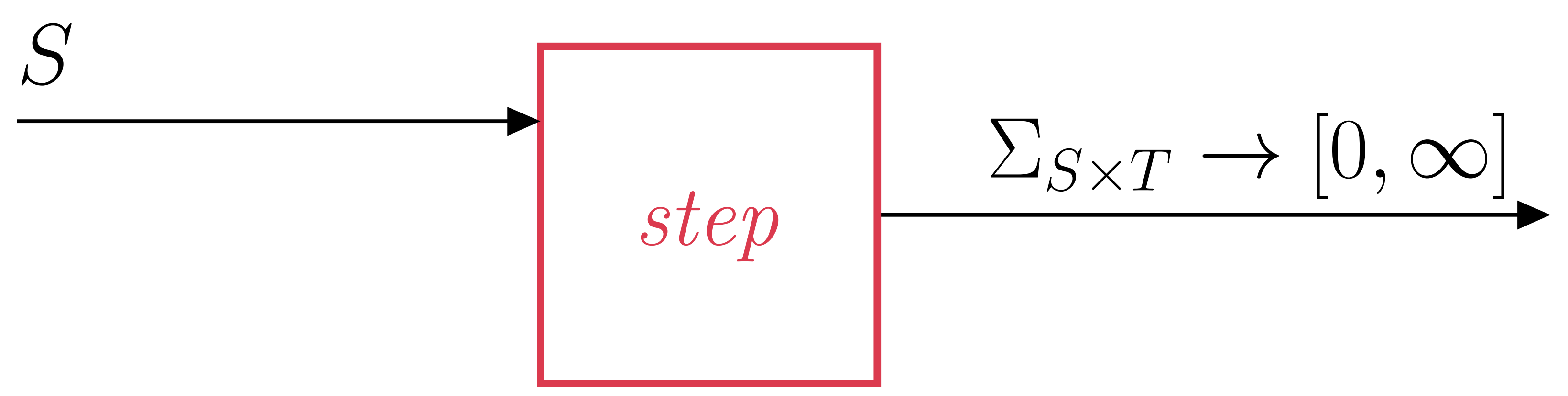


Probabilistic semantics

allocation: S transition: $S \to \Sigma_{S\times T} \to [0,\infty]$

$$\{\!\![e]\!\!\} = \{\!\![e]\!\!\}_{\gamma}^{\text{init}} , \{\!\![e]\!\!\}_{\gamma}^{\text{step}}$$

environment



if e is deterministic

$$
\begin{aligned}
\{\!\![e]\!\!\}_{\gamma}^{\text{init}} &= [\![e]\!]_{\gamma}^{\text{init}}\\
\{\!\![e]\!\!\}_{\gamma}^{\text{step}}(s) &= \textit{let } s', v = [\![e]\!]_{\gamma}^{\text{step}}(s) \textit{ in } \delta_{s',v}\\[1em]
\{\!\![\texttt{sample}(e)]\!\!\}_{\gamma}^{\text{init}} &= [\![e]\!]_{\gamma}^{\text{init}}\\
\{\!\![\texttt{sample}(e)]\!\!\}_{\gamma}^{\text{step}}(s) &= \textit{let } s', \mu = [\![e]\!]_{\gamma}^{\text{step}}(s) \textit{ in } \int \mu(dv)\, \delta_{s',v}\\[1em]
\{\!\![\texttt{observe}(e_1, e_2)]\!\!\}_{\gamma}^{\text{init}} &= [\![e_1]\!]_{\gamma}^{\text{init}}, [\![e_2]\!]_{\gamma}^{\text{init}}\\
\{\!\![\texttt{observe}(e_1, e_2)]\!\!\}_{\gamma}^{\text{step}}(s_1, s_2) &= \textit{let } s_1', \mu = [\![e_1]\!]_{\gamma}^{\text{step}}(s_1) \textit{ in}\\
&\quad\ \textit{let } s_2', v = [\![e_2]\!]_{\gamma}^{\text{step}}(s_2) \textit{ in}\\
&\quad\ \mu_{\text{pdf}}(v) * \delta_{(s_1', s_2'),()}
\end{aligned}
$$

Probabilistic equations

$$\left\{\!\!\left[\begin{array}{l} e\ \texttt{where rec init}\ x = c \\ \qquad\qquad \texttt{and}\ x = e_x \\ \qquad\qquad \texttt{and}\ y = e_y \end{array}\right]\!\!\right\}^{\mathsf{init}}_{\gamma} \quad = \quad c, \left(\{\![e]\!\}^{\mathsf{init}}_{\gamma}, \{\![e_x]\!\}^{\mathsf{init}}_{\gamma}, \{\![e_y]\!\}^{\mathsf{init}}_{\gamma}\right)$$

$$\left\{\!\!\left[\begin{array}{l} e\ \texttt{where rec init}\ x = c \\ \qquad\qquad \texttt{and}\ x = e_x \\ \qquad\qquad \texttt{and}\ y = e_y \end{array}\right]\!\!\right\}^{\mathsf{step}}_{\gamma} (p_x, (m, m_x, m_y)) \quad = \quad \int \{\![e_x]\!\}^{\mathsf{step}}_{\gamma+[x.\texttt{last}\leftarrow p_x]} (m_x)(dm'_x, dv_x) \int \{\![e_y]\!\}^{\mathsf{step}}_{\gamma+[x.\texttt{last}\leftarrow p_x, x\leftarrow v_x]} (m_y)(dm'_y, dv_y) \int \{\![e]\!\}^{\mathsf{step}}_{\gamma+[x.\texttt{last}\leftarrow p_x, x\leftarrow v_x, y\leftarrow v_y]} (m)(dm', d_v)\ \delta_{(v_x, (m', m'_x, m'_y)), v}$$

Probabilistic equations

$$\left\{\!\!\left[\begin{array}{l} e\ \texttt{where rec init}\ x = c \\ \qquad\quad \texttt{and}\ x = e_x \\ \qquad\quad \texttt{and}\ y = e_y \end{array}\right]\!\!\right\}^{\text{init}}_{\gamma} = c, \left(\{\![e]\!\}^{\text{init}}_{\gamma}, \{\![e_x]\!\}^{\text{init}}_{\gamma}, \{\![e_y]\!\}^{\text{init}}_{\gamma}\right)$$

$$\left\{\!\!\left[\begin{array}{l} e\ \texttt{where rec init}\ x = c \\ \qquad\quad \texttt{and}\ x = e_x \\ \qquad\quad \texttt{and}\ y = e_y \end{array}\right]\!\!\right\}^{\text{step}}_{\gamma} (p_x, (m, m_x, m_y)) = \int \{\![e_x]\!\}^{\text{step}}_{\gamma+[x.\texttt{last}\leftarrow p_x]} (m_x)(dm'_x, dv_x)$$

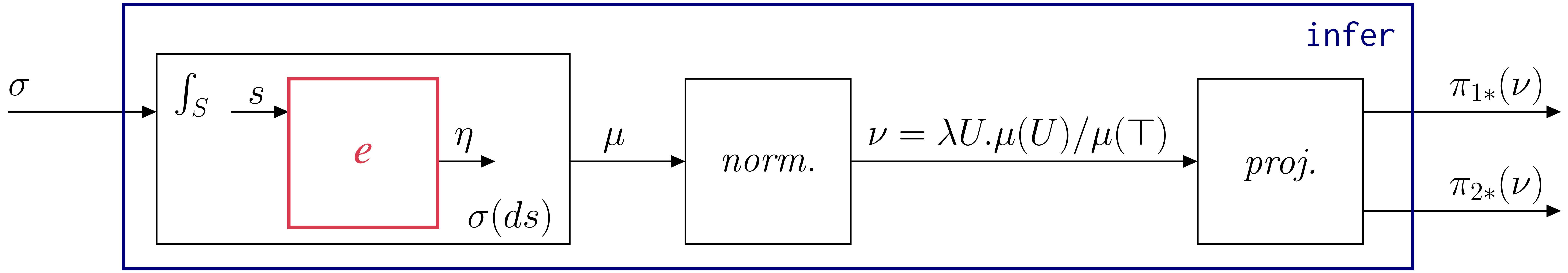
$$\int \{\![e_y]\!\}^{\text{step}}_{\gamma+[x.\texttt{last}\leftarrow p_x, x\leftarrow v_x]} (m_y)(dm'_y, dv_y)$$

$$\int \{\![e]\!\}^{\text{step}}_{\gamma+[x.\texttt{last}\leftarrow p_x, x\leftarrow v_x, y\leftarrow v_y]} (m)(dm', d_v)$$

$$\delta_{(v_x, (m', m'_x, m'_y)), v}$$

Nested integrals require a fixed schedule

Semantics of `infer`



$$\begin{aligned}
[\![\texttt{infer}(e)]\!]^{\text{init}}_{\gamma} &= \delta_{[\![e]\!]^{\text{init}}_{\gamma}} \\
[\![\texttt{infer}(e)]\!]^{\text{step}}_{\gamma}(\sigma) &= \textit{let } \nu = \int \sigma(dm)\, \{\![e]\!\}^{\text{step}}_{\gamma}(m) \textit{ in let } \overline{\nu} = \nu/\nu(\top) \textit{ in} \\
&\quad (\pi_{1*}(\overline{\nu}), \pi_{2*}(\overline{\nu}))
\end{aligned}$$

Compilation

Reactive Probabilistic Programming

Target

Simplified syntax

```
x ::= variables
c ::= constants

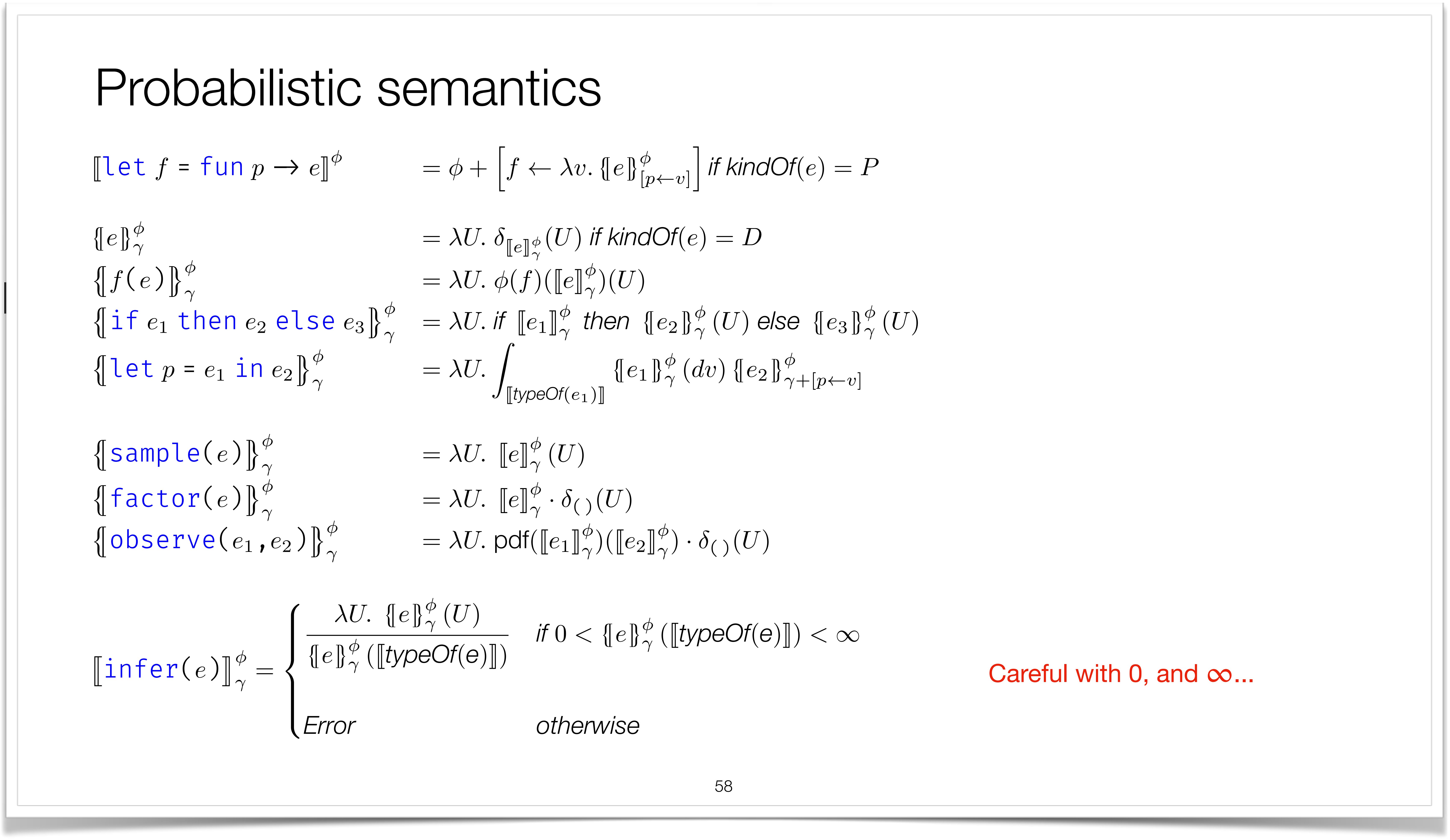
d ::= let p = e | let f = fun p → e | d d
p ::= x | (p, p)
e ::= c | x | (e, e) | op (e) | f (e)
    | if e then e else e | let p = e in e
    | sample (e) | factor (e) | observe (e, e)
    | infer (e)
```

Target

Simplified syntax

```
x ::= variables
c ::= constants

d ::= let p = e | let f =
p ::= x | (p, p)
e ::= c | x | (e, e) | op
    | if e then e else e |
    | sample (e) | factor
    | infer (e)
```



Probabilistic semantics

$$[\![\texttt{let } f \texttt{ = fun } p \to e]\!]^{\phi} = \phi + \left[f \leftarrow \lambda v.\ \{\!\{e\}\!\}^{\phi}_{[p\leftarrow v]}\right] \textit{ if kindOf}(e) = P$$

$$\{\!\{e\}\!\}^{\phi}_{\gamma} = \lambda U.\ \delta_{[\![e]\!]^{\phi}_{\gamma}}(U) \textit{ if kindOf}(e) = D$$

$$\{\!\{f(e)\}\!\}^{\phi}_{\gamma} = \lambda U.\ \phi(f)([\![e]\!]^{\phi}_{\gamma})(U)$$

$$\{\!\{\texttt{if } e_1 \texttt{ then } e_2 \texttt{ else } e_3\}\!\}^{\phi}_{\gamma} = \lambda U.\ \textit{if } [\![e_1]\!]^{\phi}_{\gamma} \textit{ then } \{\!\{e_2\}\!\}^{\phi}_{\gamma}(U) \textit{ else } \{\!\{e_3\}\!\}^{\phi}_{\gamma}(U)$$

$$\{\!\{\texttt{let } p \texttt{ = } e_1 \texttt{ in } e_2\}\!\}^{\phi}_{\gamma} = \lambda U.\ \int_{[\![\textit{typeOf}(e_1)]\!]} \{\!\{e_1\}\!\}^{\phi}_{\gamma}(dv)\ \{\!\{e_2\}\!\}^{\phi}_{\gamma+[p\leftarrow v]}$$

$$\{\!\{\texttt{sample}(e)\}\!\}^{\phi}_{\gamma} = \lambda U.\ [\![e]\!]^{\phi}_{\gamma}(U)$$

$$\{\!\{\texttt{factor}(e)\}\!\}^{\phi}_{\gamma} = \lambda U.\ [\![e]\!]^{\phi}_{\gamma} \cdot \delta_{()}(U)$$

$$\{\!\{\texttt{observe}(e_1, e_2)\}\!\}^{\phi}_{\gamma} = \lambda U.\ \mathrm{pdf}([\![e_1]\!]^{\phi}_{\gamma})([\![e_2]\!]^{\phi}_{\gamma}) \cdot \delta_{()}(U)$$

$$[\![\texttt{infer}(e)]\!]^{\phi}_{\gamma} = \begin{cases} \dfrac{\lambda U.\ \{\!\{e\}\!\}^{\phi}_{\gamma}(U)}{\{\!\{e\}\!\}^{\phi}_{\gamma}([\![\textit{typeOf(e)}]\!])} & \textit{if } 0 < \{\!\{e\}\!\}^{\phi}_{\gamma}([\![\textit{typeOf(e)}]\!]) < \infty \\ \textit{Error} & \textit{otherwise} \end{cases}$$

Careful with 0, and ∞...

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Compilation = Allocation + Transition

Synchronous languages

- Static analyses (typing, causality, initialization)
- Normalization, scheduling (`rec x = y + 1 and y = 0 -> pre x`)
- Compilation

Memory can be statically allocated

$$
\begin{tikzcd}
 & \text{transition: } S \to S \times T \\
e \mapsto \mathcal{A}(e), \mathcal{C}(e) \arrow[ur] \arrow[d] & \\
\text{allocation: } S &
\end{tikzcd}
$$

Allocation

$$\begin{aligned}
\mathcal{A}(x) &= (\,) \\
\mathcal{A}((e_1, e_2)) &= (\mathcal{A}(e_1), \mathcal{A}(e_2)) \\
\mathcal{A}(\texttt{present}\ e \rightarrow e_1\ \texttt{else}\ e_2) &= (\mathcal{A}(e), \mathcal{A}(e_1), \mathcal{A}(e_2)) \\
\mathcal{A}(\mathit{op}(e)) &= \mathcal{A}(e) \\
\mathcal{A}(f(e)) &= (f\texttt{\_init}, \mathcal{A}(e)) \\
\mathcal{A}(\texttt{sample}(e)) &= \mathcal{A}(e) \\
\mathcal{A}(\texttt{factor}(e)) &= \mathcal{A}(e) \\
\mathcal{A}(\texttt{observe}(e_1, e_2)) &= (\mathcal{A}(e_1), \mathcal{A}(e_2)) \\
\mathcal{A}(\texttt{infer}(e)) &= \mathcal{A}(e)
\end{aligned}$$

Allocation

$$
\begin{array}{lcl}
\mathcal{A}(x) & = & (\,) \\
\mathcal{A}((e_1, e_2)) & = & (\mathcal{A}(e_1), \mathcal{A}(e_2)) \\
\mathcal{A}(\texttt{present}\ e \rightarrow e_1\ \texttt{else}\ e_2) & = & (\mathcal{A}(e), \mathcal{A}(e_1), \mathcal{A}(e_2)) \\
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\mathcal{A}(f(e)) & = & (f\texttt{\_init}, \mathcal{A}(e)) \\
\mathcal{A}(\texttt{sample}(e)) & = & \mathcal{A}(e) \\
\mathcal{A}(\texttt{factor}(e)) & = & \mathcal{A}(e) \\
\mathcal{A}(\texttt{observe}(e_1, e_2)) & = & (\mathcal{A}(e_1), \mathcal{A}(e_2)) \\
\mathcal{A}(\texttt{infer}(e)) & = & \mathcal{A}(e)
\end{array}
$$

$$
\mathcal{A}\left(\begin{array}{l}
e\ \texttt{where} \\
\quad \texttt{rec init}\ x = c_x \\
\quad \texttt{and init}\ y = c_y \\
\quad \texttt{and}\ x = e_x \\
\quad \texttt{and}\ y = e_y
\end{array}\right)
=
\left(\begin{array}{c}
(c_x, c_y), \\
(\mathcal{A}(e_1), \mathcal{A}(e_2)), \\
\mathcal{A}(e)
\end{array}\right)
$$

Transition

```
C(c) = fun s -> (c, s)
C(x) = fun s -> (x, s)
C(last x) = fun s -> (x_last, s)

C((e1, e2)) = fun (s1,s2) ->
  let v1,s1' = C(e1)(s1) in
  let v2,s2' = C(e2)(s2) in
  ((v1,v2), (s1',s2'))

C(op(e)) = fun s ->
  let v,s' = C(e)(s) in
  (op(v), s')

C(f(e)) = fun (s1,s2) ->
  let v1,s1' = C(e)(s1) in
  let v2,s2' = f_step(s2,v) in
  (v2, (s1',s2'))

C(present e -> e1 else e2) =
fun (s,s1,s2) ->
  let v, s' = C(e)(s) in
  if v then let v1,s1' = C(e1)(s1) in
    (v1, (s',s1',s2))
  else let v2,s2' = C(e2)(s2) in
    (v2, (s',s1,s2'))
```

```
C(sample(e)) = fun s ->
  let mu,s' = C(e)(s) in
  let v = sample(mu) in (v, s')

C(observe(e1, e2)) = fun (s1,s2) ->
  let v1,s1' = C(e1)(s1) in
  let v2,s2' = C(e2)(s2) in
  let _ = observe(v1,v2) in
  ((), (s1',s2'))

C(factor(e)) = fun s ->
  let v,s' = C(e)(s) in
  let _ = factor(v) in ((), s')

C(infer(e)) = fun sigma ->
  let mu,sigma' = infer(C(e), sigma) in
  (mu, sigma')
```

Transition

```
C(c) = fun s -> (c, s)
C(x) = fun s -> (x, s)
C(last x) = fun s -> (x_last, s)

C((e1, e2)) = fun (s1,s2) ->
  let v1,s1' = C(e1)(s1) in
  let
  ((v

C(op(
  let
  (op

C(f(e
  let
  let
  (v2
```

```
C(present e -> e1 else e2) =
fun (s,s1,s2) ->
  let v, s' = C(e)(s) in
  if v then let v1,s1' = C(e1)(s1) in
    (v1, (s',s1',s2))
  else let v2,s2' = C(e2)(s2) in
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```

```
C(present e -> e1 else e2) =
fun (s,s1,s2) ->
  let v, s' = C(e)(s) in
  if v then let v1,s1' = C(e1)(s1) in
    (v1, (s',s1',s2))
  else let v2,s2' = C(e2)(s2) in
    (v2, (s',s1,s2'))
```

```
C(sample(e)) = fun s ->
  let mu,s' = C(e)(s) in
  let v = sample(mu) in (v, s')

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  let _ = observe(v1,v2) in
  ((), (s1',s2'))

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```

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C(c) = fun s -> (c, s)
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  let
  ((v

C(op(
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  (op

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  let
  let
  (v2
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  let v, s' = C(e)(s) in
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    (v1, (s',s1',s2))
  else let v2,s2' = C(e2)(s2) in
    (v2, (s',s1,s2'))
```

```
C(sample(e)
  let mu,s'
  let v = s

C(observe(e
  let v1,s1
  let v2,s2
  let _ = c
  ((), (s1'

factor(e)
  let v,s'
  let _ = f

infer(e))
  let mu,si
  (mu, sigm
```

```
C(sample(e)) = fun s ->
  let mu,s' = C(e)(s) in
  let v = sample(mu) in (v, s')

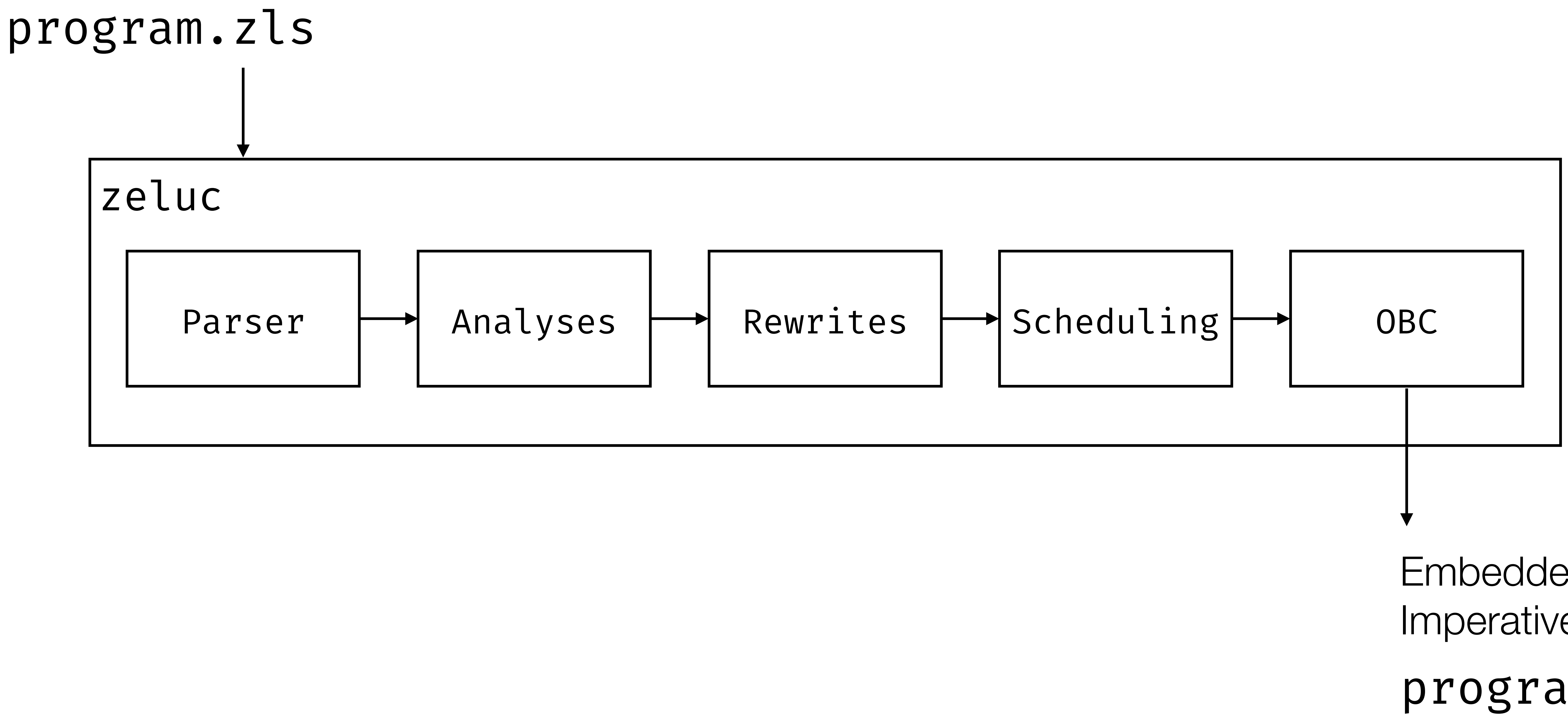
C(observe(e1, e2)) = fun (s1,s2) ->
  let v1,s1' = C(e1)(s1) in
  let v2,s2' = C(e2)(s2) in
  let _ = observe(v1,v2) in
  ((), (s1',s2'))

C(factor(e)) = fun s ->
  let v,s' = C(e)(s) in
  let _ = factor(v) in ((), s')

C(infer(e)) = fun sigma ->
  let mu,sigma' = infer(C(e), sigma) in
  (mu, sigma')
```

```
C(present e -> e1 else e2) =
fun (s,s1,s2) ->
  let v, s' = C(e)(s) in
  if v then let v1,s1' = C(e1)(s1) in
    (v1, (s',s1',s2))
  else let v2,s2' = C(e2)(s2) in
    (v2, (s',s1,s2'))
```

The Zelus compiler



Bourke, Pouzet 2013

Generated code

```
(* a synchronous stream function with type 'a -D→ 'b *)
(* is represented by an OCaml value of type ('a, 'b) node *)
type ('a, 'b) node =
    Node:
      { alloc : unit → 's; (* allocate the state *)
        step : 's → 'a → 'b; (* compute a step *)
        reset : 's → unit; (* reset/initialize the state *)
      } → ('a, 'b) cnode

(*
  let m = alloc () in
  reset m;
  while true do
    let o = step m i in ...
  done
*)
```

https://github.com/INRIA/zelus/blob/main/lib/std/ztypes.ml

Streaming inference

Reactive Probabilistic Programming

Particle filter

Approximate inference algorithm : importance sampling, but...

- Add a resampling step at each `observe`
- Compute the score of the particles to compute a distribution
- Re-sample a new set of particles from this distribution

How can we duplicate a particle during execution?

- Continuation Passing Style (CPS)?
- Clone the memory state?

Particle filter

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- Compute the score of the particles to compute a distribution
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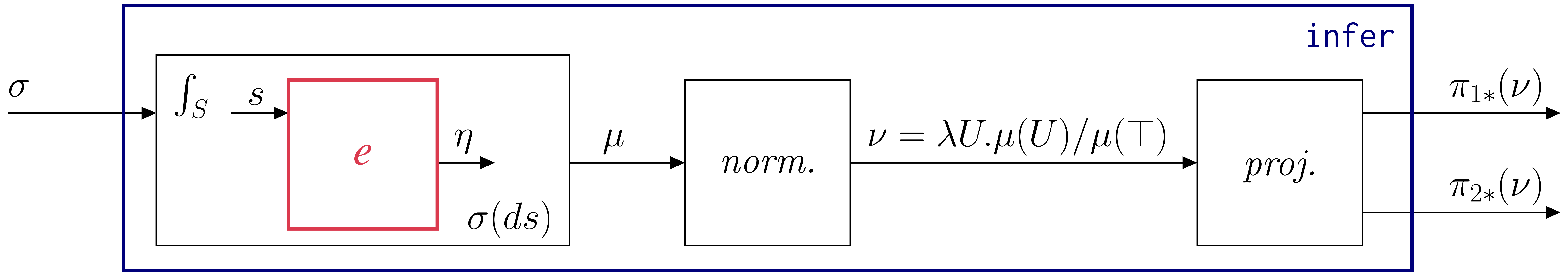
How can we duplicate a particle during execution?

- Continuation Passing Style (CPS)?
- Clone the memory state?

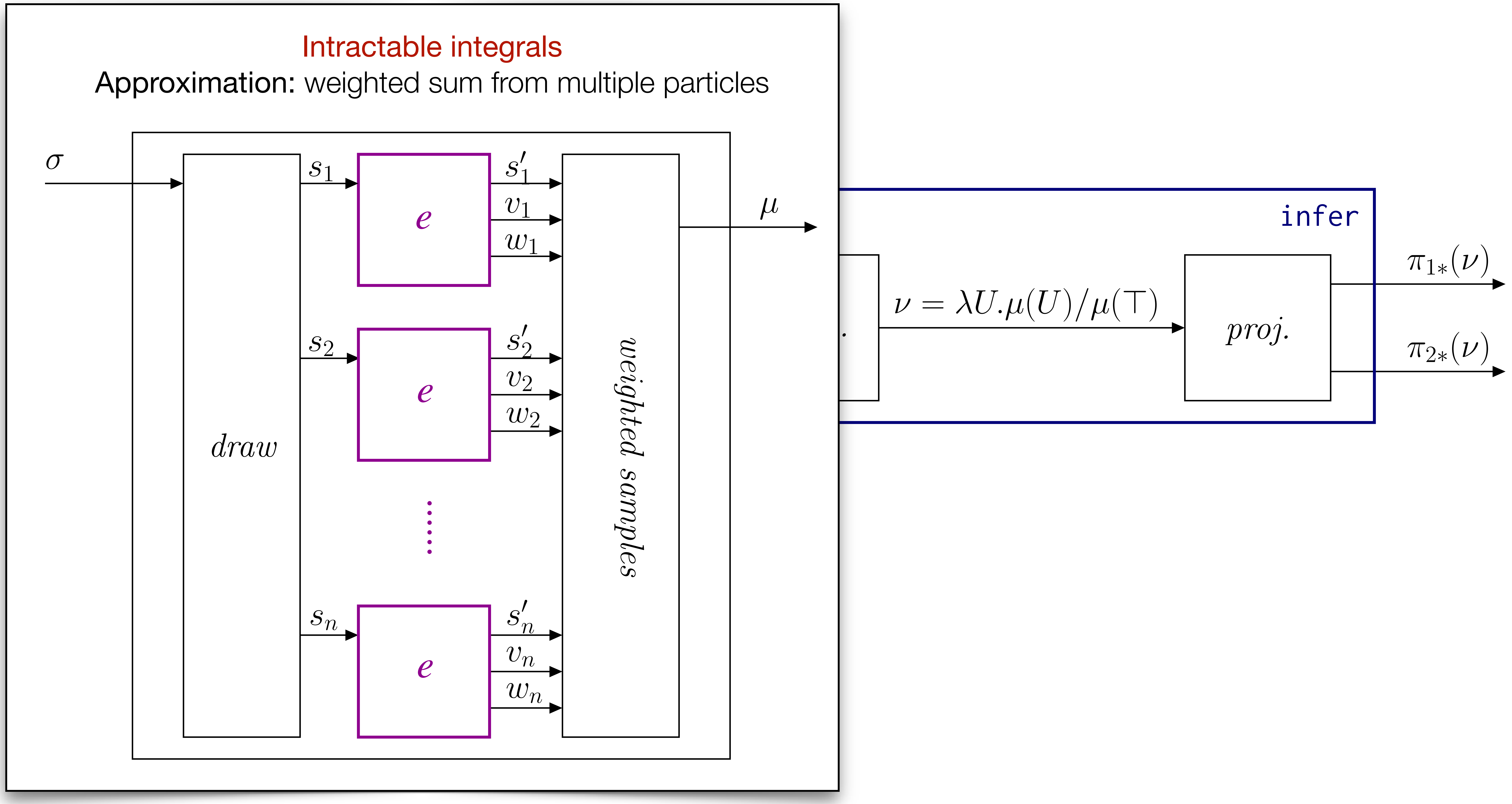
```
(* the same with a method copy *)
type ('a, 'b) cnode =
    Cnode:
      { alloc : unit → 's; (* allocate the state *)
        copy : 's → 's → unit; (* copy the source into the destination *)
        step : 's → 'a → 'b; (* compute a step *)
        reset : 's → unit; (* reset/initialize the state *)
      } → ('a, 'b) cnode
```

https://github.com/INRIA/zelus/blob/main/lib/std/ztypes.ml

Particle filter



Particle filter



Particle filter

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

Particle filter

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) -> gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

Particle filter

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10)) -> gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

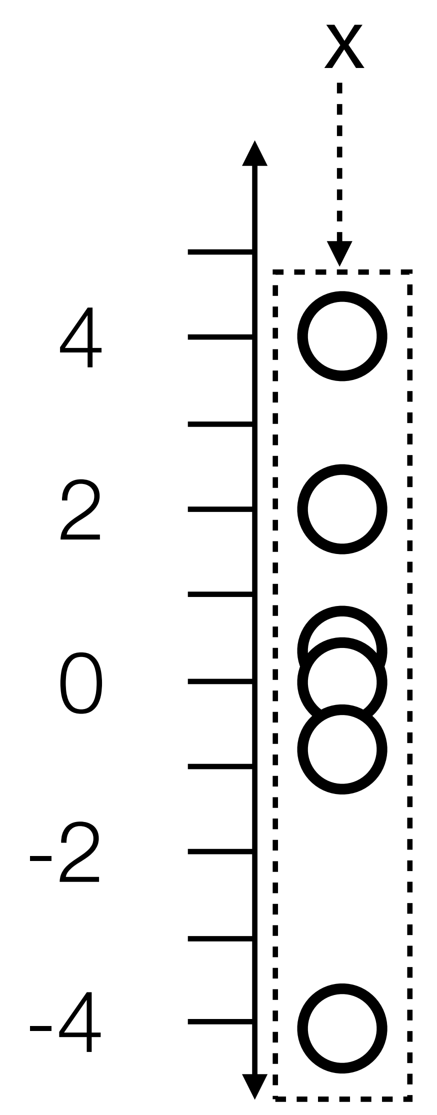
```
sample (gaussian (0, 10))
```

Particle filter

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10)) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
```

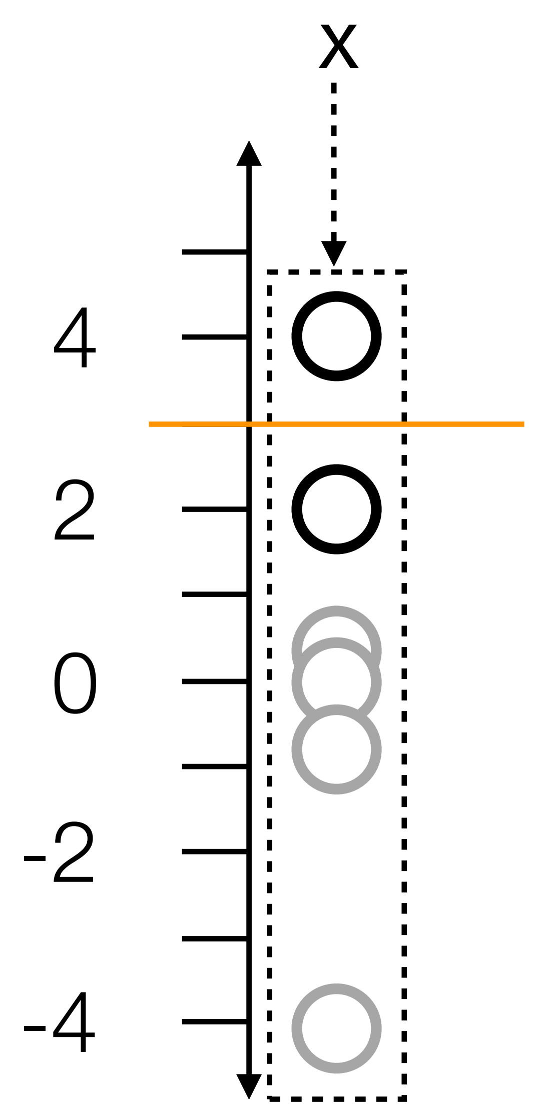


Particle filter

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```



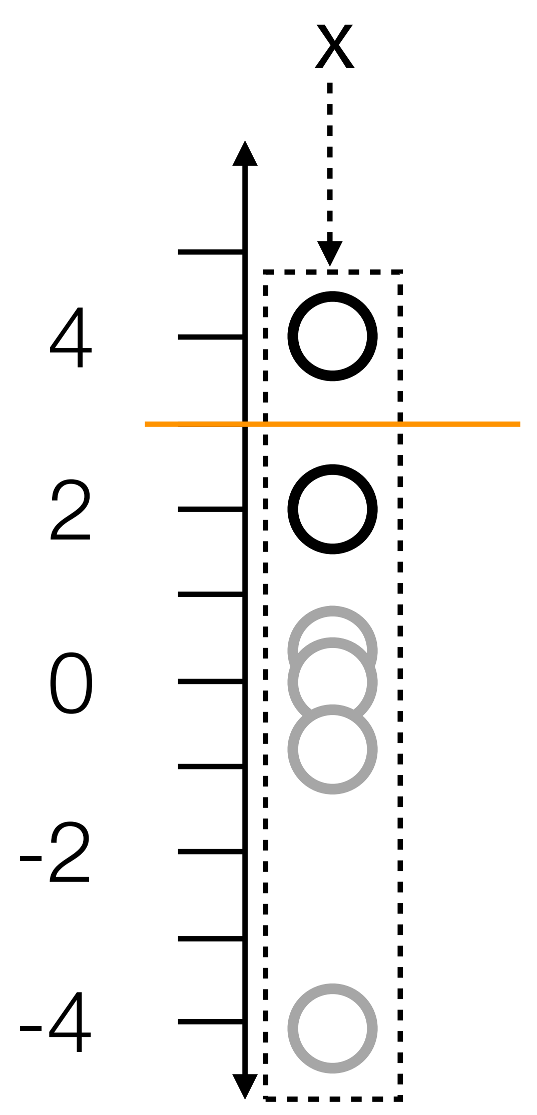
Particle filter

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

$t = 1$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```



Particle filter

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

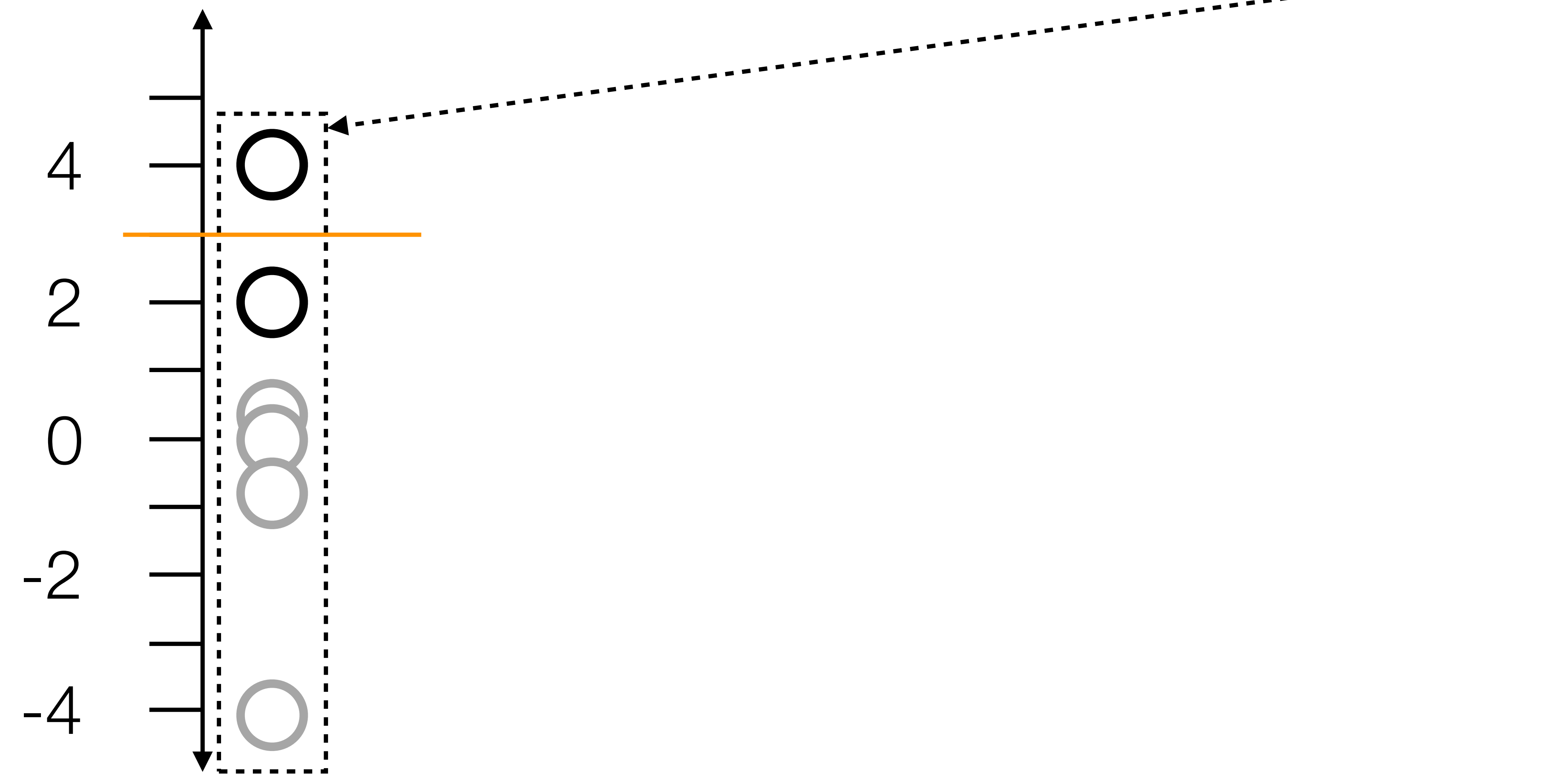
```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
```

x

pre x   x



Particle filter

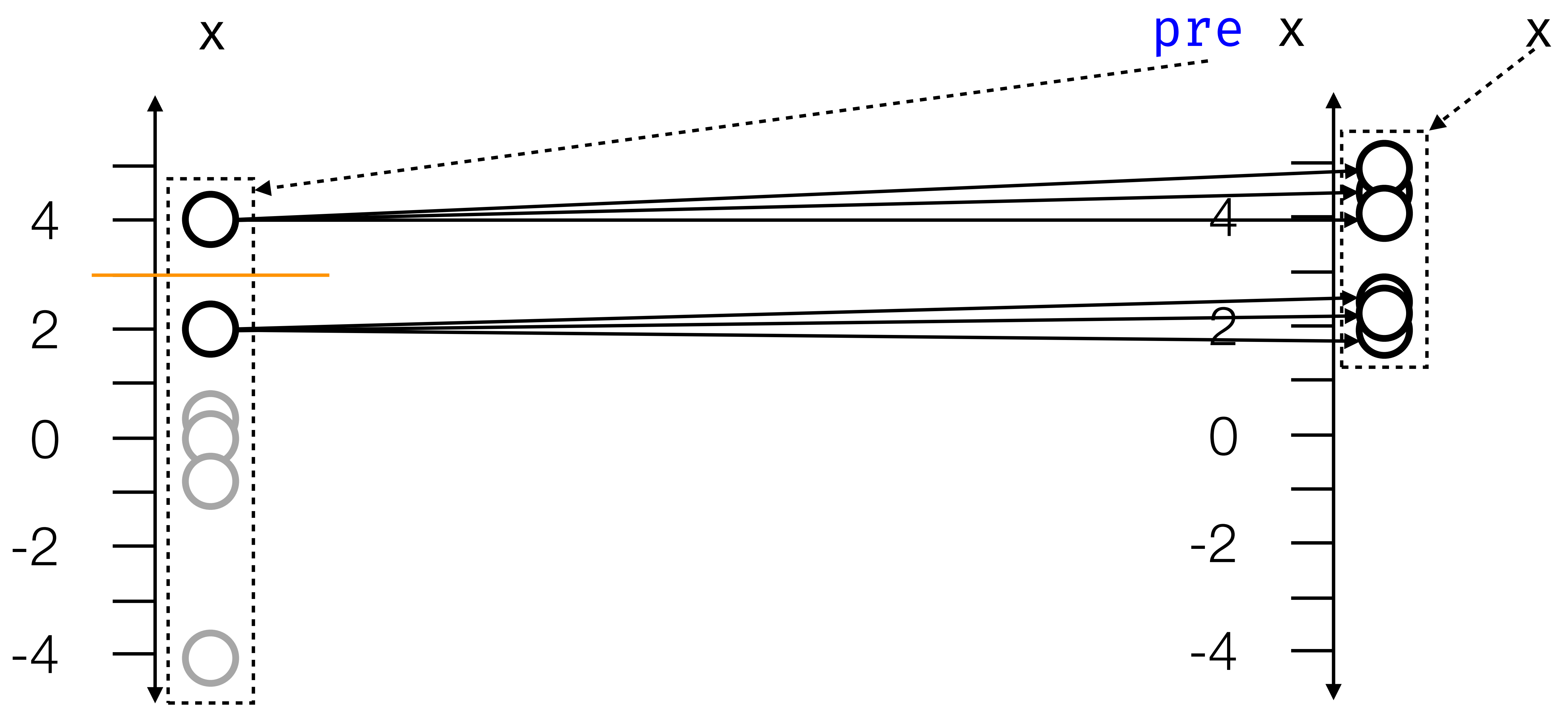
```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
```



Particle filter

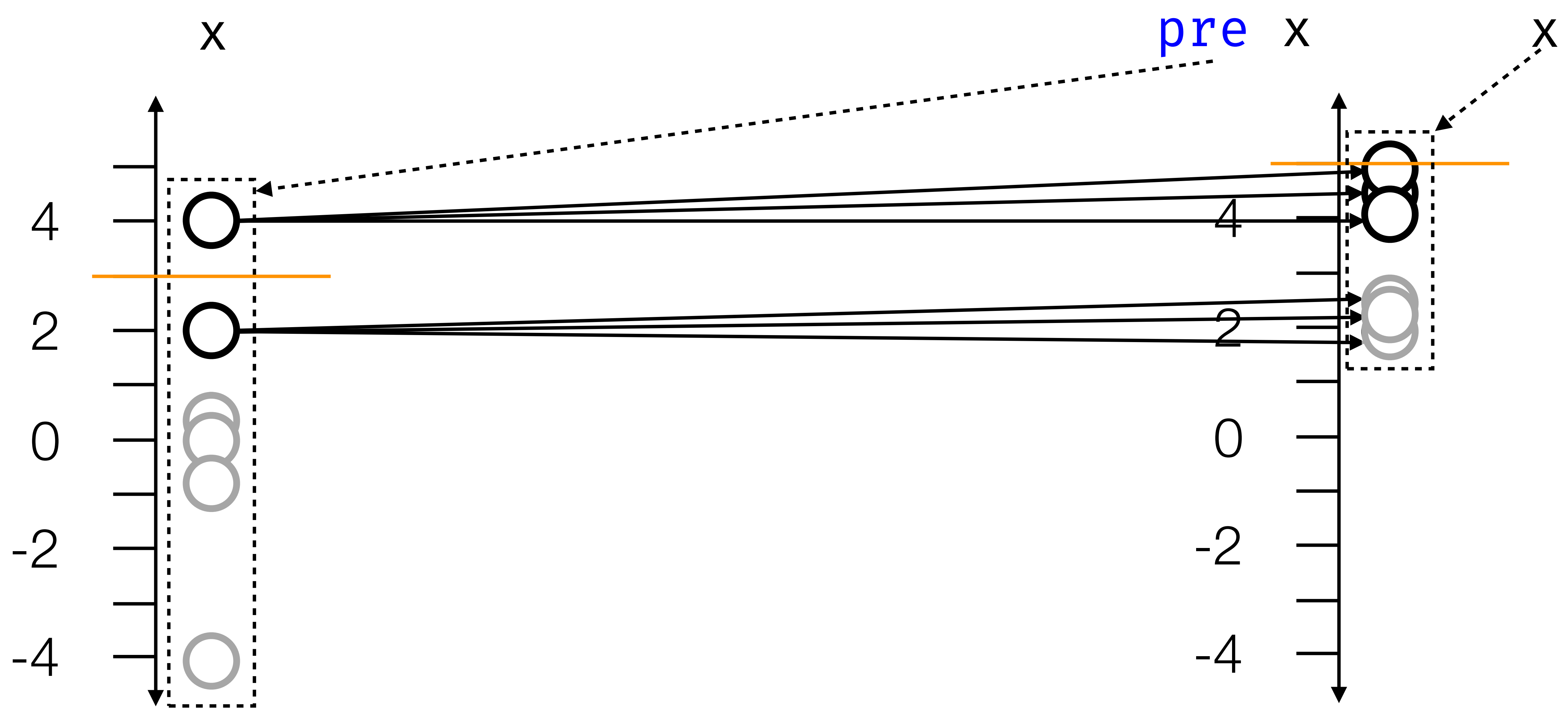
```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), 5)
```



Particle filter

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

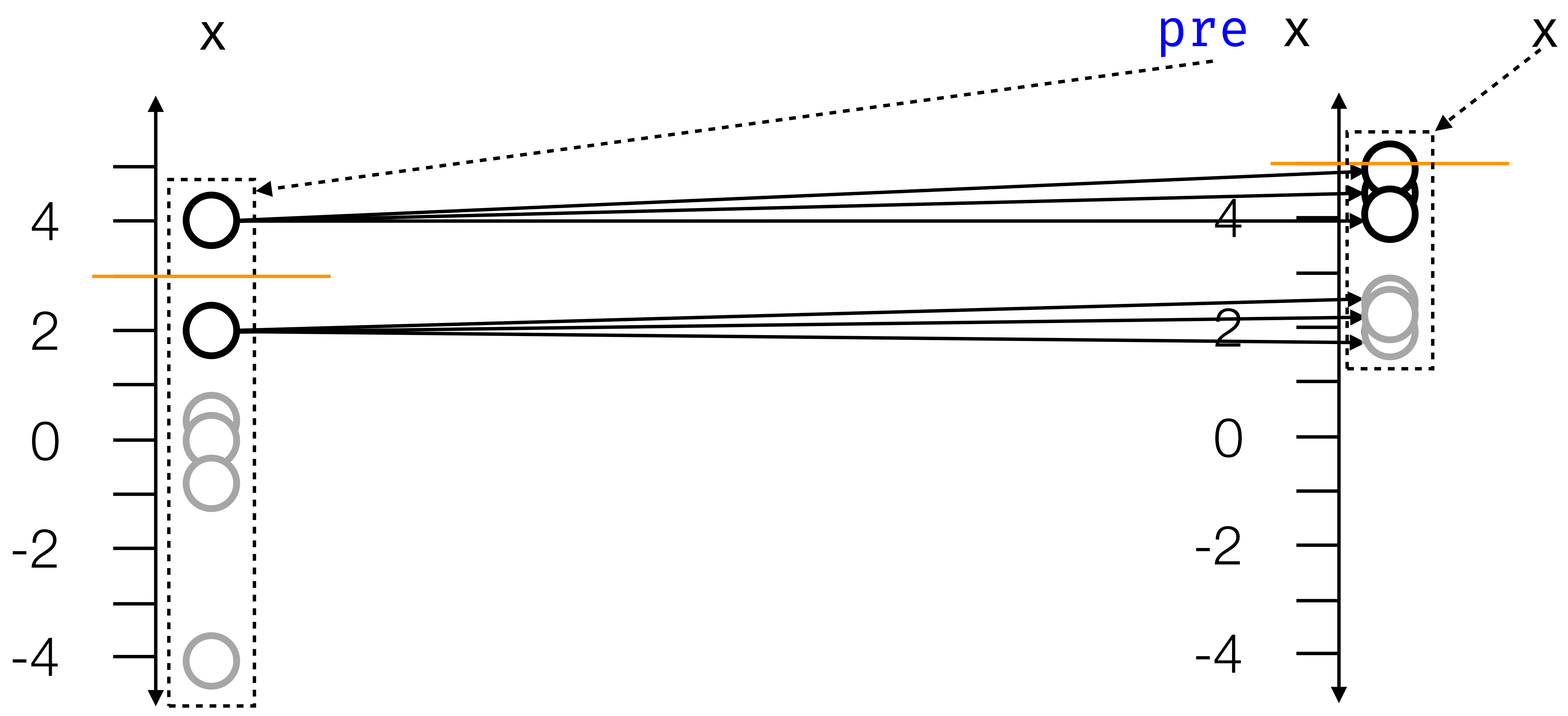
```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), 5)
```

$t = 2$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), …)
```



…

Delayed sampling

Simple Particles Filters can be impractical

- Require lot of computing power
- Poor approximation

Exact inference is often possible

Semi-Symbolic inference

- Perform as much exact computation as possible
- Fall back to a Particle Filter when symbolic computation fails

Main idea

- Keep track of conjugacy relationships
- Incorporate observations analytically
- Sample only when necessary

Murray et al. 2018

Delayed sampling

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Main idea

- Keep track of conjugacy relationships
- Incorporate observations analytically
- Sample only when necessary

Example: Conjugate Gaussians

$$x \sim \mathcal{N}(\mu_0, \sigma_0)$$
$$y \sim \mathcal{N}(x, \sigma)$$

$$x \mid (y = v) \sim \mathcal{N}(\mu_1, \sigma_1)$$
$$\mu_1 = \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2}\right)^{-1} \left(\frac{\mu_0}{\sigma_0^2} + \frac{v}{\sigma^2}\right)$$
$$\sigma_1 = \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2}\right)^{-2}$$

Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10)) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
```

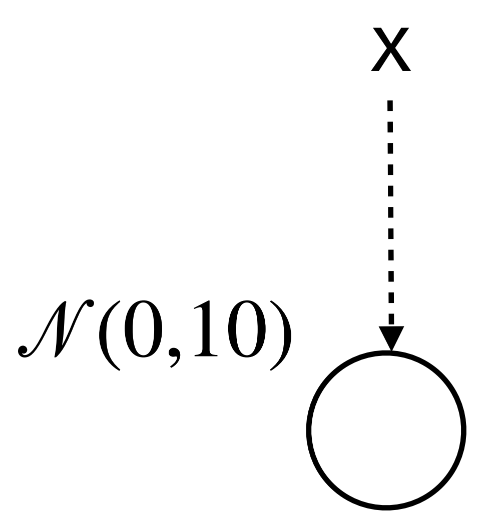
Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10)) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
```



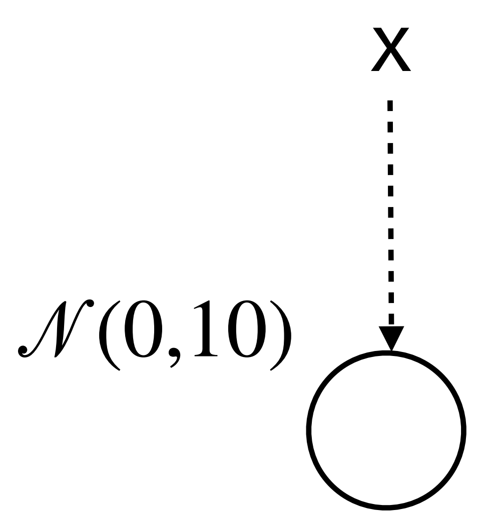
Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```



Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
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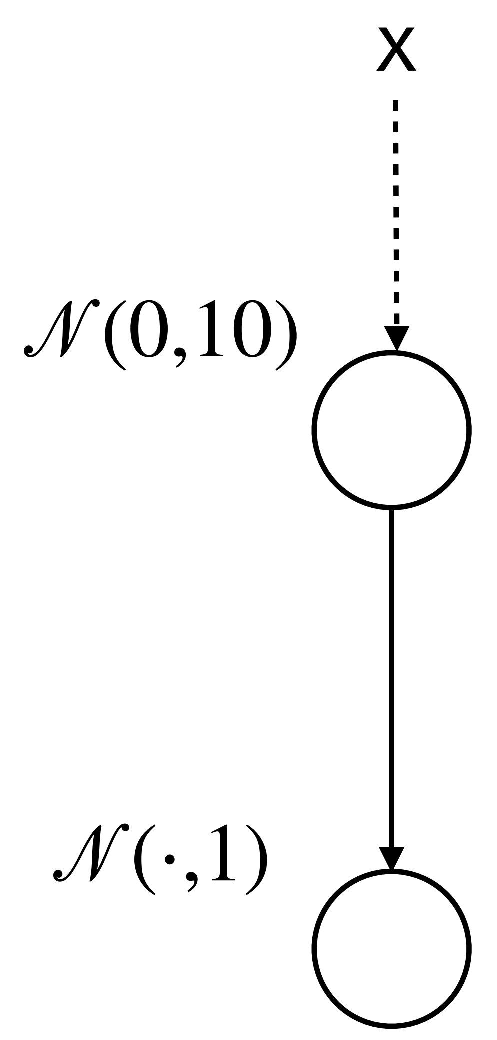
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```



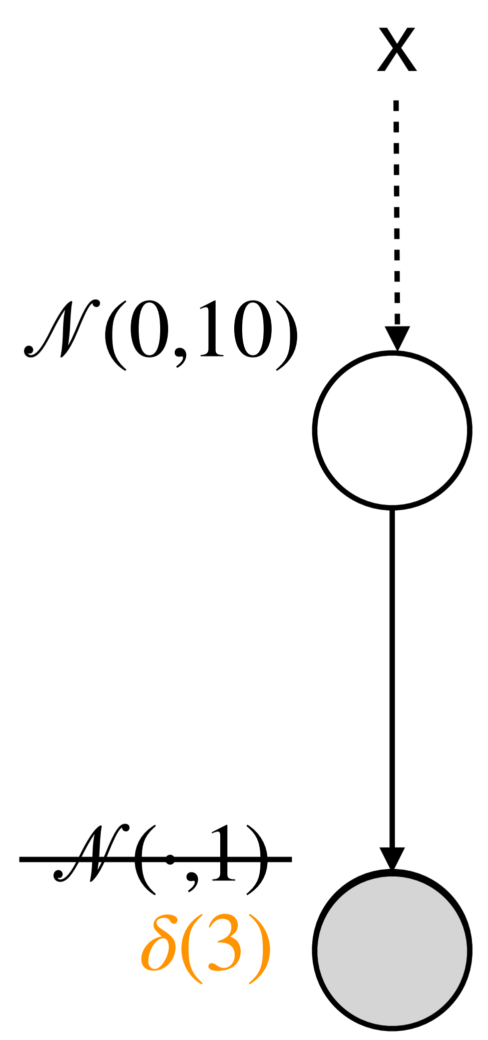
Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```



Murray et al. 2018

Delayed sampling

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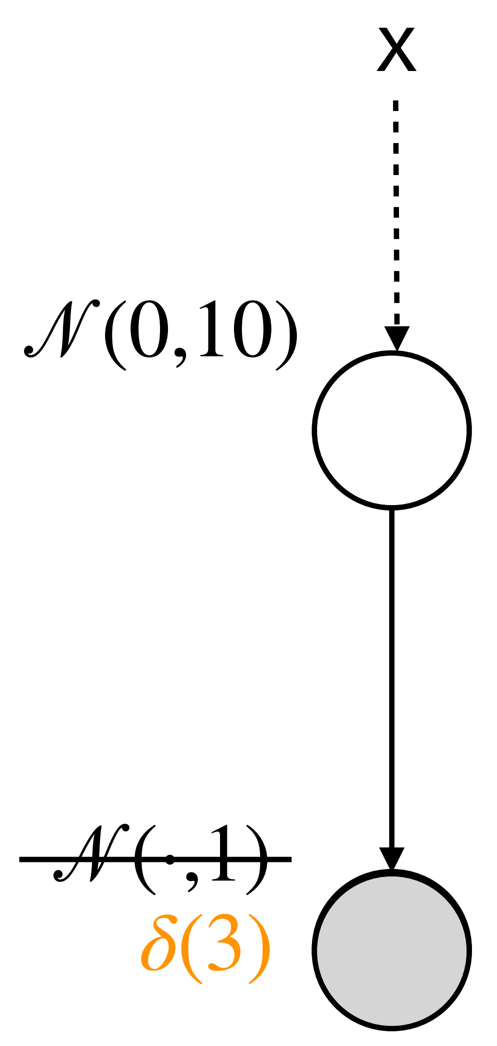
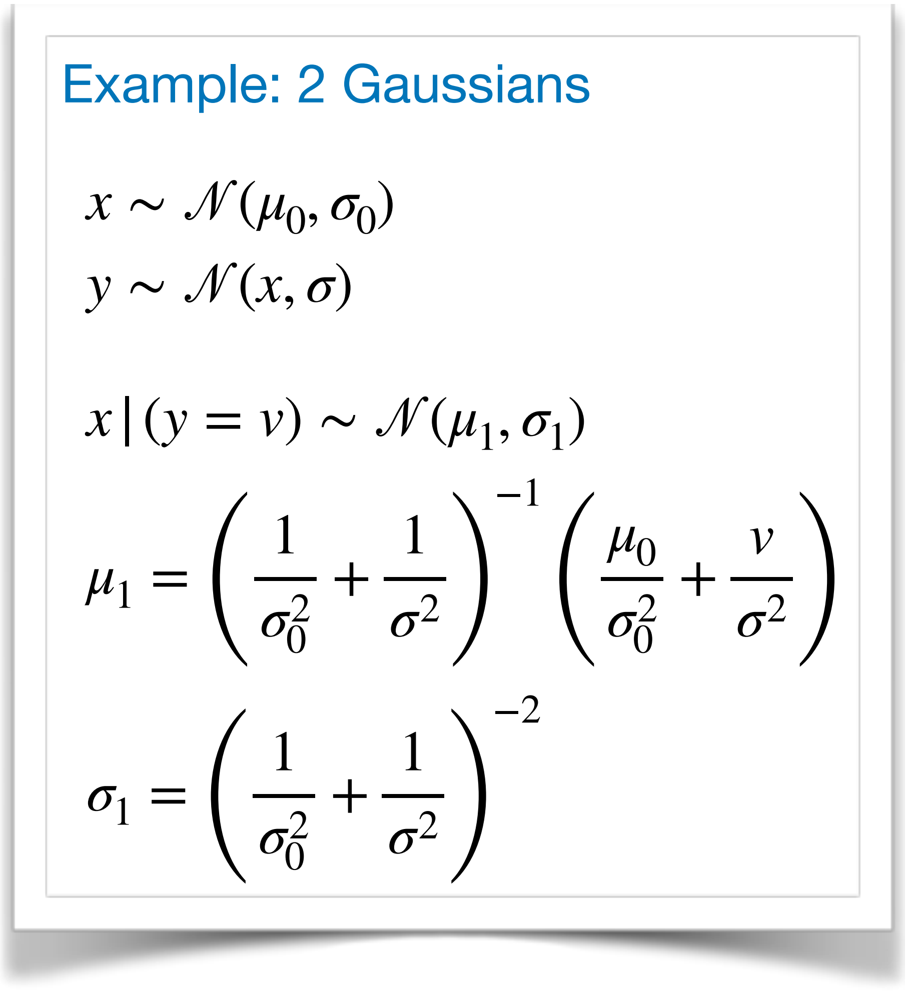
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```




Example: 2 Gaussians

$$x \sim \mathcal{N}(\mu_0, \sigma_0)$$
$$y \sim \mathcal{N}(x, \sigma)$$

$$x \mid (y = v) \sim \mathcal{N}(\mu_1, \sigma_1)$$

$$\mu_1 = \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2}\right)^{-1} \left(\frac{\mu_0}{\sigma_0^2} + \frac{v}{\sigma^2}\right)$$

$$\sigma_1 = \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2}\right)^{-2}$$

Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
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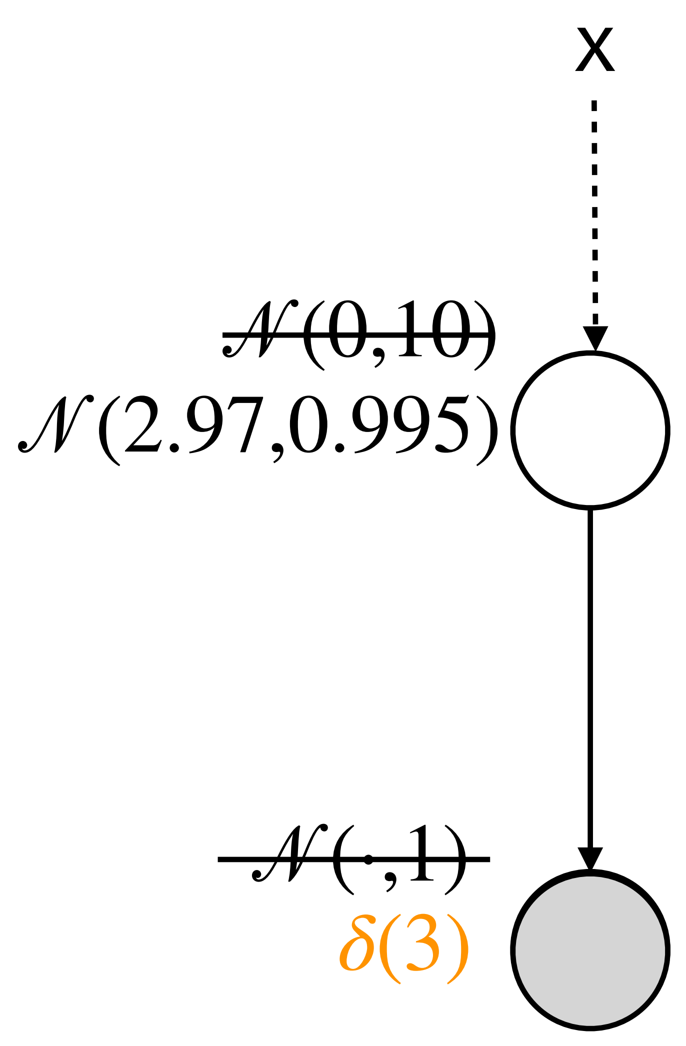
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```



Example: 2 Gaussians

$x \sim \mathcal{N}(\mu_0, \sigma_0)$
$y \sim \mathcal{N}(x, \sigma)$

$x \mid (y = v) \sim \mathcal{N}(\mu_1, \sigma_1)$

$$\mu_1 = \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2}\right)^{-1} \left(\frac{\mu_0}{\sigma_0^2} + \frac{v}{\sigma^2}\right)$$

$$\sigma_1 = \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2}\right)^{-2}$$

Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
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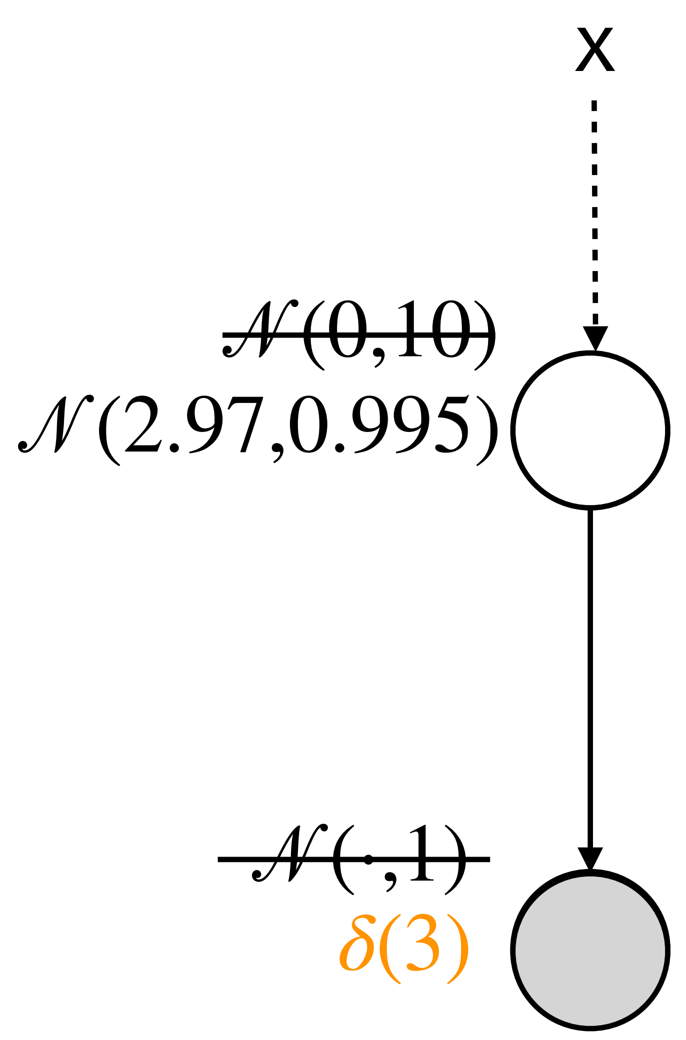
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
```



Murray et al. 2018

Delayed sampling

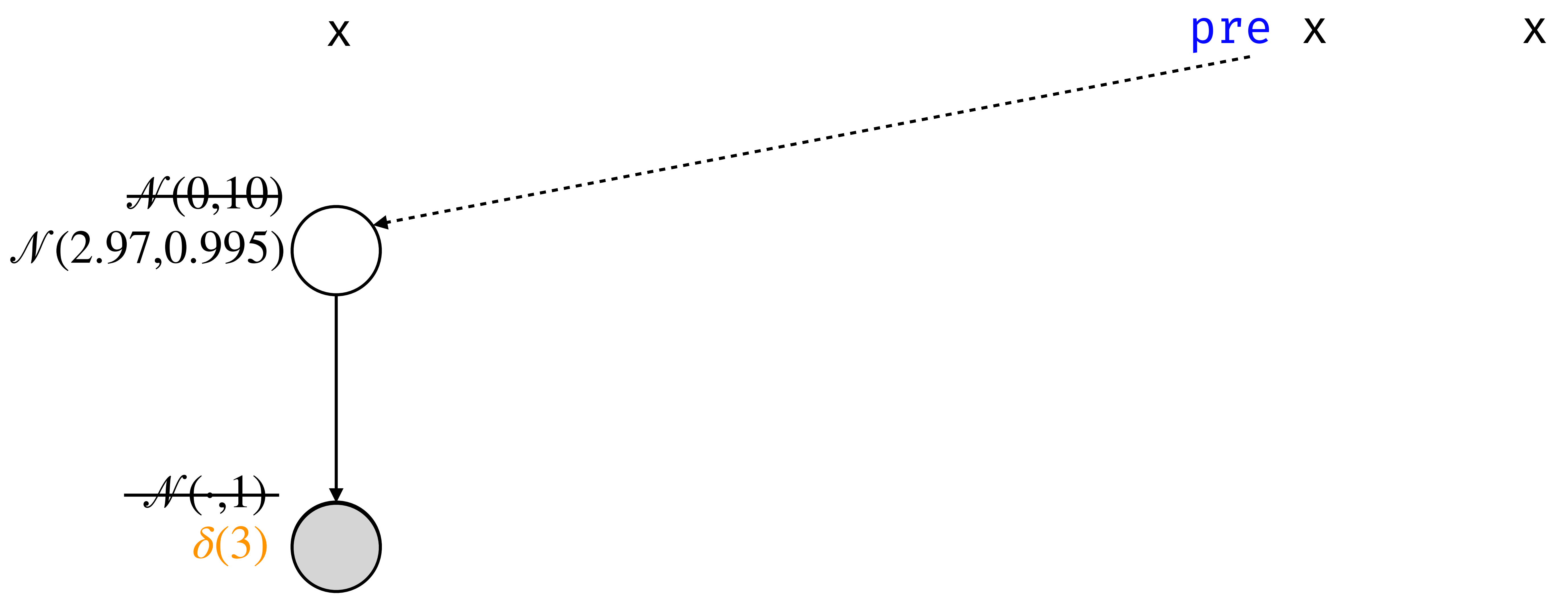
```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10)) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
```



Murray et al. 2018

Delayed sampling

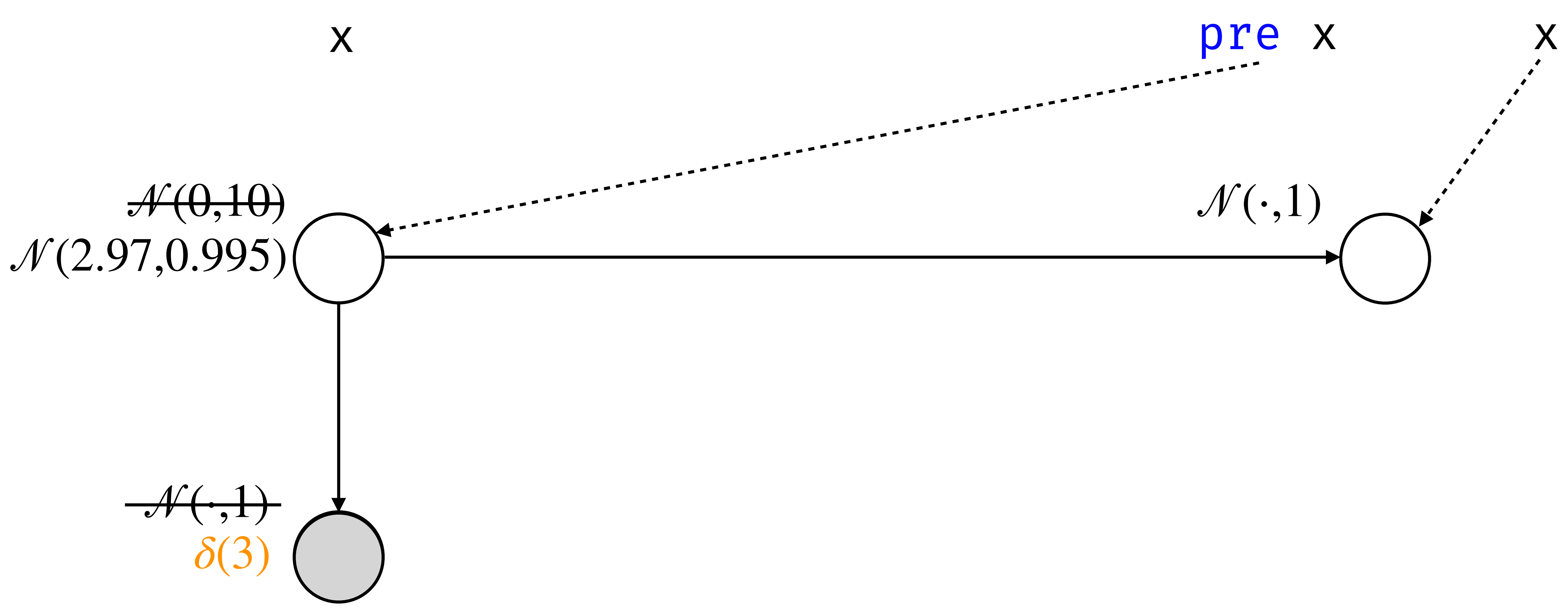
```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10)) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
```



Murray et al. 2018

Delayed sampling

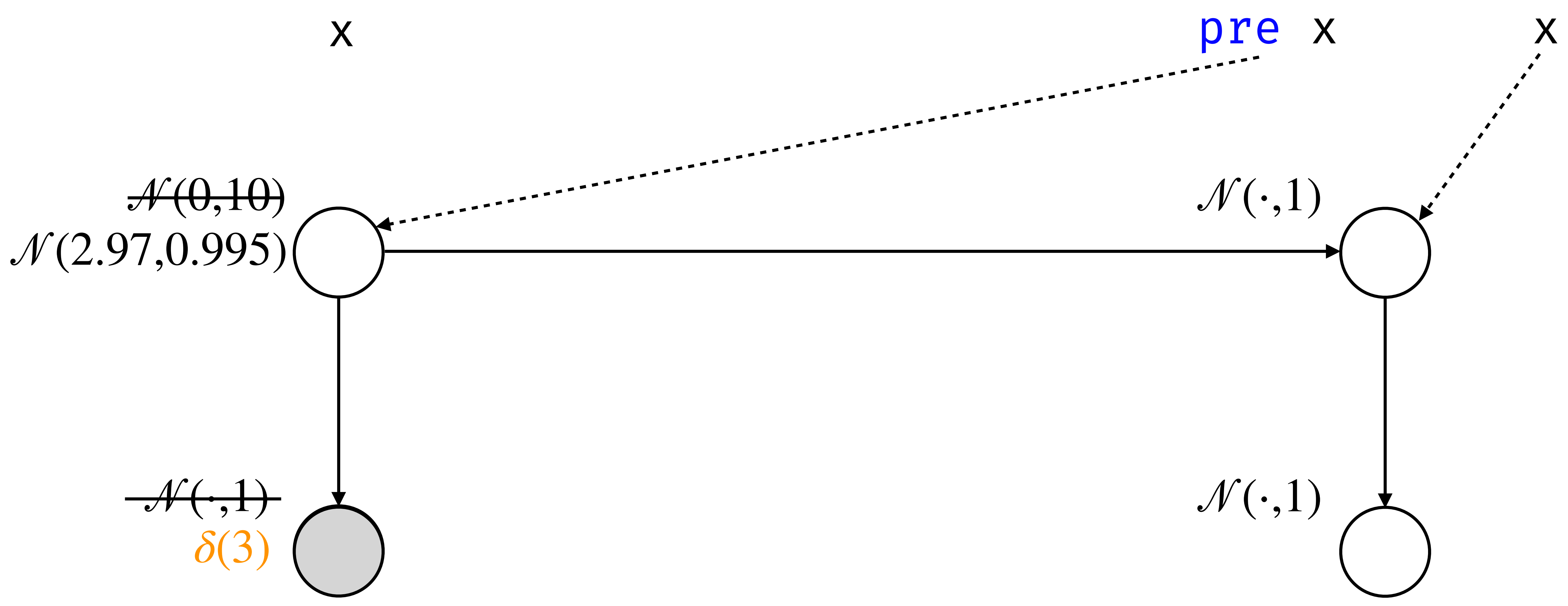
```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), 5)
```



Murray et al. 2018

Delayed sampling

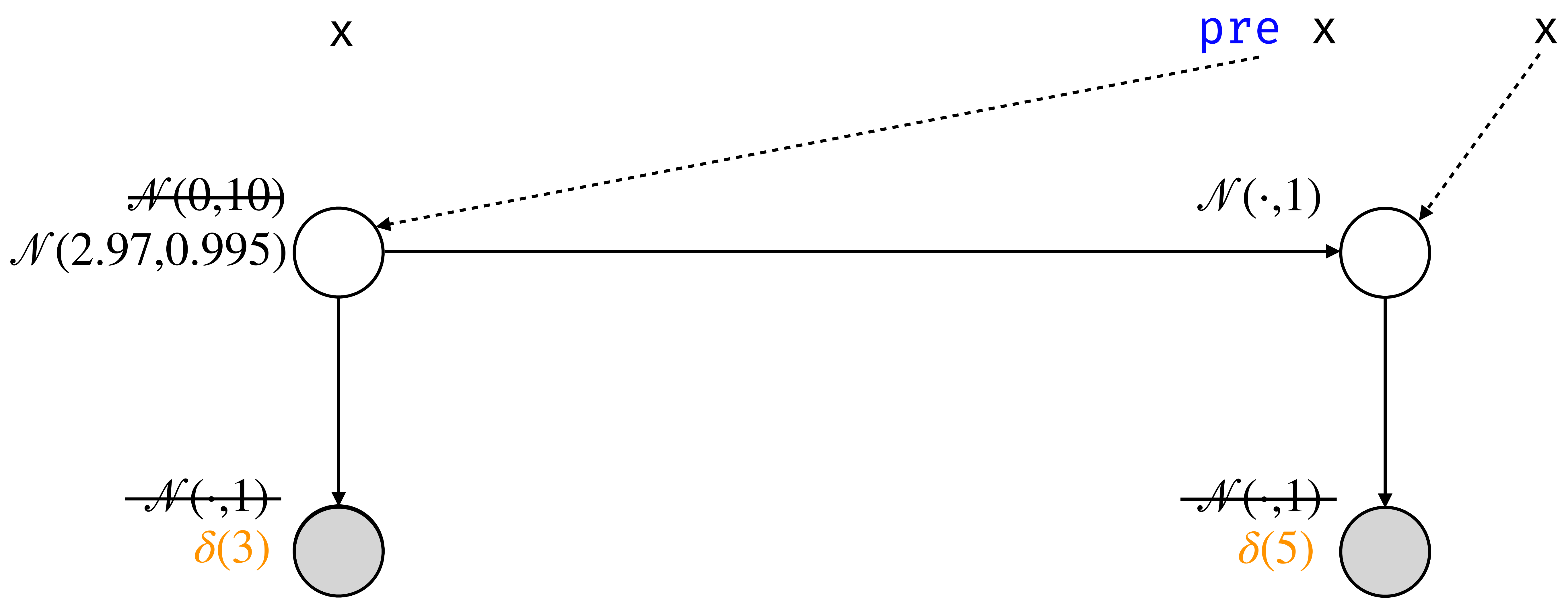
```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), 5)
```


x
pre x
x
~~$\mathcal{N}(0,10)$~~
$\mathcal{N}(2.97,0.995)$
$\mathcal{N}(\cdot,1)$
~~$\mathcal{N}(\cdot,1)$~~
$\delta(3)$
~~$\mathcal{N}(\cdot,1)$~~
$\delta(5)$

Murray et al. 2018

Delayed sampling

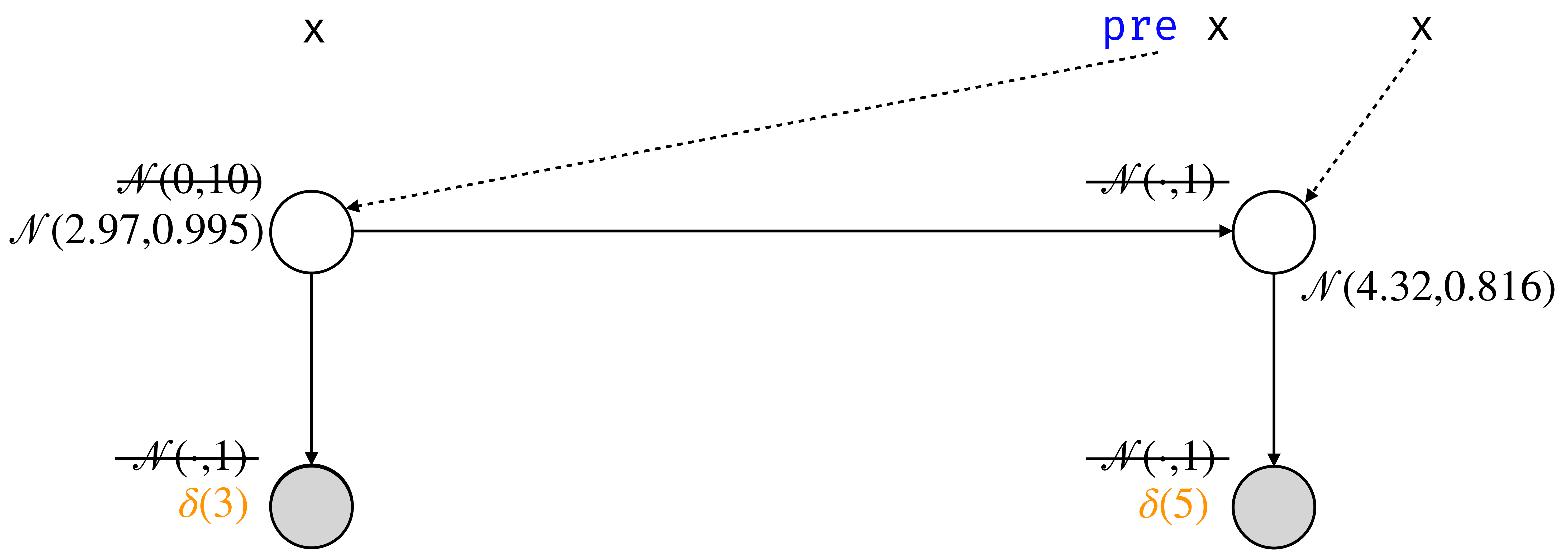
```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), 5)
```



Murray et al. 2018

Delayed sampling

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

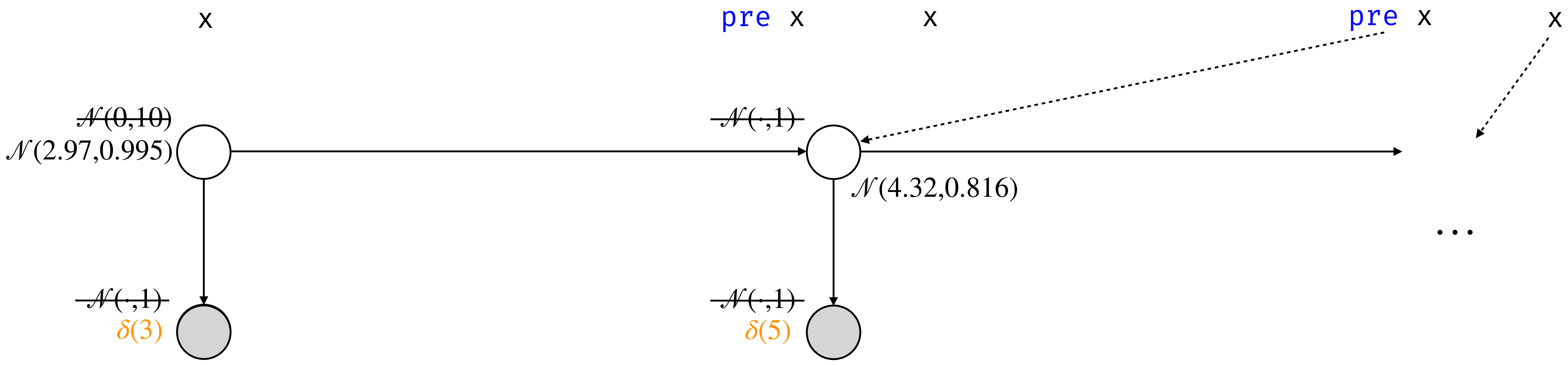
```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), 5)
```

$t = 2$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), …)
```


x
pre x
x
pre x
x
~~$\mathcal{N}(0,10)$~~
$\mathcal{N}(2.97,0.995)$
~~$\mathcal{N}(\cdot,1)$~~
$\mathcal{N}(4.32,0.816)$
~~$\mathcal{N}(\cdot,1)$~~
$\delta(3)$
~~$\mathcal{N}(\cdot,1)$~~
$\delta(5)$
…

Murray et al. 2018

Delayed sampling

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observe (gaussian (x, 1), 5)
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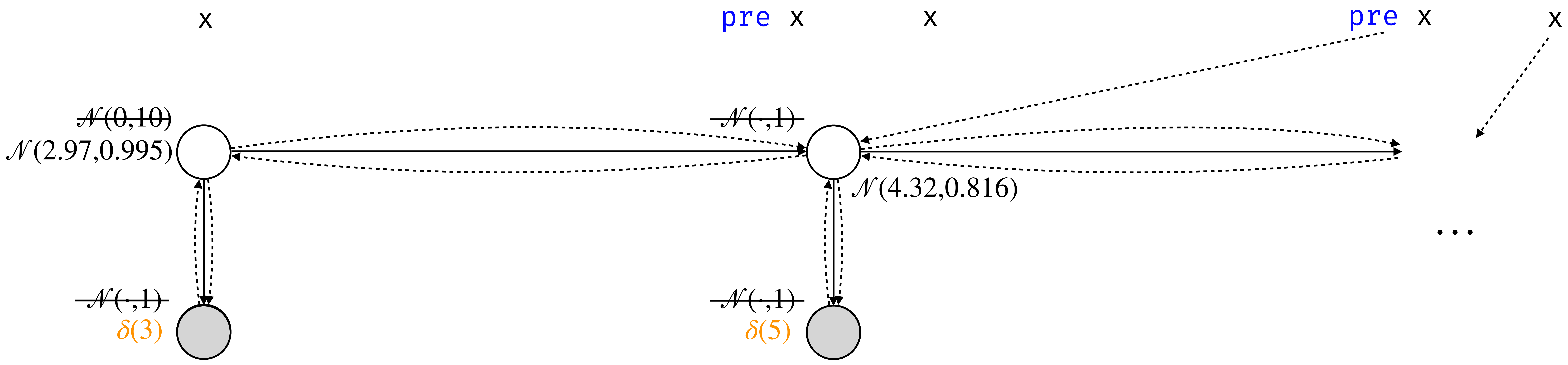
$t = 2$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), …)
```



Murray et al. 2018

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sample (gaussian (0, 10))
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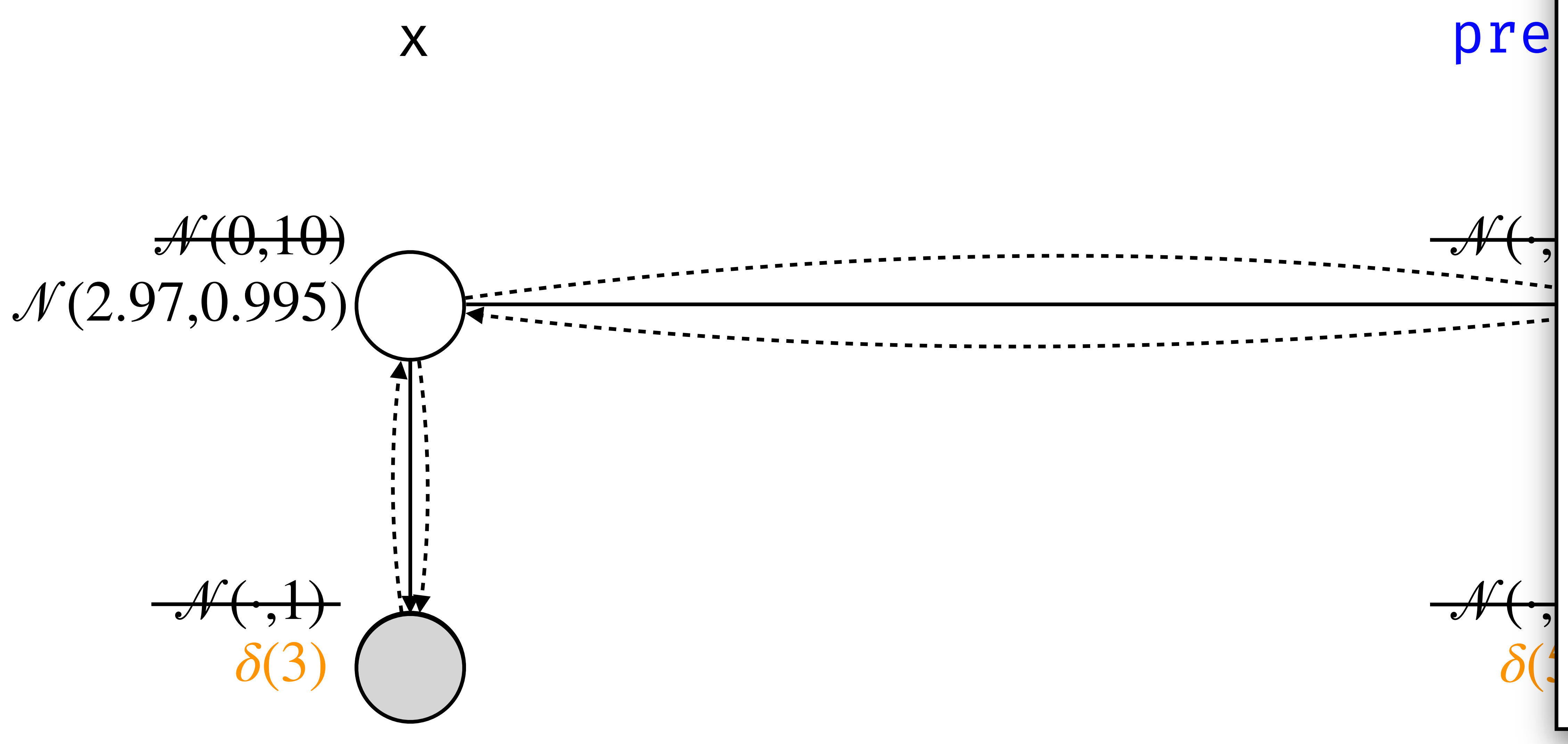
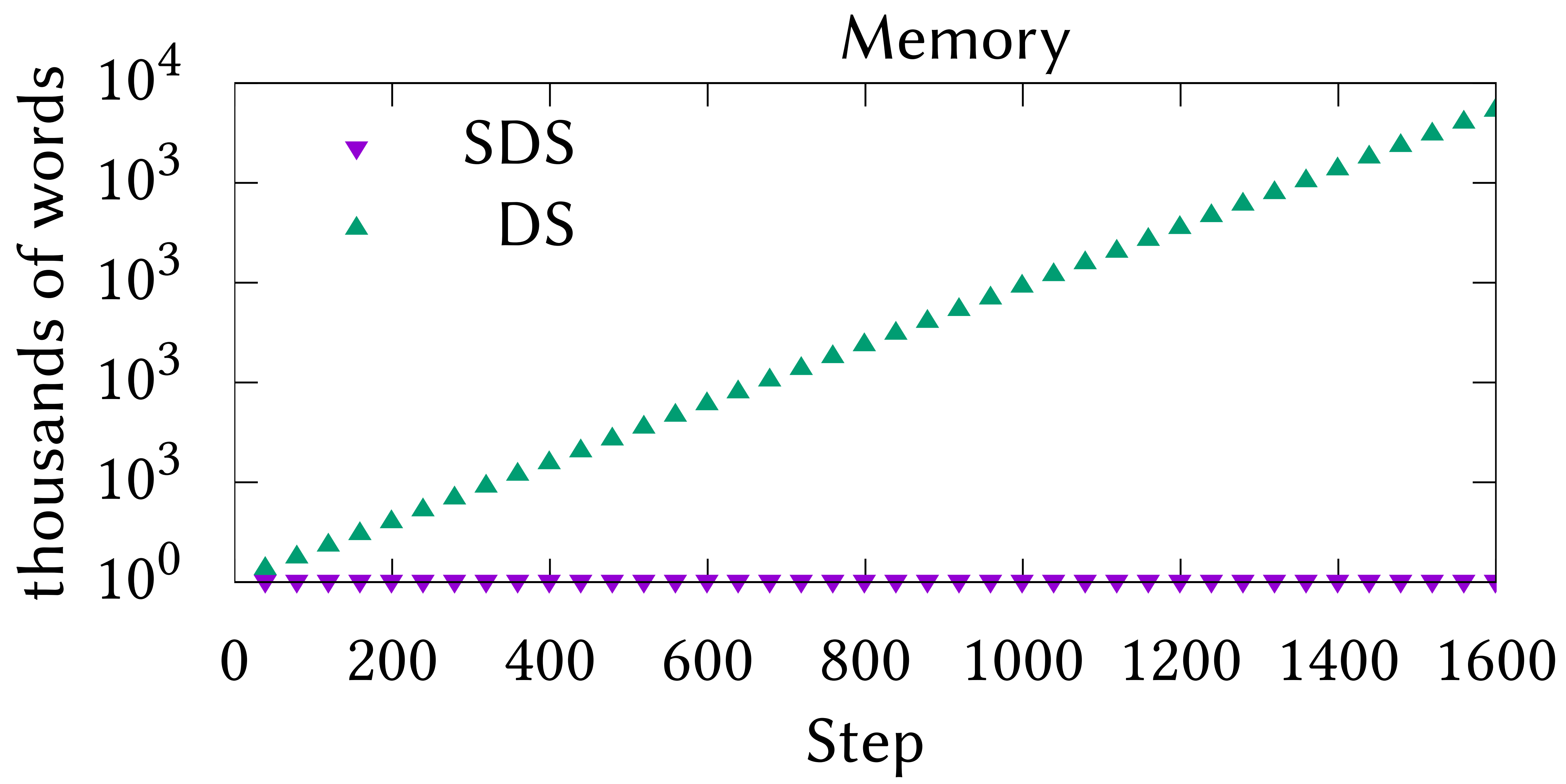
```
sample (gaus
observe (gau
```



Unbounded resources



Murray et al. 2018

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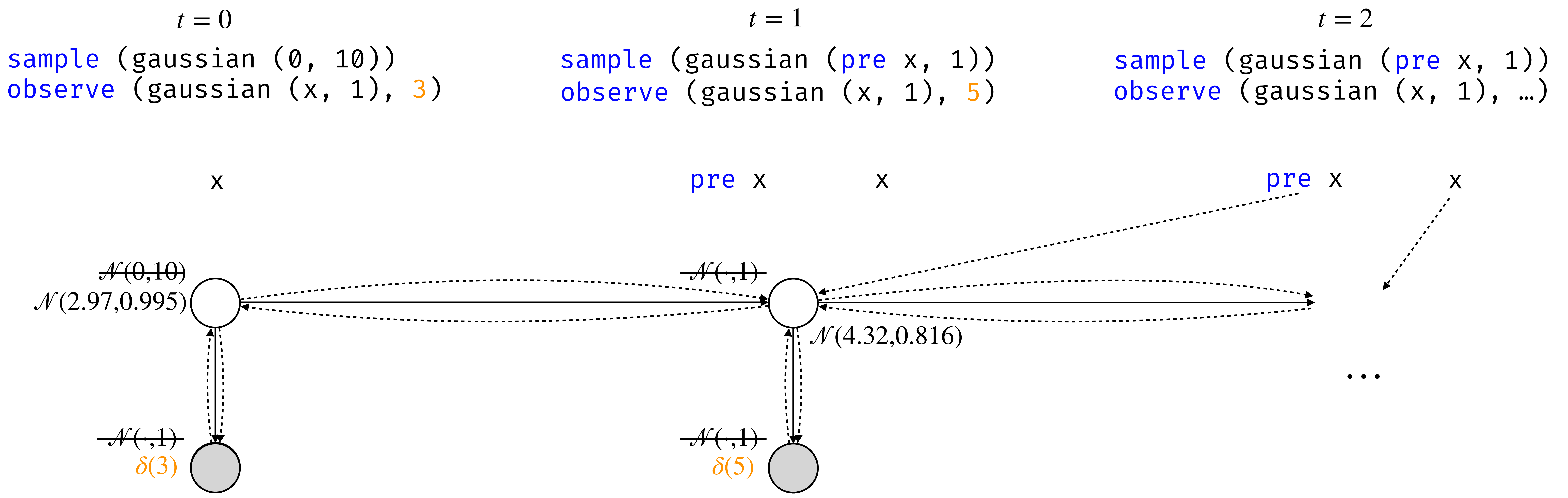

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observe (gaussian (x, 1), 5)
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observe (gaussian (x, 1), …)
x
pre x
x
pre x
x
~~$\mathcal{N}(0,10)$~~
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Murray et al. 2018

Streaming
Delayed sampling

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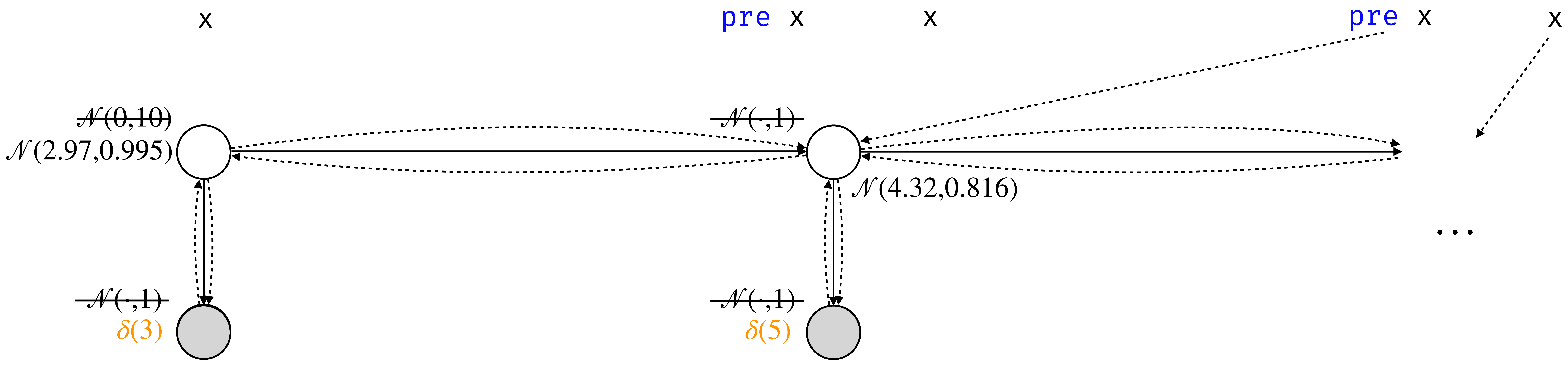

x
pre x
x
pre x
x
~~$\mathcal{N}(0{,}10)$~~
$\mathcal{N}(2.97{,}0.995)$
~~$\mathcal{N}(\cdot{,}1)$~~
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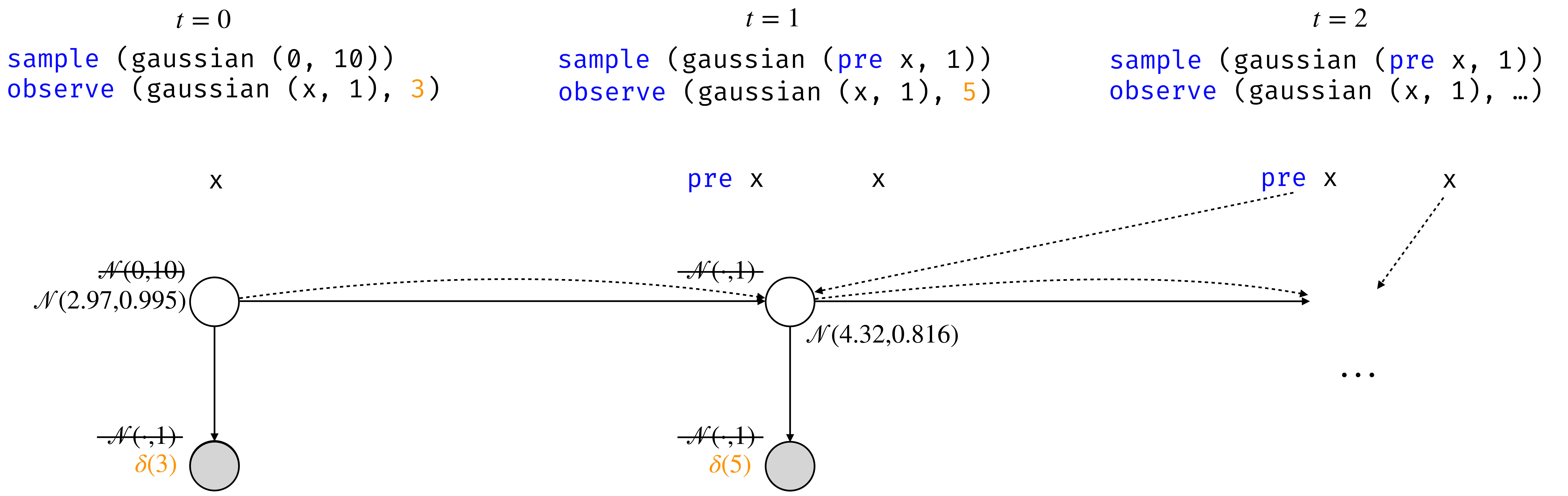

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x
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```

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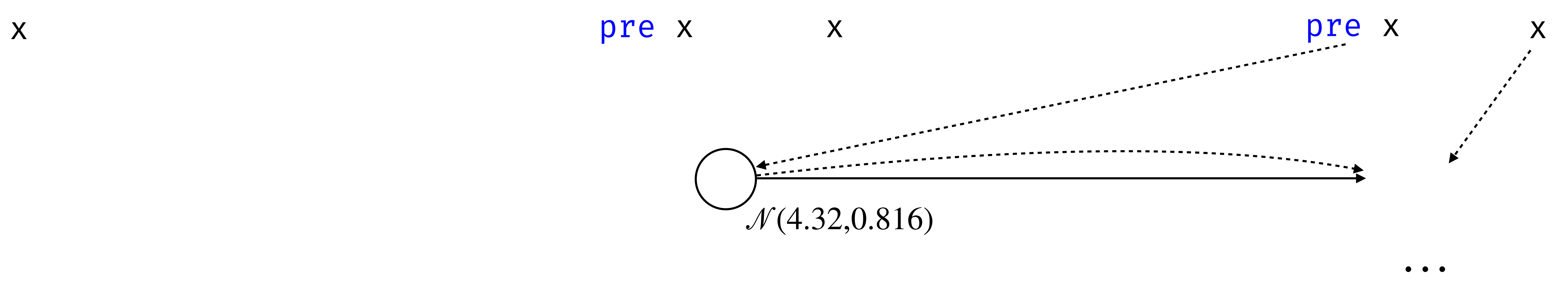
```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), …)
```



Delayed Sampling semantics

$$\{\!\!\{op(e)\}\!\!\}_{\gamma,g,w} =$$
$$\quad let\ (e', g_e, w_e) = \{\!\!\{e\}\!\!\}_{\gamma,g,w}\ in\ (\mathsf{app}(op, e'), g_e, w_e)$$

$$\{\!\!\{\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2\}\!\!\}_{\gamma,g,w} =$$
$$\begin{aligned}&let\ e', g_e, w_e = \{\!\!\{e\}\!\!\}_{\gamma,g,w}\ in\\ &let\ v, g_v = value(e', g_e)\ in\\ &if\ v\ then\ \{\!\!\{e_1\}\!\!\}_{\gamma,g_v,w_e}\ else\ \{\!\!\{e_2\}\!\!\}_{\gamma,g_v,w_e}\end{aligned}$$

$$\{\!\!\{\texttt{sample}(e)\}\!\!\}_{\gamma,g,w} =$$
$$\begin{aligned}&let\ \mu, g_e, w' = \{\!\!\{e\}\!\!\}_{\gamma,g,w}\ in\\ &let\ X, g' = assume(\mu, g_e)\ in\ (X, g', w')\end{aligned}$$

$$\{\!\!\{\texttt{observe}(e_1, e_2)\}\!\!\}_{\gamma,g,w} =$$
$$\begin{aligned}&let\ \mu, g_1, w_1 = \{\!\!\{e_1\}\!\!\}_{\gamma,g,w}\ in\ let\ X, g_x = assume(\mu, g_1)\ in\\ &let\ e_2', g_2, w_2 = \{\!\!\{e_2\}\!\!\}_{\gamma,g_x,w_1}\ in\ let\ v, g_v = value(e_2', g_2)\ in\\ &let\ g' = observe(X, v, g_v)\ in\ ((), g', w_2 * \mu_{\mathrm{pdf}}(v))\end{aligned}$$

$$[\![\texttt{infer(fun}\ s\ \texttt{->}\ e,\ \sigma)]\!]_\gamma =$$
$$\begin{aligned}&let\ \mu = \lambda U.\ \sum_{i=1}^{N}\ let\ s_i, g_i = \mathrm{draw}([\![\sigma]\!]_\gamma)\ in\\ &\qquad let\ (e_i, s_i'), w_i, g_i' = \{\!\!\{\texttt{fun}\ s\ \texttt{->}\ e\}\!\!\}_{\gamma,1,g_i}(s_i)\ in\\ &\qquad let\ d_i = distribution(e_i, g_i')\ in\\ &\qquad \overline{w_i} * d_i(\pi_1(U)) * \delta_{s_i', g_i'}(\pi_2(U))\\ &in\ (\pi_{1*}(\mu), \pi_{2*}(\mu))\end{aligned}$$

$$\overline{w_i} = w_i / \sum_{i=1}^{N} w_i$$

Delayed Sampling semantics

$$\{\!\!\{ op(e) \}\!\!\}_{\gamma,g,w} = \quad let\ (e', g_e, w_e) = \{\!\!\{ e \}\!\!\}_{\gamma,g,w}\ in\ (\mathrm{app}(op, e'), g_e, w_e)$$

$$\begin{aligned}\{\!\!\{ \texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 \}\!\!\}_{\gamma,g,w} = \\ &let\ e', g_e, w_e = \{\!\!\{ e \}\!\!\}_{\gamma,g,w}\ in \\ &let\ v, g_v = value(e', g_e)\ in \\ &if\ v\ then\ \{\!\!\{ e_1 \}\!\!\}_{\gamma,g_v,w_e}\ else\ \{\!\!\{ e_2 \}\!\!\}_{\gamma,g_v,w_e}\end{aligned}$$

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Manipulate symbolic terms (e.g., app(+, ...))

$$\overline{w_i} = w_i / \sum_{i=1}^{N} w_i$$

Delayed Sampling semantics

$$
\begin{aligned}
&\{\!\![op(e)]\!\!\}_{\gamma,g,w} \ = \\
&\qquad \mathit{let}\ (e', g_e, w_e) = \{\!\![e]\!\!\}_{\gamma,g,w}\ \mathit{in}\ (\mathsf{app}(op, e'), g_e, w_e)
\end{aligned}
$$

$$
\begin{aligned}
&\{\!\![\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2]\!\!\}_{\gamma,g,w} \ = \\
&\qquad \mathit{let}\ e', g_e, w_e = \{\!\![e]\!\!\}_{\gamma,g,w}\ \mathit{in} \\
&\qquad \mathit{let}\ v, g_v = \mathit{value}(e', g_e)\ \mathit{in} \\
&\qquad \mathit{if}\ v\ \mathit{then}\ \{\!\![e_1]\!\!\}_{\gamma,g_v,w_e}\ \mathit{else}\ \{\!\![e_2]\!\!\}_{\gamma,g_v,w_e}
\end{aligned}
$$

$$
\begin{aligned}
&\{\!\![\texttt{sample}(e)]\!\!\}_{\gamma,g,w} \ = \\
&\qquad \mathit{let}\ \mu, g_e, w' = \{\!\![e]\!\!\}_{\gamma,g,w}\ \mathit{in} \\
&\qquad \mathit{let}\ X, g' = \mathit{assume}(\mu, g_e)\ \mathit{in}\ (X, g', w')
\end{aligned}
$$

$$
\begin{aligned}
&\{\!\![\texttt{observe}(e_1, e_2)]\!\!\}_{\gamma,g,w} \ = \\
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&\qquad \mathit{let}\ e_2', g_2, w_2 = \{\!\![e_2]\!\!\}_{\gamma,g_x,w_1}\ \mathit{in}\ \mathit{let}\ v, g_v = \mathit{value}(e_2', g_2)\ \mathit{in} \\
&\qquad \mathit{let}\ g' = \mathit{observe}(X, v, g_v)\ \mathit{in}\ ((), g', w_2 * \mu_{\mathrm{pdf}}(v))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\texttt{infer}(\texttt{fun}\ s\ \texttt{->}\ e,\ \sigma)]\!]_{\gamma} = \\
&\quad \mathit{let}\ \mu = \lambda U.\ \sum_{i=1}^{N}\
\begin{aligned}[t]
&\mathit{let}\ s_i, g_i = \mathrm{draw}([\![\sigma]\!]_{\gamma})\ \mathit{in} \\
&\mathit{let}\ (e_i, s_i'), w_i, g_i' = \{\!\![\texttt{fun}\ s\ \texttt{->}\ e]\!\!\}_{\gamma,1,g_i}(s_i)\ \mathit{in} \\
&\mathit{let}\ d_i = \mathit{distribution}(e_i, g_i')\ \mathit{in} \\
&\overline{w_i} * d_i(\pi_1(U)) * \delta_{s_i', g_i'}(\pi_2(U))
\end{aligned} \\
&\quad \mathit{in}\ (\pi_{1*}(\mu), \pi_{2*}(\mu))
\end{aligned}
$$

Manipulate symbolic terms (e.g., app(+, ...))

High-level API: graph manipulations
assume, observe, value

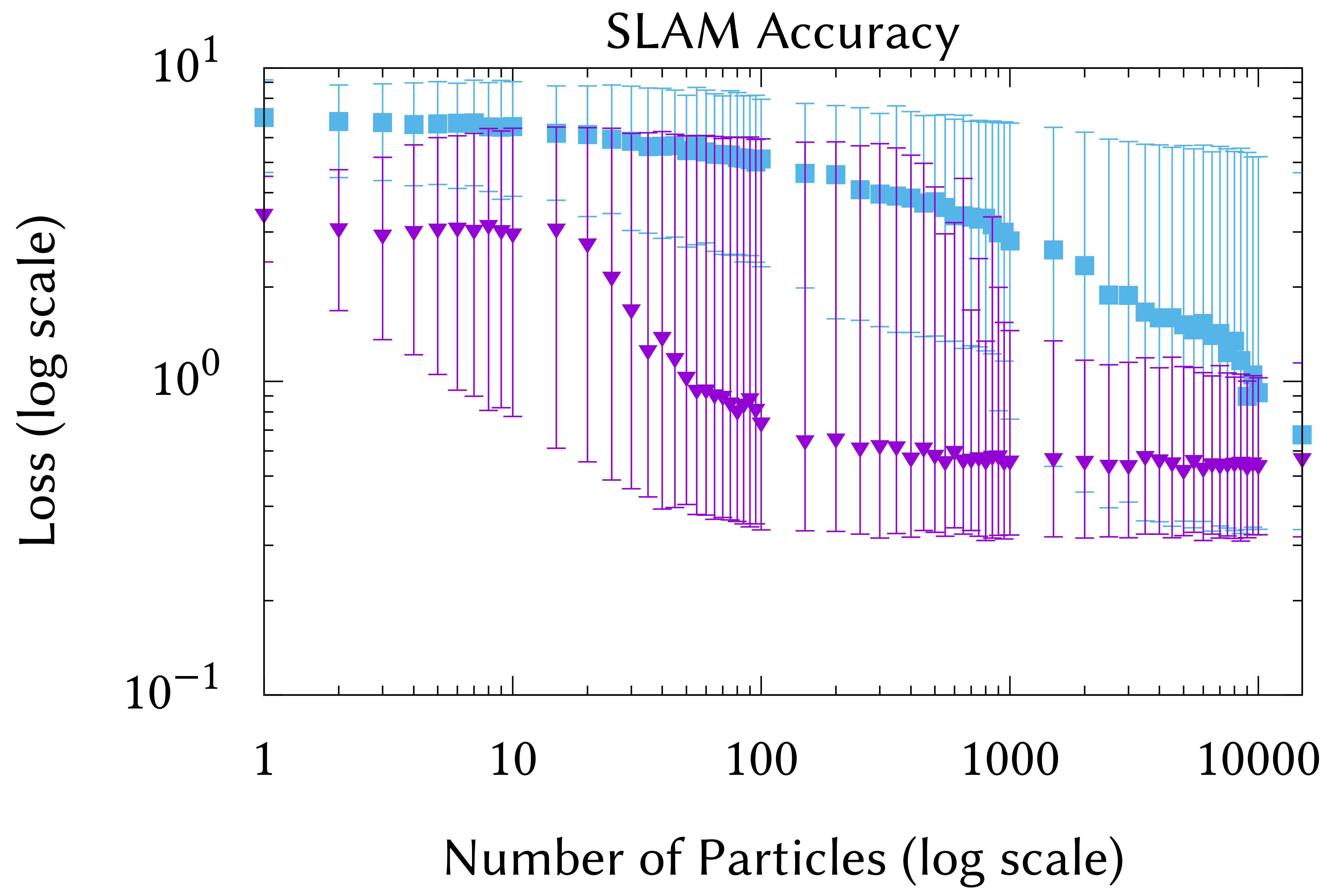
$$\overline{w_i} = w_i / \sum_{i=1}^{N} w_i$$

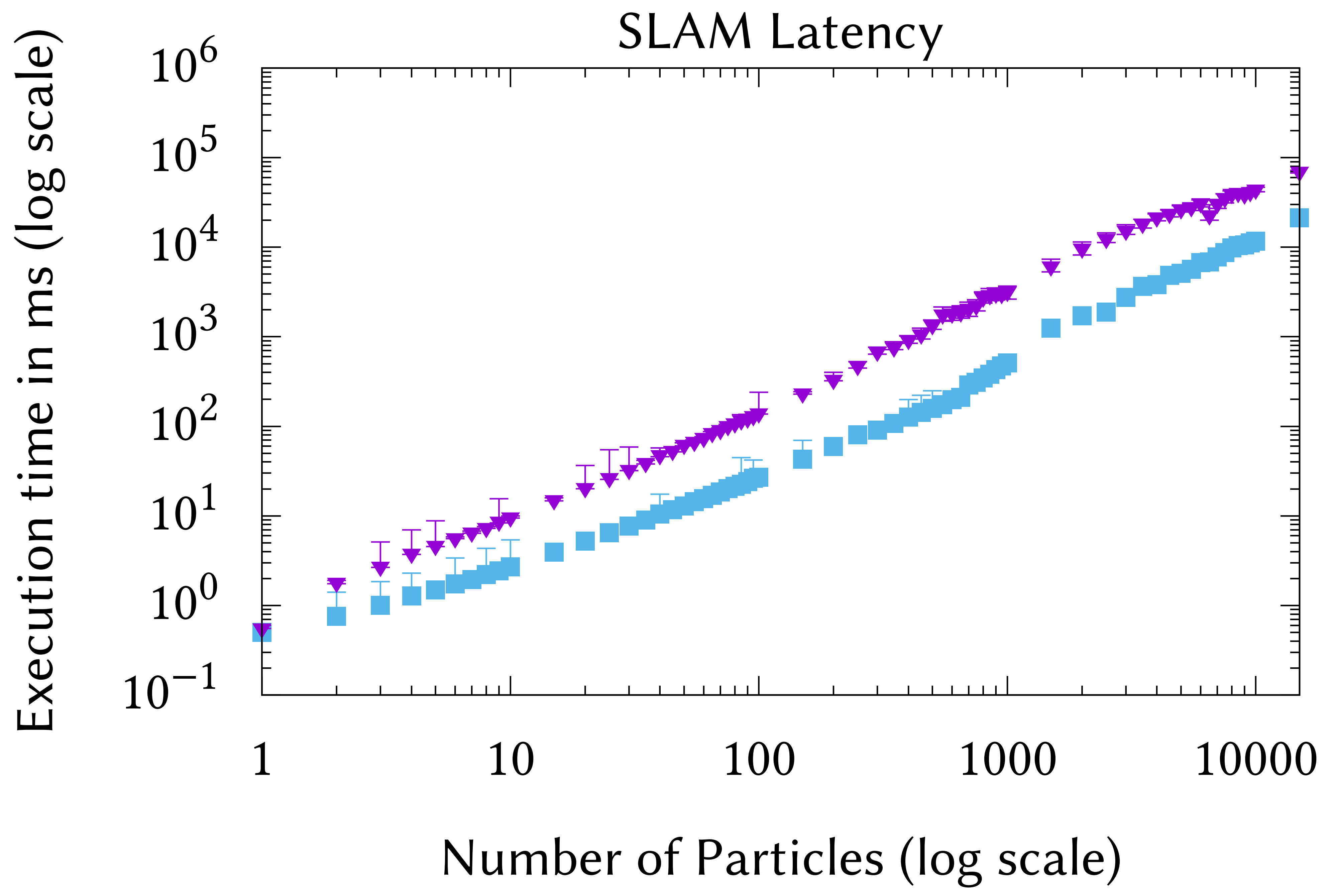
Evaluation

Algorithms comparison

■ PF Particle Filtering

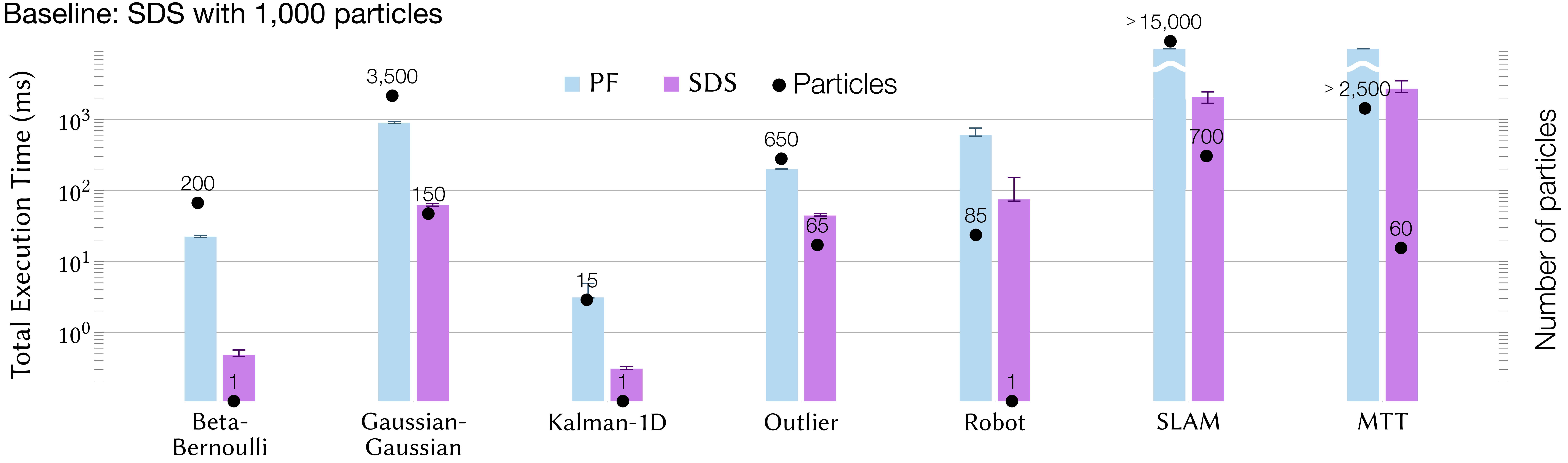
▼ SDS Streaming Delayed Sampling





Benchmarks

Baseline: SDS with 1,000 particles



Conclusions

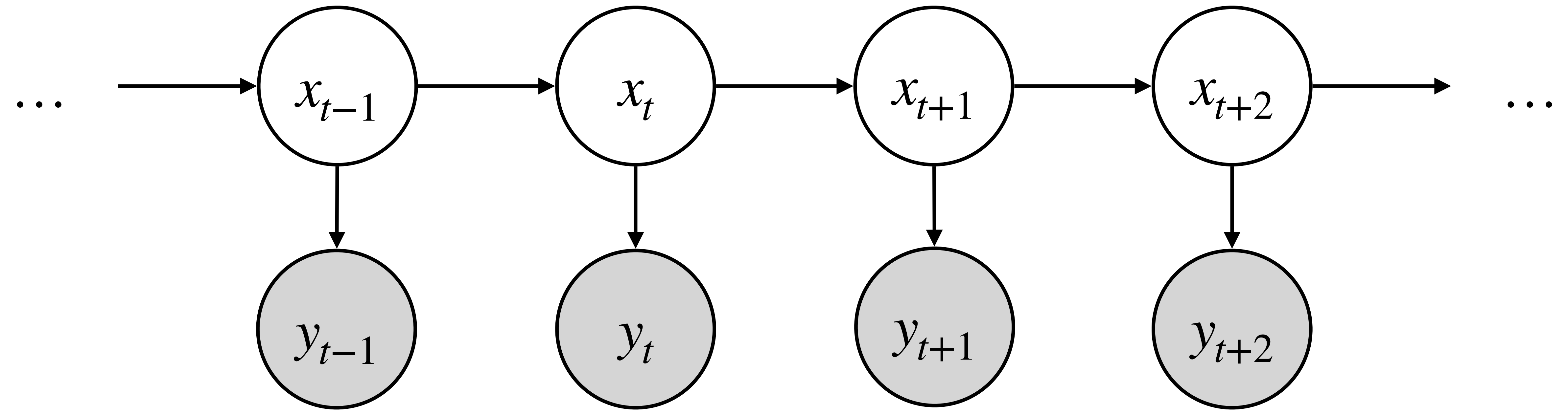
- SDS is always faster to match accuracy
- Reduction in particle count outweighs symbolic overhead
- SDS can be exact (1 particle)
- PF is impractical for advanced examples

Static analysis

Reactive Probabilistic Programming

Bounded memory delayed sampling?

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```



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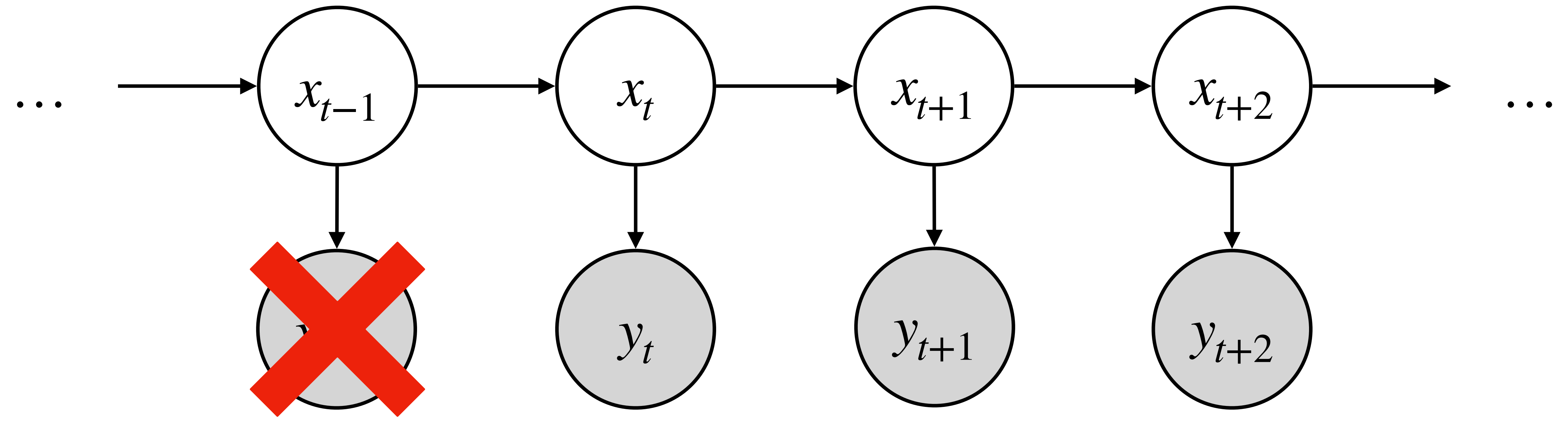

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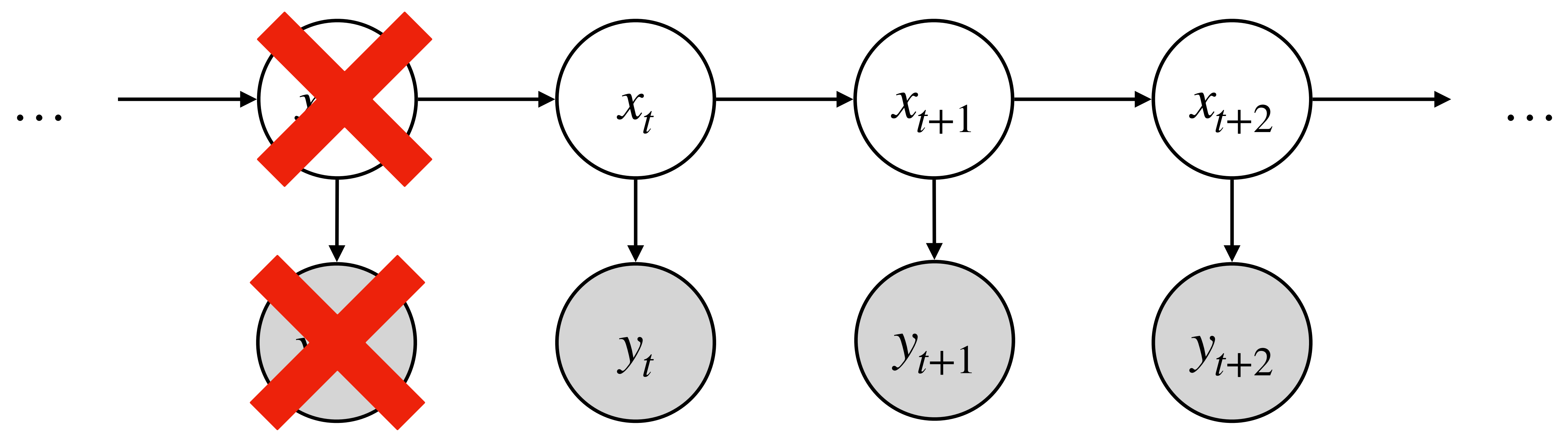

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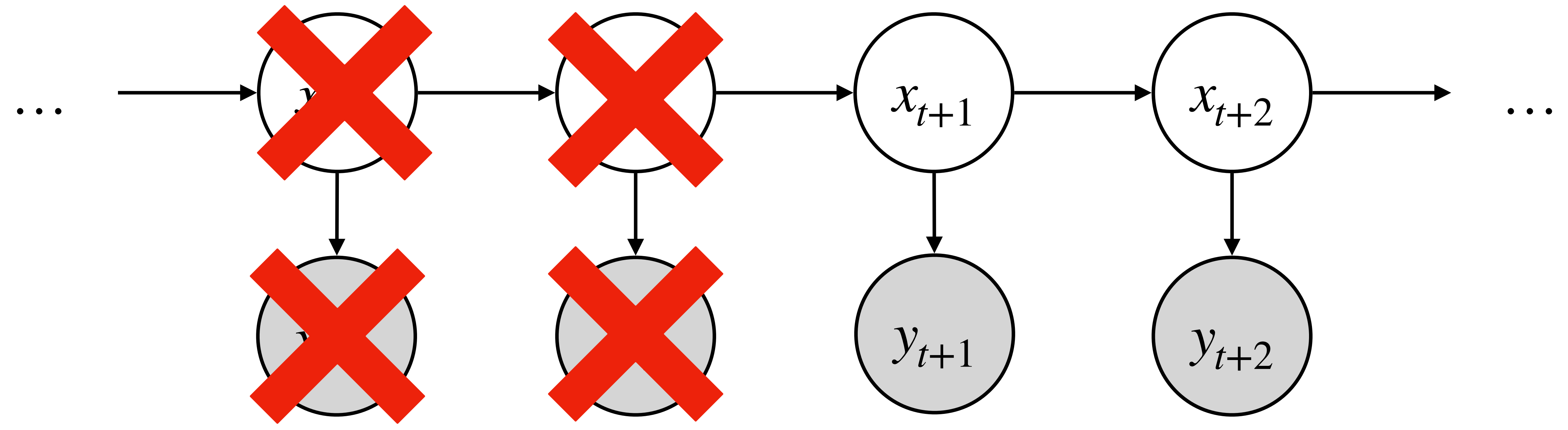

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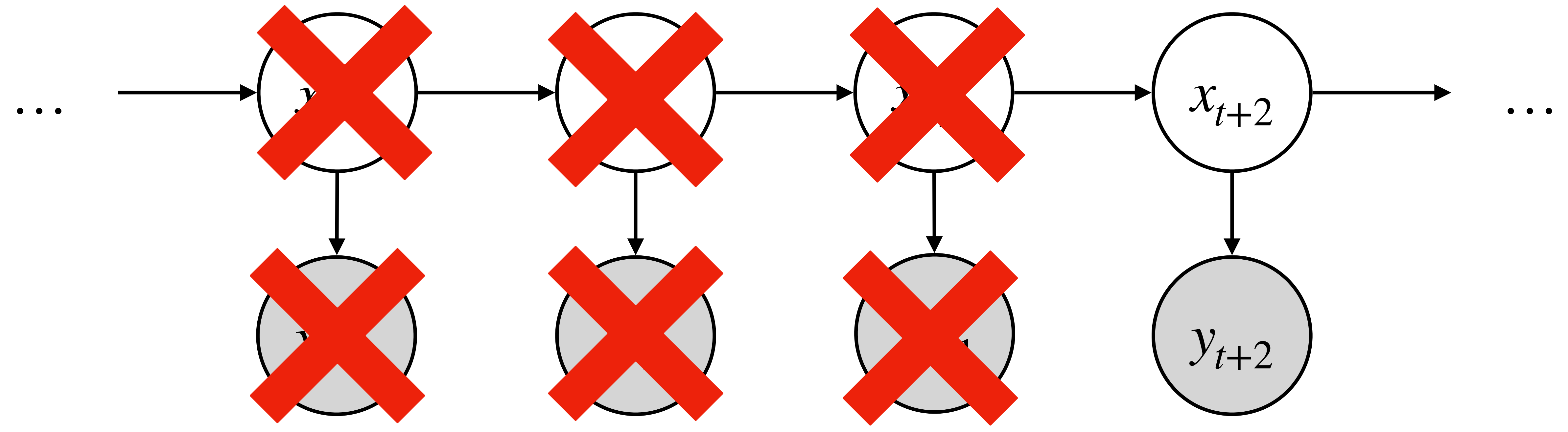

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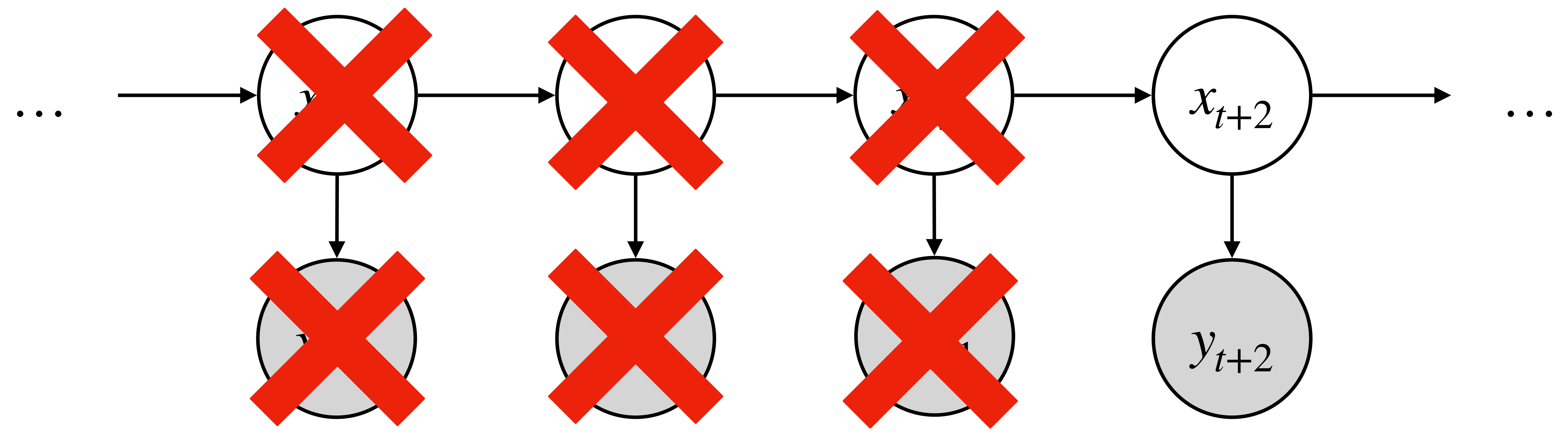
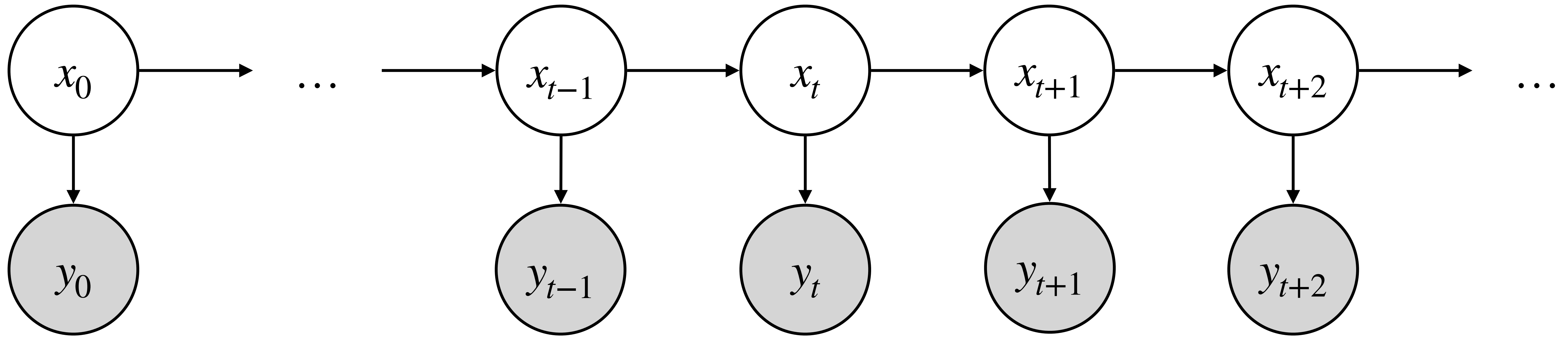


Yes!

Bounded memory delayed sampling?

```
let proba tracker (y) = x, x0 where
  rec init x0 = sample (gaussian (0, 10))
  and x   = x0 → sample (gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```



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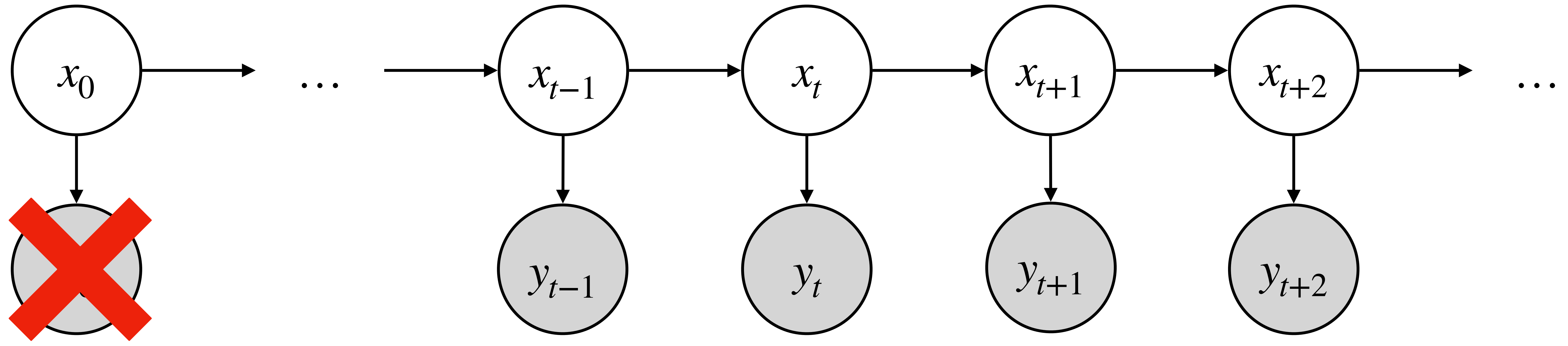

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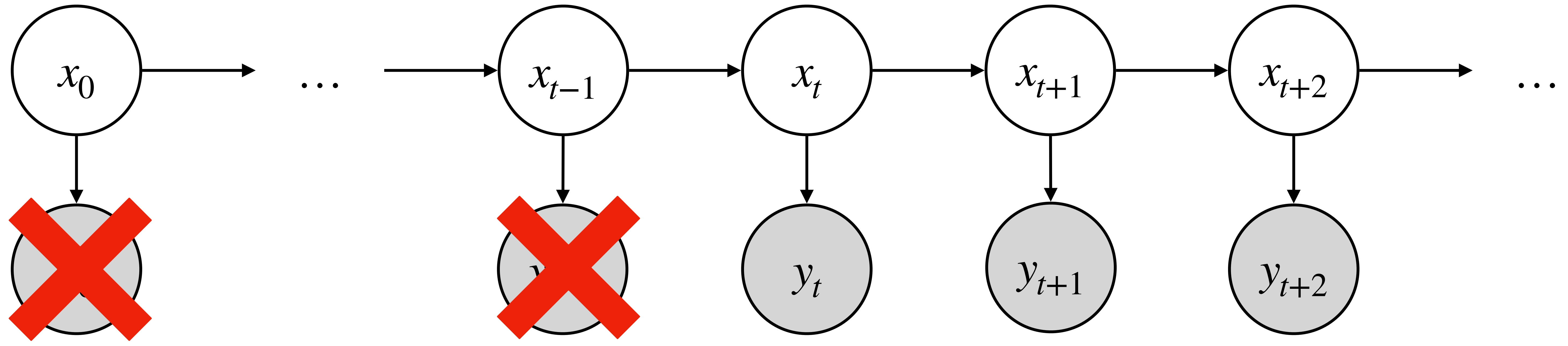

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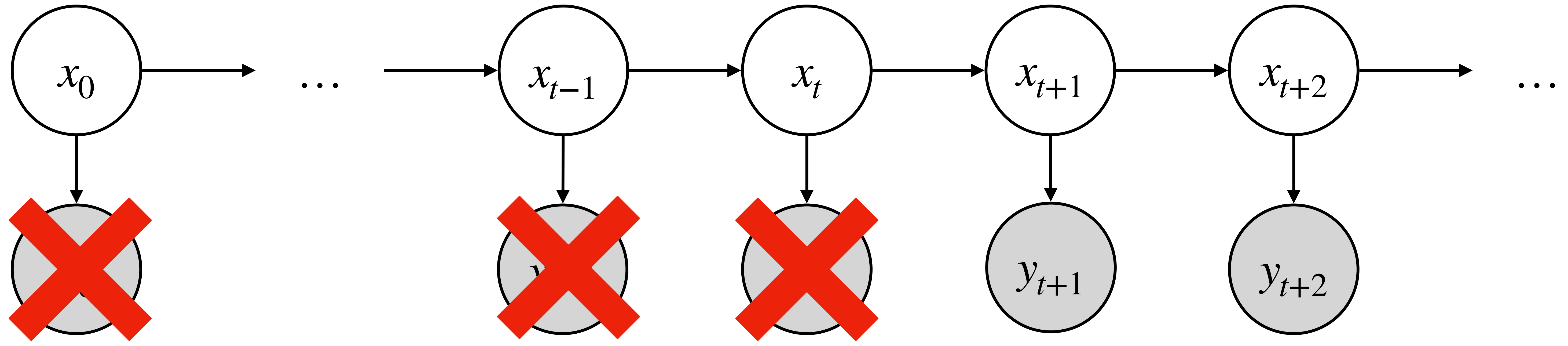

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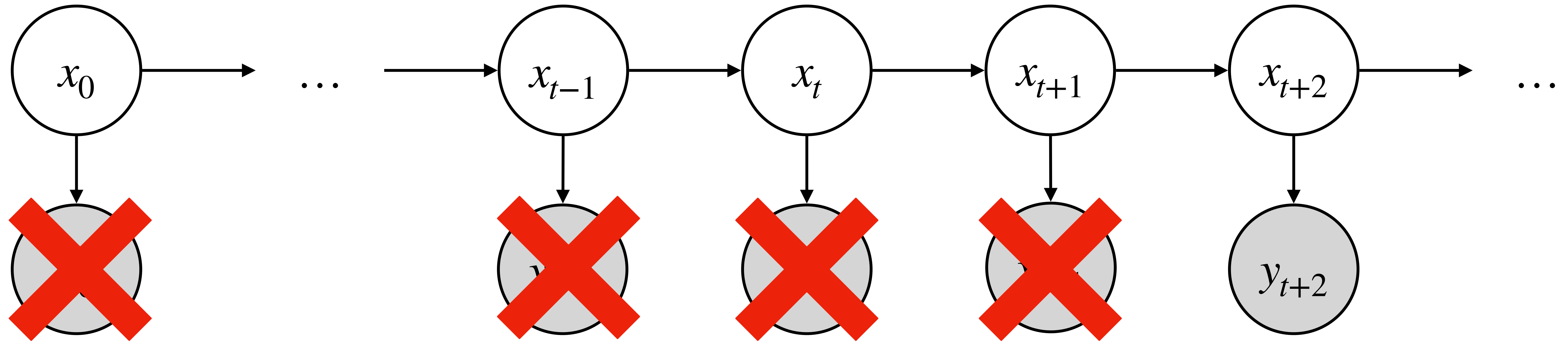

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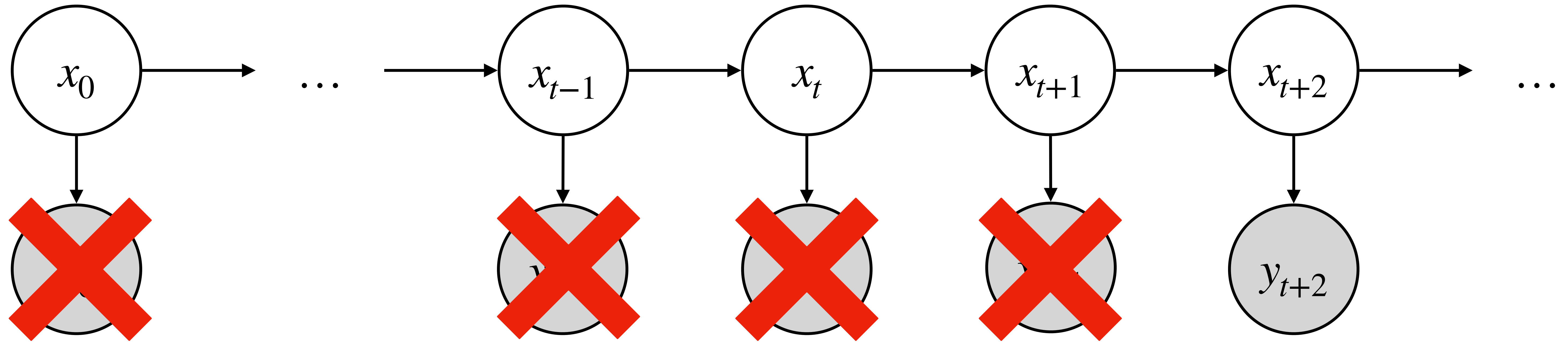

No!

Bounded memory delayed sampling?

```
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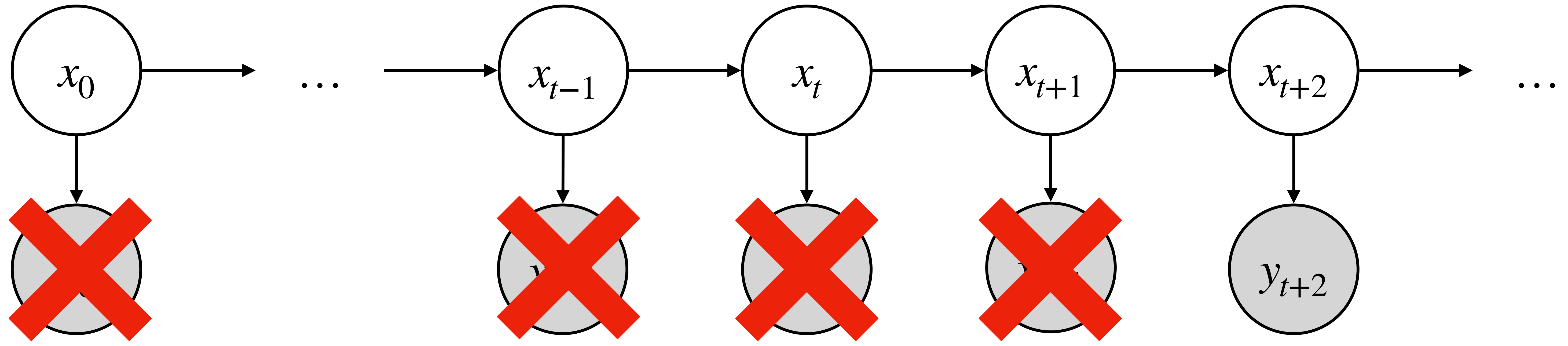
  rec init x0 = sample (gaussian (0, 10))
  and x   = x0 → sample (gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

Can we determine if a given program will run in bounded memory?



No!

Trace: abstract execution

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

trace	state	time
$x_0 \leftarrow \perp$ :: $y_0 \leftarrow x_0$ :: observe y_0 ::	`x = ` x_0	$t = 0$
$x_1 \leftarrow x_0$ :: $y_1 \leftarrow x_1$ :: observe y_1 ::	`x = ` x_1`, pre x = ` x_0	$t = 1$
$x_2 \leftarrow x_1$ :: $y_2 \leftarrow x_2$ :: …	`x = ` x_2`, pre x = ` x_1	$t = 2$

Trace: abstract execution

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

random variable →

trace	state	time
$x_0 \leftarrow \bot$ ::	x = x_0	$t = 0$
$y_0 \leftarrow x_0$ ::		
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	x = x_1, pre x = x_0	$t = 1$
$y_1 \leftarrow x_1$ ::		
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	x = x_2, pre x = x_1	$t = 2$
$y_2 \leftarrow x_2$ ::		
$\ldots$		

Trace: abstract execution

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

	trace	state	time
random variable →	$x_0 \leftarrow \perp$ ::	`x = ` x_0	$t = 0$
	$y_0 \leftarrow x_0$ ::		
observation →	observe y_0 ::		
	$x_1 \leftarrow x_0$ ::	`x = ` x_1`, pre x = ` x_0	$t = 1$
	$y_1 \leftarrow x_1$ ::		
	observe y_1 ::		
	$x_2 \leftarrow x_1$ ::	`x = ` x_2`, pre x = ` x_1	$t = 2$
	$y_2 \leftarrow x_2$ ::		
	…		

Static analysis for delayed sampling

Semantic properties

m-consumed property
Chains of variables before an observe are bounded

unseparated paths property
Chains of variables referenced in the state are bounded

Theorem: *The program satisfies these two properties iff it executes in bounded memory*

Static analysis for delayed sampling

Semantic properties

m-consumed property
Chains of variables before an observe are bounded

unseparated paths property
Chains of variables referenced in the state are bounded

Theorem: *The program satisfies these two properties iff it executes in bounded memory*

Static analysis

Track variables introduced but not used yet

Track maximal path between pairs of variable in the state

Theorem: Any *program that passes the analysis executes in bounded memory*

m-consumed property

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

trace	state	time
$x_0 \leftarrow \bot$ ::	`x = ` x_0	$t = 0$
$y_0 \leftarrow x_0$ ::		
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	`x = ` x_1`, pre x = ` x_0	$t = 1$
$y_1 \leftarrow x_1$ ::		
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	`x = ` x_2`, pre x = ` x_1	$t = 2$
$y_2 \leftarrow x_2$ ::		
…		

m-consumed property

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gauss
  and () = observe (gaussian (x, 1), y)
```

Chains of variables before an observe are bounded

trace	state	time
$x_0 \leftarrow \perp$ ::	`x = `x_0	$t = 0$
$y_0 \leftarrow x_0$ ::		
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	`x = `x_1`, pre x = `x_0	$t = 1$
$y_1 \leftarrow x_1$ ::		
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	`x = `x_2`, pre x = `x_1	$t = 2$
$y_2 \leftarrow x_2$ ::		
…		

m-consumed property

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gauss
  and () = observe (gaussian (x, 1), y)
```

Chains of variables before an observe are bounded

y_0 is 0-consumed $\longrightarrow$

trace	state	time
$x_0 \leftarrow \bot$ ::	`x = `x_0	$t = 0$
$y_0 \leftarrow x_0$ ::		
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	`x = `x_1`, pre x = `x_0	$t = 1$
$y_1 \leftarrow x_1$ ::		
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	`x = `x_2`, pre x = `x_1	$t = 2$
$y_2 \leftarrow x_2$ ::		
…		

m-consumed property

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gauss
  and () = observe (gaussian (x, 1), y)
```

Chains of variables before an observe are bounded

	trace	state	time
	$x_0 \leftarrow \bot$ ::	`x = ` x_0	$t = 0$
x_0 is 1-consumed $\longrightarrow$	$y_0 \leftarrow x_0$ ::		
y_0 is 0-consumed $\longrightarrow$	observe y_0 ::		
	$x_1 \leftarrow x_0$ ::	`x = ` x_1`, pre x = ` x_0	$t = 1$
	$y_1 \leftarrow x_1$ ::		
	observe y_1 ::		
	$x_2 \leftarrow x_1$ ::	`x = ` x_2`, pre x = ` x_1	$t = 2$
	$y_2 \leftarrow x_2$ ::		
	…		

m-consumed property

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gauss
  and () = observe (gaussian (x, 1), y)
```

Chains of variables before an observe are bounded

	trace	state	time
	$x_0 \leftarrow \bot$ ::	`x = `x_0	$t = 0$
x_0 is 1-consumed $\longrightarrow$	$y_0 \leftarrow x_0$ ::		
y_0 is 0-consumed $\longrightarrow$	observe y_0 ::		
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	$x_2 \leftarrow x_1$ ::	`x = `x_2`, pre x = `x_1	$t = 2$
	$y_2 \leftarrow x_2$ ::		
	$\ldots$		

m-consumed property

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proba tracker (y) = x where
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  and () = observe (gaussian (x, 1), y)
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Chains of variables before an observe are bounded

	trace	state	time
	$x_0 \leftarrow \bot$ ::	`x = `x_0	$t = 0$
x_0 is 1-consumed $\longrightarrow$	$y_0 \leftarrow x_0$ ::		
y_0 is 0-consumed $\longrightarrow$	observe y_0 ::		
	$x_1 \leftarrow x_0$ ::	`x = `x_1`, pre x = `x_0	$t = 1$
x_1 is 1-consumed $\longrightarrow$	$y_1 \leftarrow x_1$ ::		
y_1 is 0-consumed $\longrightarrow$	observe y_1 ::		
	$x_2 \leftarrow x_1$ ::	`x = `x_2`, pre x = `x_1	$t = 2$
	$y_2 \leftarrow x_2$ ::		
	…		

Yes!

Unseparated paths property

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

trace	state	time
$x_0 \leftarrow \perp$ ::	`x = `x_0	$t = 0$
$y_0 \leftarrow x_0$ ::		
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	`x = `x_1`, pre x = `x_0	$t = 1$
$y_1 \leftarrow x_1$ ::		
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	`x = `x_2`, pre x = `x_1	$t = 2$
$y_2 \leftarrow x_2$ ::		
$\dots$		

Unseparated paths property

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gauss
  and () = observe (gaussian (x, 1), y)
```

Chains of variables referenced in the state are bounded

trace	state	time
$x_0 \leftarrow \perp$ ::	`x = `x_0	$t = 0$
$y_0 \leftarrow x_0$ ::		
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	`x = `x_1`, pre x = `x_0	$t = 1$
$y_1 \leftarrow x_1$ ::		
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	`x = `x_2`, pre x = `x_1	$t = 2$
$y_2 \leftarrow x_2$ ::		
…		

Unseparated paths property

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gauss
  and () = observe (gaussian (x, 1), y)
```

Chains of variables referenced in the state are bounded

trace	state	time
$x_0 \leftarrow \perp$ ::	`x` = x_0	$t = 0$
$y_0 \leftarrow x_0$ ::		
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	`x` = x_1, `pre x` = x_0	$t = 1$
$y_1 \leftarrow x_1$ ::		
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	`x` = x_2, `pre x` = x_1	$t = 2$
$y_2 \leftarrow x_2$ ::		
…		

Unseparated paths property

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gauss
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trace	state	time
$x_0 \leftarrow \perp$ ::	`x = ` x_0	$t = 0$
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$x_1 \leftarrow x_0$ ::	`x = ` x_1`, pre x = ` x_0	$t = 1$
$y_1 \leftarrow x_1$ ::		
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	`x = ` x_2`, pre x = ` x_1	$t = 2$
$y_2 \leftarrow x_2$ ::		
…		

Yes!

Unseparated paths property

```
proba tracker (y) = x where
  rec init x0 = sample (gaussian (0, 10))
  and x   = x0 → sample (gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

Chains of variables referenced in the state are bounded

trace	state	time
$x_0 \leftarrow \perp$ ::	x = x_0	$t = 0$
$y_0 \leftarrow x_0$ ::	x0 = x_0	
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	x = x_1, pre x = x_0	$t = 1$
$y_1 \leftarrow x_1$ ::	x0 = x_0	
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	x = x_2, pre x = x_1	$t = 2$
$y_2 \leftarrow x_2$ ::	x0 = x_0	
…		

Unseparated paths property

```
proba tracker (y) = x where
  rec init x0 = sample (gaussian (0, 10))
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Chains of variables referenced in the state are bounded

trace	state	time
$x_0 \leftarrow \perp$ ::	x = x_0	$t = 0$
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observe y_0 ::		
$x_1 \leftarrow x_0$ ::	x = x_1, pre x = x_0	$t = 1$
$y_1 \leftarrow x_1$ ::	x0 = x_0	
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	x = x_2, pre x = x_1	$t = 2$
$y_2 \leftarrow x_2$ ::	x0 = x_0	
…		

Unseparated paths property

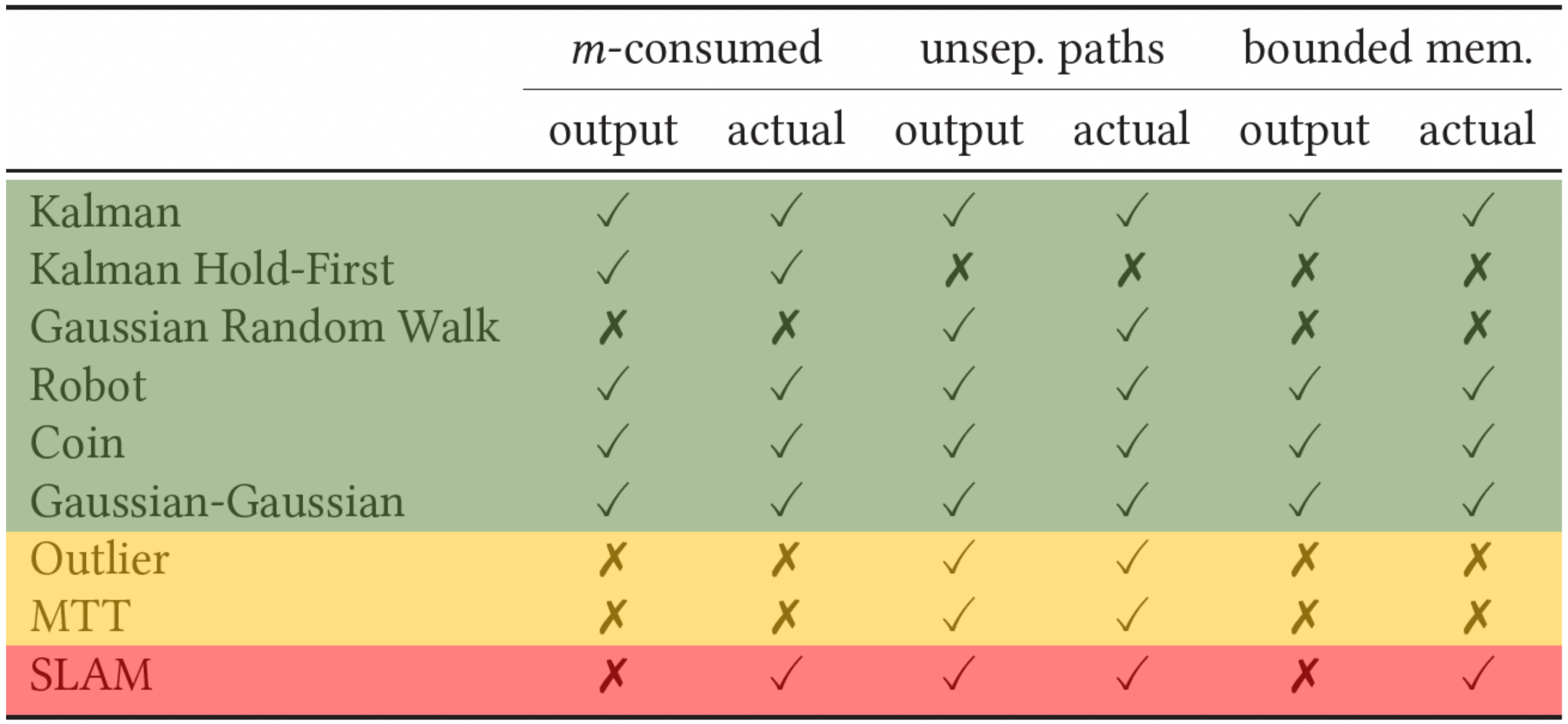
```
proba tracker (y) = x where
  rec init x0 = sample (gaussian (0, 10))
  and x   = x0 → sample (gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

Chains of variables referenced in the state are bounded

trace	state	time
$x_0 \leftarrow \bot$ ::	`x = ` x_0	$t = 0$
$y_0 \leftarrow x_0$ ::	`x0 = ` x_0	
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	`x = ` x_1`, pre x = ` x_0	$t = 1$
$y_1 \leftarrow x_1$ ::	`x0 = ` x_0	
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	`x = ` x_2`, pre x = ` x_1	$t = 2$
$y_2 \leftarrow x_2$ ::	`x0 = ` x_0	
…		

No!

Evaluation



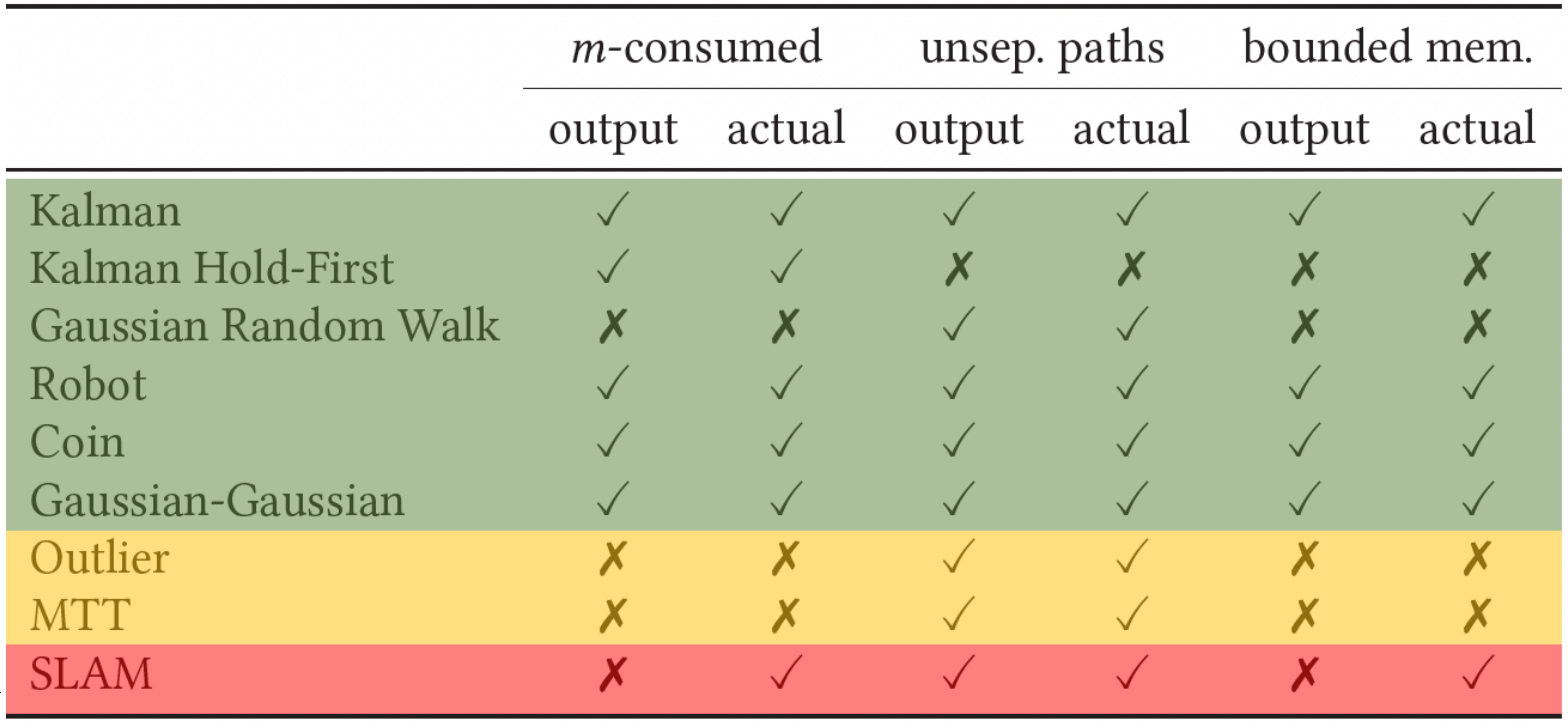
	m-consumed		unsep. paths		bounded mem.	
	output	actual	output	actual	output	actual
Kalman	✓	✓	✓	✓	✓	✓
Kalman Hold-First	✓	✓	✗	✗	✗	✗
Gaussian Random Walk	✗	✗	✓	✓	✗	✗
Robot	✓	✓	✓	✓	✓	✓
Coin	✓	✓	✓	✓	✓	✓
Gaussian-Gaussian	✓	✓	✓	✓	✓	✓
Outlier	✗	✗	✓	✓	✗	✗
MTT	✗	✗	✓	✓	✗	✗
SLAM	✗	✓	✓	✓	✗	✓

Evaluation

memory is probabilistically bounded

	m-consumed		unsep. paths		bounded mem.	
	output	actual	output	actual	output	actual
Kalman	✓	✓	✓	✓	✓	✓
Kalman Hold-First	✓	✓	✗	✗	✗	✗
Gaussian Random Walk	✗	✗	✓	✓	✗	✗
Robot	✓	✓	✓	✓	✓	✓
Coin	✓	✓	✓	✓	✓	✓
Gaussian-Gaussian	✓	✓	✓	✓	✓	✓
Outlier	✗	✗	✓	✓	✗	✗
MTT	✗	✗	✓	✓	✗	✗
SLAM	✗	✓	✓	✓	✗	✓

Evaluation



	m-consumed		unsep. paths		bounded mem.	
	output	actual	output	actual	output	actual
Kalman	✓	✓	✓	✓	✓	✓
Kalman Hold-First	✓	✓	✗	✗	✗	✗
Gaussian Random Walk	✗	✗	✓	✓	✗	✗
Robot	✓	✓	✓	✓	✓	✓
Coin	✓	✓	✓	✓	✓	✓
Gaussian-Gaussian	✓	✓	✓	✓	✓	✓
Outlier	✗	✗	✓	✓	✗	✗
MTT	✗	✗	✓	✓	✗	✗
SLAM	✗	✓	✓	✓	✗	✓

memory is probabilistically bounded

memory is always bounded

Applications: Control

Reactive Probabilistic Programming

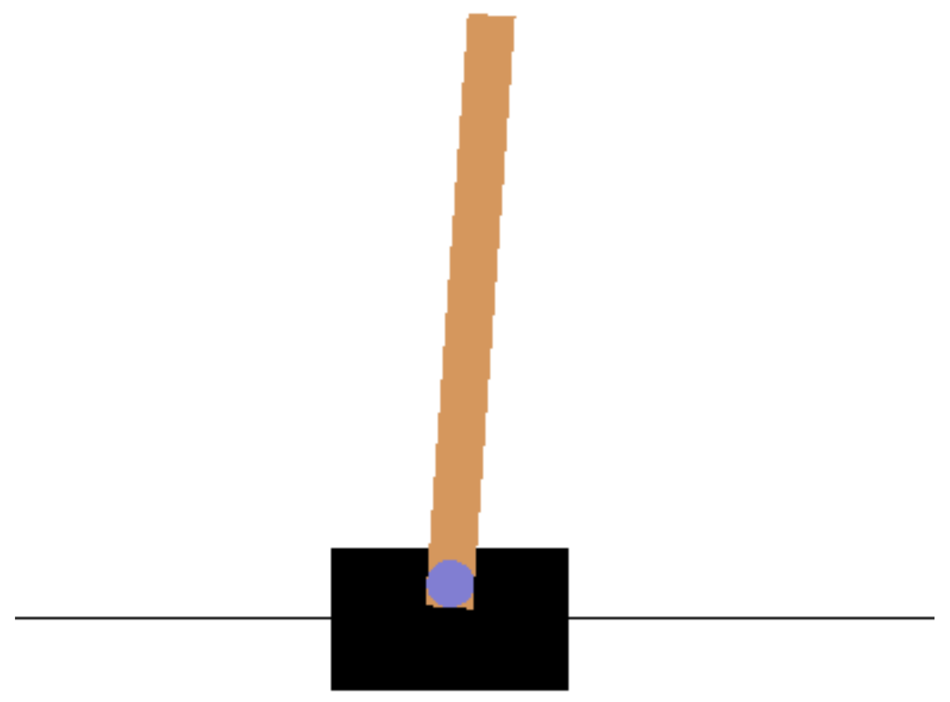
Cartpole PID

exec_simple_pid

```
let node controller (angle, (p,i,d)) = action where
  rec e = angle -. (0.0 → pre theta)
  and theta = p *. e +. i *. integr(0., e) +. d *. deriv(e)
  and action = if theta > 0. then Right else Left

let p = 0.0403884114239
let i = 0.041460471604
let d = 0.0705417538223

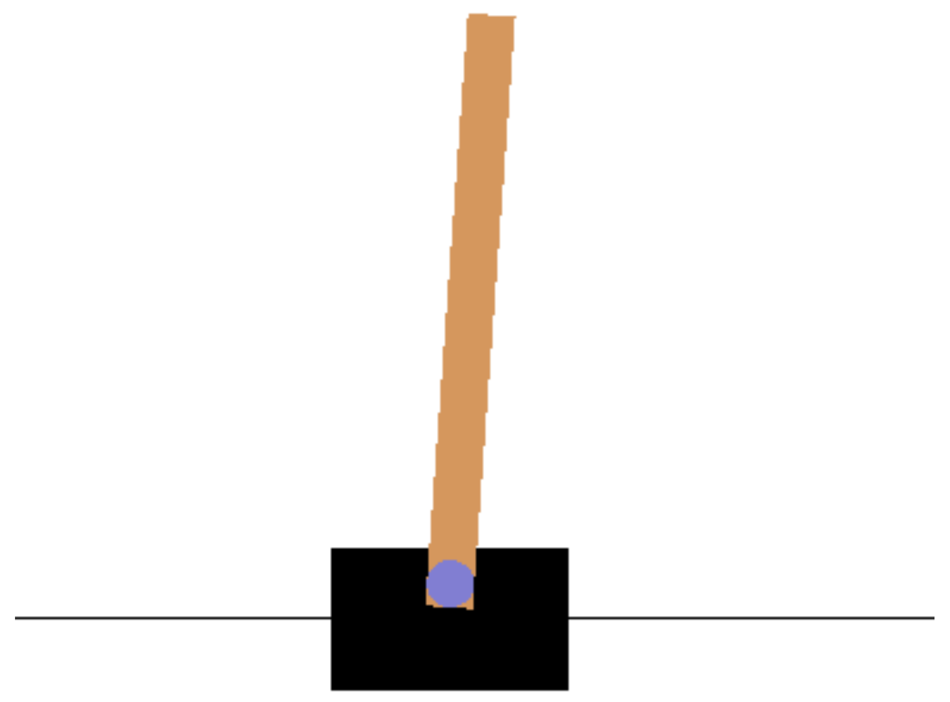
let node main () = () where
  rec obs, _, stop = cart_pole_gym true (Right → pre action)
  and reset action = controller (obs.pole_angle, (p, i, d))
      every stop
```



Cartpole learn from angle

```
(* Learn the coefficients that minimize the angle *)
let proba model obs_init = p, (i, d) where
  rec init p = sample (gaussian 0. 0.1)
  and init i = sample (gaussian 0. 0.1)
  and init d = sample (gaussian 0. 0.1)
  and action = controller (obs.pole_angle, (p,i,d))
  and obs = simple_pendulum (obs_init,  Right → pre action)
  and () = factor (-10. *. abs_float (obs.pole_angle))

let node main () = () where
  rec obs, _, stop = cart_pole_gym true (Right → pre action)
  and reset action = controller (obs.pole_angle, (p, i, d))
      every stop
  and pid_dist = infer 1000 model obs
  and p, (i, d) = draw pid_dist
```

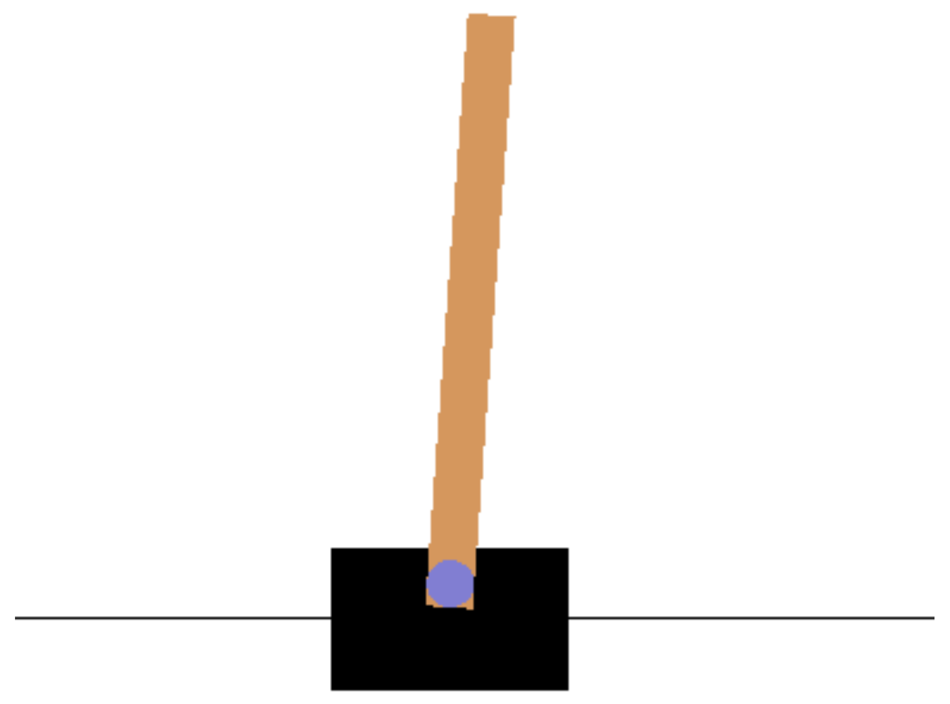


Cartpole learn from example

exec_smart_pid

```
(* Favor action similar to example *)
let proba model (obs, ctrl_action) = p, (i, d) where
  rec init p = sample (gaussian 0. 0.1)
  and init i = sample (gaussian 0. 0.1)
  and init d = sample (gaussian 0. 0.1)
  and action = controller (obs.pole_angle, (p,i,d))
  and () = factor (if action = ctrl_action then 0. else -0.2)

let node main () = () where
  rec obs, _, stop = cart_pole_gym true (Right → pre action)
  and reset action = controller (obs.pole_angle, (p, i, d))
      every stop
  and pid_dist = infer 1000 model obs
  and p, (i, d) = draw pid_dist
```

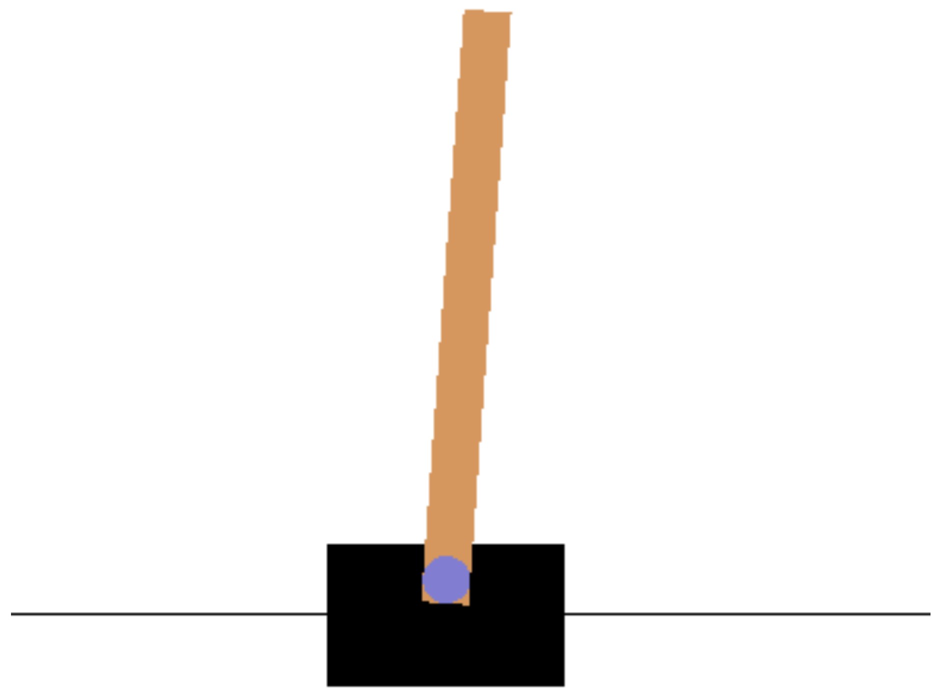


Cartpole learn from example

exec_smart_pid

```
(* Favor action similar to example *)
let proba model (obs, ctrl_action) = p, (i, d) where
  rec init p = sample (gaussian 0. 0.1)
  and init i = sample (gaussian 0. 0.1)
  and init d = sample (gaussian 0. 0.1)
  and action = controller (obs.pole_angle, (p,i,d))
  and () = factor (if action = ctrl_action then 0. else -0.2)

let node main fix = () where
  rec obs, _, stop = cart_pole_gym true (Right → pre action)
  and reset action = controller (obs.pole_angle, (p, i, d))
      every stop
  and automaton
      | Learn → do pid_dist = infer 1000 model obs
                and p, (i, d) = draw pid_dist
        until fix then Fix
      | Fix → do until (not fix) then Learn
  end
```



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