Probabilistic Programming Languages

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Probabilistic programming

Programming and reasoning with uncertainty

- Sample from probability distributions
- Condition on observed data

Bayesian Inference: learn parameters from data

- Latent parameter θ
- Observed data $x_1, \ldots, x_n$

$$\boxed{p(\theta \mid x_1, \ldots x_n)} = \frac{p(\theta)\, p(x_1, \ldots, x_n \mid \theta)}{p(x_1, \ldots, x_n)} \qquad \text{(Bayes' theorem)}$$

posterior

$$\propto \boxed{p(\theta)}\, \boxed{p(x_1, \ldots, x_n \mid \theta)} \qquad \text{(Data are constants)}$$

prior *likelihood*

Thomas Bayes (1701-1761)

https://en.wikipedia.org/wiki/Thomas_Bayes

Example: Coin

Consider a series of coin tosses

- Observations: head or tail
- Each toss is independant
- What is the probability of getting head at the next toss?

Probabilistic model

- Prior: $z \sim \mathit{Uniform}(0, 1)$
- Observations: for $i \in [1, n]$, $x_i \sim \mathit{Bernoulli}(z)$
- Posterior: $p(z \mid x_1, \ldots, x_n)$?

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- Posterior: $p(z \mid x_1, \dots, x_n)$?

$$
\begin{aligned}
p(x_1, \dots, x_n \mid z) &= \prod_{i=1}^{n} p(x_i \mid z) \\
&= \prod_{i=1}^{n} z^{x_i}\ (1-z)^{1-x_i} \\
&= z^{\sum_{i=1}^{n} x_1}\ (1-z)^{\sum_{i=1}^{n}(1-x_i)} \\
&= z^{\text{\#heads}}\ (1-z)^{\text{\#tails}}
\end{aligned}
$$

$$
\begin{aligned}
p(z \mid x_1, \dots, x_n) &= \frac{p(x_1, \dots, x_n \mid z) p(z)}{p(x_1, \dots, x_n)} \\
&= \frac{p(x_1, \dots, x_n \mid z) p(z)}{\displaystyle\int_z p(x_1, \dots, x_n \mid z)}
\end{aligned}
$$

$$
\begin{aligned}
p(z \mid x_1, \dots, x_n) &= \frac{z^{\text{\#heads}}\ (1-z)^{\text{\#tails}}}{\displaystyle\int_z z^{\text{\#heads}}\ (1-z)^{\text{\#tails}}} \\
&= \frac{z^{\text{\#heads}}\ (1-z)^{\text{\#tails}}}{B(\text{\#heads}+1,\ \text{\#tails}+1)} \\
&= \text{pdf}(\mathit{Beta}(\text{\#heads}+1,\ \text{\#tails}+1))
\end{aligned}
$$

Example: Coin

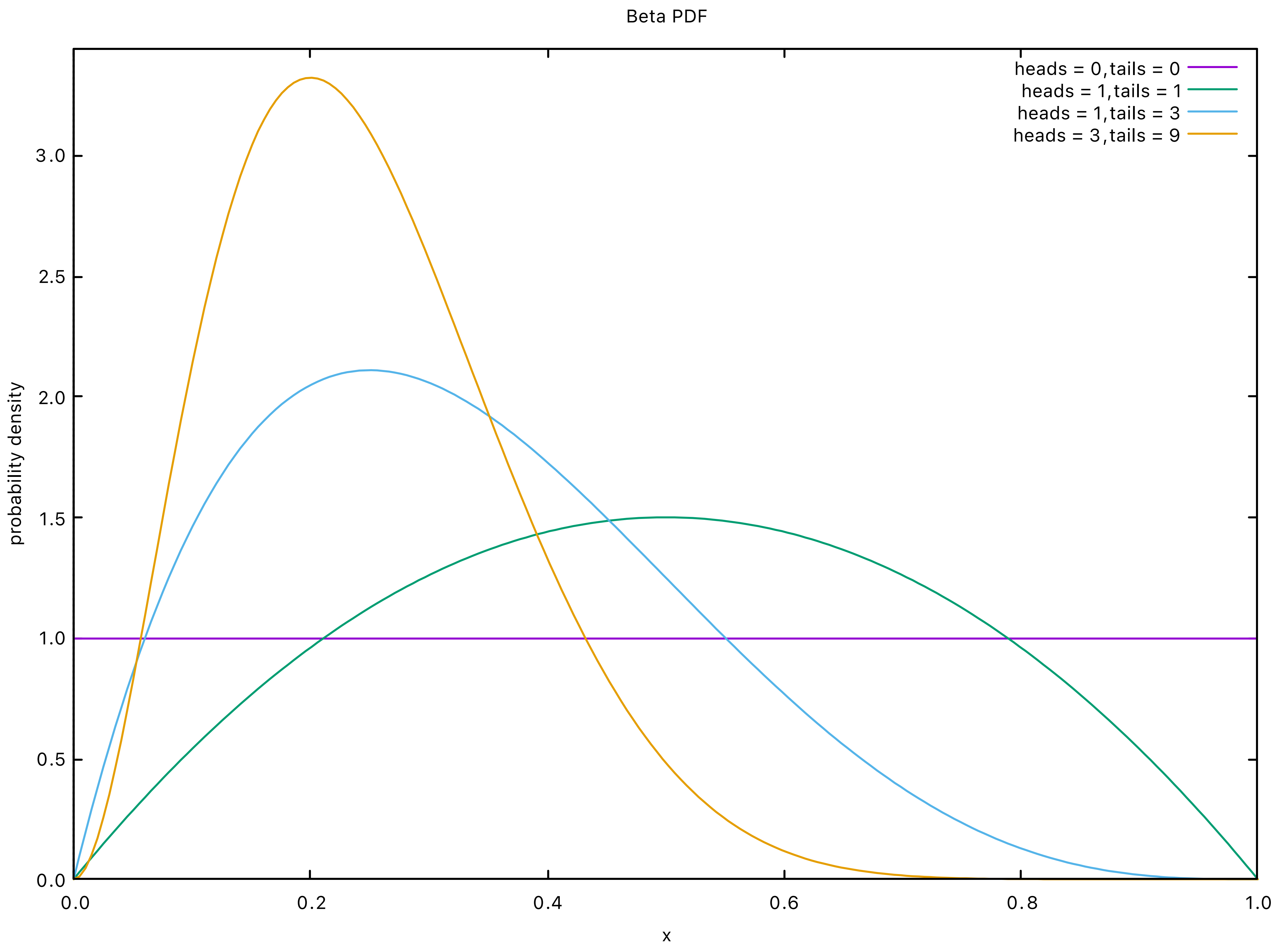
Consider a series of coin tosses

- Observations: head or tail
- Each toss is independant
- What is the probability of getting head at the next toss?

Probabilistic model

- Prior: $z \sim \mathit{Uniform}(0, 1)$
- Observations: for $i \in [1, n]$, $x_i \sim \mathit{Bernoulli}(z)$
- Posterior: $p(z \mid x_1, \ldots, x_n)$?

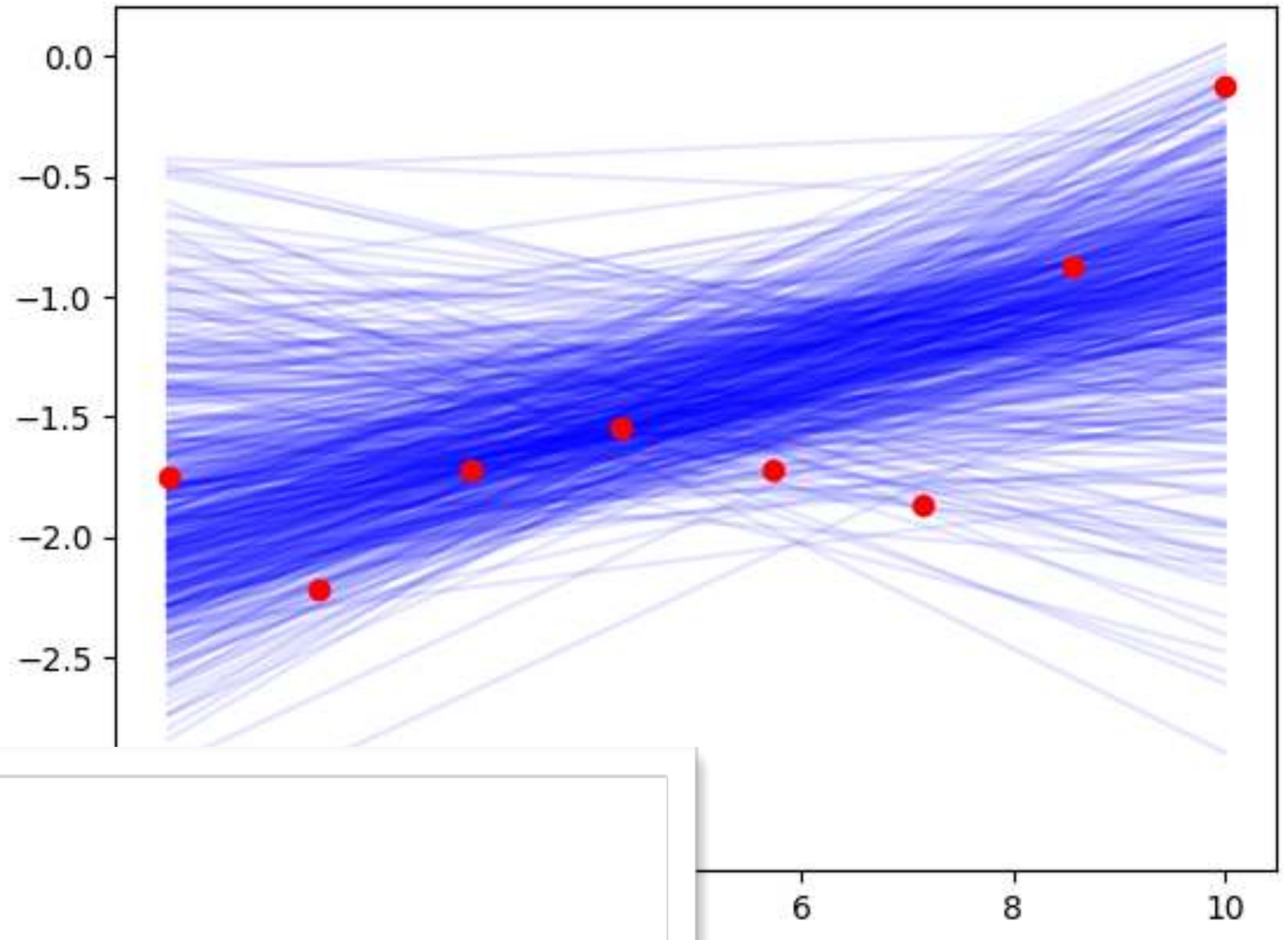
$z \sim \mathit{Beta}(\#\text{heads} + 1,\ \#\text{tails} + 1)$



Example: Linear Regression

Consider a series of observations

- Observation: point (x, y)
- Each point is independent from others
- Find the distribution of possible regressions



Probabilistic model

- Prior: $a \sim \mathcal{N}(0, 1)$, and $b \sim \mathcal{N}(0, 1)$
- Observations: for $i \in [1, n]$, $y_i \sim \mathcal{N}(a \times x_i + b, \sigma)$
- Posterior: $p(a, b \mid (x_1, y_1) \ldots, (x_n, y_n))$?

What if the model is much more complex?

What if we use arbitrary control flow?

Can we compute the posterior automatically?

Probabilistic programming languages

General purpose programming languages extended with probabilistic constructs

- `sample`: draw a sample from a distribution
- `assume`, `factor`, `observe`: condition the model on inputs (e.g., observed data)
- `infer`: compute the posterior distribution of a model given the inputs

Multiple examples:

- Church, Anglican (lisp, clojure), 2008
- WebPPL (javascript), 2014
- Pyro/NumPyro (python), 2017/2019
- Gen (julia), 2018
- ProbZelus (Zelus), 2019
- ...

More and more, incorporating new ideas:

- New inference techniques, e.g., stochastic variational inference (SVI)
- Interaction with neural nets (deep probabilistic programming)

Bayesian reasoning

Notation: $x \sim \mu$
- parameter (output): `sample(mu)`
- observation (input): `observe(mu)`

Bayesian Inference: learn parameters from data
- Latent parameter x
- Observed data $y_1, \ldots, y_n$

$$p(x \mid y_1, \ldots, y_n) = \frac{p(x)\, p(y_1, \ldots, y_n \mid x)}{p(y_1, \ldots, y_n)} \qquad \text{(Bayes' theorem)}$$

posterior

$$\propto p(x)\, p(y_1, \ldots, y_n \mid x) \qquad \text{(Data are constants)}$$

prior *likelihood*

Probabilistic constructs
- `x = sample(d)`: introduce a random variable `x` of distribution `d`
- `observe(d, y)`: condition on the fact that `y` was sampled from `d`
- `infer(m, y)`: compute posterior distribution of `m` given `y`

Thomas Bayes (1701-1761)

https://en.wikipedia.org/wiki/Thomas_Bayes

Bayesian reasoning

Bayesian Inference: learn parameters from data

- Latent parameter x
- Observed data $y_1, \ldots, y_n$

Notation: $x \sim \mu$

- parameter (output): `sample(mu)`
- observation (input): `observe(mu)`

$$\boxed{p(x \mid y_1, \ldots, y_n)} = \frac{p(x)\, p(y_1, \ldots, y_n \mid x)}{p(y_1, \ldots, y_n)} \qquad \text{(Bayes' theorem)}$$

posterior

$$\propto \boxed{p(x)}\, \boxed{p(y_1, \ldots, y_n \mid x)} \qquad \text{(Data are constants)}$$

prior *likelihood*

```
let model (y1, ... , yn) =
  let x = sample prior in
  let () = observe ((likelihood x), (y1, ... , yn)) in
  x

infer model (y1, ... , yn)
```

Thomas Bayes (1701-1761)

https://en.wikipedia.org/wiki/Thomas_Bayes

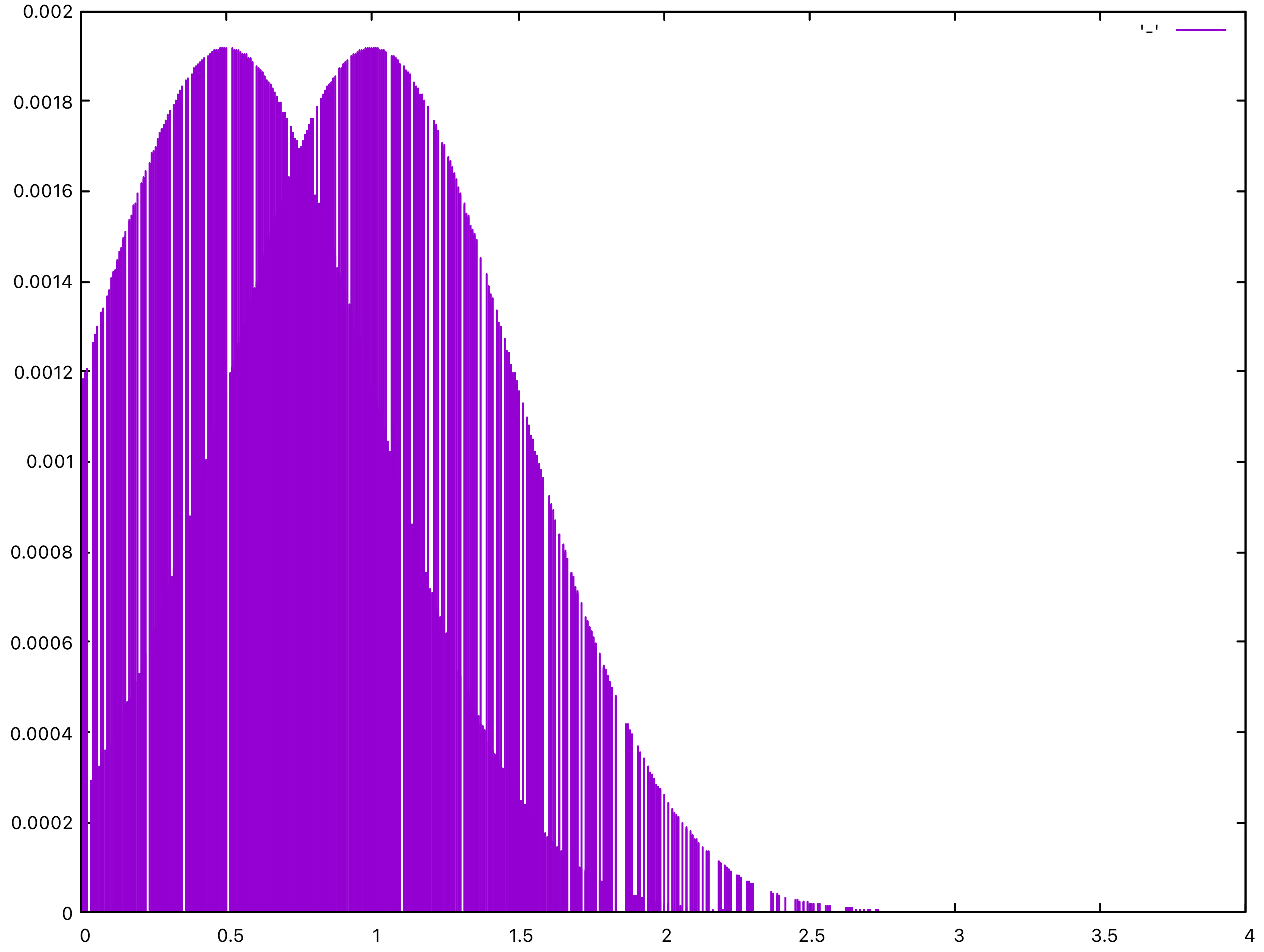
Probabilistic programming

Probabilistic constructs

- `x = sample(d)`: introduce a random variable `x` of distribution `d`
- `observe(d, y)`: condition on the fact that `y` was sampled from `d`
- `infer(m,y)`: compute posterior distribution of `m` given `y`

More general than classic Bayesian Reasoning

```
let rec weird () =
  let b = ample(Bernoulli(0.5)) in
  let mu = 0.5 if (b = 1) else 1.0 in
  let theta = sample(Gaussian(mu, 1.0)) in
  if theta > 0.:
    let () = observe (Gaussian(mu, 0.5), theta) in
    theta
  else:
    weird ()
```



Outline

I - Language

- Syntax: language and types
- Types and kinds: deterministic vs. probabilistic

II - Runtime: basic inference

- Rejection sampling (hard)
- Importance sampling

III - Kernel Semantics

- Types as measurable spaces
- Expressions as measures

Language

Probabilistic Programming Languages

Language and types

Simplified syntax

```
x ::= variables
c ::= constants

d ::= let p = e | let f = fun p → e | d d
p ::= x | (p, p)
e ::= c | x | (e, e) | op (e) | f (e)
    | if e then e else e | let p = e in e
    | sample (e) | factor (e) | observe (e, e) | infer (e)
```

Types

t ::= unit | bool | float | t dist | t dist* | $t \times t$ | $t \to t$

- t dist : distribution over values of type t
- t dist*: distribution with densities (pdf$(d) : V \to [0, \infty)$ is defined)

Types and kinds

expression

Context

type

$$G \vdash^{k} e : t$$

kind: D (deterministic) | P (probabilistic)

Kind P guards what can be expressed in a probabilistic model

Typing declarations

$$\frac{G \vdash^{D} e : t}{G \vdash^{D} \texttt{let}\ p\ \texttt{=}\ e : G + [p \leftarrow t]}$$

$$\frac{k \in \{D, P\} \qquad G + [p \leftarrow t_1] \vdash^{k} e : t_2}{G \vdash^{D} \texttt{let}\ f\ \texttt{=}\ \texttt{fun}\ p \rightarrow e : G + \left[f \leftarrow (t_1 \rightarrow^{k} t_2)\right]}$$

Declarations are deterministic
Functions can be D or P

$$\frac{G \vdash^{D} d_1 : G_1 \qquad G_1 \vdash^{D} d_2 : G_2}{G \vdash^{D} d_1\ d_2 : G_2}$$

Typing probabilistic constructs

$$\frac{G \vdash^P e : t}{G \vdash^D \texttt{infer}(e) : t\ \texttt{dist}}$$

$$\frac{G \vdash^D e : t\ \texttt{dist}}{G \vdash^P \texttt{sample}(e) : t}$$

$$\frac{G \vdash^D e : \texttt{float}}{G \vdash^P \texttt{factor}(e) : \texttt{unit}}$$

$$\frac{G \vdash^D e_1 : t\ \texttt{dist}^* \qquad G \vdash^D e_2 : t}{G \vdash^P \texttt{observe}(e_1, e_2) : \texttt{unit}}$$

$$\frac{G \vdash^D e : t}{G \vdash^P e : t}$$

$$\frac{G \vdash^D e : t\ \texttt{dist}^*}{G \vdash^D e : t\ \texttt{dist}}$$

Subtyping

Typing expressions

$$\frac{\mathit{typeOf}(c) = t}{G \vdash^D c : t}$$

$$\frac{G(x) = t}{G \vdash^D x : t}$$

$$\frac{G \vdash^D e_1 : t_1 \qquad G \vdash^D e_2 : t_2}{G \vdash^D (e_1, e_2) : t_1 \times t_2}$$

$$\frac{\mathit{typeOf}(op) = t_1 \to^D t_2 \qquad G \vdash^D e : t_1}{G \vdash^D \mathit{op}(e) : t_2}$$

$$\frac{G(f) = t_1 \to^k t_2 \qquad G \vdash^D e : t_1}{G \vdash^k f(e) : t_2}$$

$$\frac{G \vdash^D e_1 : \texttt{bool} \qquad G \vdash^k e_2 : t \qquad G \vdash^k e_3 : t}{G \vdash^k \texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3 : t}$$

$$\frac{G \vdash^k e_1 : t_1 \qquad G + [p \leftarrow t_1] \vdash^k e_2 : t_2}{G \vdash^k \texttt{let}\ p\ \texttt{=}\ e_1\ \texttt{in}\ e_2 : t_2}$$

Polymorphic kind

Example : Coin

coin.ml

```
let coin (x1, … , xn) =
  let z = sample (uniform (0., 1.)) in
  observe (bernoulli (z), x1);
  … ;
  observe (bernoulli (z), xn);
  z

let _ =
  let d = infer (coin (1; 1; 0; 0; … )) in
  plot (d)
```

$[\texttt{coin} : ???]$

$[\texttt{x1} : \alpha_1, \ldots, \texttt{xn} : \alpha_n] \vdash^P \texttt{z} : \texttt{float}$

$[\texttt{x1} : \texttt{int}, \ldots, \texttt{xn} : \alpha_n, z : \texttt{float}] \vdash^P \_ : \texttt{unit}$

$[\texttt{x1} : \texttt{int}, \ldots, \texttt{xn} : \texttt{int}, z : \texttt{float}] \vdash^P \_ : \texttt{unit}$

$[\texttt{x1} : \texttt{int}, \ldots, \texttt{xn} : \texttt{int}, z : \texttt{float}] \vdash^P \_ : \texttt{float}$

Example : Coin

coin.ml

```
let coin (x1, … , xn) =
  let z = sample (uniform (0., 1.)) in
  observe (bernoulli (z), x1);
  … ;
  observe (bernoulli (z), xn);
  z
```

$[\mathtt{coin} : (\mathtt{int} \times \cdots \times \mathtt{int}) \rightarrow^P \mathtt{float}]$

$[\mathtt{x1} : \alpha_1, \ldots, \mathtt{xn} : \alpha_n] \vdash^P \mathtt{z} : \mathtt{float}$

$[\mathtt{x1} : \mathtt{int}, \ldots, \mathtt{xn} : \alpha_n, z : \mathtt{float}] \vdash^P \_ : \mathtt{unit}$

$[\mathtt{x1} : \mathtt{int}, \ldots, \mathtt{xn} : \mathtt{int}, z : \mathtt{float}] \vdash^P \_ : \mathtt{unit}$

$[\mathtt{x1} : \mathtt{int}, \ldots, \mathtt{xn} : \mathtt{int}, z : \mathtt{float}] \vdash^P \_ : \mathtt{float}$

```
let _ =
  let d = infer (coin (1; 1; 0; 0; … )) in
  plot (d)
```

$[\mathtt{coin} : (\mathtt{int} \times \cdots \times \mathtt{int}) \rightarrow^P \mathtt{float}] \vdash^D \mathtt{d} : \mathtt{float\ dist}$

$[\mathtt{coin} : (\mathtt{int} \times \cdots \times \mathtt{int}) \rightarrow^P \mathtt{float}, \mathtt{d} : \mathtt{float\ dist}] \vdash^D \_ : \mathtt{unit}$

Runtime

Probabilistic Programming Languages

Hands-on: BYO-PPL

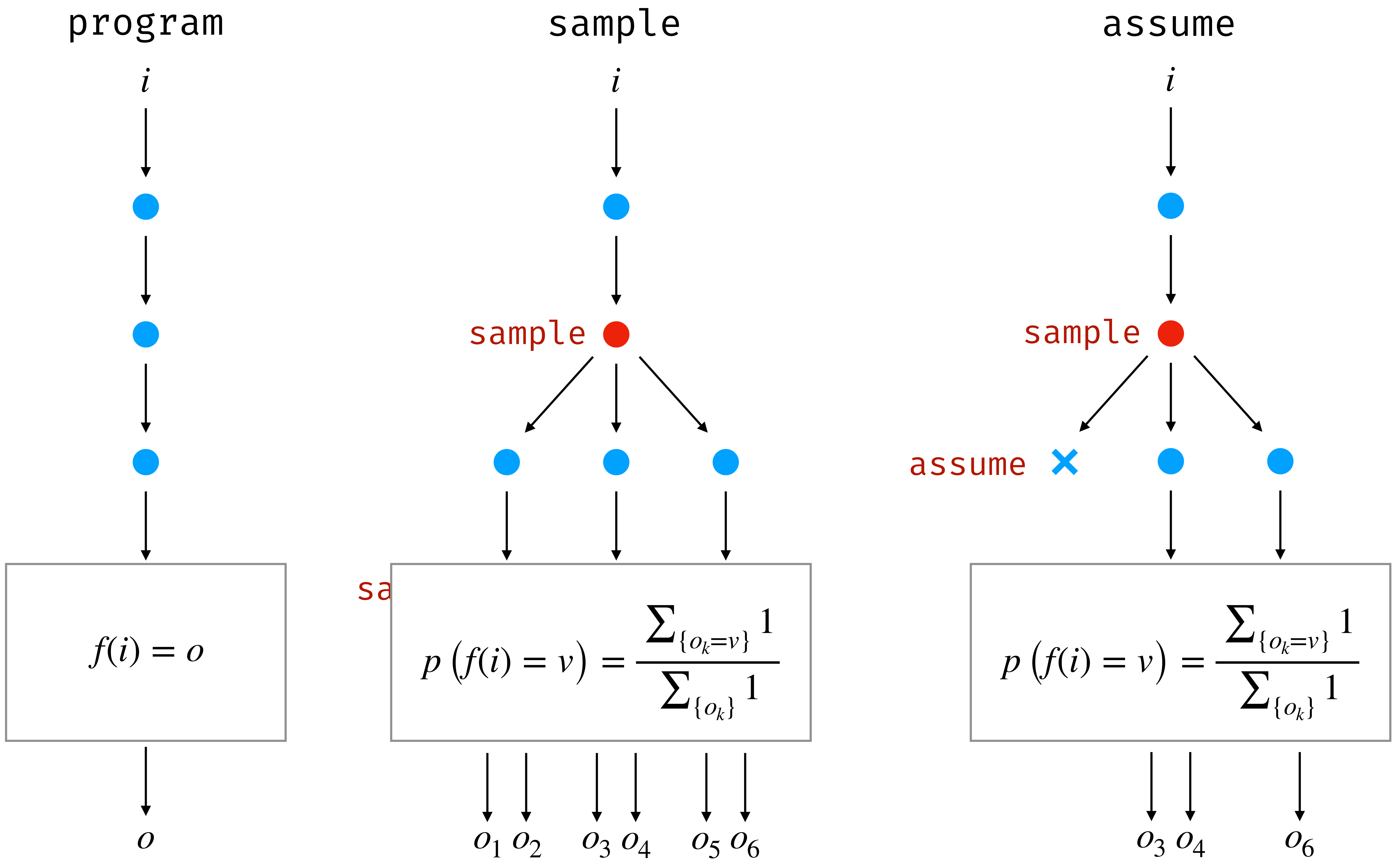
Install

- Clone https://github.com/mpri-probprog/probprog-24-25
- `cd byo-ppl`
- `opam install --deps-only .`

TODO

- Add a new distribution to distribution.ml (e.g, exponential, Poisson)
- Complete the code of `Rejection_sampling_hard` and `Importance_sampling`
- Implement and test the two models `coin` and `regression`

`infer` : $(\alpha \to \beta) \to \alpha \to \beta$ `dist`



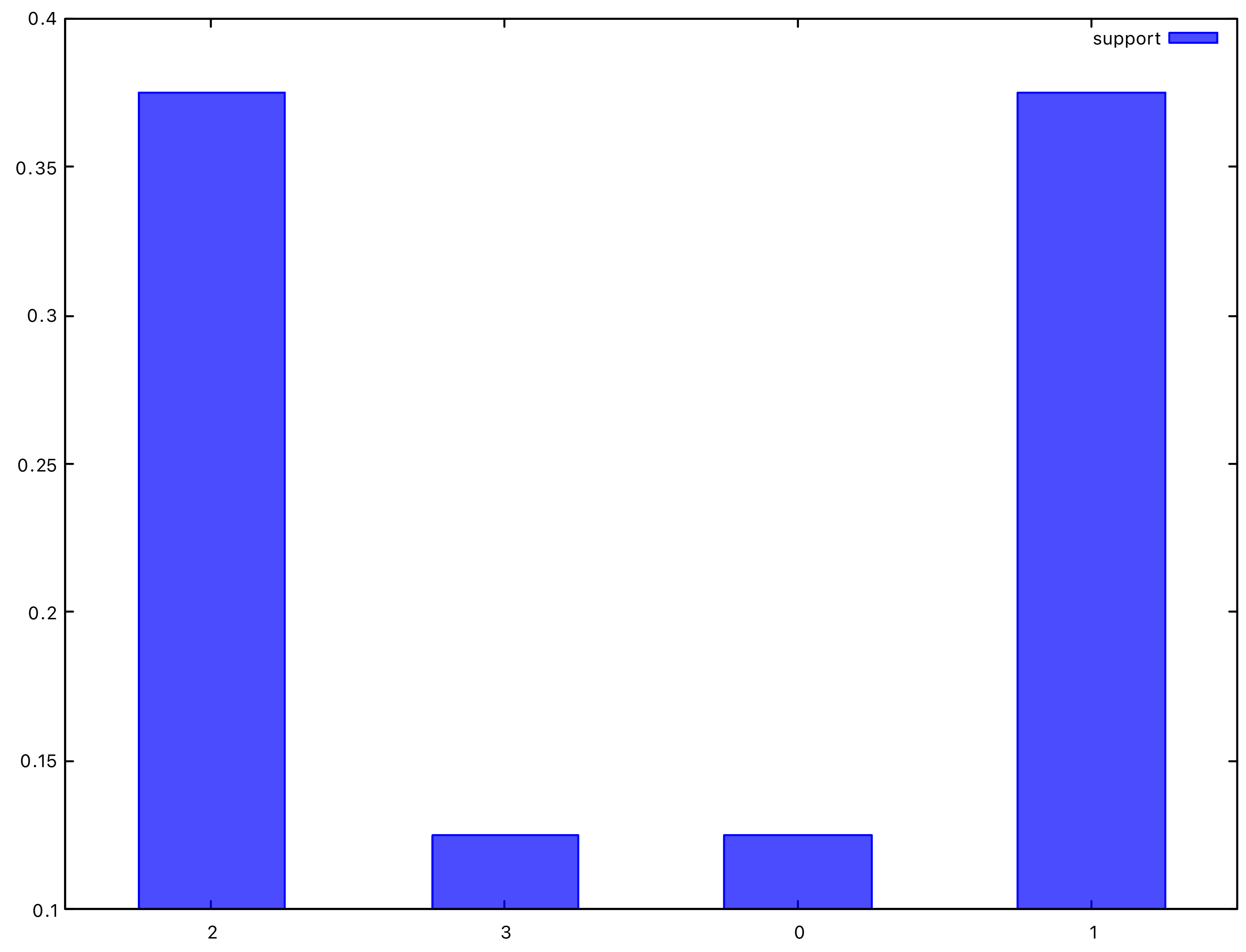
Rejection Sampling

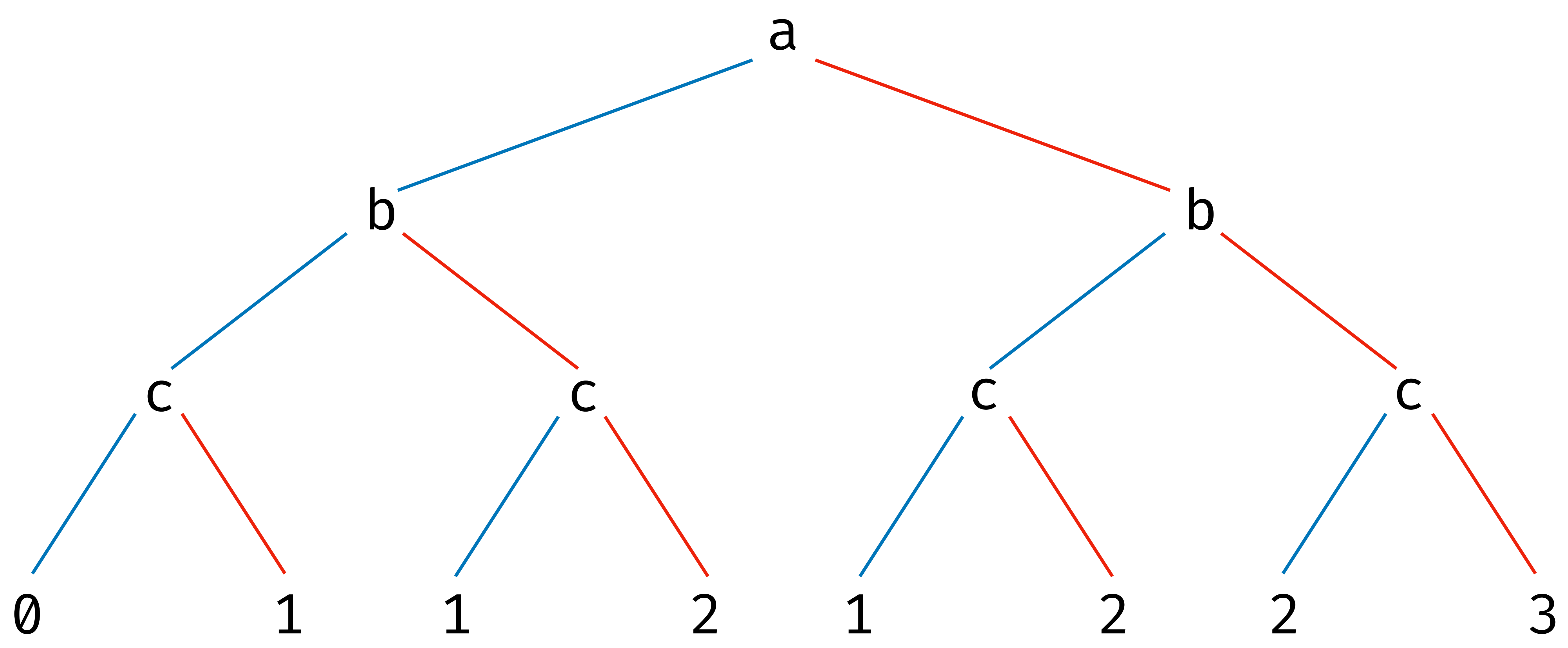
Runtime

Example: Funny Bernoulli

funny_bernoulli.ml

```
let funny_bernoulli () =
  let a = sample (bernoulli ~p:0.5) in
  let b = sample (bernoulli ~p:0.5) in
  let c = sample (bernoulli ~p:0.5) in
  a + b + c
```

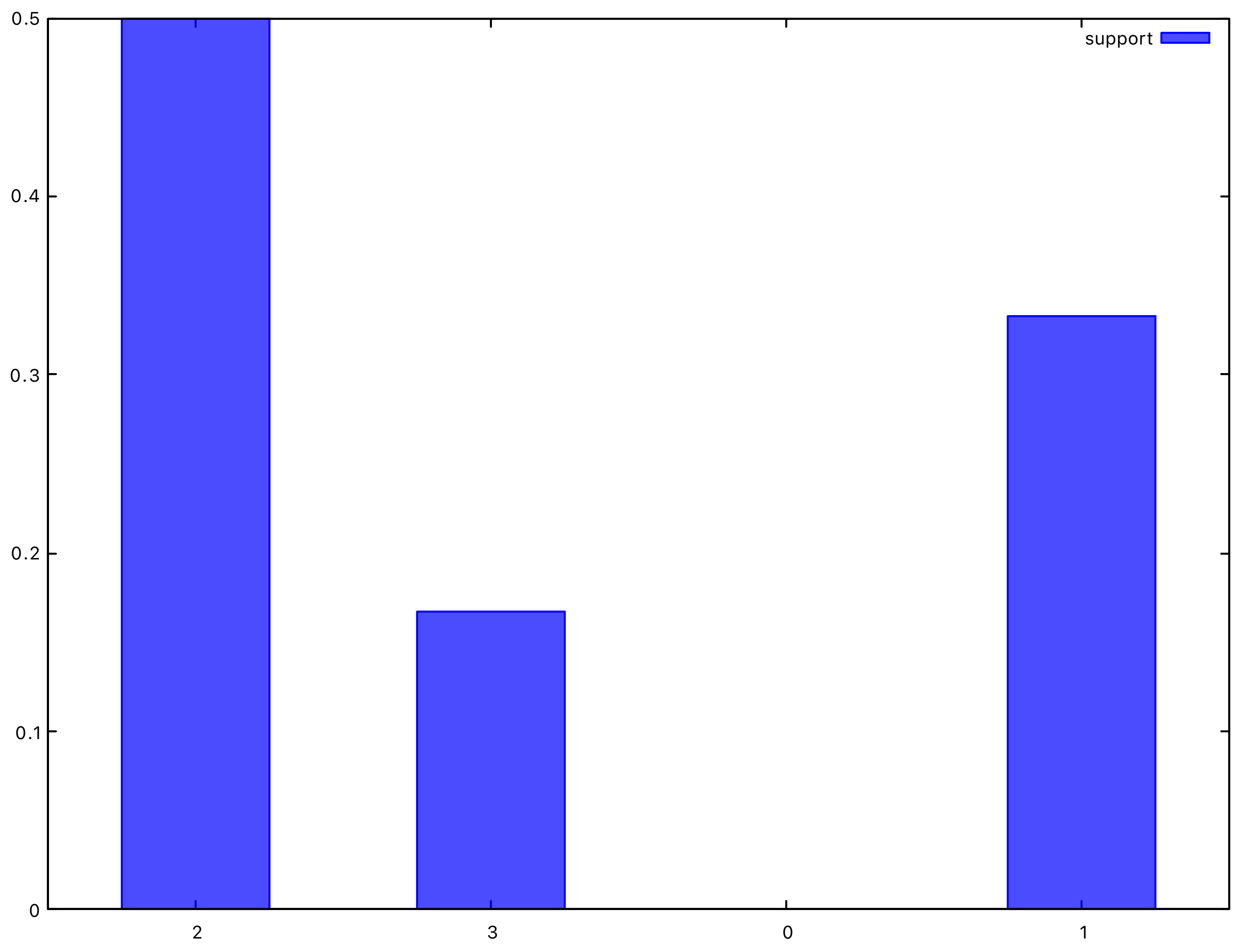


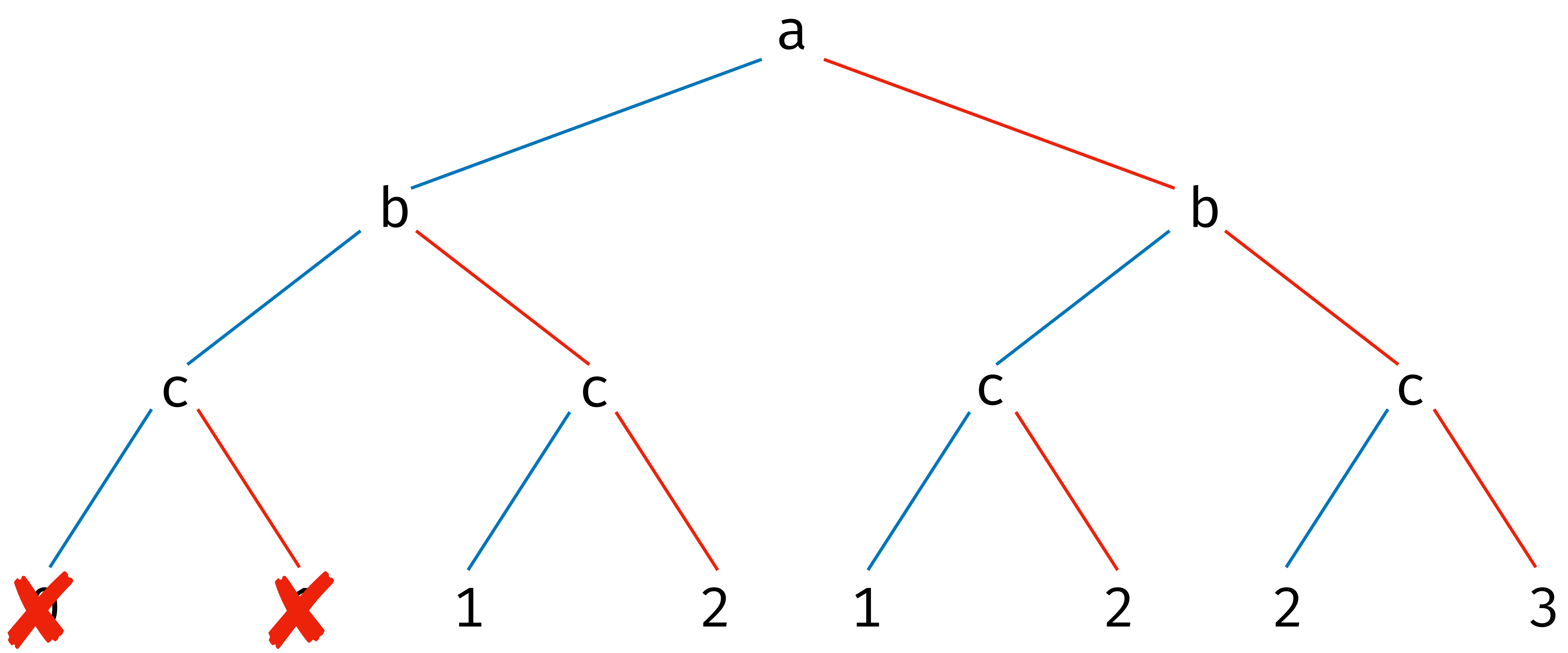


Example: Funny Bernoulli

funny_bernoulli.ml

```
let funny_bernoulli () =
  let a = sample (bernoulli ~p:0.5) in
  let b = sample (bernoulli ~p:0.5) in
  let c = sample (bernoulli ~p:0.5) in
  let () = assume (a = 1 || b = 1) in
  a + b + c
```





Rejection sampling (hard)


```
module Rejection_sampling_hard : sig
  val sample : 'a Distribution.t → 'a
  val assume : bool → unit
  val infer : ?n:int → ('a → 'b) → 'a → 'b Distribution.t
end = struct ... end
```

Inference algorithm

- Run the model to get a sample
- `sample` : draw a value from a distribution
- `assume` : accept / reject a sample
- If a sample is rejected, re-run the model to get another sample

Hard conditioning

- `val observe : 'a Distribution.t → 'a → unit`
- Assume that a value was sampled from a distribution (??)

Rejection sampling (hard)

basic.ml

```
module Rejection_sampling_hard = struct

  let sample d = assert false
  let assume p = assert false
  let observe d x = assert false

  let infer ?(n = 1000) model obs = assert false
end
```

Rejection sampling (hard)

basic.ml

```
module Rejection_sampling_hard = struct
  exception Reject

  let sample d = Distribution.draw d
  let assume p = if not p then raise Reject
  let observe d x = assume (Distribution.draw d = x)


  let infer ?(n = 1000) model obs =
    let rec gen i = try model obs with Reject → gen i in
    let values = List.init n gen in
    Distribution.empirical ~values
end
```

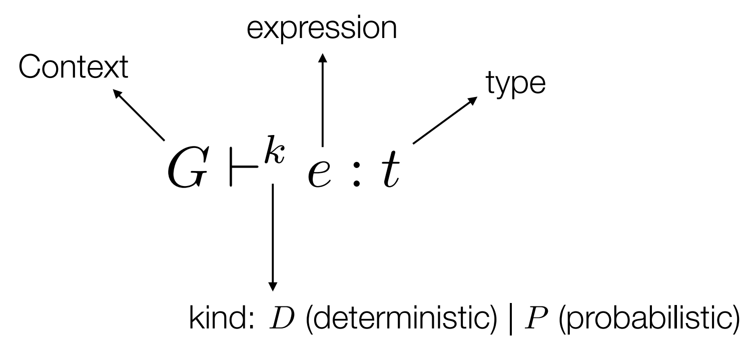
The type `prob` trick

```
module Rejection_sampling_hard : sig
  type prob
  val sample : prob → 'a Distribution.t → 'a
  val assume : prob → bool → unit
  val observe : prob → 'a Distribution.t → 'a → unit
  val infer : ?n:int → (prob → 'a → 'b) → 'a → 'b Distribution.t
end = struct ... end
```

Forbid the use of probabilistic construct outside a model

- Define a simple abstract type `prob`
- Probabilistic constructs and models all require an argument of type `prob`
- Such a value can only be build by `infer`

Types and kinds



Kind P guards what can be expressed in a probabilistic model

Rejection sampling (hard)

basic.ml

```
module Rejection_sampling_hard = struct
  type prob = Prob

  exception Reject

  let sample _prob d = Distribution.draw d
  let assume _prob p = if not p then raise Reject
  let observe _prob d x = assume (Distribution.draw d = x)

  let infer ?(n = 1000) model obs =
    let rec exec i = try model Prob obs with Reject → exec i in
    let values = Array.init n exec in
    Distribution.uniform_support ~values
end
```

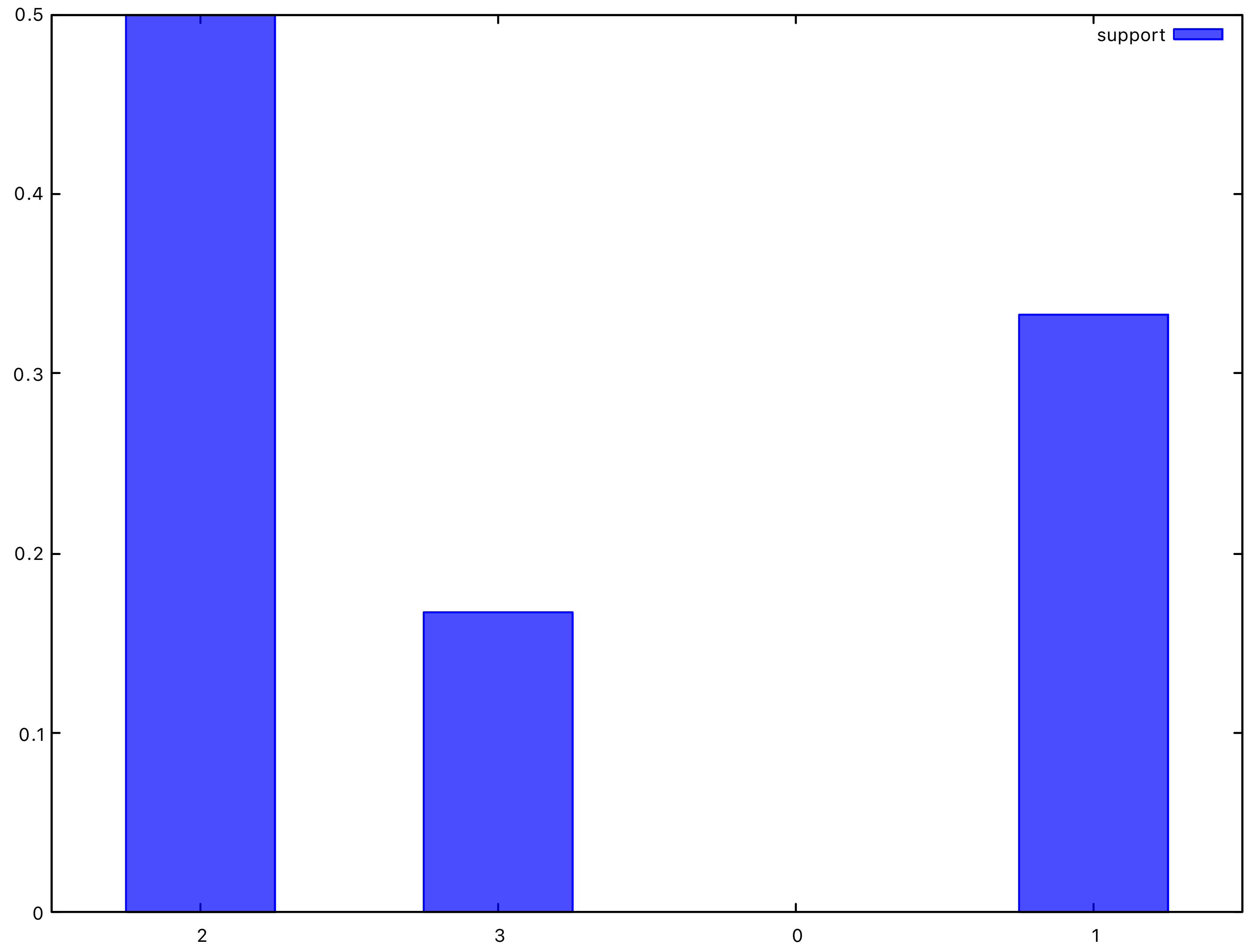
Example: Funny Bernoulli

funny_bernoulli.ml

```
open Byoppl
open Distribution
open Basic.Rejection_sampling_hard

let funny_bernoulli prob () =
  let a = sample prob (bernoulli ~p:0.5) in
  let b = sample prob (bernoulli ~p:0.5) in
  let c = sample prob (bernoulli ~p:0.5) in
  let () = assume prob (a = 1 || b = 1) in
  a + b + c

let _ =
  let dist = infer funny_bernoulli () in
  let support = categorical_to_list dist in
  List.iter (fun (v, w) -> Format.printf "%d %f@." v w) support
```




```
❯ dune exec ./examples/funny_bernoulli.exe
```

Example: Coin

coin.ml

```
open Basic.Rejection_sampling_hard

let coin prob data =
  let z = sample prob (uniform ~a:0. ~b:1.) in
  let tosses = List.map (fun _ -> sample prob (bernoulli ~p:z)) data in
  let () = assume prob (data = tosses) in
  z

let data = [false; true; true; false; false; false; false; false; false; false]

let _ =
  let dist = infer coin data in
  let m, s = Distribution.stats dist in
  Format.printf "Coin bias, mean:%f std:%f@." m s
```

$\longmapsto$ `observe d x`

```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.246161, std:0.119687
```

Example: Coin

coin.ml

```
open Basic.Rejection_sampling_hard

let coin prob data =
  let z = sample prob (uniform ~a:0. ~b:1.) in
  let () = List.iter (observe prob (bernoulli ~p:z)) data in
  z

let data = [false; true; true; false; false; false; false; false; false; false]

let _ =
  let dist = infer coin data in
  let m, s = Distribution.stats dist in
  Format.printf "Coin bias, mean:%f std:%f@." m s
```

```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.246161, std:0.119687
```

Example: Coin

coin.ml

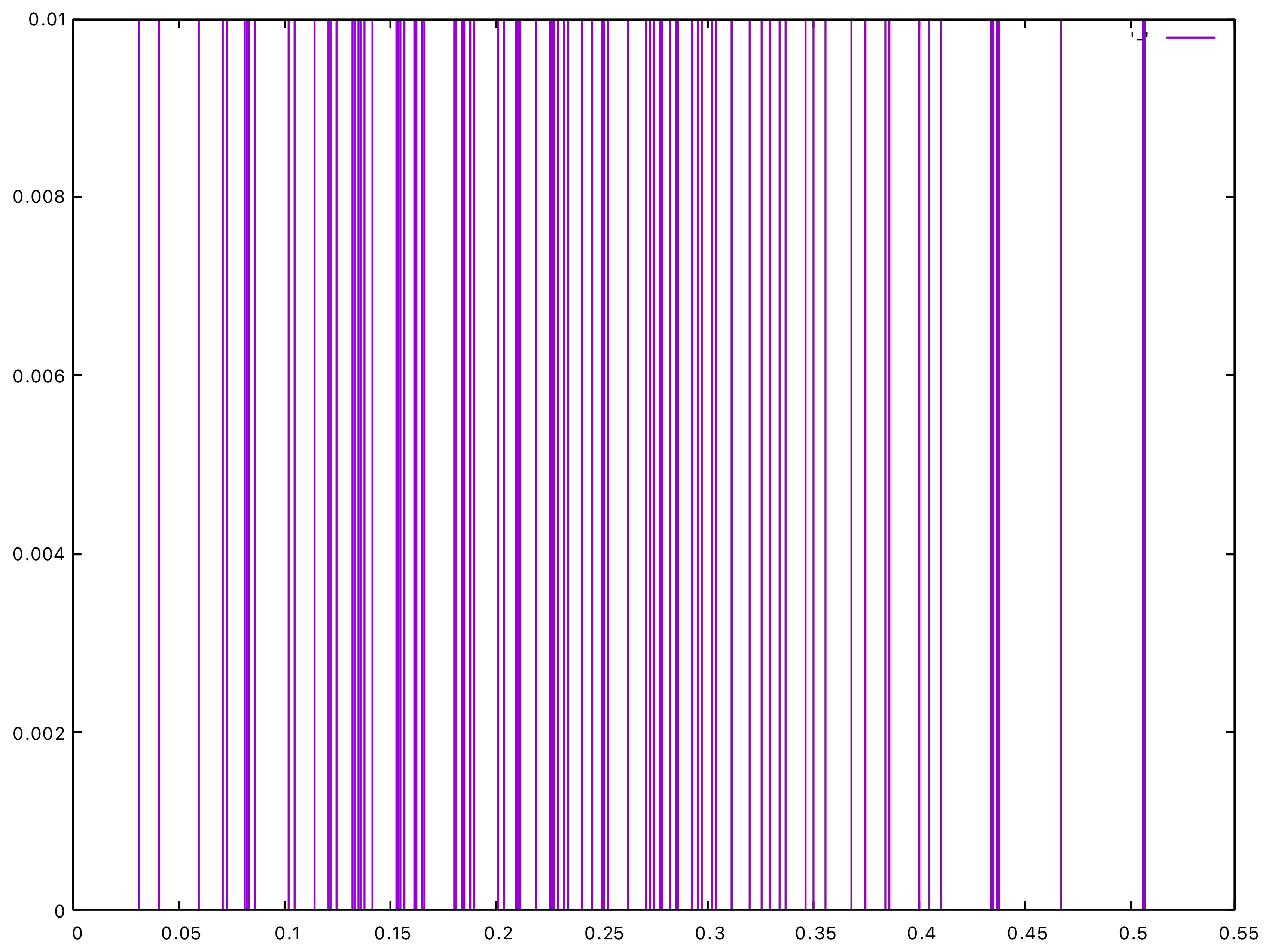
```
open Basic.Rejection_sampling_hard

let coin prob data =
  let z = sample prob (uniform ~a:0. ~b:1.) in
  let () = List.iter (observe prob (bernoulli ~p:z)) data in
  z

let data = [false; true; true; false; false; false; false; false; false; false]

let _ =
  let dist = infer coin data in
  let m, s = Distribution.stats dist in
  Format.printf "Coin bias, mean:%f std:%f@." m s
```

100 particles




```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.246161, std:0.119687
```

Example: Coin

coin.ml

```
open Basic.Rejection_sampling_hard

let coin prob data =
  let z = sample prob (uniform ~a:0. ~b:1.) in
  let () = List.iter (observe prob (bernoulli ~p:z)) data in
  z

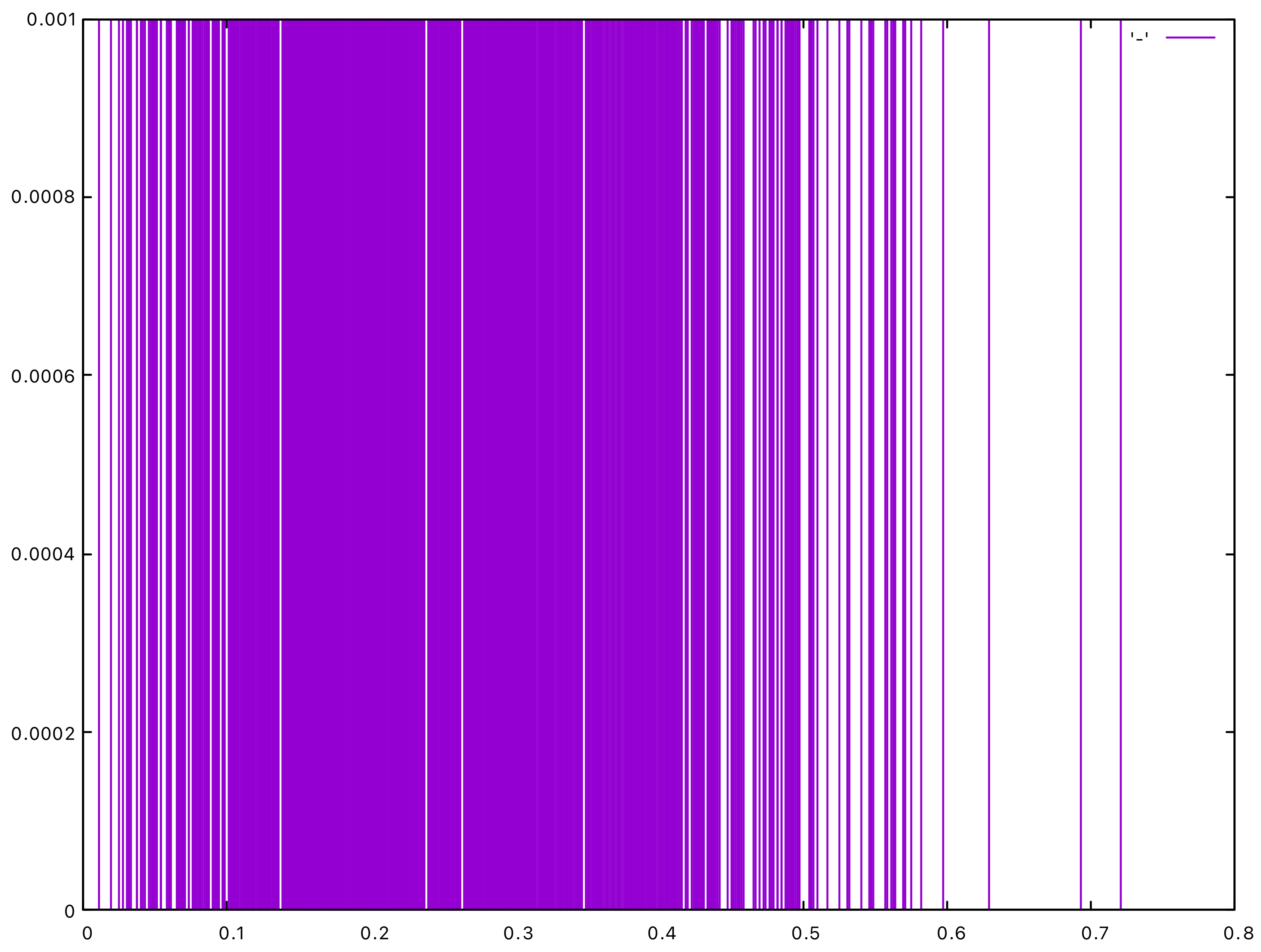
let data = [false; true; true; false; false; false; false; false; false; false]

let _ =
  let dist = infer coin data in
  let m, s = Distribution.stats dist in
  Format.printf "Coin bias, mean:%f std:%f@." m s
```

```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.246161, std:0.119687
```

1000 particles



Example: Coin

coin.ml

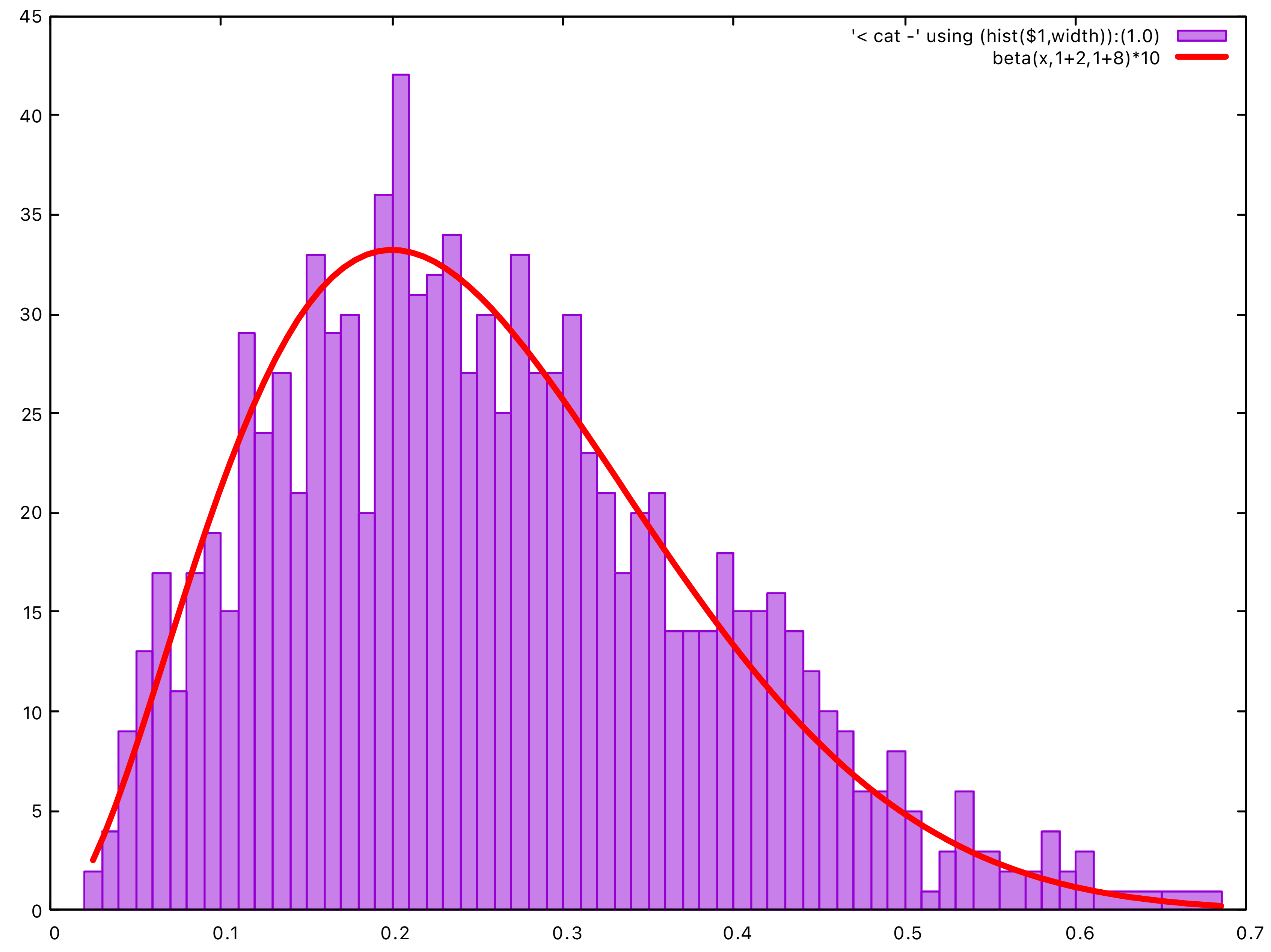
```
open Basic.Rejection_sampling_hard

let coin prob data =
  let z = sample prob (uniform ~a:0. ~b:1.) in
  let () = List.iter (observe prob (bernoulli ~p:z)) data in
  z

let data = [false; true; true; false; false; false; false; false; false; false]

let _ =
  let dist = infer coin data in
  let m, s = Distribution.stats dist in
  Format.printf "Coin bias, mean:%f std:%f@." m s
```

1000 particles




```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.246161, std:0.119687
```

Slow!

Example: Laplace and gender bias

laplace.ml

```
open Basic.Rejection_sampling_hard

let laplace prob () =
  let p = sample prob (uniform ~a:0. ~b:1.) in
  let g = sample prob (binomial ~p ~n:493_472) in
  let () = assume prob (g = 241_945) in
  p

let _ =
  let dist = infer ~n:1000 laplace () in
  let m, s = Distribution.stats dist in
  Format.printf "Gender bias, mean:%f std:%f@." m s
```

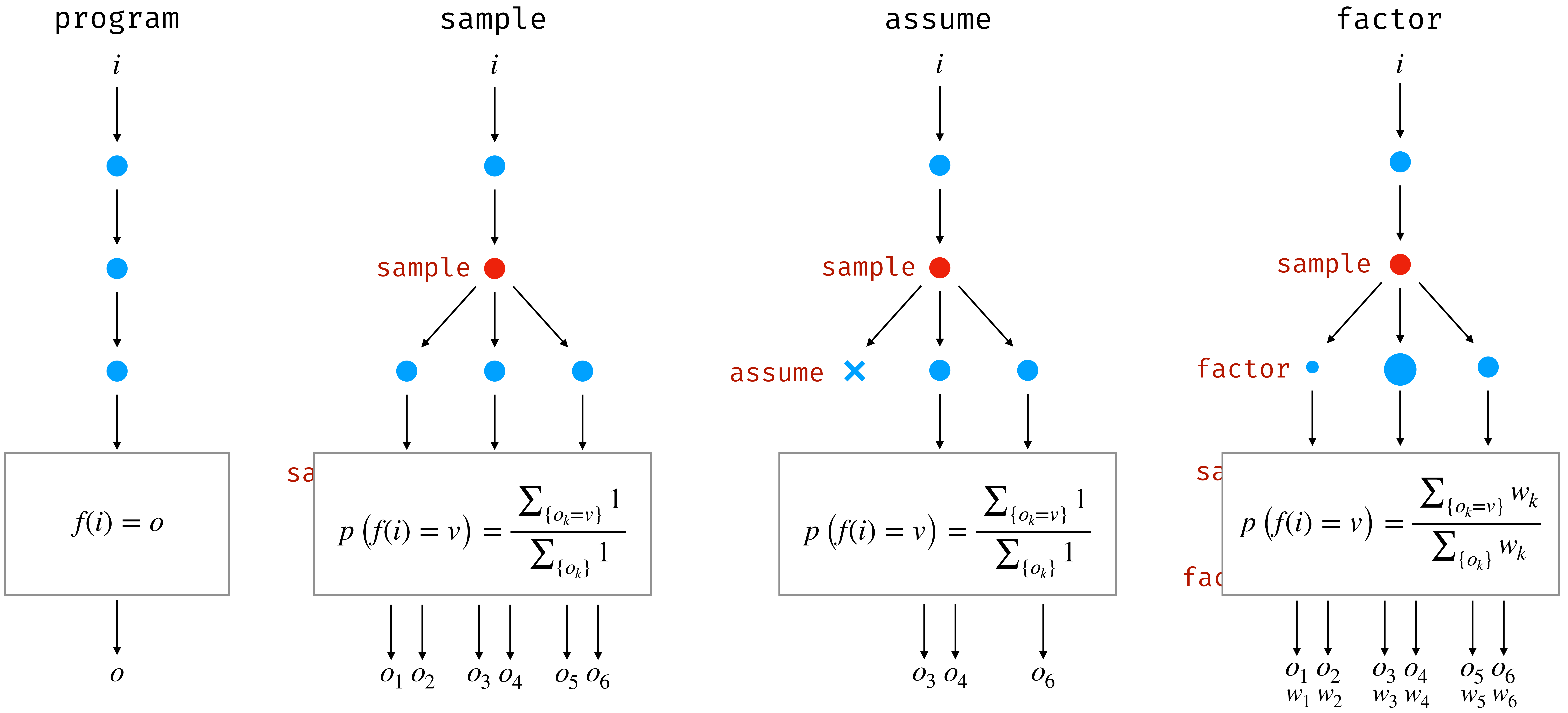
↦

```
observe prob
  (binomial ~p ~n:493_472) 241_945
```

```
❯ dune exec ./examples/laplace.exe
```

Never terminate!

infer : $(\alpha \to \beta) \to \alpha \to \beta$ dist



Importance Sampling

Runtime

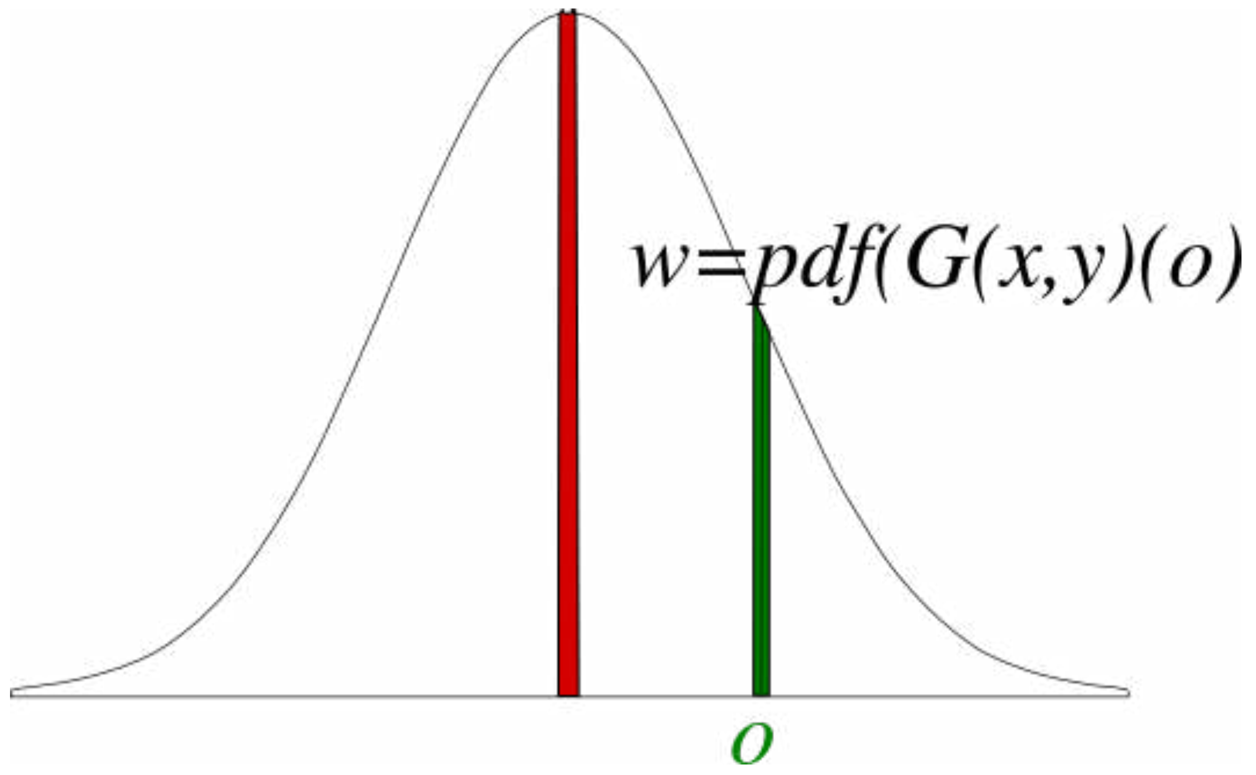
Importance sampling

basic.ml

```
module Importance_sampling : sig
  type prob
  val sample : prob → 'a Distribution.t → 'a
  val factor : prob → float → unit
  val infer : ?n:int → (prob → 'a → 'b) → 'a → 'b Distribution.t
end = struct ... end
```

Inference algorithm

- Run a set of n independent executions
- `sample`: draw a sample from a distribution
- `factor`: associate a score to the current execution
- Gather output values and score to approximate the posterior distribution



Likelihood weighting

- `observe d x := factor (logpdf d x)`

Importance sampling


```
module Importance_sampling = struct
  type prob =  ...

  let sample prob d = assert false
  let factor prob s = assert false
  let observe prob d x = factor prob (Distribution.logpdf d x)

  let infer ?(n = 1000) model obs = assert false
end
```

Importance sampling

basic.ml

```
module Importance_sampling = struct
  type prob = { mutable score : float }

  let sample _prob d = Distribution.draw d
  let factor prob s = prob.score ← prob.score +. s
  let observe prob d x = factor prob (Distribution.logpdf d x)

  let infer ?(n = 1000) model obs =
    let gen _ =
      let prob = { score = 0. } in
      let value = model prob data in
      (value, prob.score)
    in
    let support = List.init n gen in
    Distribution.categorical ~support
end
```

Example: Coin

coin.ml

```
open Basic.Importance_sampling

let coin prob data =
  let z = sample prob (uniform ~a:0. ~b:1.) in
  let () = List.iter (observe prob (bernoulli ~p:z)) data in
  z

let data = [false; true; true; false; false; false; false; false; false; false]

let _ =
  let dist = infer coin data in
  let m, s = Distribution.stats dist in
  Format.printf "Coin bias, mean:%f, std:%f@." m s
```

```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.247876, std:0.118921
Beta(2+1, 8+1), mean:0.250000, std:0.120096
```

Example: Coin

coin.ml

```
open Basic.Importance_sampling

let coin prob data =
  let z = sample prob (uniform ~a:0. ~b:1.) in
  let () = List.iter (observe prob (bernoulli ~p:z)) data in
  z

let data = [false; true; true; false; false; false; false; false; false; false]

let _ =
  let dist = infer coin data in
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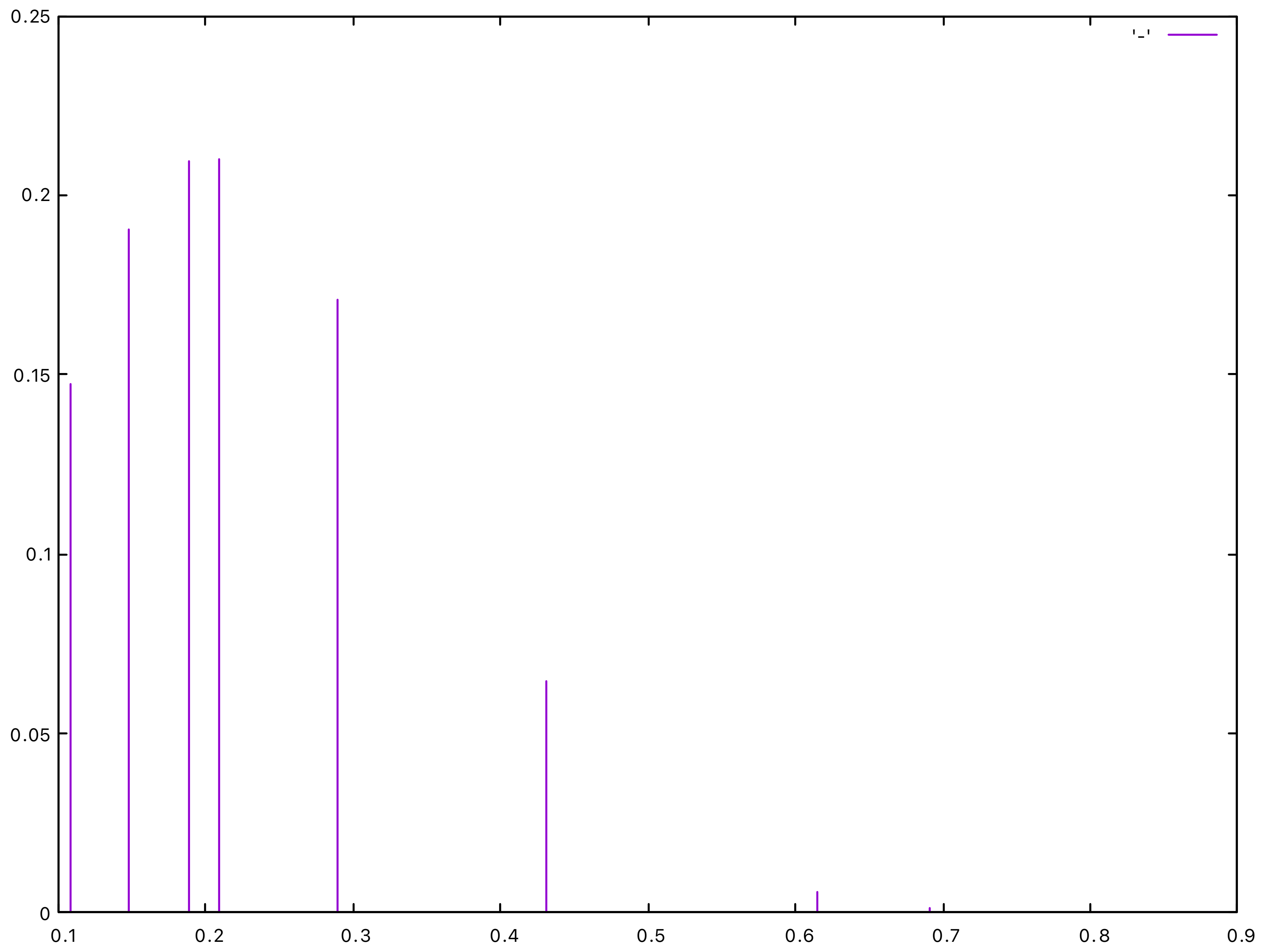
  Format.printf "Coin bias, mean:%f, std:%f@." m s
```

```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.247876, std:0.118921
Beta(2+1, 8+1), mean:0.250000, std:0.120096
```

10 particles



Example: Coin

coin.ml

```
open Basic.Importance_sampling

let coin prob data =
  let z = sample prob (uniform ~a:0. ~b:1.) in
  let () = List.iter (observe prob (bernoulli ~p:z)) data in
  z

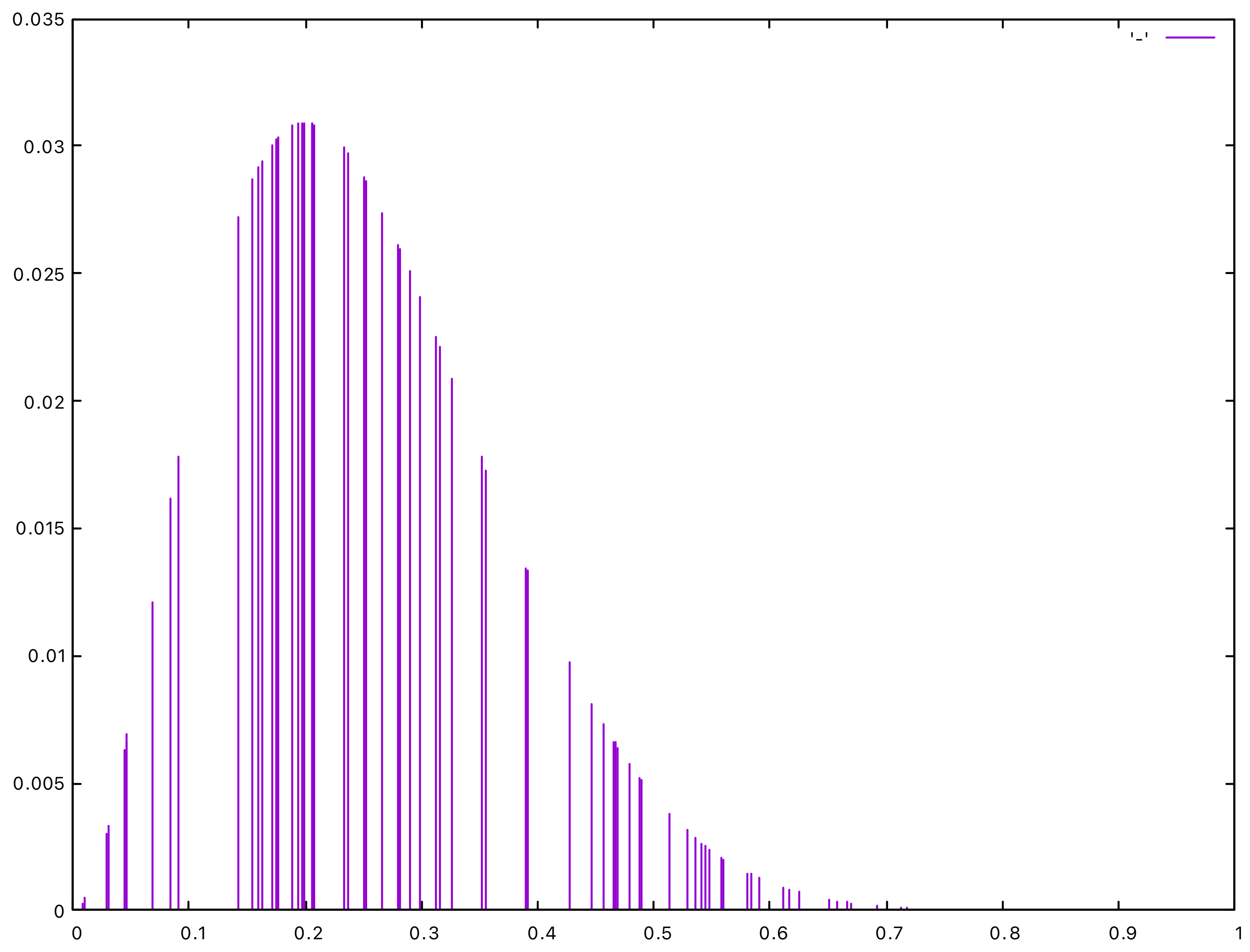
let data = [false; true; true; false; false; false; false; false; false; false]

let _ =
  let dist = infer coin data in
  let m, s = Distribution.stats dist in
  Format.printf "Coin bias, mean:%f, std:%f@." m s
```

```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.247876, std:0.118921
Beta(2+1, 8+1), mean:0.250000, std:0.120096
```

100 particles



Example: Coin

coin.ml

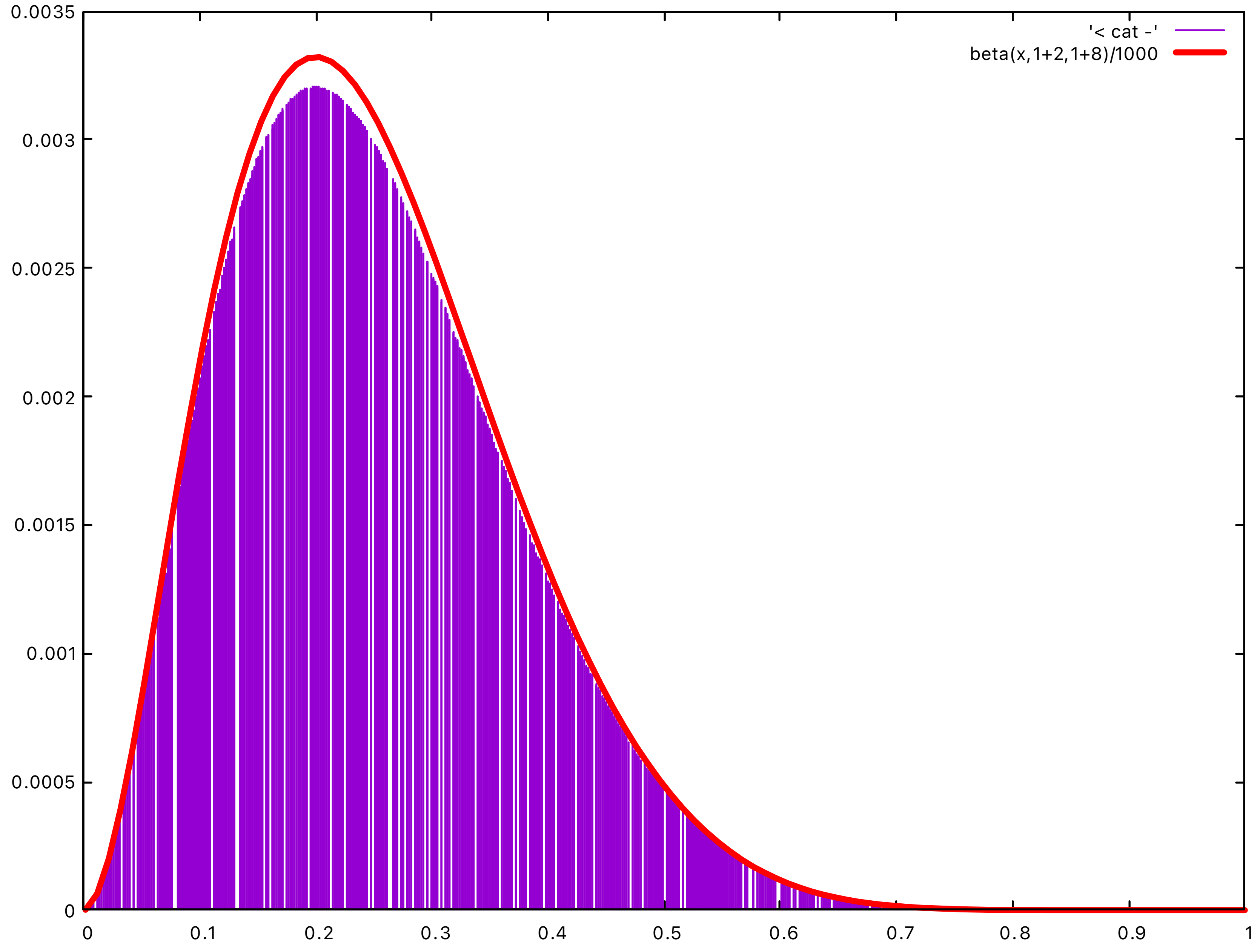
```
open Basic.Importance_sampling

let coin prob data =
  let z = sample prob (uniform ~a:0. ~b:1.) in
  let () = List.iter (observe prob (bernoulli ~p:z)) data in
  z

let data = [false; true; true; false; false; false; false; false; false; false]

let _ =
  let dist = infer coin data in
  let m, s = Distribution.stats dist in
  Format.printf "Coin bias, mean:%f, std:%f@." m s
```

1000 particles



```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.247876, std:0.118921
Beta(2+1, 8+1), mean:0.250000, std:0.120096
```

Conditioning

basic.ml

```
module Rejection_sampling_hard = struct ...

  (* Reject if [p] is not true. *)
  let assume prob p =
    if not p then raise Reject

  (* Assume [x] was sampled from [d]. *)
  let observe prob d x =
    let v = sample d in
    assume prob (v = x)
```

Hard conditioning

```
module Importance_sampling = struct ...

  (* Update the (log)score. *)
  let factor prob s =
    prob.score ← prob.score +. s

  (* Assume [x] was sampled from [d]. *)
  let observe prob d x =
    prob.score ← prob.score +. (logpdf d x)
```

Soft conditioning

Kernel Semantics

Probabilistic Programming Languages

Types as mesurable spaces

A ground type t is interpreted as a measurable space $[\![t]\!]$

- $[\![\texttt{unit}]\!]$: discrete measurable space over the unique value ( )
- $[\![\texttt{bool}]\!]$: discrete measurable space with the two values $\texttt{true}$, $\texttt{false}$
- $[\![\texttt{float}]\!]$: reals with its Borel sets (intervals)

- $A \times B$ product space $[\![A]\!] \times [\![B]\!]$
  with the rectangles $U \times V$ for $U \in \Sigma_A$ and $B \in \Sigma_B$

- $[\![t\ \texttt{dist}]\!]$: set of probability measures on $[\![t]\!]$
  with the sets $\{\mu \mid \mu(U) < r\}$ for $U \in \Sigma_{[\![t]\!]}$ and $r \in [0, 1]$ (Giry monad)

- A context $G = [x_1 : A_1, \ldots, x_n : A_n]$ maps variables to types
  $[\![G]\!] = \prod_{i=1}^{n} [\![A_i]\!]$ is also a measurable space (product of all variables spaces)

What about function types?

Deterministic vs. probabilistic

Deterministic semantics $G \vdash^{D} e : t$

- Classic denotational semantics
- Environments: ϕ (global declarations), γ (local variables)
- Given the declarations ϕ, $[\![e]\!]^{\phi} : \Gamma \to t$ is a measurable function
- $[\![e]\!]^{\phi}_{\gamma}$ is a value of type t

Probabilistic semantics $G \vdash^{P} e : t$

- Given the declarations ϕ, expressions are interpreted as kernels
- $\{\![e]\!\}^{\phi} : \Gamma \times \Sigma_{[\![t]\!]} \to [0, \infty)$
- $\{\![e]\!\}^{\phi}_{\gamma}$ is a measure on values of type t

(Un)normalized measures

set of possible outcomes

model

$$[\![e]\!]^{\phi}_{\gamma} : \Sigma_{[\![t]\!]} \rightarrow [0, \infty)$$

score

environments

Unnormalized measure

(Un)normalized measures

set of possible outcomes

model

$$\{\!|e|\!\}^{\phi}_{\gamma} : \Sigma_{[\![t]\!]} \to [0, \infty)$$

Unnormalized measure

score

environments

$$[\![\texttt{infer}(e)]\!]^{\phi}_{\gamma} = \frac{\{\!|e|\!\}^{\phi}_{\gamma}}{\{\!|e|\!\}^{\phi}_{\gamma}\left([\![\mathit{typeOf}(e)]\!]\right)}$$

Distribution

normalize over all possible values

Deterministic semantics

$$
\begin{aligned}
[\![\texttt{let}\ p\ \texttt{=}\ e]\!]^{\phi} &= \phi + \left[p \leftarrow [\![e]\!]^{\phi}_{\emptyset}\right] \\
[\![\texttt{let}\ f\ \texttt{=}\ \texttt{fun}\ p \rightarrow e]\!]^{\phi} &= \phi + \left[f \leftarrow \lambda v.\, [\![e]\!]^{\phi}_{[p \leftarrow v]}\right] \\
[\![d_1\ d_2]\!]^{\phi} &= \textit{let}\ \phi_1 = \phi + [\![d_1]\!]^{\phi}\ \textit{in}\ [\![d_2]\!]^{\phi_1}
\end{aligned}
$$

$$
\begin{aligned}
[\![c]\!]^{\phi}_{\gamma} &= c \\
[\![x]\!]^{\phi}_{\gamma} &= (\gamma + \phi)(x) \\
[\![(e_1, e_2)]\!]^{\phi}_{\gamma} &= ([\![e_1]\!]^{\phi}_{\gamma}, [\![e_2]\!]^{\phi}_{\gamma}) \\
[\![\textit{op}(e)]\!]^{\phi}_{\gamma} &= \textit{op}([\![e]\!]^{\phi}_{\gamma}) \\
[\![f(e)]\!]^{\phi}_{\gamma} &= \phi(f)([\![e]\!]^{\phi}_{\gamma}) \\
[\![\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3]\!]^{\phi}_{\gamma} &= \textit{if}\ [\![e_1]\!]^{\phi}_{\gamma}\ \textit{then}\ [\![e_2]\!]^{\phi}_{\gamma}\ \textit{else}\ [\![e_3]\!]^{\phi}_{\gamma} \\
[\![\texttt{let}\ p\ \texttt{=}\ e_1\ \texttt{in}\ e_2]\!]^{\phi}_{\gamma} &= \textit{let}\ v = [\![e_1]\!]^{\phi}_{\gamma}\ \textit{in}\ [\![e_2]\!]^{\phi}_{\gamma + [p \leftarrow v]}
\end{aligned}
$$

Probabilistic semantics

$$[\![\texttt{let}\ f\ \texttt{=}\ \texttt{fun}\ p \to e]\!]^{\phi} = \phi + \left[f \leftarrow \lambda v.\ \{\![e]\!\}^{\phi}_{[p \leftarrow v]}\right] \textit{ if kindOf}(e) = P$$

$$\begin{aligned}
\{\![e]\!\}^{\phi}_{\gamma} &= \lambda U.\ \delta_{[\![e]\!]^{\phi}_{\gamma}}(U) \textit{ if kindOf}(e) = D \\
\{\![f(e)]\!\}^{\phi}_{\gamma} &= \lambda U.\ \phi(f)([\![e]\!]^{\phi}_{\gamma})(U) \\
\{\![\texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3]\!\}^{\phi}_{\gamma} &= \lambda U.\ \textit{if } [\![e_1]\!]^{\phi}_{\gamma} \textit{ then } \{\![e_2]\!\}^{\phi}_{\gamma}(U) \textit{ else } \{\![e_3]\!\}^{\phi}_{\gamma}(U) \\
\{\![\texttt{let}\ p\ \texttt{=}\ e_1\ \texttt{in}\ e_2]\!\}^{\phi}_{\gamma} &= \lambda U.\ \int_{[\![\textit{typeOf}(e_1)]\!]} \{\![e_1]\!\}^{\phi}_{\gamma}(dv)\ \{\![e_2]\!\}^{\phi}_{\gamma + [p \leftarrow v]}
\end{aligned}$$

$$\begin{aligned}
\{\![\texttt{sample}(e)]\!\}^{\phi}_{\gamma} &= \lambda U.\ [\![e]\!]^{\phi}_{\gamma}(U) \\
\{\![\texttt{factor}(e)]\!\}^{\phi}_{\gamma} &= \lambda U.\ [\![e]\!]^{\phi}_{\gamma} \cdot \delta_{()}(U) \\
\{\![\texttt{observe}(e_1, e_2)]\!\}^{\phi}_{\gamma} &= \lambda U.\ \mathrm{pdf}([\![e_1]\!]^{\phi}_{\gamma})([\![e_2]\!]^{\phi}_{\gamma}) \cdot \delta_{()}(U)
\end{aligned}$$

$$[\![\texttt{infer}(e)]\!]^{\phi}_{\gamma} = \begin{cases} \dfrac{\lambda U.\ \{\![e]\!\}^{\phi}_{\gamma}(U)}{\{\![e]\!\}^{\phi}_{\gamma}([\![\textit{typeOf}(e)]\!])} & \textit{if } 0 < \{\![e]\!\}^{\phi}_{\gamma}([\![\textit{typeOf}(e)]\!]) < \infty \\ \textit{Error} & \textit{otherwise} \end{cases}$$

Careful with 0, and ∞...

Example : Gaussian

my_gaussian.ml

```
let my_gaussian (mu, sigma) =
  let x = sample (gaussian (mu, sigma)) in
  x
```

$$
\begin{aligned}
\{\!|\texttt{my\_gaussian (mu, sigma)}|\!\}_{\emptyset}(U) &= \int_{\mathbb{R}} \{\!|\texttt{sample (gaussian (mu, sigma))}|\!\}_{[\texttt{mu}\leftarrow\mu,\texttt{sigma}\leftarrow\sigma]}(dx)\ \{\!|\texttt{x}|\!\}_{[\texttt{mu}\leftarrow\mu,\texttt{sigma}\leftarrow\sigma,\texttt{x}\leftarrow x]}(U) \\
&= \int_{\mathbb{R}} \mathit{Gaussian}(\mu,\sigma)(dx)\ \delta_x(U) \\
&= \int_U \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx \\
&= \mathit{Gaussian}(\mu,\sigma)(U)
\end{aligned}
$$

Example : Beta


```
let my_beta (a, b) =
  let x = sample (uniform (0., 1.)) in
  let () = observe (beta (a, b), x) in
  x
```

$$
\begin{aligned}
\{\!\!\{\texttt{my\_beta (a, b)}\}\!\!\}_{\emptyset}(U) &= \int_0^1 \{\!\!\{\texttt{sample (uniform (0, 1))}\}\!\!\}_{[\texttt{a}\leftarrow a, \texttt{b}\leftarrow b]}(dx) \\
&\qquad \int_{()} \{\!\!\{\texttt{observe (beta (a, b), x)}\}\!\!\}_{[\texttt{a}\leftarrow a, \texttt{b}\leftarrow b, \texttt{x}\leftarrow x]}(du)\, \{\!\!\{\texttt{x}\}\!\!\}_{[\texttt{a}\leftarrow a, \texttt{b}\leftarrow b, \texttt{x}\leftarrow x]}(U) \\
&= \int_0^1 \mathit{Uniform}(dx)\ \mathrm{pdf}(\mathit{Beta}(a,b))(x)\ \delta_x(U) \\
&= \int_U \mathrm{pdf}(\mathit{Beta}(a,b))(x) dx \\
&= \mathit{Beta}(a,b)(U)
\end{aligned}
$$

Example : Coin

coin.ml

```
let coin (x1, … , xn) =
  let z = sample (uniform (0., 1.)) in
  observe (bernoulli (z), x1); … ; observe (bernoulli (z), xn);
  z
```

$$
\begin{aligned}
\{\!|\texttt{coin (x1, … , xn)}|\!\}_{\emptyset}(U) = \int_0^1 & \{\!|\texttt{sample (uniform (0, 1))}|\!\}_{[\texttt{x1}\leftarrow x_1,\ldots,\texttt{xn}\leftarrow x_n]}(dz) \\
& \int_{()} \{\!|\texttt{observe (bernoulli (z), x1)}|\!\}_{[\texttt{z}\leftarrow z,\texttt{x1}\leftarrow x_1,\ldots,\texttt{xn}\leftarrow x_n]}(du_0) \\
& \quad \int_{()} \{\!|\texttt{observe (bernoulli (z), x2)}|\!\}_{[\texttt{z}\leftarrow z,\texttt{x1}\leftarrow x_1,\ldots,\texttt{xn}\leftarrow x_n]}(du_1) \\
& \quad \ldots \\
& \qquad \int_{()} \{\!|\texttt{observe (bernoulli (z), xn)]}|\!\}_{[\texttt{z}\leftarrow z,\texttt{x1}\leftarrow x_1,\ldots,\texttt{xn}\leftarrow x_n]}(du_n) \\
& \qquad\quad \{\!|\texttt{z}|\!\}_{[\texttt{z}\leftarrow z,\texttt{x1}\leftarrow x_1,\ldots,\texttt{xn}\leftarrow x_n]}(U) \\
= \int_0^1 & \mathit{Uniform}(0,1)(dz) \prod_{i=1}^{n} \mathrm{pdf}(\mathit{Bernoulli}(z))(x_i)\ \delta_z(U) \\
= \int_U & z^{\#\mathrm{heads}}\ (1-z)^{\#\mathrm{tails}}\ dz
\end{aligned}
$$

Unnormalized!

Example : Coin

coin.ml

```
let coin (x1, ... , xn) =
  let z = sample (uniform (0., 1.)) in
  observe (bernoulli (z), x1); ... ; observe (bernoulli (z), xn);
  z

let d = infer (coin (data))
```

$$\{\!\![\texttt{coin (x1, ... , xn)}]\!\!\}_{\emptyset}(U) = \int_U z^{\text{\#heads}}\,(1-z)^{\text{\#tails}}\,dz$$

$$[\![\texttt{infer (coin (x1, ... , xn))}]\!]_{[\texttt{coin}]} = \frac{\int_U z^{\text{\#heads}}\,(1-z)^{\text{\#tails}}\,dz}{\int_0^1 z^{\text{\#heads}}\,(1-z)^{\text{\#tails}}\,dz} = \frac{\int_U z^{\text{\#heads}}\,(1-z)^{\text{\#tails}}\,dz}{\mathrm{B}(\text{\#heads}+1,\text{\#tails}+1)} = \mathit{Beta}(\text{\#heads}+1,\text{\#tails}+1)(U)$$

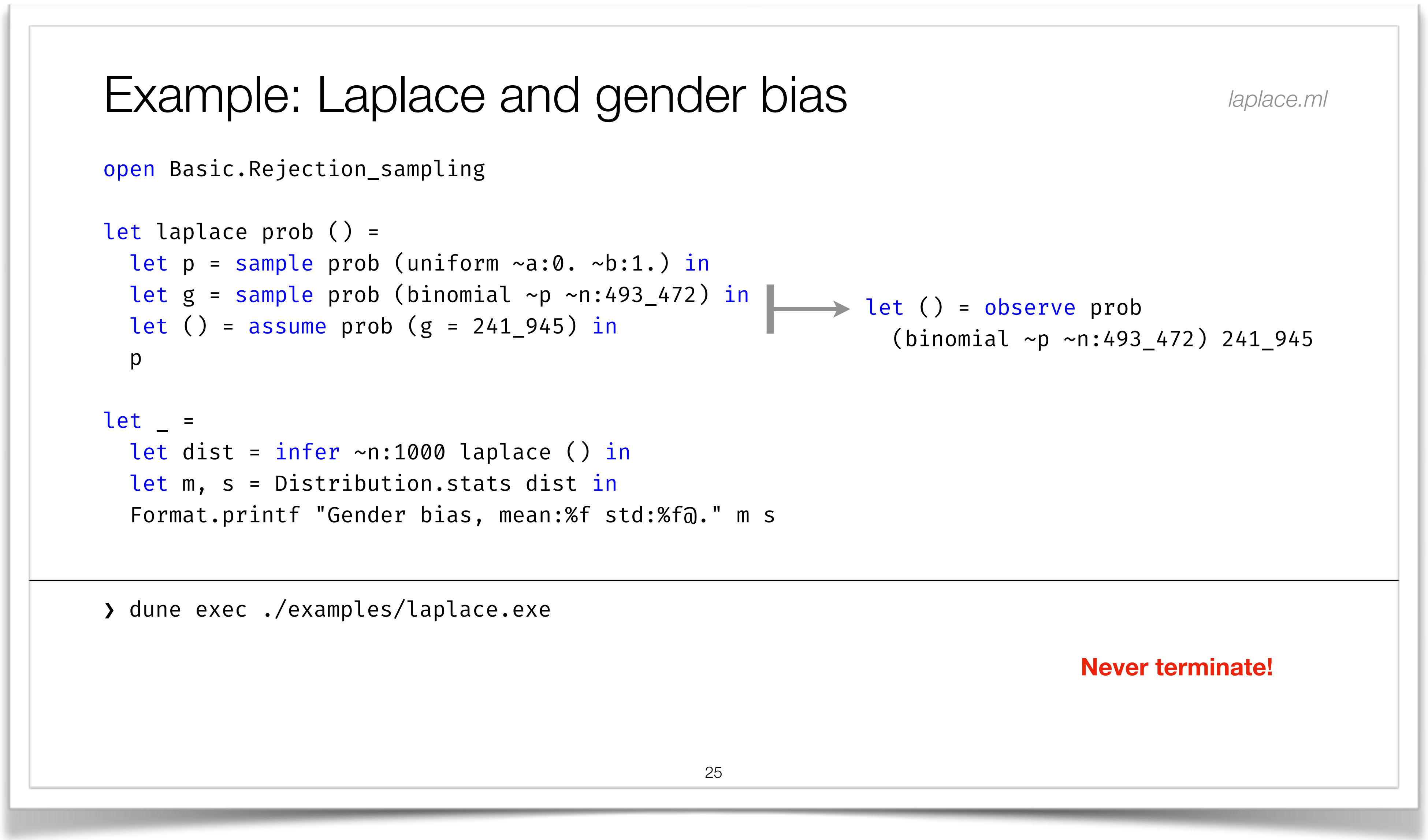
Exercises

Prove the following properties

- ```
  sample mu (* where mu is defined on [a, b] *)
  ≡
  let x = sample (uniform (a, b)) in
  let () = observe (mu, x) in
  x
  ```

- ```
  observe (mu, x) (* where mu is a discrete distribution *)
  ≡
  let y = sample mu in
  assume x = y
  ```

- ```
  sample (bernoulli (0.5))
  ≡
  let x = sample (gaussian (0., 1.)) in
  x > 0
  ```


  ```

Improper priors

Uniform priors on bounded domains

- If $\mu : t\ \mathtt{dist}^*$ is defined on $[a, b]$ and has a density
- $[\![\mathtt{sample}(\mu)]\!] = [\![\mathtt{let}\ x = \mathtt{sample}(\mathit{Uniform}(a,b))\ \mathtt{in}\ \mathtt{observe}(\mu, x);\ x]\!]$

Improper priors

```
let improper =
  let x = sample (gaussian 0 1) in
  factor (1. /. (pdf (gaussian 0 1) x));
  x
```

$$
\begin{aligned}
[\![\mathtt{improper}]\!]_\emptyset (U) &= \int_U \mathit{Gaussian}(0,1)(dx)\ \frac{1}{f(x)}\ dx \\
&= \int_U f(x)\ \frac{1}{f(x)}\ dx \\
&= \lambda(U)
\end{aligned}
$$

Staton 2017

References

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Jan-Willem van de Meent, Brooks Paige, Hongseok Yang, Frank Wood
https://arxiv.org/abs/1809.10756

Semantics of probabilistic programs.
Dexter Kozen
Journal of Computer and System 1981

Commutative semantics for probabilistic programming
Sam Staton
ESOP 2017

Semantics of Probabilistic Programs using s-Finite Kernels in Coq
Reynald Affeldt, Cyril Cohen, Ayumu Saito
CPP 2023

TP : A short introduction to Stan

Everything is on Github: https://github.com/mpri-probprog/probprog-24-25

- Go to `td/td4-stan`
- Launch `jupyter notebook` (or `jupyter lab`)

Requirements

- Pandas
- CmdStanPy
- Jupyter
- Matplotlib

https://mc-stan.org/