Probabilistic Programming Languages

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MPRI 2024-2025

Reminder: Exact Inference

Probabilistic Programming Languages

Example: Coin

Consider a series of coin tosses

- Observations: head or tail
- Each toss is independant
- What is the probability of getting head at the next toss?

Probabilistic model

- Prior: $z \sim \mathit{Uniform}(0, 1)$
- Observations: for $i \in [1, n]$, $x_i \sim \mathit{Bernoulli}(z)$
- Posterior: $p(z \mid x_1, \ldots, x_n)$?

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$$
\begin{aligned}
p(x_1, \ldots, x_n \mid z) &= \prod_{i=1}^{n} p(x_i \mid z) \\
&= \prod_{i=1}^{n} z^{x_i} \ (1-z)^{1-x_i} \\
&= z^{\sum_{i=1}^{n} x_1} \ (1-z)^{\sum_{i=1}^{n} (1-x_i)} \\
&= z^{\#\text{heads}} \ (1-z)^{\#\text{tails}}
\end{aligned}
$$

$$
\begin{aligned}
p(z \mid x_1, \ldots, x_n) &= \frac{p(x_1, \ldots, x_n \mid z) p(z)}{p(x_1, \ldots, x_n)} \\
&= \frac{p(x_1, \ldots, x_n \mid z) p(z)}{\int_z p(x_1, \ldots, x_n \mid z)}
\end{aligned}
$$

$$
\begin{aligned}
p(z \mid x_1, \ldots, x_n) &= \frac{z^{\#\text{heads}} \ (1-z)^{\#\text{tails}}}{\int_z z^{\#\text{heads}} \ (1-z)^{\#\text{tails}}} \\
&= \frac{z^{\#\text{heads}} \ (1-z)^{\#\text{tails}}}{B(\#\text{heads} + 1, \ \#\text{tails} + 1)} \\
&= \text{pdf}(\textit{Beta}(\#\text{heads} + 1, \ \#\text{tails} + 1))
\end{aligned}
$$

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$$
\begin{aligned}
p(z \mid x_1, \ldots, x_n) &= \\
&= \\
&= \mathit{eta}(\#\text{heads} + 1, \ \#\text{tails} + 1))
\end{aligned}
$$

What if the model is much more complex?

What if we use arbitrary control flow?

Can we compute the posterior automatically?

Semi-Symbolic Inference

Probabilistic Programming Languages

Semi-symbolic inference

Approximate inference can be impractical

- Require lot of computing power
- Can yield poor approximation

Exact inference is often possible

Rao-Blackwellized inference

- Each execution of the model maintains symbolic distributions
- Perform as much exact computation as possible
- Fall back to sampling when symbolic computation fails

Key elements

- Marginalization: remove dependencies
- Conditioning: incorporate observations

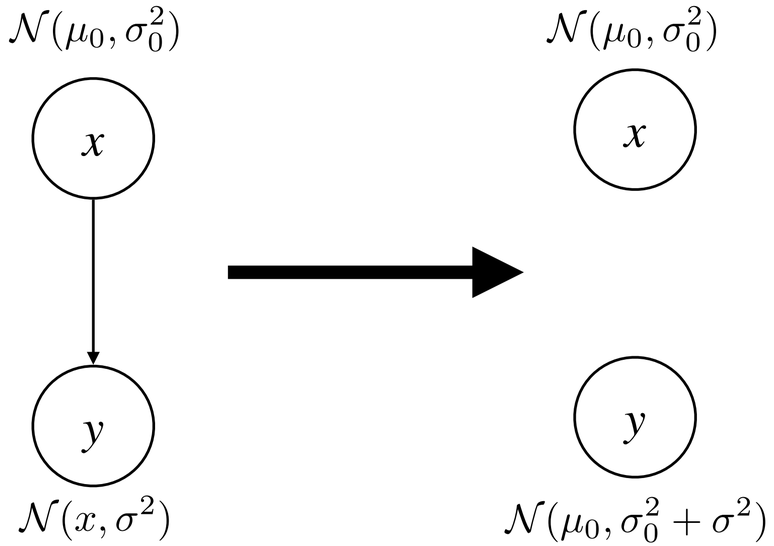
Marginalization

Remove dependencies

- Parent distribution: $p(x)$
- Child distribution: $p(x \mid y)$
- Marginal: $p(x) = \int p(x \mid y)p(x)$

Gaussians

- Parent distribution: $x \sim \mathcal{N}(\mu_0, \sigma_0^2)$
- Child distribution: $(x \mid y) \sim \mathcal{N}(x, \sigma^2)$
- Marginal: $y \sim \mathcal{N}(\mu_0, \sigma_0^2 + \sigma^2)$



Conjugate priors

Bayesian Inference: learn parameters from data

- Latent parameter θ
- Observed data $x_1, \ldots, x_n$

$$p(\theta \mid x_1, \ldots x_n) = \frac{p(\theta)\ p(x_1, \ldots, x_n \mid \theta)}{p(x_1, \ldots, x_n)} \qquad \text{(Bayes' theorem)}$$

$$\propto p(\theta)\ p(x_1, \ldots, x_n \mid \theta) \qquad \text{(Data are constants)}$$

If the posterior $p(\theta \mid x)$ is in the same distribution family as the prior $p(\theta)$

- Prior and posterior are called *conjugate distributions* w.r.t., the likelihood
- The prior is the *conjugate prior* for the likelihood function

Conjugate priors examples

Beta-Bernoulli

- Prior: $x \sim \mathit{Beta}(a, b)$
- Likelihood: $(y \mid x) \sim \mathit{Bernoulli}(x)$
- Posterior: $(x \mid x = v) \sim \mathit{Beta}(a + v, b + (1 - v))$

Gaussians

- Prior: $x \sim \mathcal{N}(\mu_0, \sigma_0^2)$
- Likelihood: $(y \mid x) \sim \mathcal{N}(x, \sigma^2)$
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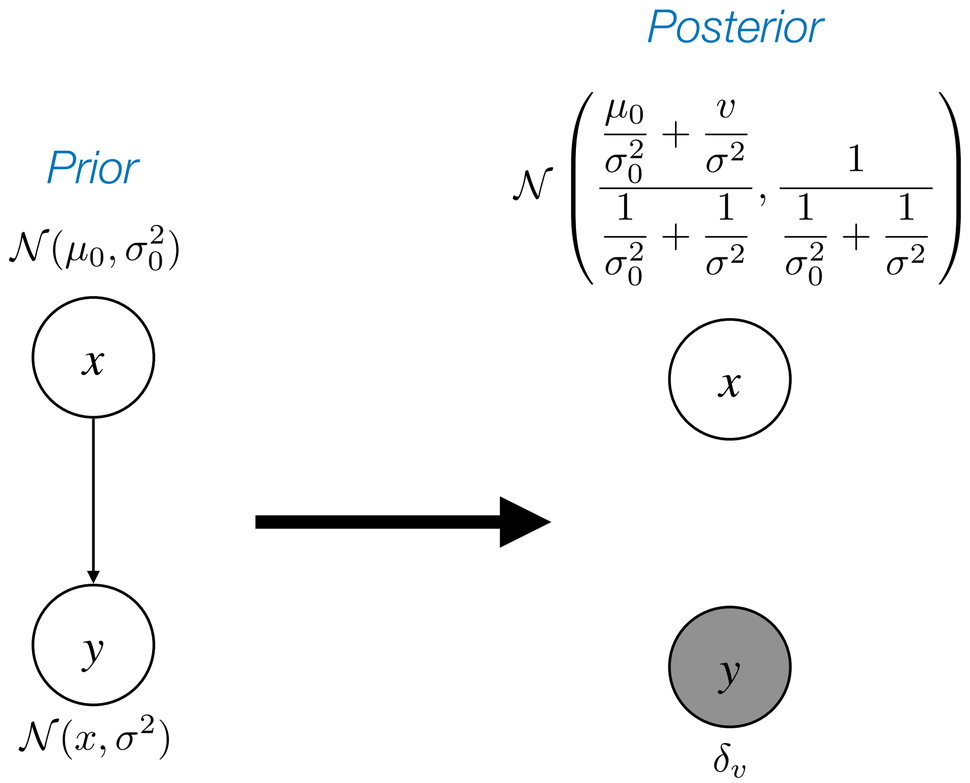
$$\begin{aligned} \mu_1 &= \left(\tfrac{1}{\sigma_0^2} + \tfrac{1}{\sigma^2}\right)^{-1} \left(\tfrac{\mu_0}{\sigma_0^2} + \tfrac{v}{\sigma^2}\right) \\ \sigma_1^2 &= \left(\tfrac{1}{\sigma_0^2} + \tfrac{1}{\sigma^2}\right)^{-1} \end{aligned}$$



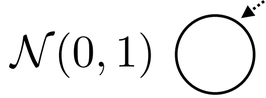
Try it in BYO-PPL!

Delayed Sampling

Probabilistic Programming Languages

Delayed sampling

```
let gauss () =
  let a = sample gaussian 0. 1. in
  observe (gaussian z 1.) 2.;
  observe (gaussian z 1.) 2.;
  observe (gaussian z 1.) 2.;
  a
```



Murray et al. 2018

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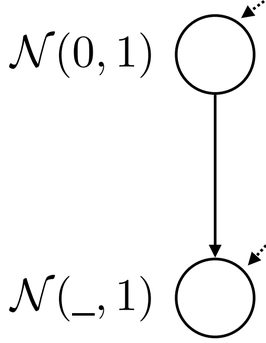

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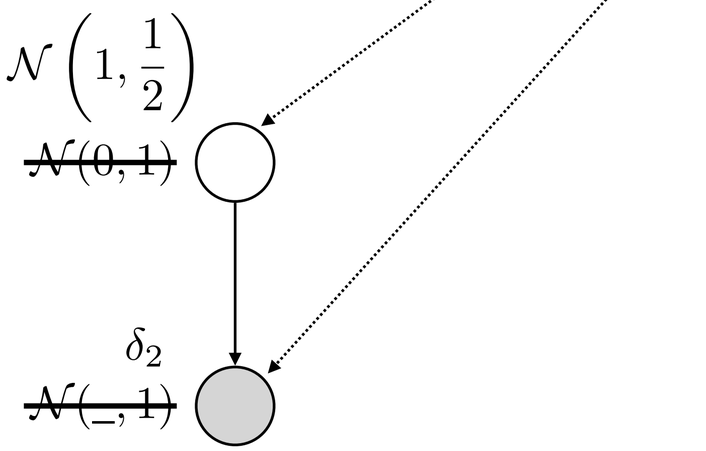

Conditionning

Gaussians

- Prior: $x \sim \mathcal{N}(\mu_0, \sigma_0^2)$
- Likelihood: $(y \mid x) \sim \mathcal{N}(x, \sigma^2)$
- Posterior: $(y \mid x = v) \sim \mathcal{N}(\mu_1, \sigma_1^2)$

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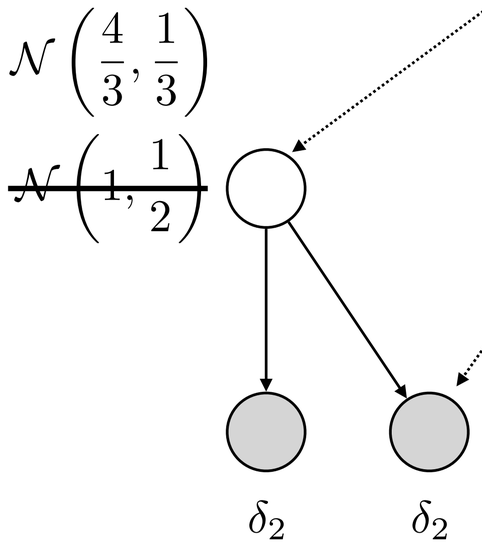

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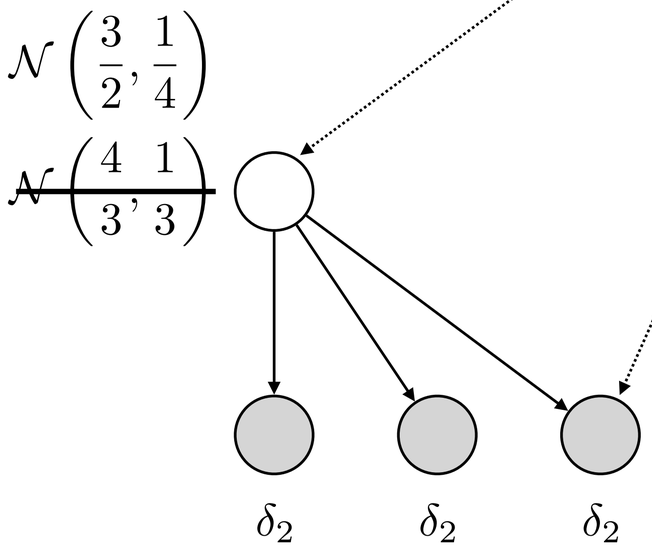

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Murray et al. 2018

Delayed sampling

```
let gauss () =
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  let b = sample gaussian a 1. in
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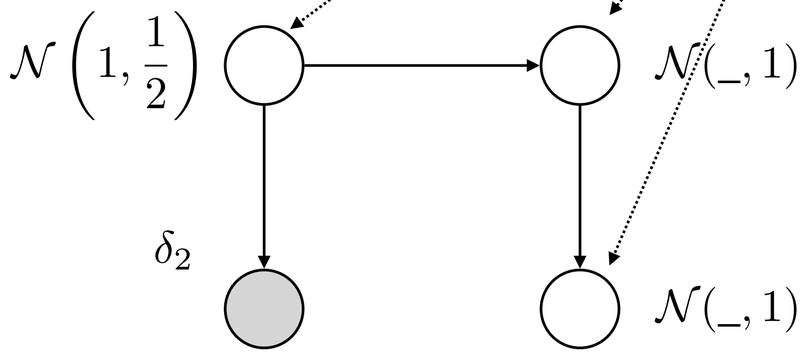
```



Murray et al. 2018

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let gauss () =
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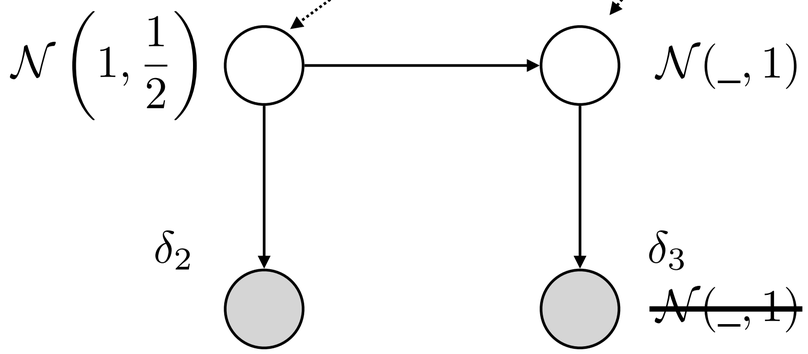
  let b = sample gaussian a 1. in
  observe (gaussian b 1.) 3.;
  b
```



Conditional distribution
Cannot condition yet.

Murray et al. 2018

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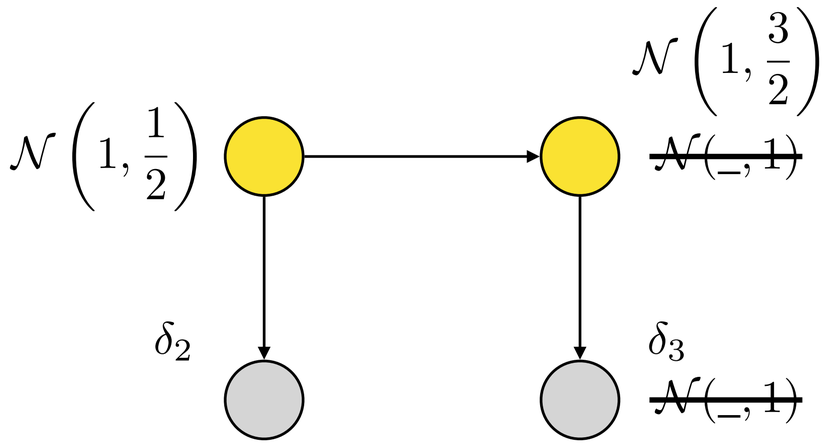

Marginalization

Gaussians
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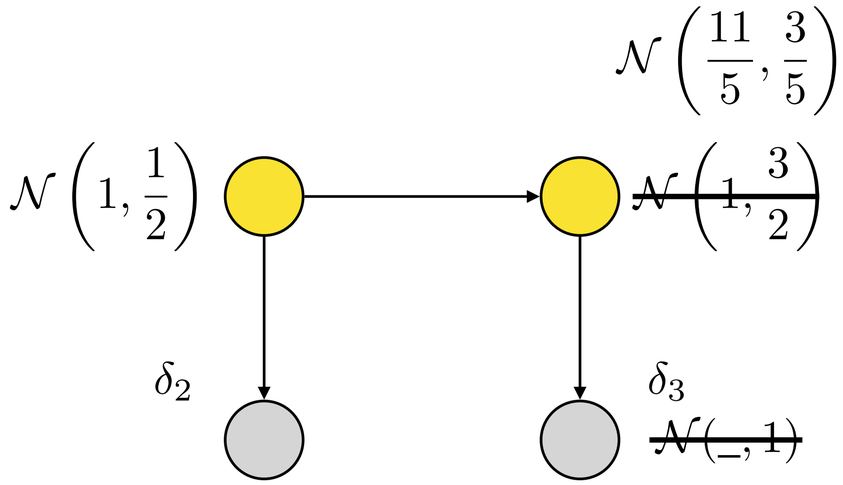

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Murray et al. 2018

Delayed sampling: summary

`infer model data`

- Launch n independent executions (approximate inference)
- Maintain a graph (Bayesian model for the current execution)
- Returns a mixture of symbolic distributions

`sample d`

- add a node to the graph (random variable)
- keep track of dependencies

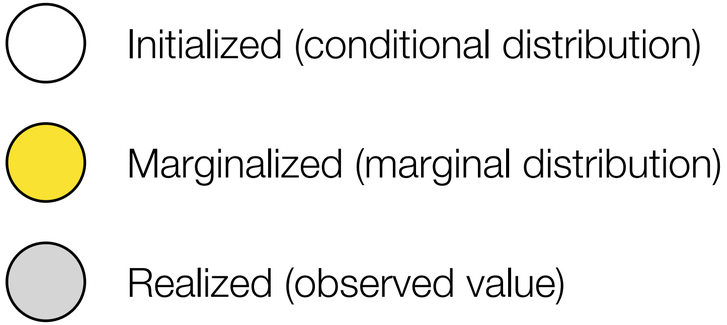
`observe`

- add a node to the graph (observed random variable)
- marginalize parent (recursively until the root)
- condition the parent on the observation

If error (conditioning or marginalization fails)

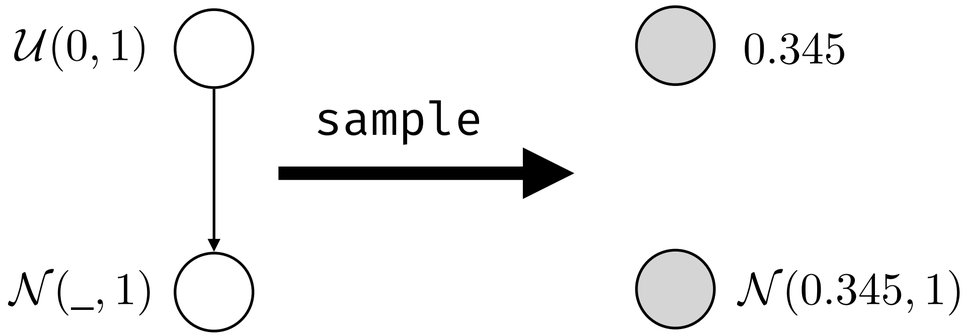
- Sample parent
- Update the children

Node state a runtime

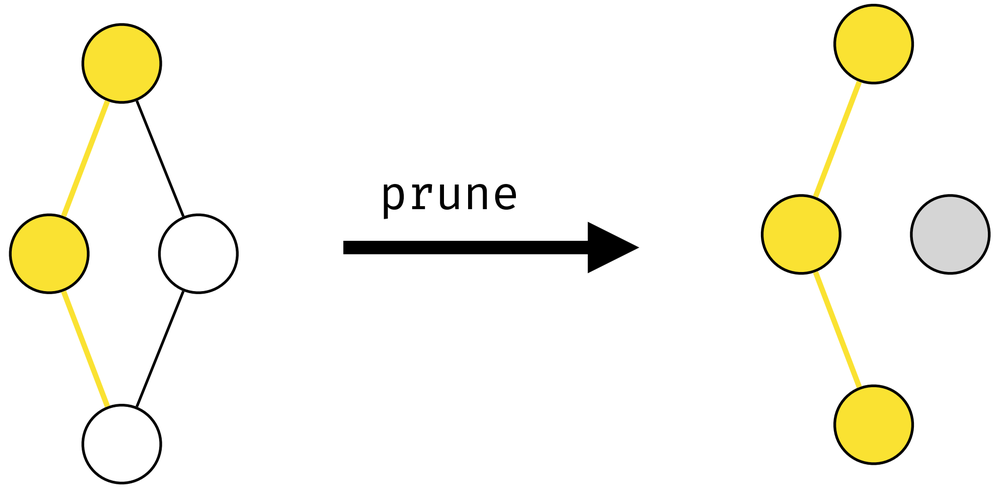


Murray et al. 2018

Delayed sampling: actual sampling



no conjugacy



single m-path

Delayed sampling is limited to forests.
Can we do better?

Semi-symbolic inference

Probabilistic Programming Languages

Semi-symbolic inference

Gaussians

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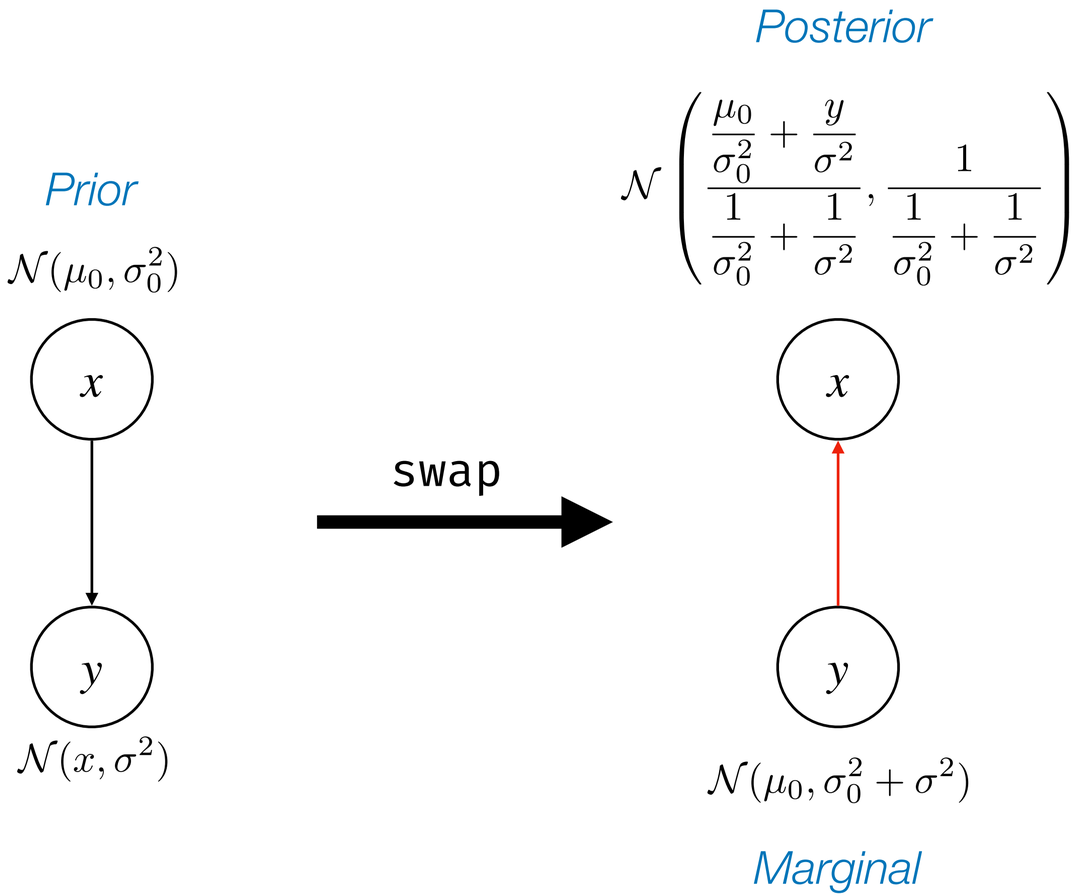


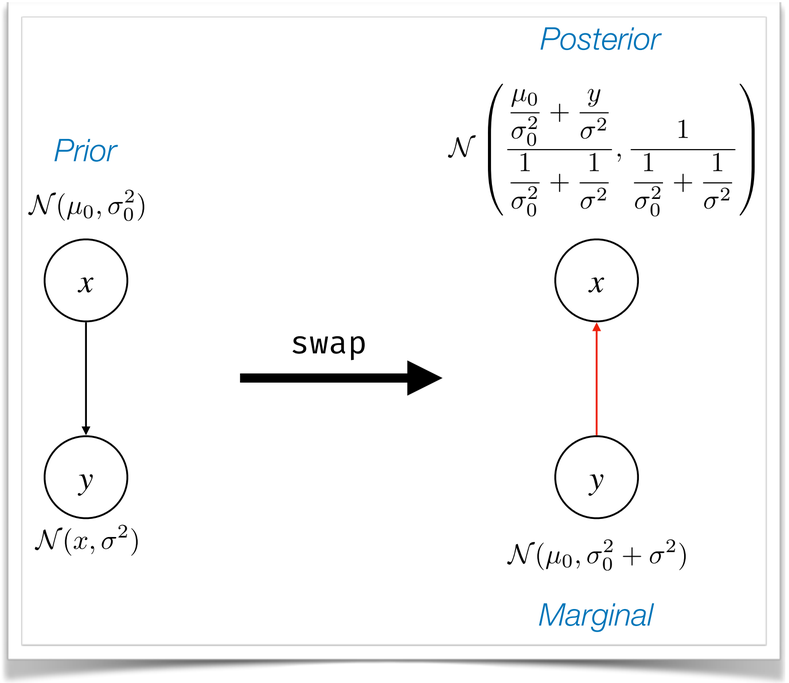
Semi-symbolic inference

Main idea

- Maintain symbolic expressions involving random variables
- Each operation triggers a sequence of swaps
- Swap change the dependency between two variables but preserves the symbolic state
- Only root nodes (with no parent) can be observed

Closed families

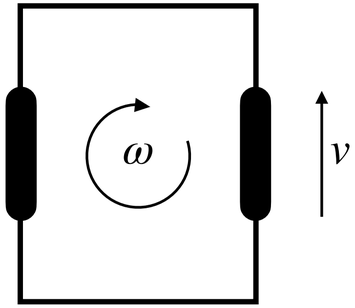
- A swap always succeed and the resulting state remains in the family.
- E.g., linear Gaussians models, and finite discrete models.

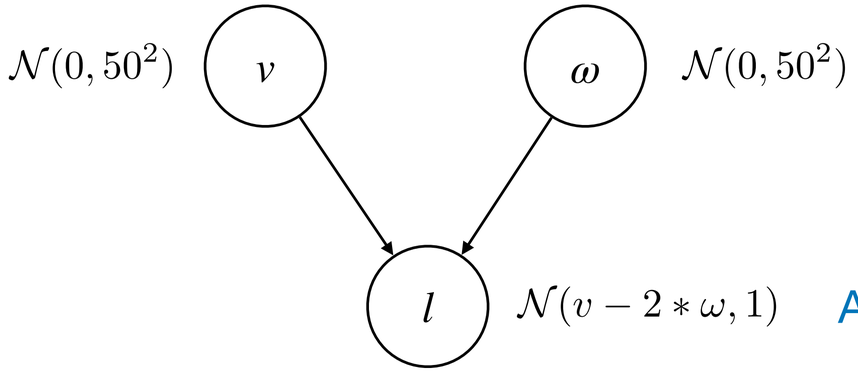


Atkinson et al. 2022

Semi-symbolic inference

```
let wheels(l_rate, r_rate) =
  let omega = sample (gaussian 0. 50.0) in
  let vel = sample (gaussian 0. 50.0) in
  observe (gaussian(vel - 2.0 * omega, 1.0) l_rate;
  observe (gaussian(vel + 2.0 * omega, 1.0) r_rate;
  omega, vel
```





Affine Gaussians

Atkinson et al. 2022

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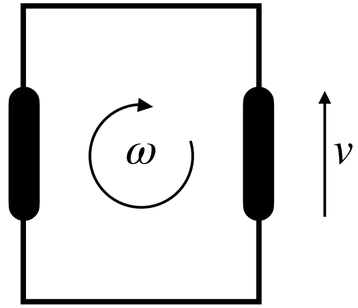
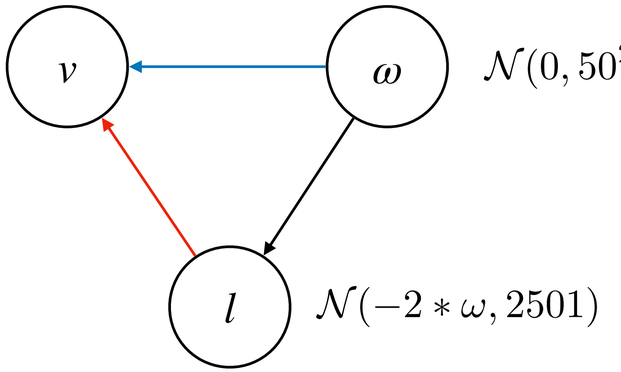


$$\mathcal{N}\left(\frac{2500}{2501} * l + \frac{5000}{2501} * w, \frac{2500}{2501}\right)$$




```
swap(l, v)
```


Atkinson et al. 2022

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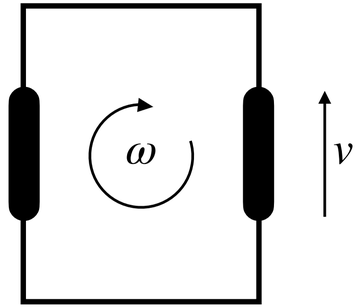

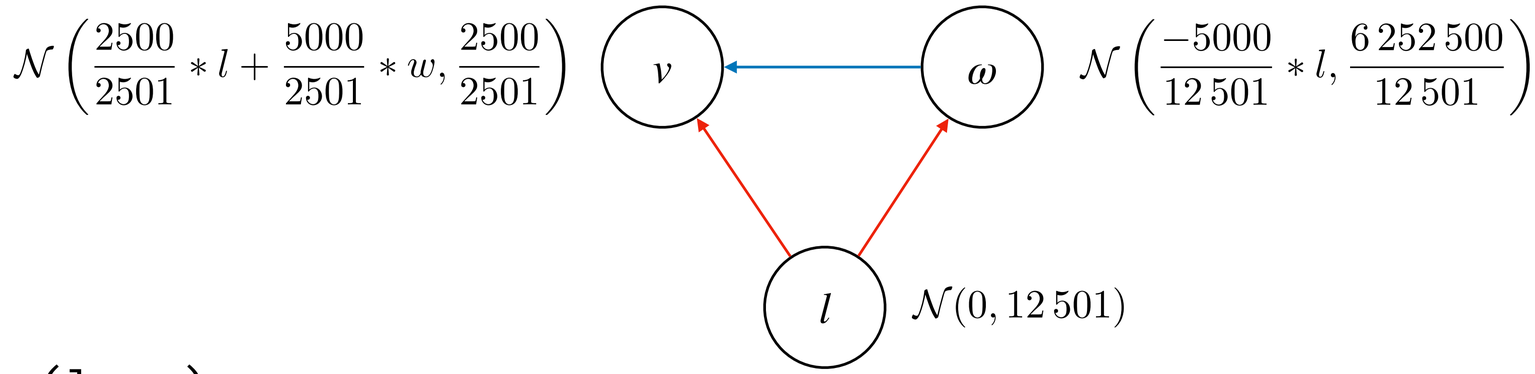


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Atkinson et al. 2022

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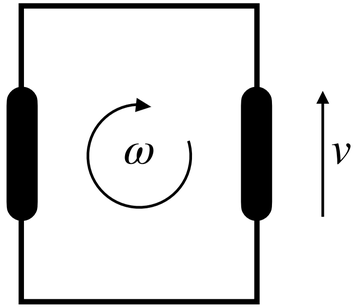
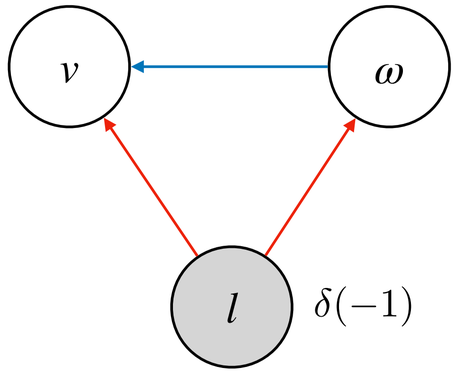


$$\mathcal{N}\left(\frac{-2500}{2501} + \frac{5000}{2501} * w, \frac{2500}{2501}\right)$$



$$\mathcal{N}\left(\frac{5000}{12\,501}, \frac{6\,252\,500}{12\,501}\right)$$

```
value(l)
```


Atkinson et al. 2022

Semi-symbolic: summary

`infer model data`

- Launch n independent executions (approximate inference)
- Maintain a graph (Bayesian model for the current execution)
- Returns a mixture of symbolic distributions

`sample d`

- add a node to the graph (random variable)
- keep track of dependencies

`observe`

- add a node to the graph (observed random variable)
- swaps to turn the node into a root
- condition an simplify the graph

If error (swap fails)

- Sample (some) parents
- Update the children

More general than delayed sampling

Atkinson et al. 2022

Reactive Probabilistic Programming

Probabilistic Programming Languages

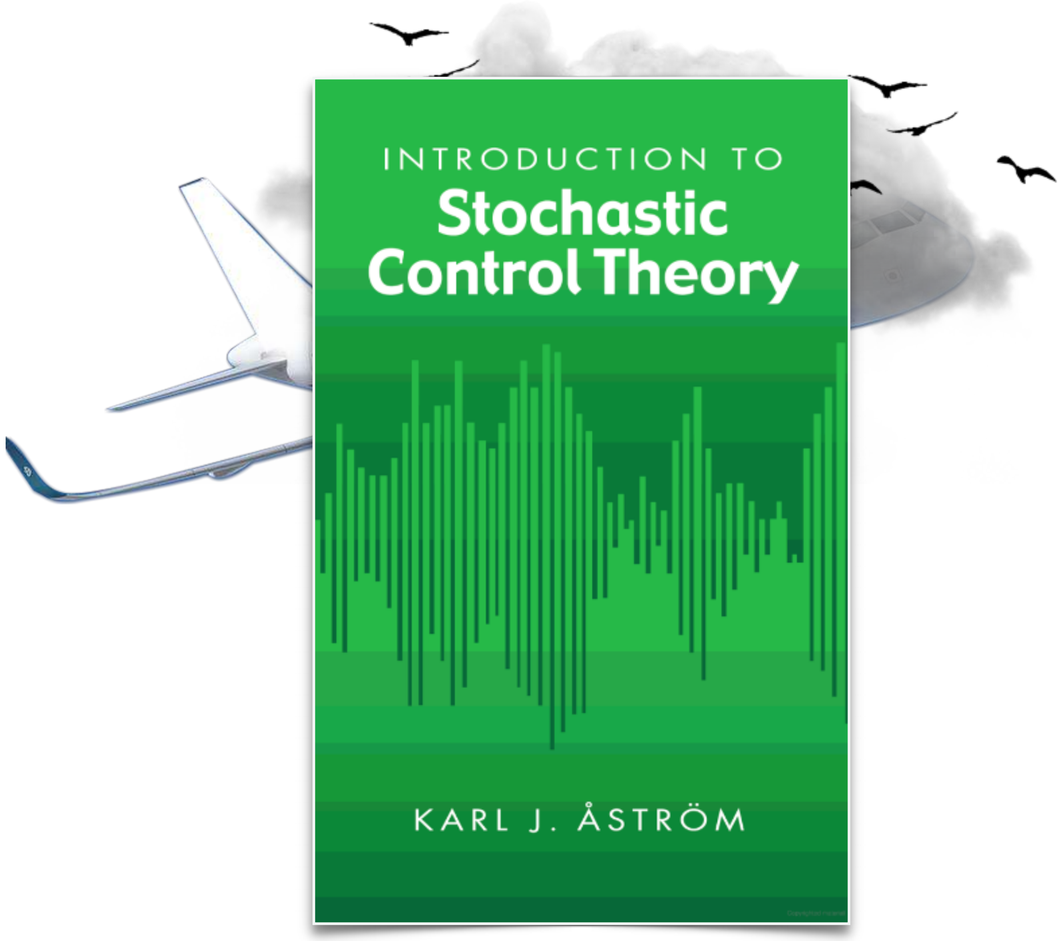
Uncertainty in embedded systems

Synchronous languages
- High-level specification language
- Generate correct-by-construction embedded code
- Industrial tool: ANSYS Scade

Challenges
- Noisy environment, perceived through noisy sensors
- Interaction with other autonomous entities

Existing approaches
- Manually implement stochastic controller: *Can be error prone*
- Offline statistical tests: *Requires up-to-date offline data*

Reactive Probabilistic Programming
- Synchronous languages with probabilistic constructs
- Make the probabilistic model explicit
- Automatically learn posterior distributions from observations

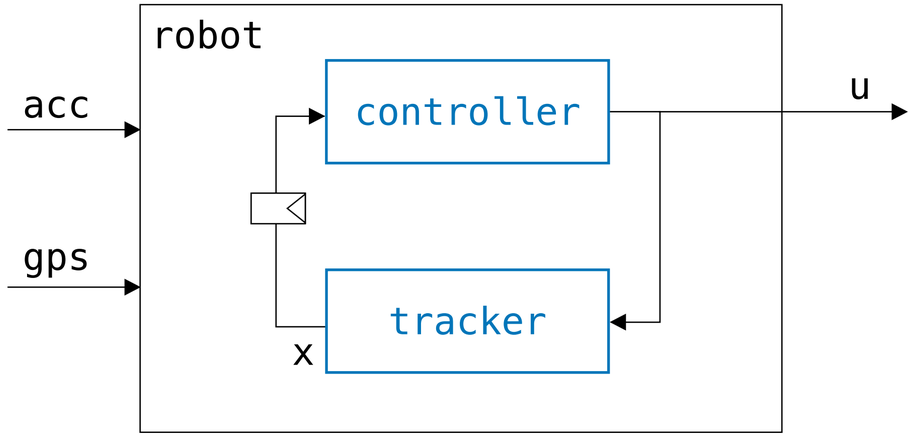


Reactive systems

Synchronous data-flow languages and block diagrams

- Signal: stream of values
- System: stream processor

State: `x:` $(position \times velocity \times acceleration)$



Reactive probabilistic systems

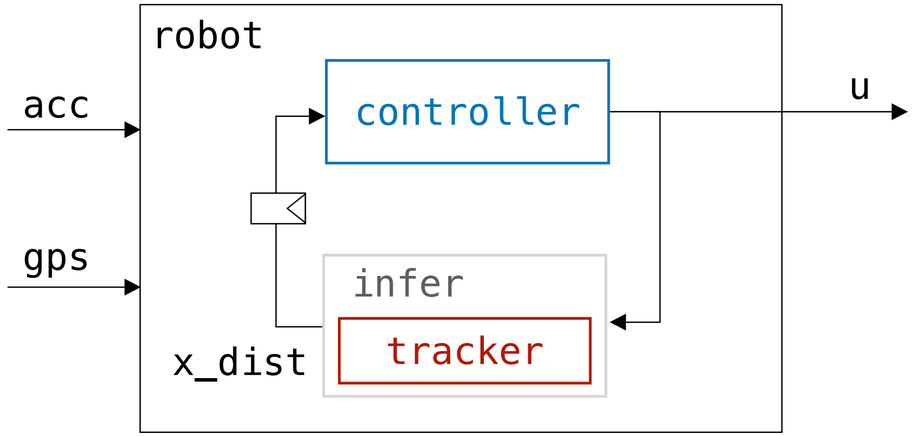
Synchronous data-flow languages and block diagrams

- Signal: stream of values
- System: stream processor

ProbZelus: add support to deal with uncertainty

- Extend a synchronous language
- Parallel composition: deterministic/probabilistic
- Inference-in-the-loop
- Streaming inference

State: `x_dist:` $(position \times velocity \times acceleration)$ `dist`



Synchronous Programming

Reactive Probabilistic Programming

Example: tracker

latent

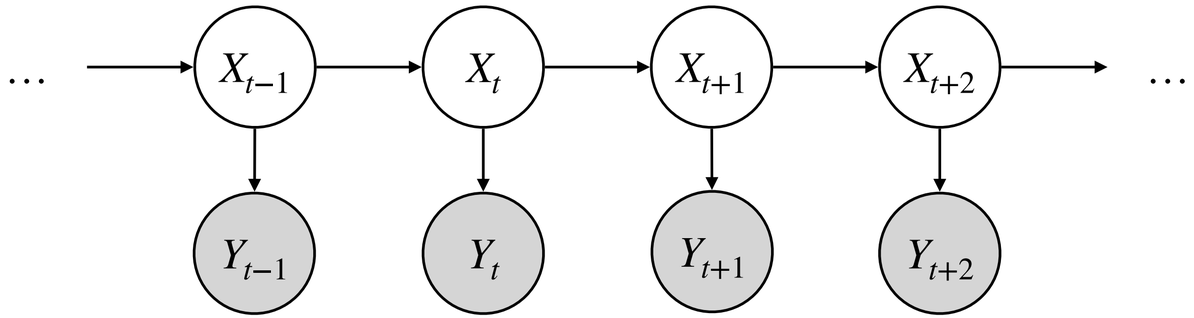
observed

Model

- Linear motion: $X_k \sim \mathcal{N}(FX_{k-1}, Q)$
- Observation: $Y_k \sim \mathcal{N}(HX_k, R)$

E.g., with Q and R constant noise matrices

- $X_k = \begin{pmatrix} p_k \\ v_k \end{pmatrix}$ (position, velocity)
- $F = \begin{pmatrix} 1 & dt \\ 0 & 1 \end{pmatrix}$ (discrete integration)
- $H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (projection)



$$
\begin{aligned}
X_k^{\text{pred}} &= FX_{k-1}^{\text{est}} \\
X_k^{\text{est}} &= X_k^{\text{pred}} + K_k(Y_k - HX_k^{\text{pred}}) \\
S_k &= (R + HP_k^{\text{pred}}H^T)^{-1} \\
K_k &= P_k^{\text{pred}}H^T S_k \\
P_k^{\text{pred}} &= Q + FP_{k-1}^{\text{est}}F^T \\
P_k^{\text{est}} &= P_k^{\text{pred}} - K_k H P_k^{\text{pred}}
\end{aligned}
$$

Solution: Kalman filter

Reactive synchronous programming

Dataflow synchronous programming

- Set of stream equations
- Discrete logical time steps
- At each step, compute the current value given inputs and previous values

```
let node kalman(y) = x_est where
  rec x_pred = f * (x0 → pre x_est)
  and x_est  = x_pred + k * (y - h * x_pred)
  and s      = r + h * p_pred * (transpose h)
  and k      = p_pred * (transpose h) * (inv s)
  and p_pred = q + f * (p0 → pre p_est) * (transpose f)
  and p_est  = p_pred - k * h * p_pred
```

$$
\begin{aligned}
X_k^{\text{pred}} &= F X_{k-1}^{\text{est}} \\
X_k^{\text{est}} &= X_k^{\text{pred}} + K_k(Y_k - H X_k^{\text{pred}}) \\
S_k &= R + H P_k^{\text{pred}} H^T \\
K_k &= P_k^{\text{pred}} H^T S_k^{-1} \\
P_k^{\text{pred}} &= Q + F P_{k-1}^{\text{est}} F^T \\
P_k^{\text{est}} &= P_k^{\text{pred}} - K_k H P_k^{\text{pred}}
\end{aligned}
$$

Solution: Kalman filter

Bourke, Pouzet 2013

Reactive synchronous programming

```
let node kalman(y) = x_est where
  rec x_pred = f * (x0 → pre x_est)
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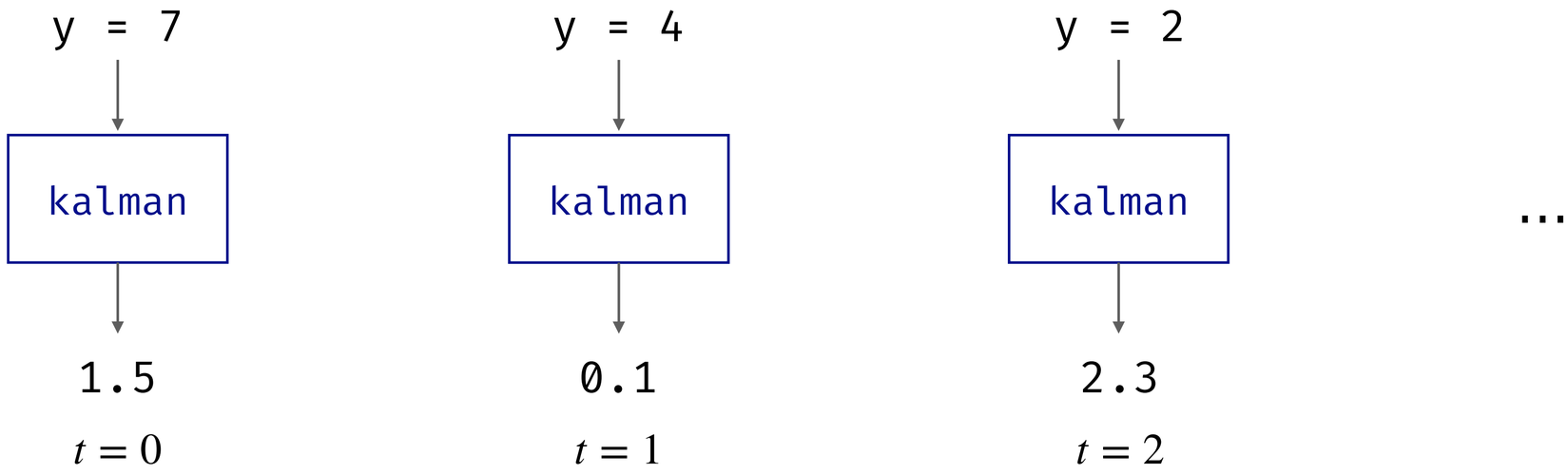
  and s      = r + h * p_pred * (transpose h)
  and k      = p_pred * (transpose h) * (inv s)
  and p_pred = q + f * (p0 → pre p_est) * (transpose f)
  and p_est  = p_pred - k * h * p_pred
```

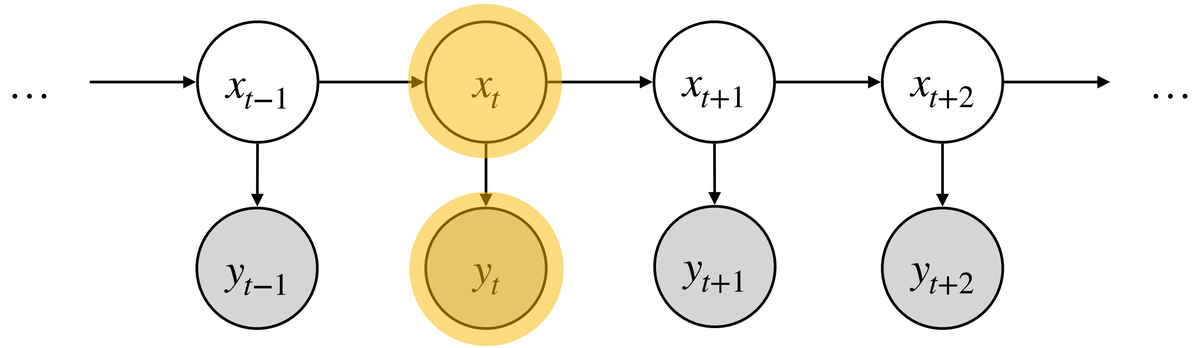
What if the assumptions change?
What if the model is not linear?

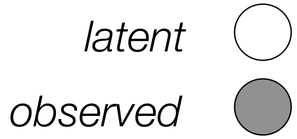


Reactive probabilistic programming

Probabilistic constructs

- `x = sample(d)`: introduce a random variable `x` of distribution `d`
- `observe(d, y)`: condition on the fact that `y` was sampled from `d`
- `infer m y`: compute posterior distribution of `m` given `y`



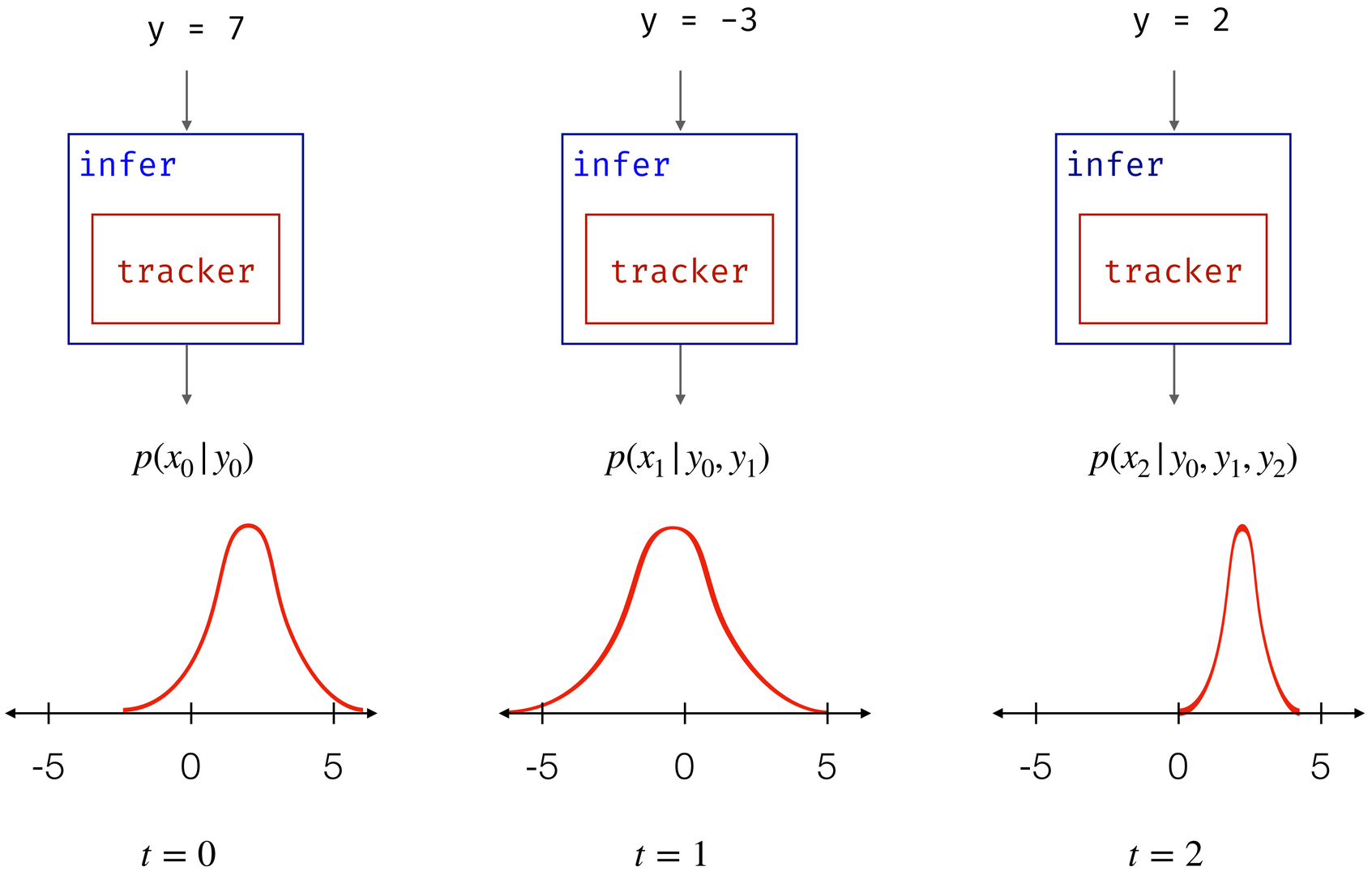



```
let proba tracker (y) = x where
  rec x = x0 → sample(mv_gaussian(f * (pre x), q))
  and () = observe(mv_gaussian(h * x, r), y)
```

Model

- Linear motion: $X_k \sim \mathcal{N}(FX_{k-1}, Q)$
- Observation: $Y_k \sim \mathcal{N}(HX_k, R)$

Reactive probabilistic programming



Demo

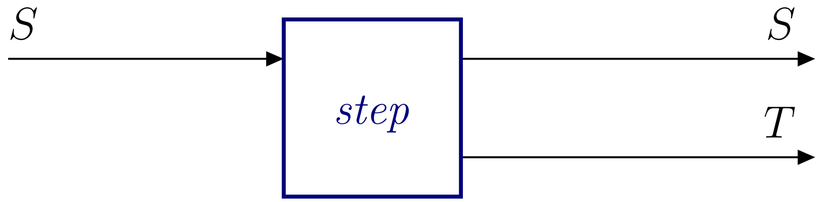
Reactive semantics

Reactive Probabilistic Programming

Deterministic streams

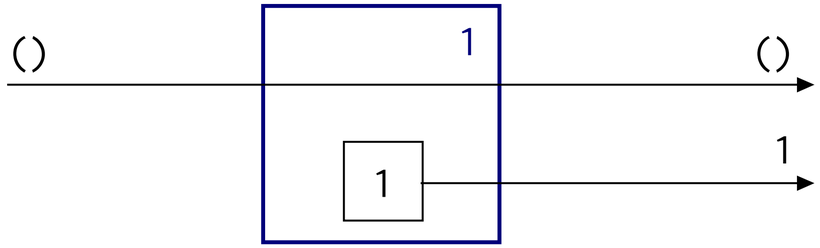
Initial state, transition function

$CoStream(T, S) = S \times (S \to S \times T)$



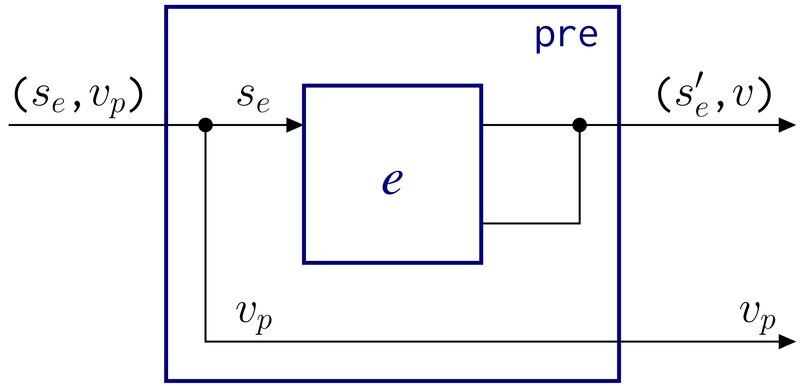
Constant 1: `unit` $\to$ `unit` $\times$ `int`

- Initial state: the value of type unit
- Step function: return the constant 1



Unit delay `pre` e: $(S \times T) \times (S \times T \to (S \times T) \times T)$

- Initial state: the initial state of e and default value
- Step function: the result of e is stored in the state and returned at the next iteration

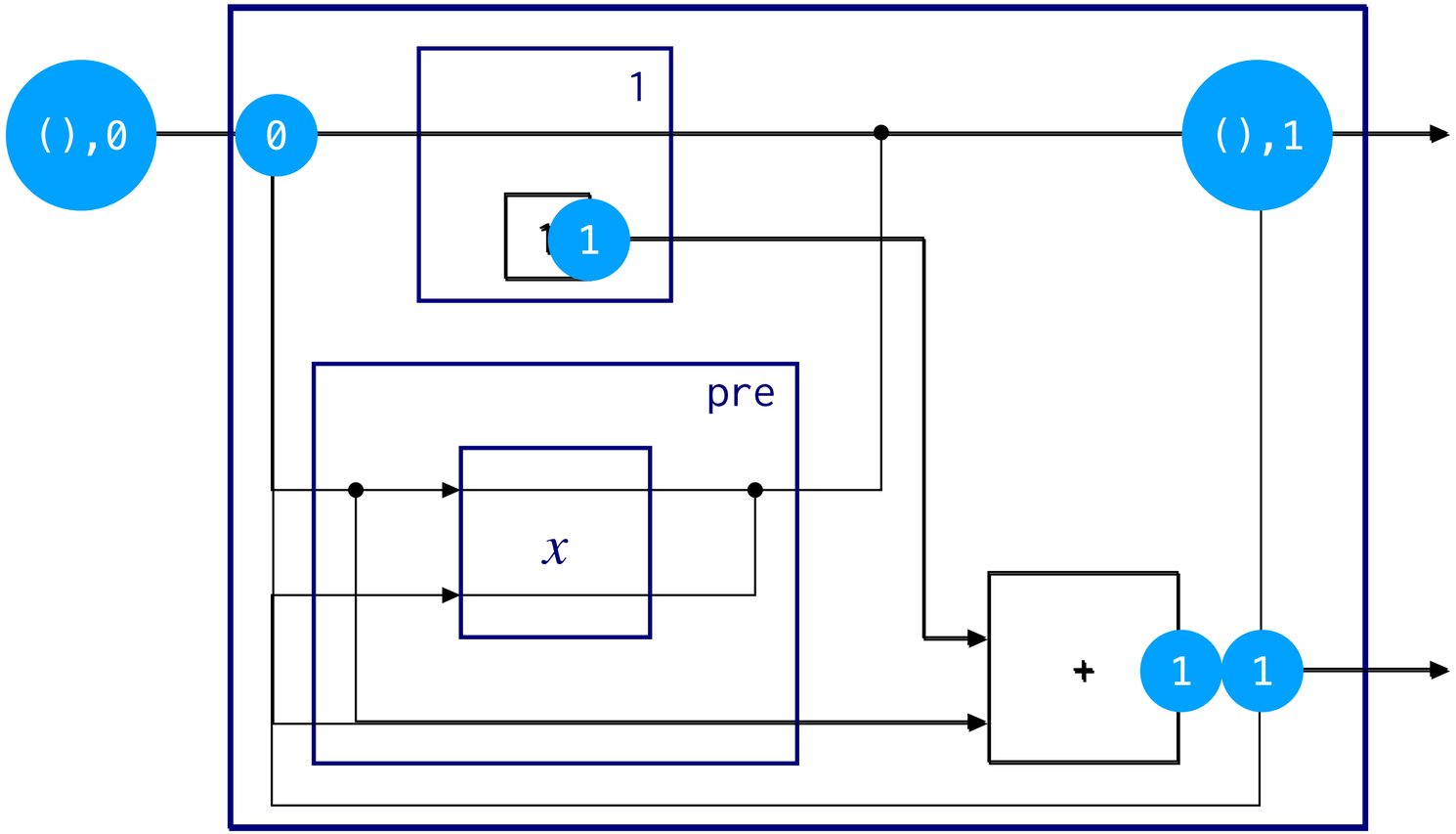


Caspi & Pouzet. 1998

Example

```
rec x = 1 + pre x
```

- Initial state: `(), 0`
- Output: `1, 2, 3, 4, ....`



Example

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- Initial state: `(), 0`
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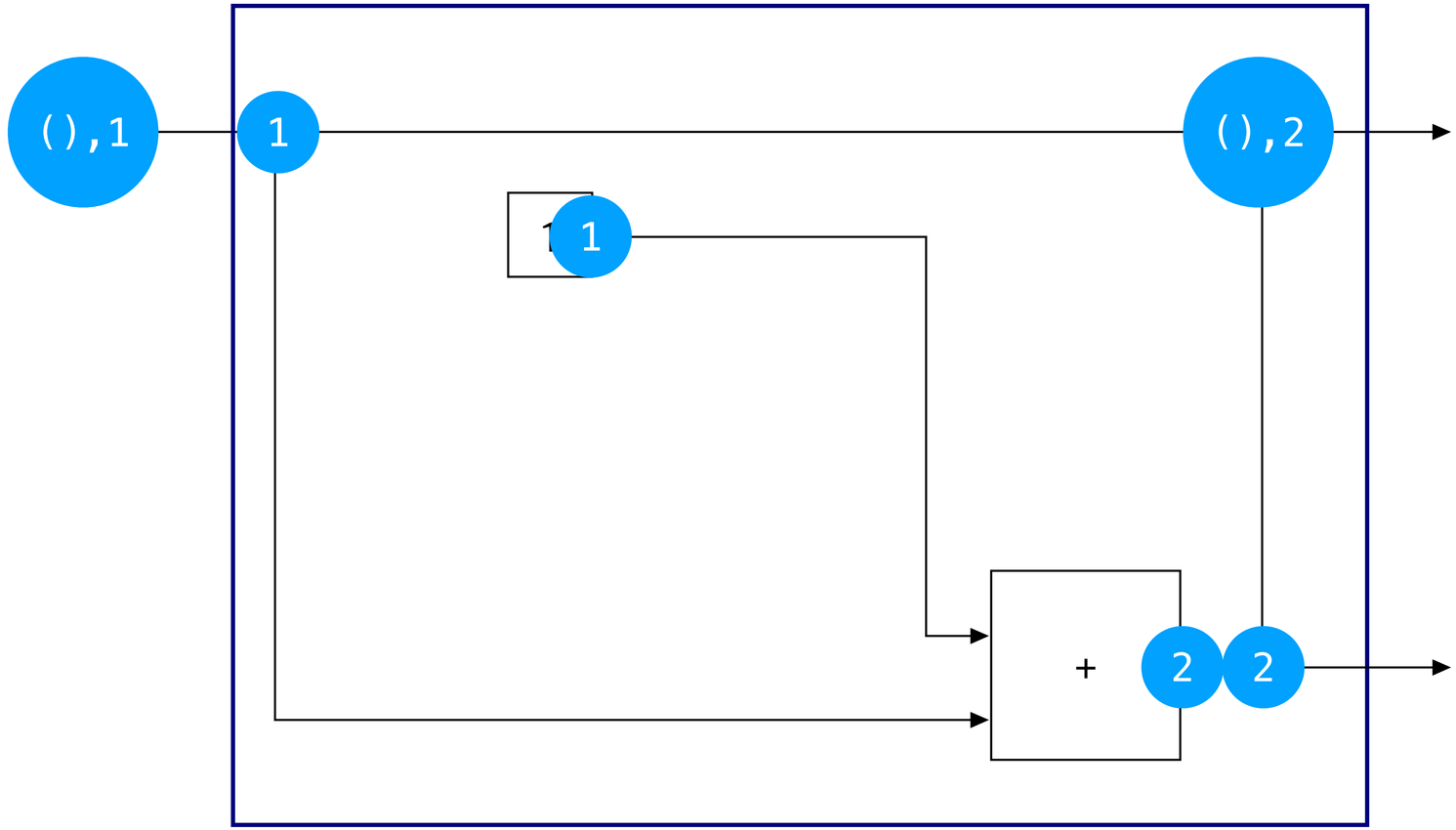
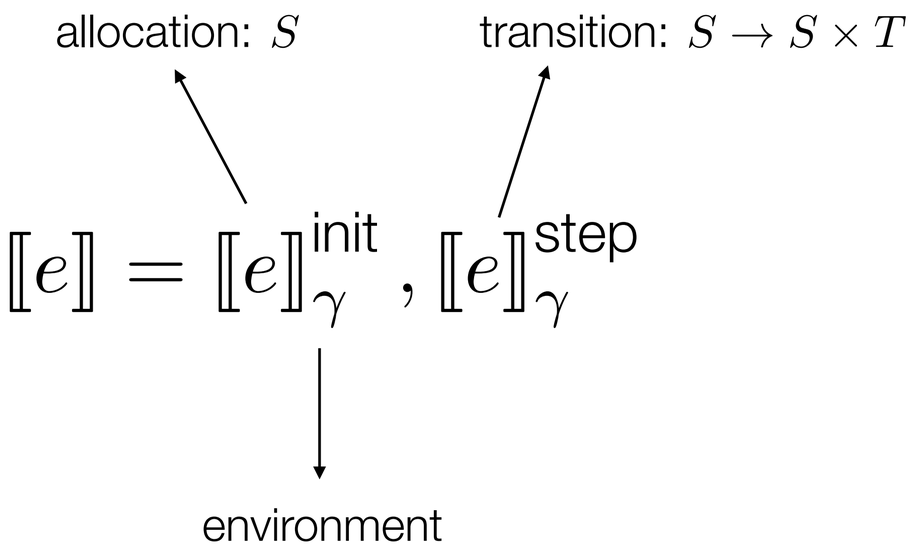
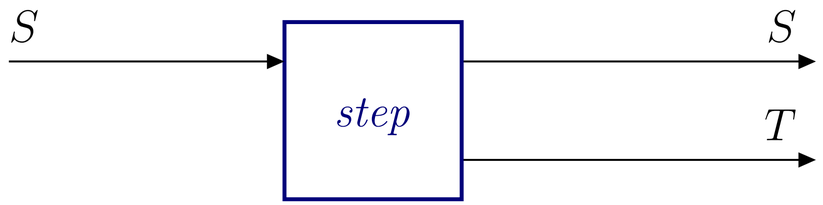


Deterministic streams





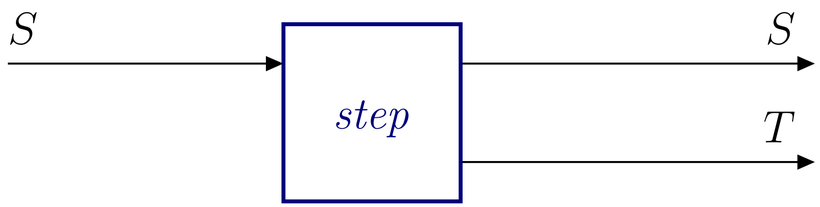
$$
\begin{array}{lcl}
[\![c]\!]_\gamma^{\text{init}} & = & ( ) \\
[\![c]\!]_\gamma^{\text{step}}\,(( )) & = & ( ), c \\
 & & \\
[\![x]\!]_\gamma^{\text{init}} & = & ( ) \\
[\![x]\!]_\gamma^{\text{step}}\,(( )) & = & ( ), \gamma(x) \\
 & & \\
[\![i \to \texttt{pre}\ e]\!]_\gamma^{\text{init}} & = & (i, [\![e]\!]_\gamma^{\text{init}}) \\
[\![i \to \texttt{pre}\ e]\!]_\gamma^{\text{step}}\,(p, s) & = & \textit{let}\ s', v = [\![e]\!]_\gamma^{\text{step}}\,(s)\ \textit{in}\ (v, s'), p
\end{array}
$$

Deterministic vs. probabilistic

Deterministic streams

Transition function returns a pair of state and value

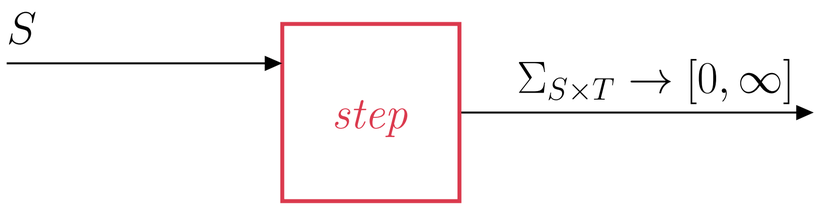
$$\mathit{CoStream}(T, S) = S \times (S \to S \times T)$$



Probabilistic streams

Transition function returns a **measure** over (state, value)

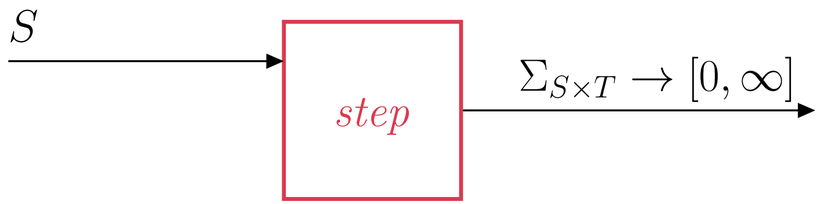
$$\mathit{CoPStream}(T, S) = S \times (S \to \Sigma_{S \times T} \to [0, \infty])$$



Probabilistic streams

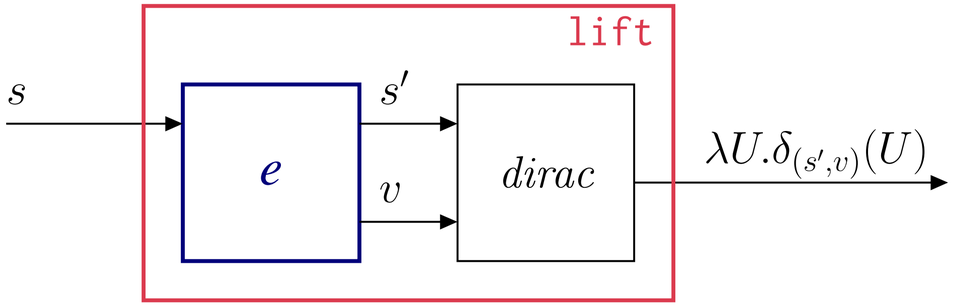
Transition function returns a *measure*

$CoPStream(T, S) = S \times (S \to \Sigma_{S \times T} \to [0,\infty])$



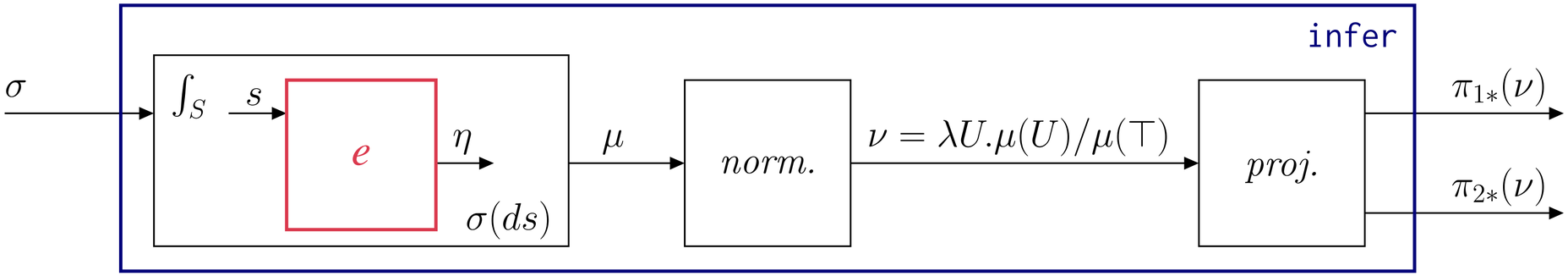
`lift` turns a deterministic expression into a probabilistic one

`lift:` $S \times (S \to S \times T) \to S \times (S \to \Sigma_{S \times T} \to [0,\infty])$



`infer` turns probabilistic expressions to a pair of distributions

`infer:` $S \times (S \to \Sigma_{S \times T} \to [0,\infty]) \to S$ `dist` $\times (S$ `dist` $\to S$ `dist` $\times T$ `dist`$)$



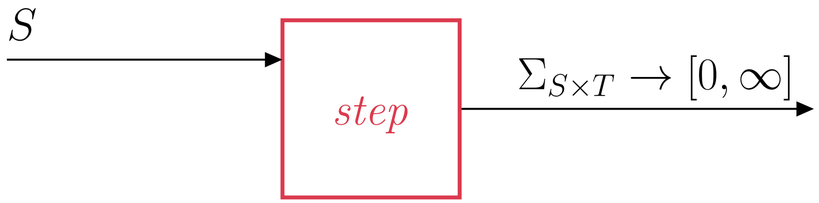
Probabilistic streams

allocation: S

transition: $S \to \Sigma_{S \times T} \to [0, \infty]$

$$\{\!|e|\!\} = \{\!|e|\!\}_\gamma^{\text{init}}, \{\!|e|\!\}_\gamma^{\text{step}}$$

environment



if e is deterministic

$$\{\!|e|\!\}_\gamma^{\text{init}} = [\![e]\!]_\gamma^{\text{init}}$$

$$\{\!|e|\!\}_\gamma^{\text{step}}(s) = \textit{let } s', v = [\![e]\!]_\gamma^{\text{step}}(s) \textit{ in } \delta_{s',v}$$

$$\{\!|\texttt{sample}(e)|\!\}_\gamma^{\text{init}} = [\![e]\!]_\gamma^{\text{init}}$$

$$\{\!|\texttt{sample}(e)|\!\}_\gamma^{\text{step}}(s) = \textit{let } s', \mu = [\![e]\!]_\gamma^{\text{step}}(s) \textit{ in } \int \mu(dv)\ \delta_{s',v}$$

$$\{\!|\texttt{observe}(e_1, e_2)|\!\}_\gamma^{\text{init}} = [\![e_1]\!]_\gamma^{\text{init}}, [\![e_2]\!]_\gamma^{\text{init}}$$

$$\{\!|\texttt{observe}(e_1, e_2)|\!\}_\gamma^{\text{step}}(s_1, s_2) = \textit{let } s_1', \mu = [\![e_1]\!]_\gamma^{\text{step}}(s_1) \textit{ in } \textit{let } s_2', v = [\![e_2]\!]_\gamma^{\text{step}}(s_2) \textit{ in } \mu_{\text{pdf}}(v) * \delta_{(s_1', s_2'),()}$$

Streaming inference

Reactive Probabilistic Programming

Particle filter

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10)) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

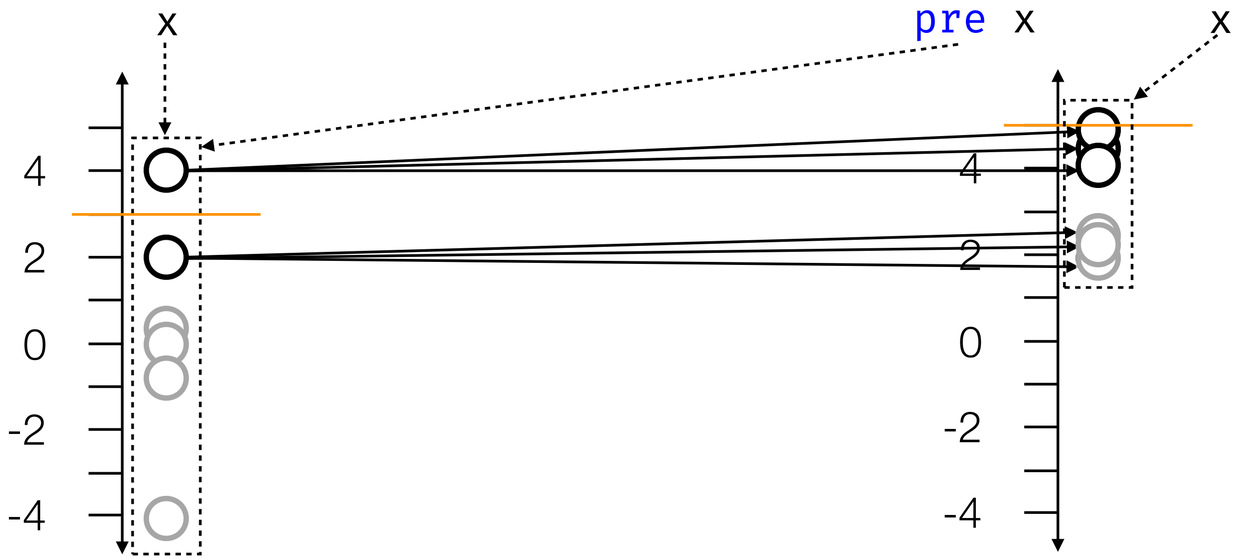
```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```

$t = 1$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), 5)
```

$t = 2$

```
sample (gaussian (pre x, 1))
observe (gaussian (x, 1), …)
```



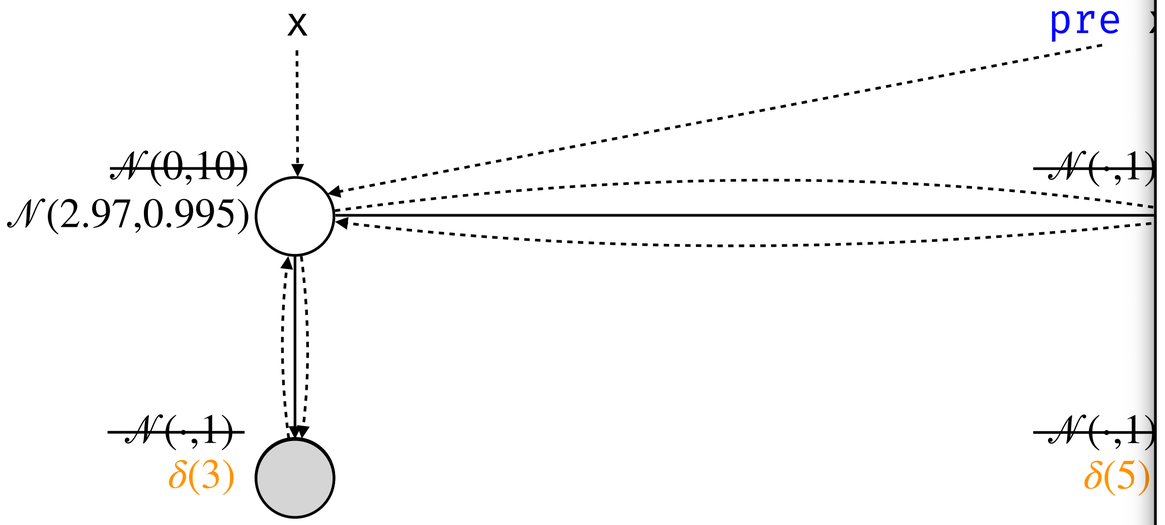
...

Delayed sampling

```
let proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10)) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

$t = 0$

```
sample (gaussian (0, 10))
observe (gaussian (x, 1), 3)
```



$t = 1$

```
sample (gauss
observe (gaus
```

pre x

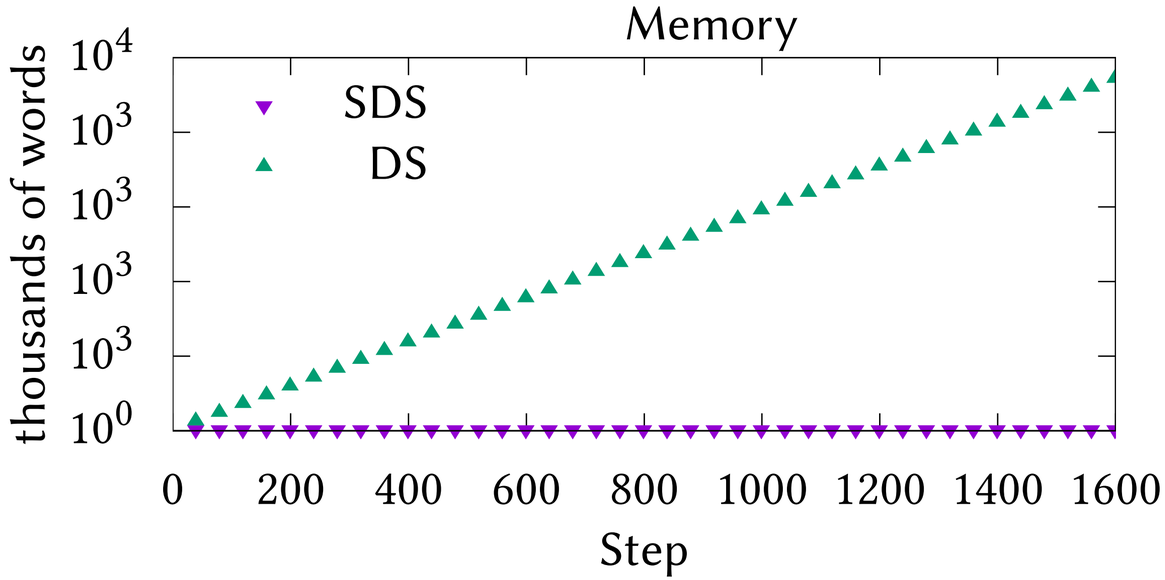
$\mathcal{N}(\cdot,1)$

$\mathcal{N}(\cdot,1)$

$\delta(5)$

$t = 2$

Unbounded resources



Benchmarks

Time to reach a target accuracy

Baseline:
SDS with 1,000 particles

SSI computations cost more,
but can outperform DS

	PF		DS		SSI	
MODEL	# PART.	TIME (MS)	# PART.	TIME (MS)	# PART.	TIME (MS)
Beta-Bernoulli	200	23.05 (22.54-23.90)	✓ 1	0.28 (0.27-0.28)	✓ 1	0.65 (0.65-0.66)
Gaussian-Gaussian	3000	877.44 (689.04-890.68)	150	62.99 (61.69-65.86)	150	182.57 (181.40-183.54)
Kalman-1D	15	3.27 (3.26-3.28)	✓ 1	0.33 (0.33-0.34)	✓ 1	1.15 (1.14-1.15)
Outlier	700	222.27 (220.76-223.84)	65	43.81 (43.45-46.48)	65	125.88 (125.27-128.08)
Robot	85	771.32 (767.98-775.51)	✓ 1	91.44 (90.94-92.07)	✓ 1	96.40 (96.21-97.53)
SLAM		✗	800	2812.55 (2755.99-2853.89)	800	5649.30 (5619.59-5675.81)
MTT		✗	60	2889.11 (2615.76-3244.30)	60	4457.79 (4068.35-4996.20)
Tree	150	35.55 (35.41-35.68)	90	58.83 (58.55-59.74)	✓ 1	2.67 (2.66-2.70)
Wheels	550	246.48 (245.06-248.75)	550	699.12 (672.25-713.64)	✓ 1	8.04 (8.00-8.10)
Delayed GPS	150	1221.00 (1218.76-1230.67)	9	304.73 (303.17-306.31)	✓ 1	108.55 (108.02-109.07)

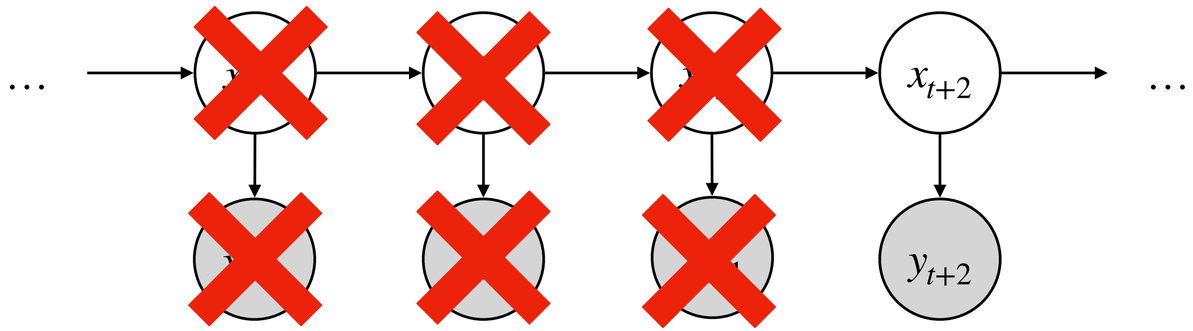
Trade-off: cost of symbolic computations vs. precision

Static Analysis

Reactive Probabilistic Programming

Bounded Memory Delayed Sampling?

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

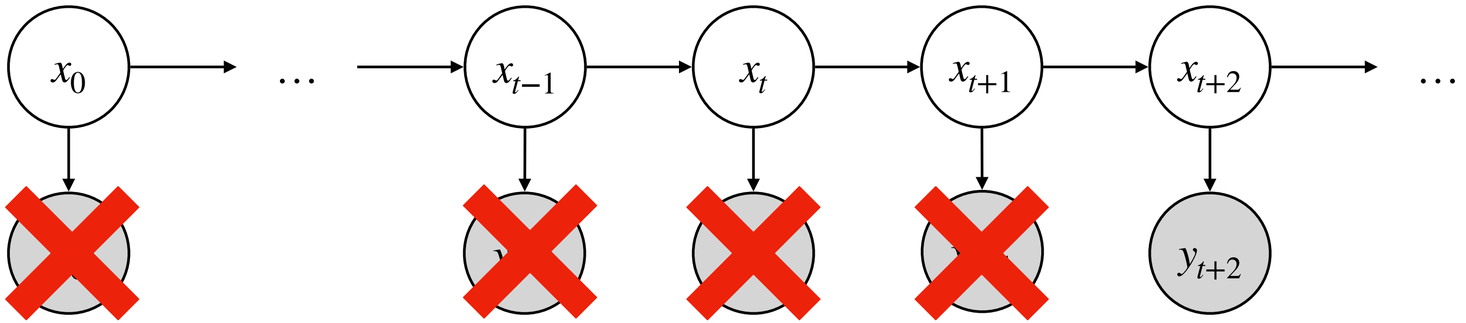


Yes!

Bounded Memory Delayed Sampling?

```
proba tracker (y) = x, x0 where
  rec init x0 = sample (gaussian (0, 10))
  and x  = x0 → sample (gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```

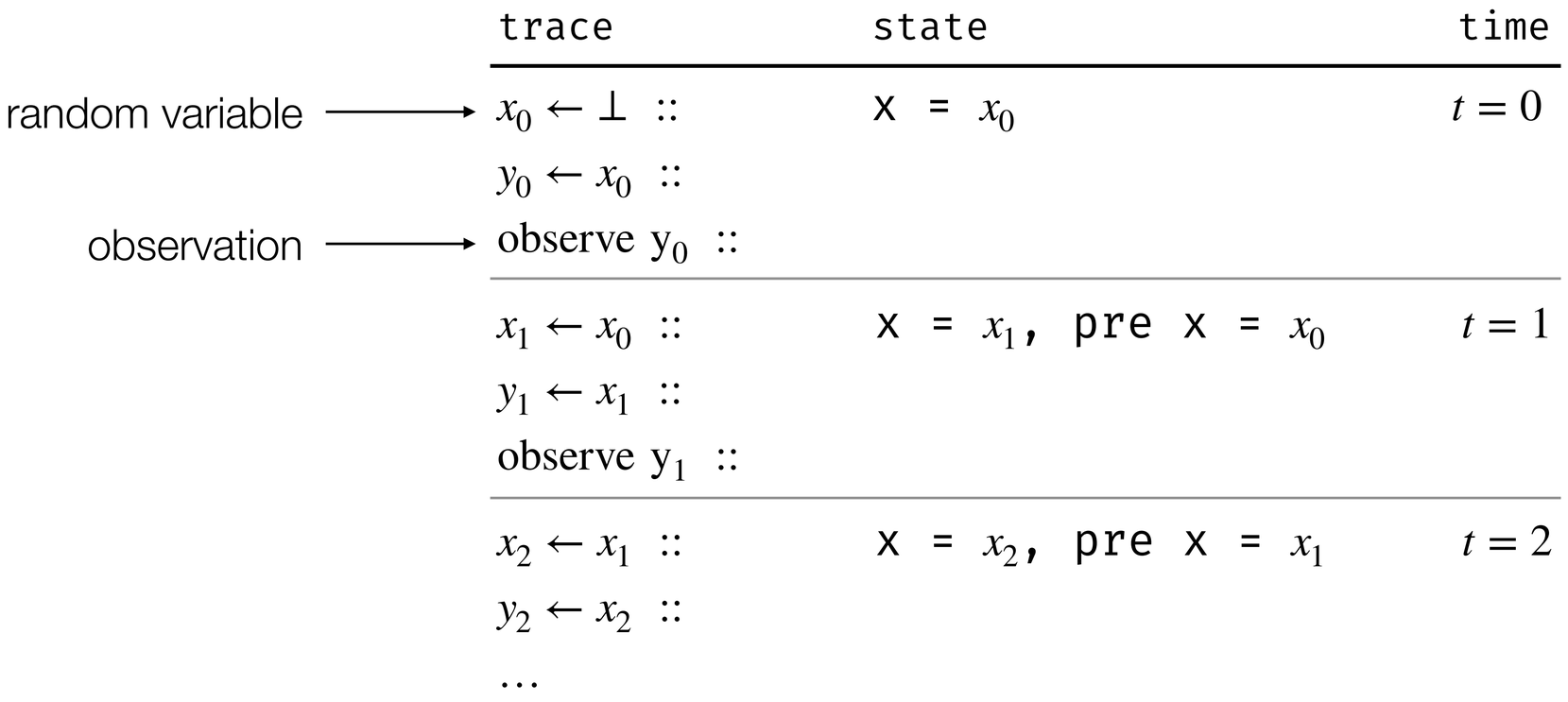
Can we determine if a given program will run in bounded memory?



No!

Trace: Abstract Execution

```
proba tracker (y) = x where
  rec x  = sample (gaussian (0, 10) → gaussian (pre x, 1))
  and () = observe (gaussian (x, 1), y)
```



Static Analysis for Delayed Sampling

Semantic properties

m-consumed property
Chains of variables before an observe are bounded

unseparated paths property
Chains of variables referenced in the state are bounded

Theorem: *The program satisfies these two properties iff it executes in bounded memory*

Static analysis

Track variables introduced but not used yet

Track maximal path between pairs of variable in the state

Theorem: *The program pass the analysis if it executes in bounded memory*

m-consumed Property

```
proba tracker (y) = x where
  rec x   = sample (gaussian (0, 10) → gauss
  and () = observe (gaussian (x, 1), y)
```

Chains of variables before an observe are bounded

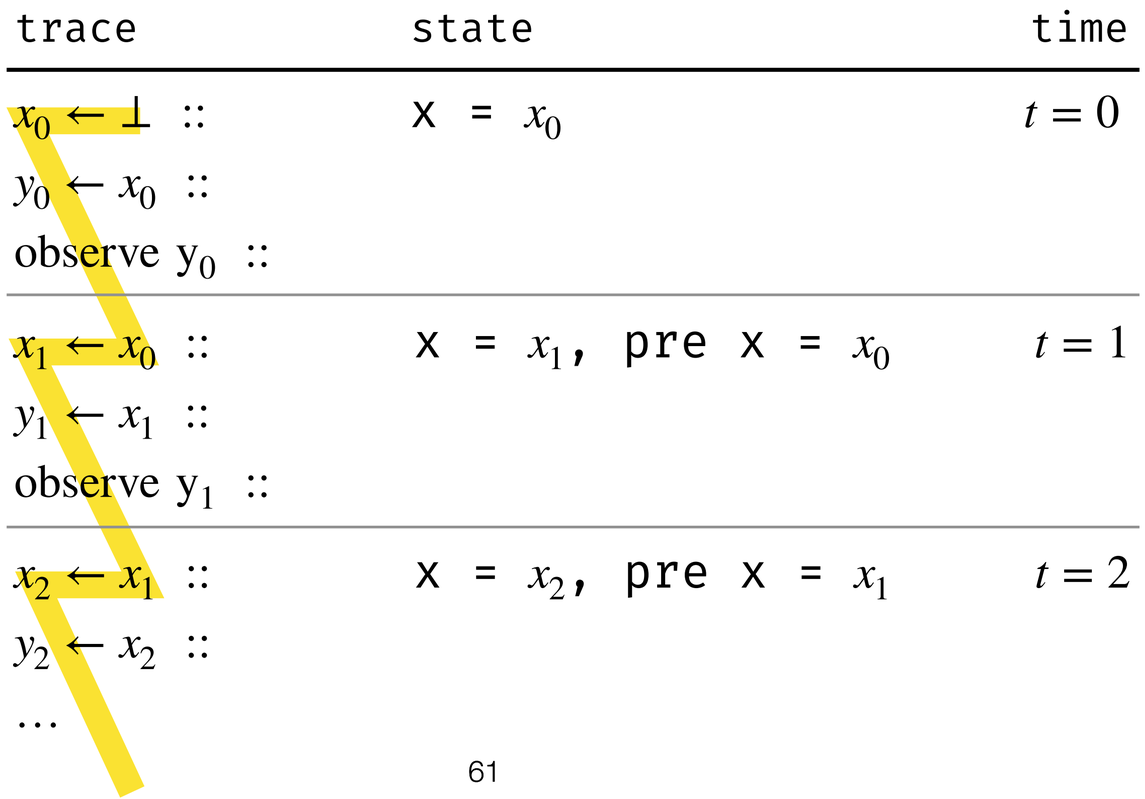
	trace	state	time
	$x_0 \leftarrow \bot$ ::	x = x_0	$t = 0$
x_0 is 1-consumed →	$y_0 \leftarrow x_0$ ::		
y_0 is 0-consumed →	observe y_0 ::		
	$x_1 \leftarrow x_0$ ::	x = x_1, pre x = x_0	$t = 1$
x_1 is 1-consumed →	$y_1 \leftarrow x_1$ ::		
y_1 is 0-consumed →	observe y_1 ::		
	$x_2 \leftarrow x_1$ ::	x = x_2, pre x = x_1	$t = 2$
	$y_2 \leftarrow x_2$ ::		
	…		

Yes!

Unseparated Paths Property

```
proba tracker (y) = x where
  rec x   = sample (gaussian (0, 10) → gauss
  and () = observe (gaussian (x, 1), y)
```

Chains of variables referenced in the state are bounded



Yes!

Unseparated Paths Property

```
proba tracker (obs) = x where
  rec init x0 = sample (gaussian (0, 10))
  and x  = x0 → sample (gaussian (pre x, 1)
  and () = observe (gaussian (x, 1), y)
```

Chains of variables referenced in the state are bounded

trace	state	time
$x_0 \leftarrow \bot$ ::	x = x_0	$t = 0$
$y_0 \leftarrow x_0$ ::	x0 = x_0	
observe y_0 ::		
$x_1 \leftarrow x_0$ ::	x = x_1, pre x = x_0	$t = 1$
$y_1 \leftarrow x_1$ ::	x0 = x_0	
observe y_1 ::		
$x_2 \leftarrow x_1$ ::	x = x_2, pre x = x_1	$t = 2$
$y_2 \leftarrow x_2$ ::	x0 = x_0	
…		

No!

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