

Probabilistic Programming & Linear Logic.

MPRI 2025 – Nov. 7 & 14

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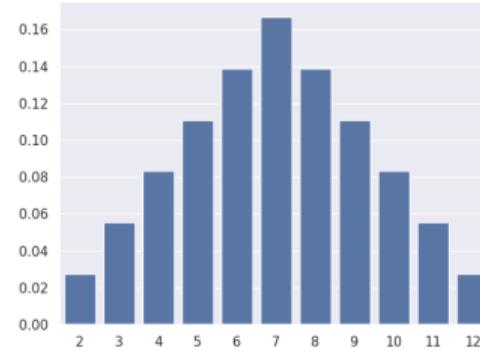


Probabilistic Programming

Programs as Random Variables

Sum of two Dice¹

```
def dice() → int:  
    a = sample(RandInt(1, 6))  
    b = sample(RandInt(1, 6))  
    return a + b  
  
with Enumeration():  
    dist: Categorical[float] = infer(dice)
```



dice is a Random Variable whose semantics is a distribution over $\{2, \dots, 12\}$ obtained by **sums**

$$\begin{aligned}\llbracket \text{dice} \rrbracket : \quad \mathbb{N} &\rightarrow \quad \mathbb{R}^+ \\ k &\mapsto \quad \sum_{a=1}^6 \sum_{b=1}^6 \frac{1}{36} \delta_{a+b}(k)\end{aligned}$$

¹Baudart 2024, *muPPL, a PPL prototype in Python*, <https://github.com/gbdrt/mu-ppl>

Historical interlude - Kolmogorov (1930) probability

An **experiment** can produce a number of results

Example : toss a coin, roll a dice, choose a point on a line

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A σ -**algebra** \mathcal{F} : set of computable events

Example : $\mathcal{P}(\{1, 2, 3, 4, 5, 6\})$, intervals of \mathbb{R} .

Theoretical interlude - Probability space

Definition

Let Ω be a set, a σ -algebra \mathcal{F} is a set of subsets of Ω such that

$\emptyset \in \mathcal{F}$ and if $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
and if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

The σ -algebra **generated** by a family of subsets is the closure by complement countable intersections and unions. For instance:

The Borel σ -algebra is generated by finite rectangles

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A **measure** on (Ω, \mathcal{F}) is a function $\mu : \mathcal{F} \rightarrow [0, 1]$

such that $\mu(\emptyset) = 0$

and if $A_1, A_2, \dots \in \mathcal{F}$ are pairwise disjoint, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

If moreover $\mathbb{P}(\Omega) = 1$, then it is a probability measure.

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A **probabilistic space** is given by $(\Omega, \mathcal{F}, \mathbb{P})$.

Random variable

Let $\Omega, \mathcal{F}, \mathbb{P}$ be a probability space.

Definition

A **random variable** is a measurable function $X : \Omega \rightarrow \mathbb{R}$ such that

$$\forall x \in \mathbb{R} \quad \{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F}.$$

The **probability distribution** μ of X , denoted $X \sim \mu$ is the pushforward probability measure of X on its outputs:

$$\forall U \subseteq \mathbb{R}, \quad \mu(U) = \mathbb{P}(X^{-1}(U))$$

For a real random variable, the **cumulative distribution function** (CDF) of X is $F_X(x) = \mathbb{P}(X \leq x)$ characterizes its distribution μ . The inverted CDF is denoted idf_X .

Exercises: describe the universe and the random variable

```
def SqDist() → float:  
    x = sample(Uniform(-1, 1))  
    y = sample(Gaussian(0,1))  
    return x**2 + y**2
```

```
def RandomWalk(n: int, d) → int:  
    s = 0  
    for k in range(n):  
        s += sample(d)  
    return s
```

```
def StoppingTime(d) → int:  
    time = 1  
    while sample(d):  
        time = time +1  
    return time
```

```
def model():  
    m = sample(Gaussian(0.0, 3.0))  
    b = sample(Gaussian(0.0, 3.0))  
    f = lambda x: m*x +b  
    return f
```

Discrete Random Variable

Definition

A random variable $X : \Omega \rightarrow \mathbb{R}$ is discrete if it takes a countable number of values.

The probability mass function (PMF) of a discrete random variable X is a function $f : \mathbb{R} \rightarrow [0, 1]$ such that: $f(x) = \mathbb{P}(X = x) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$.

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The mean (expected value), written $\mathbb{E}(X)$ of a discrete random variable X is:

$$\mathbb{E}(X) = \sum_x x \mathbb{P}(X = x) = \sum_x x f(x)$$

when the sum exists (it is always the case when $X > 0$).

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If $g : \mathbb{R} \rightarrow \mathbb{R}$ measurable and X is a discrete random variable, then $g(X)$ is a discrete random variable

$$\mathbb{E}(g(X)) = \sum_x g(x) \mathbb{P}(X = x)$$

Trace Semantics

```
def dice() → int:  
    a = sample(RandInt(1, 6))  
    b = sample(RandInt(1, 6))  
    return a + b
```

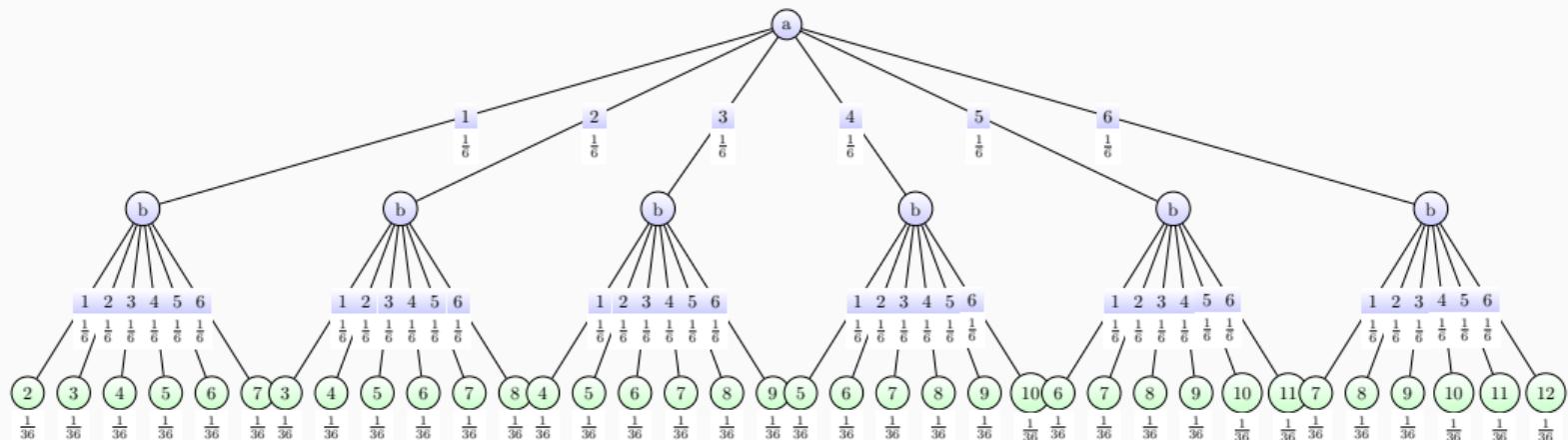
Trace Semantics

```
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This program simulates the **random variable** dice:

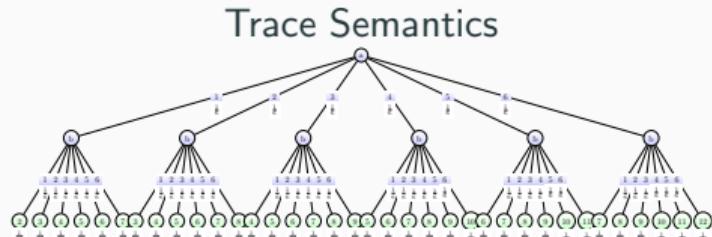
"The sum of two independent fair dice."

At each execution, the operator sample **simulates** a random variable uniform over $\{1, \dots, 6\}$.



What is the law of this program ?

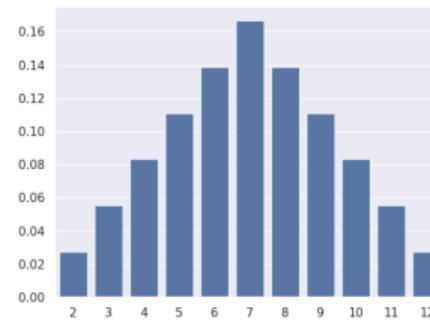
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The discrete measure:

$$\llbracket \text{dice} \rrbracket : \quad \mathbb{P}(\text{dice}() = k) \\ = \sum_{a=1}^6 \sum_{b=1}^6 \frac{1}{36} \mathbb{1}_{\{a+b=k\}}$$

Measure Semantics

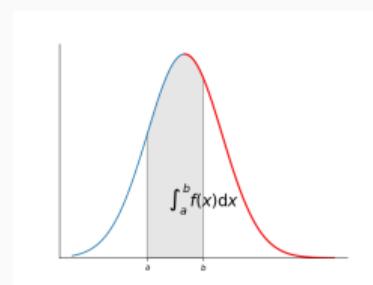


Continuous random variable with density

Definition

A random variable $X : \Omega \rightarrow \mathbb{R}$ is **continuous with density** (absolutely continuous with respect to Lebesgue's measure) if its **cumulative distribution function**: $F(x) = \mathbb{P}(\{\omega \mid X(\omega) \leq x\})$ can be described by the integral of $f : \mathbb{R} \rightarrow [0, \infty]$ called the probability **density function** (PDF) of X :

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x f(u)du$$

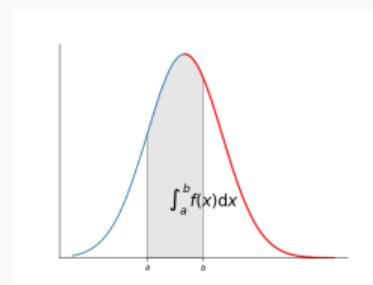


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The **mean**, denoted $\mathbb{E}(X)$ of the continuous random variable X is defined by the integral when it exists:

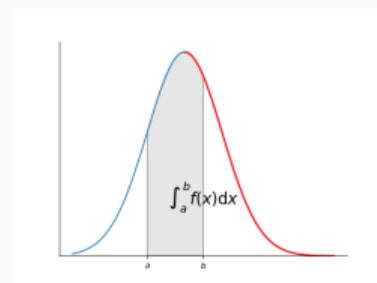
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$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(u)du \quad \mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(u) f(u)du$$

Density versus Measure semantics

```
def SqDist() → float:  
    x = sample(Uniform(-1, 1))  
    y = sample(Gaussian(0,1))  
    return x**2 + y**2
```

This program simulates the square distance from the origin to a point obtained from sampling in a uniform over $[-1, 1]$ and in a Gaussian of mean 0 and variance 1.

Density:

$$\text{pdf(SD)}(r_1, r_2) = \mathbb{1}_{[-1,1]}(r_1) \mathcal{N}(0, 1)(r_2)$$

Measure: by integration along the path

$$[\![\text{SD}]\!](U) = \int_{\substack{r_1 \in [0,1] \\ r_2 \in \mathbb{R}}} \mathcal{N}(0, 1)(r_2) \delta_{r_1^2 + r_2^2}(U) dr_1 dr_2$$

Path:

$$\alpha : r_1, r_2 \mapsto \delta_{r_1^2 + r_2^2}$$

Exercises: describe trace/density and measure semantics

```
def model():
    m = sample(Gaussian(0.0, 3.0))
    b = sample(Gaussian(0.0, 3.0))
    f = lambda x: m*x +b
    return f
```

```
def CondGauss() → float:
    x = sample(Uniform(-10,10))
    y = sample(Gaussian(x,1))
    return (x, y)
```

```
def FairCoin(d) → bool:
    a = sample(d)
    b = sample(d)
    if (a and not b):
        return True
    elif (b and not a):
        return False
    else:
        return FairCoin(d)
```

```
def StoppingTime(d) → int:
    time = 1
    while sample(d):
        time = time +1
    return time
```

Monte-Carlo simulation and the law of large numbers

A **Probabilistic program** is the [random variable](#) whose values are the outcome of the program execution. If the program has no [effect](#), then its [executions](#) are i.i.d. Then:

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Law of large numbers:

Run n times the probabilistic program

Store the outputs x_1, \dots, x_n .

Compute $\frac{x_1 + \dots + x_n}{n}$ that approximates $\mathbb{E}(X)$ and $\frac{g(x_1) + \dots + g(x_n)}{n}$ that approximates $\mathbb{E}(g(X))$

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Monte-Carlo Simulation: histograms approximate distributions:

For a random variable X ,

$\frac{1}{n} \#(\{i | x_i = x\})$ approximates $\mathbb{E}(\chi_{X=x}) = \mathbb{P}(X = x)$

$\frac{1}{n} \#(\{i | a \leq x_i \leq b\})$ approximates $\mathbb{E}(\chi_{a \leq X \leq b}) = \mathbb{P}(a \leq X \leq b)$

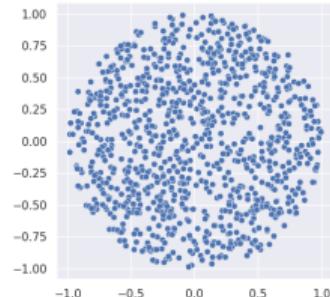
Probabilistic Programming

Bayesian Inference

Exercise: What are the Prior, Likelihood, Conditional measures

```
def HardDisk() → Tuple[float, float]:  
    x = sample(Uniform(-1, 1))  
    y = sample(Uniform(-1, 1))  
    d2 = x**2 + y**2  
    assume(d2 < 1)  
    return (x, y)
```

```
with RejectionSampling(num_samples=100):  
    dist: Empirical = infer(HardDisk)
```

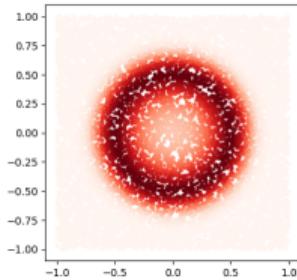


```
def SoftDisk() → Tuple[float, float]:  
    x = sample(Uniform(-1, 1))  
    y = sample(Uniform(-1, 1))  
    d2 = x**2 + y**2  
    observe(Gaussian(d2, 0.1), 0.5)  
    return(x, y)
```

```
with ImportanceSampling(num_particles  
=10000):
```

$$\frac{\int_{-1}^1 \int_{-1}^1 \text{obx}(x, y) \delta_{x,y} dx dy}{\int_{-1}^1 \int_{-1}^1 \delta_{x,y} dx dy}$$

Monte-Carlo
Simulation



Conditional Probability

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The conditional probability is defined when $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

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$$\mathbb{P}(A) = \mathbb{P}(A|B) \text{ or } \mathbb{P}(B) = \mathbb{P}(B|A) \text{ or } \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

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Properties

if $\mathbb{P}(B) > 0$, then $\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c)$.

if B_1, B_2, \dots, B_n is a partition Ω , then

$$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A|B_i)\mathbb{P}(B_i)$$

Interlude theoretic - Bayes Law

Bayes Law

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}$$

Bayes Formula

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c)}$$

Generalization, if $\mathbb{P}(A) > 0$ and B_1, B_2, \dots, B_n is a partition of Ω such that $\forall i \mathbb{P}(B_i) > 0$, then

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_{i=1}^n \mathbb{P}(A|B_i)\mathbb{P}(B_i)}$$

Conditional Probability and random variables

Discrete

Bayes Law

$$\mathbb{P}(Y = y | X = x) = \frac{\mathbb{P}(X = x | Y = y) \mathbb{P}(Y = y)}{\mathbb{P}(X = x)}$$

Conditional Probability and random variables

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Continuous

Bayes Law

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$$

Conditional Probability and random variables

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Bayes Formula

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Theoretical Interlude - Conditional Law

Joint Distribution: Let X and Y be two random variables. Their joint distribution function is given by

$$F(x, y) = \mathbb{P}(X \leq x, Y \leq y)$$

X and Y are jointly continuous with density if there is joint probability density function f such that

$$F(x, y) = \int_{-\infty}^y \int_{-\infty}^x f_{X,Y}(u, v) dudv$$

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Then, the second marginal give the density function of the law of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(u, y) du$$

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Let us define **conditional density function** by:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Exercise: What are the Prior, Likelihood, Conditional measures

```

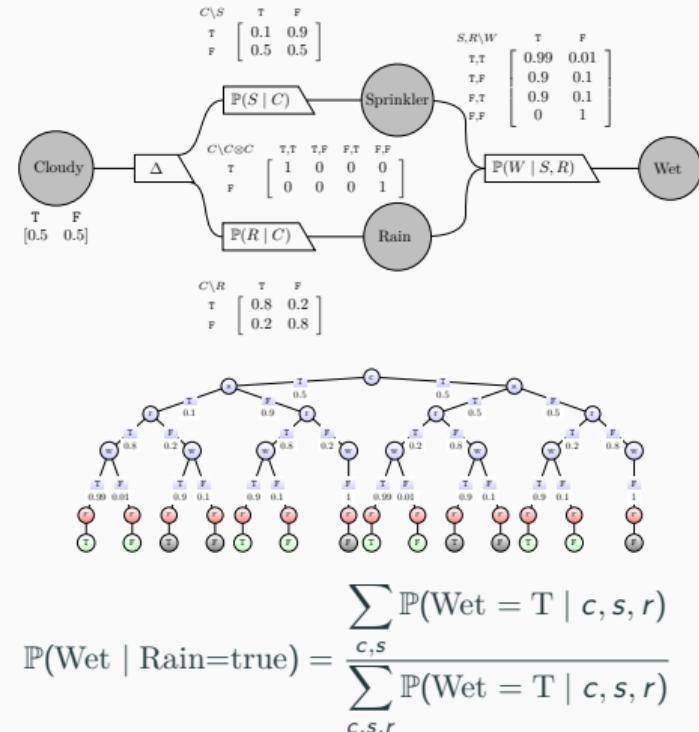
def wet() → bool:
    cloudy = sample(Bernoulli(0.5))

    p_s, p_r = (0.1, 0.8) if cloudy else (0.5, 0.2)
    sprinkle = sample(Bernoulli(p_s))
    rain = sample(Bernoulli(p_r))

    p_w = 0.99 if (sprinkle and rain) else 0.9 if
        (sprinkle != rain) else 0
    wet = sample(Bernoulli(p_w))
    assume(rain)
    return wet

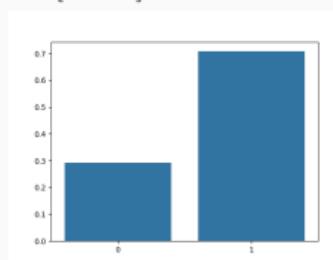
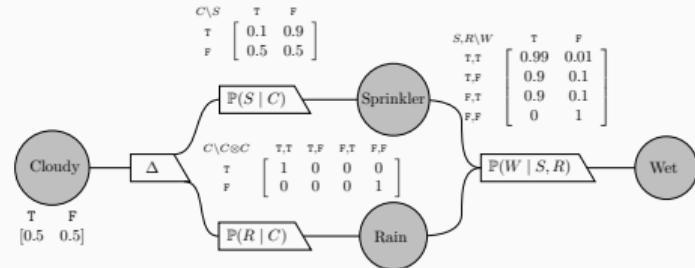
with Enumeration():
    dist: Categorical[bool] = infer(wet)

```



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    p_s, p_r = (0.1, 0.8) if cloudy else (0.5, 0.2)  
    sprinkle = sample(Bernoulli(p_s))  
    rain = sample(Bernoulli(p_r))  
  
    p_w = 0.99 if (sprinkle and rain) else 0.9 if  
        (sprinkle != rain) else 0  
    wet = sample(Bernoulli(p_w))  
    assume(rain)  
    return wet  
  
with Enumeration():  
    dist: Categorical[bool] = infer(wet)
```



Exact Mass Function

Parameterized Programs

```
def sum(d) → int:  
    a = sample(d)  
    b = sample(d)  
    return a + b
```

It is a probabilistic distribution transformer.

Discrete semantics: stochastic matrix

$$\llbracket \text{sum} \rrbracket (d) : k \mapsto \sum_a \sum_b d_a d_b \delta_{a+b}(k) = \sum_{a+b=k} d_a d_b$$

Continuous semantics: stochastic kernel

$$\llbracket \text{sum} \rrbracket (\mu) : U \mapsto \int \int \mu(da) \mu(db) \delta_{a+b}(U)$$

Probabilistic Programming

The Category of Markov Kernels

Markov Kernels (History)

Lawvere 1962, “The category of probabilistic mappings” introduces stochastic kernels to denote temporally discrete Markov processes.

Kozen 1979, “Semantics of Probabilistic Programs” proposes stochastic kernels between measurable spaces to denote operational semantics of probabilistic programs.

Giry 1982, “A categorical approach to probability theory” describes Markov kernels as the Kleisli Category of the probabilistic monad.

Panangaden 1998, “The Category of Markov Kernels” relates Markov Kernels and Relations.

Markov Kernels (Definition)

Measurable space: $\mathcal{X} = (\Omega_{\mathcal{X}}, \mathcal{F}_{\mathcal{X}})$

$\Omega_{\mathcal{X}}$ set of possible experiment outcomes, with $\mathcal{F}_{\mathcal{X}}$ the σ -algebra of measurable events.

Stochastic kernel: $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ is a function $\kappa : \Omega_{\mathcal{X}} \times \mathcal{F}_{\mathcal{Y}} \rightarrow \mathbb{R}^+$ such that

$\forall x \in \mathcal{X}, \kappa(\cdot|x) : \mathcal{F}_{\mathcal{Y}} \rightarrow \mathbb{R}^+$ is a measure

$\forall V \in \mathcal{F}_{\mathcal{Y}}, \kappa(V| \cdot) : \mathcal{X} \rightarrow \mathbb{R}^+$ is a measurable function

Composition: $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $\kappa' : \mathcal{Y} \rightsquigarrow \mathcal{Z}$.

$$\forall x \in \Omega_{\mathcal{X}} \quad \forall W \in \mathcal{F}_{\mathcal{Z}}, \quad \kappa' \circ \kappa(W|x) = \int_{\mathcal{Y}} \kappa'(W|y) \kappa(dy|x)$$

Markov Kernel (Monad of Finite Measures)

Finite Measures: Let \mathcal{X} be a measurable space.

$\mathcal{G}(\mathcal{X})$ is the set of finite measures μ

Giry Monad

$\mathcal{G}(\mathcal{X})$ is a measurable space when endowed with the σ -algebra generated by the evaluations on measurable sets $\text{ev}_U : \mu \mapsto \mu(U)$, i.e. by the collection of $\text{ev}_U^{-1}([a, b])$ for $a, b \in \mathbb{R}^+$.

Remark: $\mathcal{G}(\{0, 1\}) = \{p\delta_0 + (1 - p)\delta_1 \mid 0 \leq p \leq 1\}$

with $\{\delta_1\}$ and $\{p\delta_0 + (1 - p)\delta_1 \mid p \in [0, 1] \cap [a, b]\}$ measurable sets.

Kernel and Giry: $\text{Kern}(\mathcal{X}, \mathcal{Y}) = \text{Meas}(\mathcal{X}, \mathcal{G}(\mathcal{Y}))$

Kernels are measures transformers thanks to the giry monad bind

$$\forall \mu \in \mathcal{G}(\mathcal{X}) \quad \kappa \cdot \mu : W \mapsto \int \kappa(W|x) \mu(dx) \in \mathcal{G}(\mathcal{Y})$$

Probabilistic programs with several parameters

```
def sumd(d1, d2) → int:  
    a = sample(d1)  
    b = sample(d2)  
    return a + b
```

It is interpreted as a measure parameterized on the joint measure of two independent measures.

Monoidal unit:

$$\Omega_1 = \{*\}.$$

Monoidal product

$$\Omega_{\mathcal{X} \otimes \mathcal{Y}} = \Omega_{\mathcal{X}} \times \Omega_{\mathcal{Y}}$$

$\mathcal{F}_{\mathcal{X} \otimes \mathcal{Y}}$ generated by squares of measure sets $U \times V$ for $U \in \mathcal{F}_{\mathcal{X}}$ and $V \in \mathcal{F}_{\mathcal{Y}}$

Product of kernels

$$\kappa \otimes \kappa'(U \times V|x, y) = \kappa(U|x) \kappa'(V|y)$$

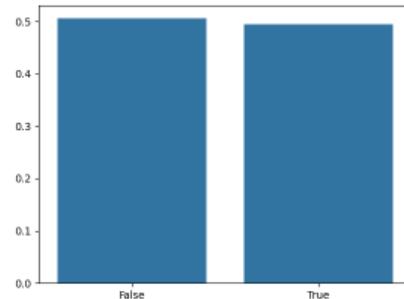
Markov Kernels is Symmetric Monoidal

Fair Coin Example: Need for Higher-Order

```
def FairCoin(d) → bool:  
    a = sample(d)  
    b = sample(d)  
    if (a and not b):  
        return True  
    elif (b and not a):  
        return False  
    else:  
        return FairCoin(d)
```

```
with ImportanceSampling(num_particles=1000):  
    FC: Categorical[bool] = infer(FairCoin,  
                                Bernoulli(0.3))
```

FC is a fair coin !



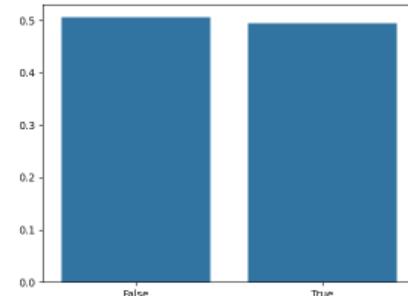
How can we prove it ?

Fair Coin Example: Need for Higher-Order

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def FairCoin(d) → bool:  
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How can we prove it ?

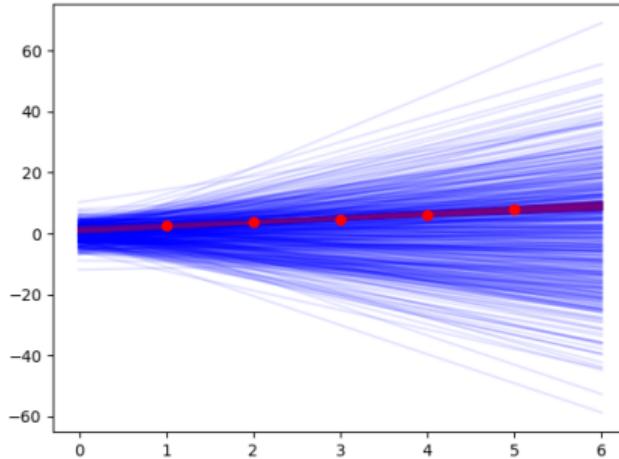
Recursive equation (if $p \notin \{0, 1\}$):

$$\begin{aligned}\mathcal{F}(t) &= p(1 - p)(\delta_T + \delta_F) \\ &\quad + (1 - 2(p(1 - p))) t \\ \llbracket \text{FC} \rrbracket &= \frac{1}{2}(\delta_T + \delta_F)\end{aligned}$$

Linear Regression

```
def model(data):
    m = sample(Gaussian(0.0, 3.0))
    b = sample(Gaussian(0.0, 3.0))
    f = lambda x: m*x +b
    for (x, y) in data:
        observe(Gaussian(f(x), 0.5), y)
    return f

data = [(1.0, 2.5), (2.0, 3.8), (3.0, 4.5),
        (4.0, 6.2), (5.0, 8.0)]
```



Wanted: Symmetric Monoidal Closed Category

- to interpret the type of parameterized programs.

Currying: $\mathbf{Kern}(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z}) = \mathbf{Kern}(\mathcal{X}, \mathcal{Y} \multimap \mathcal{Z})$

Evaluation: $\text{ev} \in \mathbf{Kern}(\mathcal{X} \otimes (\mathcal{X} \multimap \mathcal{Y}), \mathcal{Y})$

$\mathcal{X} \multimap \mathcal{Y}$ can be also denoted $\mathcal{Y}^{\mathcal{X}}$

Aumann's Lemma (1961), revisited (thanks to Ohad Kammar and Thomas Ehrhard.)

Markov kernel is not closed as there is no measurability structures that can be put on real measurable functions such that the evaluation is measurable.

Assume, by contradiction, that Markov Kernel is an SMCC:

$$\mathbf{Kern}(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z}) = \mathbf{Kern}(\mathcal{X}, \mathcal{Z}^{\mathcal{Y}})$$

where $\mathcal{Z}^{\mathcal{Y}}$ is a measurable space and the evaluation $\text{ev} : \mathcal{Z}^{\mathcal{Y}} \otimes \mathcal{Y} \rightsquigarrow \mathcal{Z}$ is a kernel

Consequence: $\mathbf{Kern}(\mathcal{Y}, \mathcal{Z}) = \mathcal{G}(\mathcal{Z}^{\mathcal{Y}})$

Proof: $\mathbf{Kern}(\mathcal{Y}, \mathcal{Z}) = \mathbf{Kern}(1 \otimes \mathcal{Y}, \mathcal{Z}) = \mathbf{Kern}(1, \mathcal{Z}^{\mathcal{Y}}) = \mathbf{Meas}(1, \mathcal{G}(\mathcal{Z}^{\mathcal{Y}})) = \mathcal{G}(\mathcal{Z}^{\mathcal{Y}})$

Consequence: the diagonal $\Delta = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$ is measurable in $\mathcal{X} \otimes \mathcal{Y}$ where

$\mathcal{X} = (\mathbb{R}, \mathcal{P}(\mathbb{R}))$ the discrete σ -algebra

$\mathcal{Y} = (\mathbb{R}, \mathcal{C}(\mathbb{R}))$ the countable-cocountable σ -algebra (*generated by countable parts and parts whose complement are countable, closed by countable unions and intersections.*)

Contradiction: the diagonal cannot be measurable in $\mathcal{X} \otimes \mathcal{Y}$

Aumann's Lemma, revisited (thanks to Ohad Kammar and Thomas Ehrhard.)

Consequence: the diagonal $\Delta = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$ is measurable in $\mathcal{X} \otimes \mathcal{Y}$ where

$\mathcal{X} = (\mathbb{R}, \mathcal{P}(\mathbb{R}))$ the discrete σ -algebra

$\mathcal{Y} = (\mathbb{R}, \mathcal{C}(\mathbb{R}))$ the countable-cocountable σ -algebra (*generated by countable parts and parts whose complement are countable, closed by countable unions and intersections.*)

Proof: Let $x \in \mathbb{R}$ and $h_x \in \mathbf{Meas}(\mathcal{Y}, \mathcal{G}(\{0, 1\})) = \mathbf{Kern}(\mathcal{Y}, \{0, 1\})$

$h_x : y \mapsto \begin{cases} \delta_1 & \text{if } x = y \\ \delta_0 & \text{otherwise} \end{cases}$ $h_x^{-1}(\{p\delta_0 + (1-p)\delta_1 \mid p \in [0, 1] \cap [a, b]\})$ can be $\emptyset, \mathbb{R}, \{x\}$ thus countable, or $\mathbb{R} \setminus \{x\}$ thus cocountable.

Since in \mathcal{X} any part is measurable, $\lambda x. h_x \in \mathbf{Meas}(\mathcal{X}, \mathbf{Kern}(\mathcal{Y}, \{0, 1\}))$, which is isomorphic to

$$\mathbf{Meas}(\mathcal{X}, \mathcal{G}(\{0, 1\}^{\mathcal{Y}})) = \mathbf{Kern}(\mathcal{X}, \{0, 1\}^{\mathcal{Y}}) = \mathbf{Kern}(\mathcal{X} \otimes \mathcal{Y}, \{0, 1\}) = \mathbf{Meas}(\mathcal{X} \otimes \mathcal{Y}, \mathcal{G}(\{0, 1\}))$$

We have built a measurable function \tilde{h} such that $\Delta = \tilde{h}^{-1}(\{\delta_1\})$.

Thus, Δ is measurable in $\mathcal{X} \otimes \mathcal{Y}$

Aumann's Lemma, revisited (thanks to Ohad Kammar and Thomas Ehrhard.)

Proposition: If W is measurable in $\mathcal{X} \otimes \mathcal{Y}$, then there is $B \subseteq \mathbb{R}$ countable such that

$$\text{If there is } (x, x') \in W \text{ such that } x' \notin B, \text{ then } \forall y \notin B, (x, y) \in W. \quad (1)$$

Proof: it is satisfied by all basic measurable sets and closed by countable union and countable intersection.

The Diagonal does not satisfy this proposition.

Proof: By contradiction, assume there is $B \subseteq \mathbb{R}$ countable satisfying (1).

Since B is countable, there are $x \notin B$ with $(x, x) \in \Delta$ and $y \notin B$ with $y \neq x$, thus $(x, y) \notin \Delta$.

Thus, B does not satisfy (1)

Contradiction: the diagonal is measurable and not measurable in $\mathcal{X} \otimes \mathcal{Y}$

Wish List for Higher-Order Probabilistic Programs with continuous distributions

Markov category:

- Measures:** $\mathcal{G}(\mathbb{R})$ to interpret close programs of type float as measures
- Stochastic kernels** (parameterized measures) to interpret programs
- Integration** to interpret sampling
- Tensor product** to interpret joint distribution
- Higher-Order**

Higher-order models of Probabilistic Programming

- Quasi-Borel Spaces: Heunen et al. 2017, “A convenient category for higher-order probability theory”
- Banach Spaces: Dahlqvist and Kozen 2020, “Semantics of higher-order probabilistic programs with conditioning”
- Cones:** Ehrhard, Pagani, and Tasson 2018b, “Measurable cones and stable, measurable functions: a model for probabilistic higher-order programming” and Ehrhard and Geoffroy 2025, “Integration in Cones”

Linear Logic and Probability

Semantics

Semantics, Probabilistic Programming and Linear Logic

Why studying semantics ?

- Beauty and nobility
- Correctness of Programs
- Sound transformation of programs
- Design new languages

Linear Logic, an inspiration for resource aware languages

In the 80's, Jean-Yves Girard recognizes, in a concrete model of the simply typed lambda-calculus, the *linear and exponential decomposition* of object of morphisms

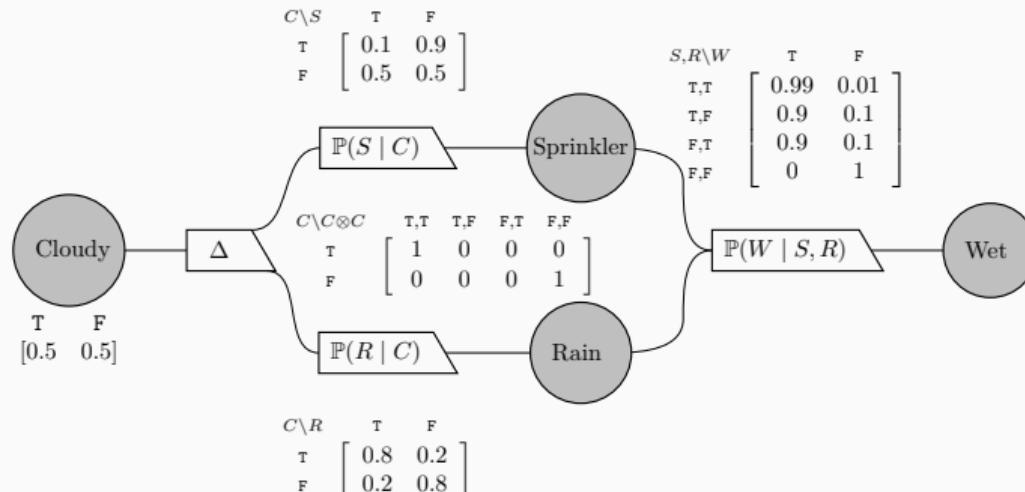
$$A \Rightarrow B \text{ versus } !A \multimap B$$

linearity and exponential are central in the semantics of Probabilistic Programming as already hinted in J.-Y. Girard 1988, "Normal functors, power series and λ -calculus".

Linear Logic and Probability

Copy, discard

Bayesian Network²: through copying³ values



$$P(C) ; \Delta ; (P(S | C) \otimes P(R | C)) ; P(W | S, R) = P(W)$$

²Pearl 1988, *Probabilistic reasoning in intelligent systems: networks of plausible inference*

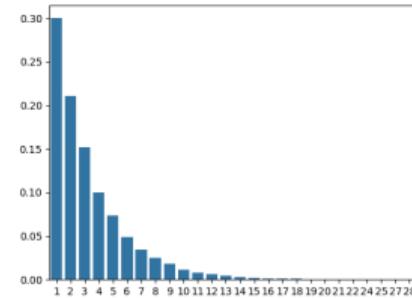
³Cho and Jacobs 2019, "Disintegration and Bayesian inversion via string diagrams"

Linear Logic and Probability

IID copies

Stopping Time: Need for bag of i.i.d. copies of a distribution

```
def StoppingTime(d) → int:  
    time = 1  
    while sample(d):  
        time = time +1  
    return time  
  
with ImportanceSampling(num_particles=10000):  
    ST: Categorical[float] = infer(StoppingTime,  
        Bernoulli(0.7))
```



Approximated Mass Function of
 $\mathbb{P}(ST(Bernoulli(0.9)))$

$ST = \text{StoppingTime}(\mu)$ is a Random Variable whose semantics is a (sub)probabilistic distribution over \mathbb{N} , with infinite support: $\text{supp}([\![ST]\!]) = \mathbb{N}$,

$$\begin{aligned} [\![\text{StoppingTime}(\mu)]\!] : \quad & \Omega_{\text{int}} \rightarrow \mathbb{R}^+ \\ & k \mapsto \mathbb{P}(ST = k \mid d = \mu) \end{aligned}$$

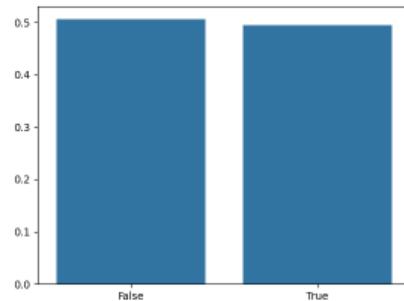
To compute the entire distribution $[\![\text{StoppingTime}(\mu)]\!]$, we sample **i.i.d. copies** of μ .

Fair Coin Example: Need for Higher-Order

```
def FairCoin(d) → bool:  
    a = sample(d)  
    b = sample(d)  
    if (a and not b):  
        return True  
    elif (b and not a):  
        return False  
    else:  
        return FairCoin(d)
```

```
with ImportanceSampling(num_particles=1000):  
    FC: Categorical[bool] = infer(FairCoin,  
                                  Bernoulli(0.3))
```

FC is a fair coin !



How can we prove it ?

Wish List for Higher-Order Probabilistic Programs

- Measures:** to interpret close programs of ground types unit, booleans, integers, reals
- Stochastic matrices or kernels** (parameterized measures) to interpret programs sampling in measures given as parameters
- Sum or Integration** to interpret sampling

$$[\![\text{let } x = \text{sample } N \text{ in } M]\!] = \sum_{a \in \mathbb{N}} [\![M]\!]_a [\![N]\!]_a \text{ or } \int_{r \in \mathbb{R}} ([\![M]\!] \circ \delta)(r) [\![N]\!](dr)$$

- Tensor product** for joint distributions
- Copy and Discard** to duplicate Samples
- Bags of i.i.d. copies** to duplicate distributions
- Higher-Order** for recursive programs and compositionality
- CPO-enriched** for loops

Linear Logic

From the probabilistic viewpoint

Origins of Probabilistic Coherence Spaces

J.-Y. Girard 1988, “Normal functors, power series and λ -calculus” define Coherence Spaces, where values of type A are given by a carrier $|X|$ and a closed program $\vdash t : A$ as a part $\llbracket t \rrbracket \in \mathcal{P}(|X|)$. An environment $x : A \vdash s : 1$ interacts deterministically with any closed program $\vdash t : A$ when:

$$\llbracket s \rrbracket \perp \llbracket t \rrbracket \iff \# \llbracket t \rrbracket \cap \llbracket s \rrbracket \leq 1.$$

J.-Y. Girard 2004, “Between Logic and Quantic: a Tract” introduces Probabilistic Coherence Spaces, as a generalization of Coherence Spaces where subsets are replaced by factors: $\llbracket t \rrbracket : |X| \rightarrow \mathbb{R}^+$. An environment $x : A \vdash s : 1$ interacts probabilistically with any closed program $\vdash t : A$ when:

$$\llbracket s \rrbracket \perp \llbracket t \rrbracket \iff \sum_{x \in |X|} \llbracket t \rrbracket_x \cdot \llbracket s \rrbracket_x \leq 1.$$

Danos and Ehrhard 2011, “Probabilistic coherence spaces as a model of higher-order probabilistic computation” gives explicit definition of all Linear Logic connectors and fixpoint of types. It studies mathematical properties of Probabilistic Coherence Spaces. It gives a model of pure lambda-calculus and PCF with binary choice and proves adequacy.

Measures of Ground types: unit, booleans and integers

The **semantics** of basic random variables is their probability distribution⁴.

$$\llbracket \vdash \text{Bernoulli}(p) : \text{bool} \rrbracket = p \cdot \delta_T + (1 - p) \cdot \delta_F \quad \llbracket \vdash \text{RandInt}(n) : \text{int} \rrbracket = \sum_{k=1}^n \frac{1}{n} \cdot \delta_k$$

The **domain** Ω_A is the set of potential output values of *terminating* probabilistic programs.

$$|1| = \{*\}$$

$$|\text{bool}| = \{\text{T}, \text{F}\}$$

$$|\text{int}| = \mathbb{N}$$

The set $P(A)$ of **random elements** is the set of sub-probability distributions.

$$P(1) = \{p \cdot \delta_* \mid p \leq 1\}$$

$$P(\text{int}) = \left\{ \sum_{k \in \mathbb{N}} x_k \cdot \delta_k \mid \sum_{k \in \mathbb{N}} x_k \leq 1 \right\}$$

$$P(\text{bool}) = \{p \cdot \delta_T + q \cdot \delta_F \mid p + q \leq 1\}$$

⚠ The probability that a probabilistic program does not terminate can be non zero. ⚠

⁴The Dirac distribution δ_a is 1 on a and 0 otherwise.

(sub)Stochastic Matrices

```
def linear(d) → bool:  
    if sample(d):  
        return True  
    else:  
        return sample(Bernoulli(0.2))
```

linear transforms a boolean distribution into a boolean distribution: $\llbracket \text{linear}(d) \rrbracket = \llbracket \text{linear} \rrbracket \cdot \llbracket d \rrbracket$, where

$$\llbracket \text{bool} \vdash \text{linear} : \text{bool} \rrbracket = \begin{pmatrix} 1 & 0.2 \\ 0 & 0.8 \end{pmatrix}$$

A **Linear Map** is a matrix $M \in (\mathbb{R}^+)^{\Omega_A \times \Omega_B}$ that preserves random elements:

$$\forall x \in P(A) \quad M \cdot x = \sum_{b \in \Omega_B} \sum_{a \in \Omega A} M_{a,b} x_a \cdot \delta_b \in P(B)$$

 Substochastic Matrices are not subprobability distributions. 

Probabilistic Coherence Spaces (PCS) from Probabilistic testing

⚠ We do **NOT** use probabilistic monad to add externally the effect to the calculus,
but probability is **intrinsic** to the model.

Probabilistic semantics $P(X)$ encodes probabilistic computation intrinsically.

$P(\text{bool})$ are subprobability boolean distributions

Ensure substochastic matrices composition with random elements is well defined, i.e. sums
are converging.

Probabilistic tests (aka orthogonality⁵): $u, u' \in (\mathbb{R}^+)^{\Omega_X}$. $u \perp u' \iff \sum_{a \in \Omega_X} u_a u'_a \leq 1$

$$P(X)^\perp = \left\{ u' \in (\mathbb{R}^+)^{\Omega_X} \mid \forall u \in P(X) \sum_{a \in \Omega_X} u_a u'_a \leq 1 \right\} = \{ u' \mid \forall u \in P(X) u' \circ u \in P(1) \}$$

⁵Reminiscent of functional analysis curves and to logical relations.

Linear Category **Pcoh**

A PCS is $X = (|X|, P(X))$ such that $|X|$ is countable, $P(X) \subseteq (\mathbb{R}^+)^{|X|}$ and $P(X)^{\perp\perp} = P(X)$,

for each $a \in |X|$ there exists $u \in P(X)$ such that $x_a > 0$ (Coverage),

for each $a \in |X|$ there exists $A > 0$ such that $\forall x \in P(X) x_a \leq A$ (Boundedness).

A morphism of PCSs from X to Y is a matrix $t \in (\mathbb{R}^+)^{|X| \times |Y|}$ which maps $P(X)$ to $P(Y)$.

$$\forall u \in P(X) \quad t u \in P(Y) \iff \forall v' \in P(Y)^\perp, \quad \sum_{(a,b) \in |X| \times |Y|} t_{a,b} u_a v'_b \leq 1.$$

Identity:

The diagonal matrix $\text{id} \in (\mathbb{R}^+)^{|X| \times |X|}$, given by $\text{id}_{a,b} = 1$ if $a = b$ and $\text{id}_{a,b} = 0$ otherwise. $A \vdash A$

Composition

Matrix multiplication, let $s \in \mathbf{Pcoh}(X, Y)$ and $t \in \mathbf{Pcoh}(Y, Z)$,
then $(t s)_{a,c} = \sum_{b \in |Y|} s_{a,b} t_{b,c}$.

$$\frac{\Gamma \vdash B \quad \Delta, B \vdash A}{\Gamma, \Delta \vdash A}$$

Linear Logic

Multiplicative fragment^a

^a J. Girard 1987, "Linear Logic"

Pcoh is Symmetric Monoidal Closed: model of intuitionistic multiplicative LL

Joint Distributions

```
def joint() → int x int:  
    a = sample(x)  
    b = sample(y)  
    return (a, b)
```

is interpreted by the joint distribution:

$$x \otimes y = \sum_{\substack{a \in \Omega_X \\ b \in \Omega_Y}} x_a y_b \cdot \delta_{(a,b)}$$

$$\frac{}{\vdash 1} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

$$\frac{\Gamma \vdash A}{\Gamma, 1 \vdash A} \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

Tensor Product (*Associativity, Symmetry, unitor*)

$$|X \otimes Y| = |X| \times |Y|$$

$$P(X \otimes Y) = \{u \otimes v \mid u \in P(X) \text{ and } v \in P(Y)\}^{\perp\perp}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

Closed Structure (*Evaluation and Curryfication*)

$$|X \multimap Y| = |X| \times |Y|$$

$$P(X \multimap Y) = \left\{ t \in (\mathbb{R}^+)^{|X| \times |Y|} \mid \forall u \in P(X) \ t u \in P(Y) \right\}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

$$\frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

Additive fragment and examples of Base Types

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \times B}$$

$$\frac{}{\Gamma \vdash \top}$$

Product: $|\top| = \emptyset$

$$\left| \prod_{i \in I} X_i \right| = \bigcup_{i \in I} \{i\} \times |X_i| \quad P\left(\prod_{i \in I} X_i\right) = \{u \mid \forall i \in I \ u(i) \in P(X_i), \text{ where } \forall a \in |X_i| \ u(i)_a = u_{(i,a)}\}$$

Enumeration: $\text{bool} = 1 \oplus 1$.

$$\left| \bigoplus_{i \in I} X_i \right| = \bigcup_{i \in I} \{i\} \times |X_i| \quad P\left(\bigoplus_{i \in I} X_i\right) = \left\{ u \mid \forall i \in I \ u(i) \in P(X_i) \text{ and } \sum_{i \in I} \|u(i)\|_{X_i} \leq 1 \right\}$$

Type Fixpoints: $\text{int} = 1 \oplus \text{int}$.

PCSs is a CPO with least element $0 = \top$ with $|0| = \emptyset$ when ordered by

$$X \subseteq Y \quad \text{iff} \quad |X| \subseteq |Y| \quad \text{and} \quad P(X) = \{v_{||X|} \mid v \in P(Y)\}$$

Properties of $P(X)$

$P(X)$ is unitary $\|u\|_X \in [0, 1]$ for all $u \in P(X)$ where the norm is defined as

$$\|u\|_X = \sup \left\{ \sum_{a \in |X|} u_a u'_a \mid u' \in P(X^\perp) \right\}$$

$P(X)$ is a cone

$$\forall u, v \in P(X) \forall \alpha, \beta \in \mathbb{R}^+ \quad \alpha + \beta \leq 1 \Rightarrow \alpha u + \beta v \in P(X).$$

$P(X)$ is an ω -continuous domain where the partial order is defined as

$$u \leq v \quad \text{iff} \quad \forall a \in |X| \quad u_a \leq v_a \in \mathbb{R}^+.$$

\mathbf{Pcoh} is CPO enriched indeed, $\mathbf{Pcoh}(X, Y) = P(X \multimap Y)$ is a CPO with 0 as least element.

Back to the StoppingTime Example

```
def StoppingTime(d) → int:  
    time = 1  
    while sample(d):  
        time = time +1  
    return time
```

Let ST be $\text{StoppingTime}(\text{Bernoulli}(p))$.

After k iterations,

$$[\![\text{ST}]\!]^1 = (1 - p)\delta_1$$

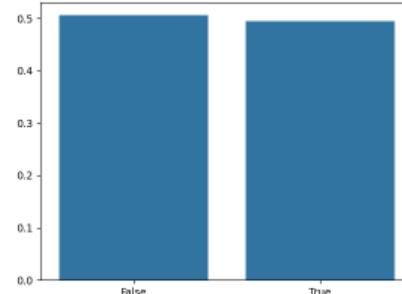
$$[\![\text{ST}]\!]_t^{k+1} = p [\![\text{ST}]\!]_{t-1}^k + (1 - p) [\![\text{ST}]\!]_t^k$$

$[\![\text{ST}]\!]$ is the lub of the increasing sequence $[\![\text{ST}]\!]^k$ in $P(\text{int}) \subseteq (\mathbb{R}^+)^{\mathbb{N}}$

Fair Coin Example: Need for Higher-Order

```
def FairCoin(d) → bool:  
    a = sample(d)  
    b = sample(d)  
    if (a and not b):  
        return True  
    elif (b and not a):  
        return False  
    else:  
        return FairCoin(d)
```

```
with ImportanceSampling(num_particles=1000):  
    FC: Categorical[bool] = infer(FairCoin,  
                                  Bernoulli(0.3))
```



Let FC be $\text{FairCoin}(\text{Bernoulli}(p))$.
Recursive Equation

$$\begin{aligned}\mathcal{F}(t) &= p(1 - p)(\delta_T + \delta_F) \\ &\quad + (1 - 2(p(1 - p))) t \\ \llbracket \text{FC} \rrbracket &= \frac{1}{2}(\delta_T + \delta_F)\end{aligned}$$

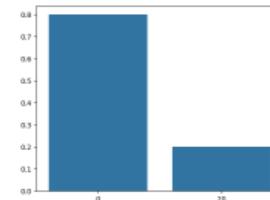
when $p \notin \{0, 1\}$

Linear Logic

Exponential fragment

Compare Linear vs Non-Linear

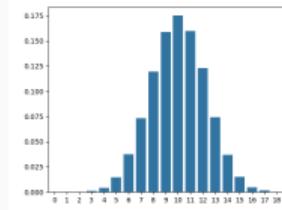
```
def FlipSum(n: int, d) → int:  
    x = sample(d)  
    sum = 0  
    for k in range(n):  
        sum += x  
    return sum
```



$$[\![\text{FlipSum}(n, \text{Bernoulli}(p))]\!] = (1 - p) \cdot \delta_0 + p \cdot \delta_n$$

`FlipSum(n, Bernoulli(p))` samples once Bernoulli(p) and copy its outcomes n times.

```
def RandomWalk(n: int, d) → int:  
    s = 0  
    for k in range(n):  
        s += sample(d)  
    return s
```



$$[\![\text{RandWalk}(n, \text{Bernoulli}(p))]\!] = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} \cdot \delta_k$$

`RandomWalk(n, Bernoulli(p))` samples n i.i.d. copies of Bernoulli(p).

Exponential of measures

The **exponential** $!X$ of a ground type represents the type of bags of *i.i.d.* copies of measures.

$$x^! = \sum_{m \in \mathfrak{M}_{\text{fin}} \Omega X} \prod_{a \in \text{supp}(m)} x_a^{m(a)} \cdot \delta_m$$

For instance,

$$[\![\text{Bernoulli}(p)]\!]^! = (p \cdot \delta_T + (1 - p) \cdot \delta_F)^! = \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} p^k (1 - p)^{n-k} \cdot \delta_{[T \mapsto k, F \mapsto n-k]}$$

Exponential PCS: $\Omega_{!X} = \mathfrak{M}_{\text{fin}} \Omega_X$ and $P(!X) = \{x^! \mid x \in P(X)\}^{\perp\perp}$.

The exponential comonad in \mathbf{Pcoh}

$$\frac{! \Gamma \vdash A}{! \Gamma \vdash !A}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

Comonad: $!$ with counit $\text{der } X \in \mathbf{Pcoh}(!X, X)$ and comultiplication $\text{dig}^X \in \mathbf{Pcoh}(!X, !!X)$.

Promotion: For $t \in \mathbf{Pcoh}(!X, Y)$, $t^! \in \mathbf{Pcoh}(!X, !Y)$ is defined such that

$$t^! u^! = (t u)^!$$

Strength given by isomorphisms from (\mathbf{Pcoh}, \times) to (\mathbf{Pcoh}, \otimes) ,

$$m^0 \in \mathbf{Pcoh}(!\top, 1) \quad \text{and} \quad m^2 \in \mathbf{Pcoh}(!!(X \& Y), !X \otimes !Y)$$

$!X$ is a Commutative Comonoid.

$$\text{contr}^{!X} \in \mathbf{Pcoh}(!X, !X \otimes !X) \quad \text{and} \quad \text{weak}^{!X} \in \mathbf{Pcoh}(!X, 1)$$

Eilenberg Moore and Kleisli Categories: copy, delete and i.i.d. copies

Eilenberg-Moore: A *Coalgebra* is a PCS Q with $h \in \mathbf{Pcoh}(Q, !Q)$ compatible with the $!$ -comonad structure.

Every coalgebra Q comes with marginalization, copy and delete

$$\pi_Q \in \mathbf{Pcoh}(Q \otimes Q', Q) \quad \text{and} \quad \Delta_Q \in \mathbf{Pcoh}(Q, Q \otimes Q) \quad \text{and} \quad \omega_Q \in \mathbf{Pcoh}(Q, 1)$$

Value types are interpreted as coalgebras

$$\varphi, \psi := \text{unit} \mid !\sigma \mid \varphi \otimes \psi \mid \varphi \oplus \psi \mid \zeta \mid \text{Rec } \zeta \varphi$$

Kleisli Category $\mathbf{Pcoh}_!$ with PCSs as objects and morphisms $\mathbf{Pcoh}_!(X, Y) = \mathbf{Pcoh}(!X, Y)$ are power series: $t \in \mathbf{Pcoh}_!(X, Y)$ iff $t \in (\mathbb{R}^+)^{\mathfrak{M}_{\text{fin}}(X) \times Y}$ and

$$\forall u \in P(X) \quad t(u) = t \ u^! = \left(\sum_{m \in \mathfrak{M}_{\text{fin}}(X)} t_{m,b} \prod_{a \in |X|} u_a^{m(a)} \right)_{b \in |Y|} \in P(Y)$$

A non-linear morphism creates an unbound bag of i.i.d. copies of its input distribution.

Linear Logic

Semantics of Probabilistic Programming

Models of Linear Logic⁶

One axiomatization:

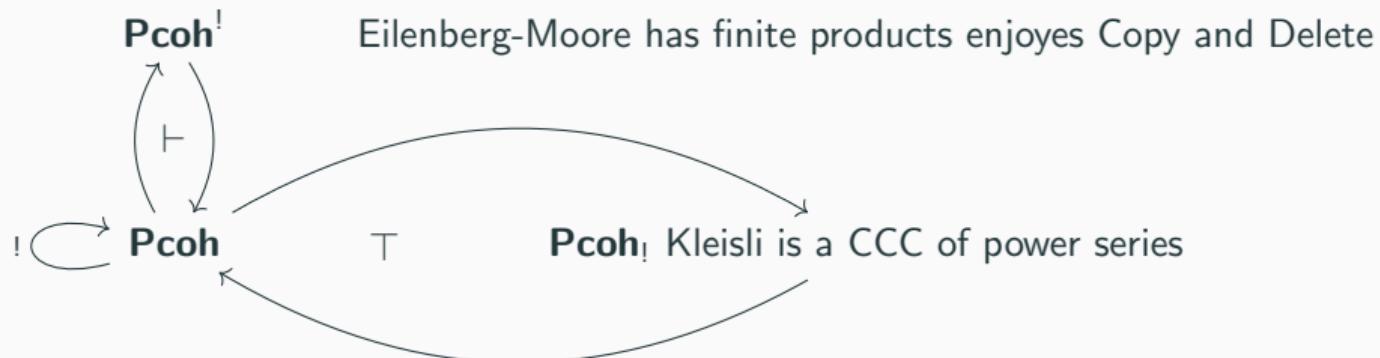
Linear Category **Pcoh**

$(\mathbf{Pcoh}, 1, \otimes, \multimap)$ is an SMCC

$(\mathbf{Pcoh}, 0, \times)$ has finite product

A comonad ! which is strong monoidal from $(\mathbf{Pcoh}, 0, \times)$ to $(\mathbf{Pcoh}, 1, \otimes)$

Adjunctions:



⁶ Melliès 2009, "Categorical semantics of linear logic"

Pcoh models probabilistic programming

Syntaxes

Probabilistic Call-By-Push-Value.

Probabilistic Call-By-Value. $A \rightarrow B$ encoded by $!(A \multimap B)$.

Probabilistic Call-By-Name. $A \Rightarrow B$ encoded by $(!A) \multimap B$.

Probabilistic PCF. Fixpoint operator: $\mathcal{Y} \in \mathbf{Pcoh}_!(X \Rightarrow X, X)$

Probabilistic Untyped Calculus. The reflexive object satisfies $D = (!D^{\mathbb{N}})^{\perp}$.

Soundness

$$\llbracket M \rrbracket = \sum_{M'} \mathbb{P}(M \rightarrow M') \llbracket M' \rrbracket$$

Adequate and Fully Abstract for pPCF⁷ and pCBPV⁸.

⁷ Ehrhard, Pagani, and Tasson 2018a, “Full Abstraction for Probabilistic PCF”

⁸ Ehrhard and Tasson 2019, “Probabilistic call by push value”

Continuous distributions

Measurable space and kernels

Wish List for Higher-Order Probabilistic Programs

- Measures:** to interpret close programs of ground types unit, booleans, integers, reals
- Stochastic kernels** (parameterized measures) to interpret programs sampling in measures given as parameters
- Integration** to interpret sampling

$$[\![\text{let } x = \text{sample } N \text{ in } M]\!] = \sum_N [\![M]\!]_a [\![N]\!]_a \text{ or } \int_{\mathbb{R}} ([\![M]\!] \circ \delta)(r) [\![N]\!](dr)$$

- Tensor product** to interpret joint distributions
- Copy and Discard** to duplicate Samples
- Bags of i.i.d. copies** to duplicate distributions at will
- Higher-order** for recursive programs

Markov Kernels (History)

Lawvere 1962, “The category of probabilistic mappings” introduces stochastic kernels to denote temporally discrete Markov processes.

Kozen 1979, “Semantics of Probabilistic Programs” proposes stochastic kernels between measurable spaces to denote operational semantics of probabilistic programs.

Giry 1982, “A categorical approach to probability theory” describes Markov kernels as the Kleisli Category of the probabilistic monad.

Panangaden 1998, “The Category of Markov Kernels” relates Markov Kernels and Relations.

Markov Kernels (Definition)

Measurable space: $\mathcal{X} = (\Omega_{\mathcal{X}}, \mathcal{F}_{\mathcal{X}})$

$\Omega_{\mathcal{X}}$ set of possible experiment outcomes, with $\mathcal{F}_{\mathcal{X}}$ the σ -algebra of measurable events.

Stochastic kernel: $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ is a function $\kappa : \Omega_{\mathcal{X}} \times \mathcal{F}_{\mathcal{Y}} \rightarrow \mathbb{R}^+$ such that

$\forall x \in \mathcal{X}, \kappa(\cdot|x) : \mathcal{F}_{\mathcal{Y}} \rightarrow \mathbb{R}^+$ is a measure

$\forall V \in \mathcal{F}_{\mathcal{Y}}, \kappa(V| \cdot) : \mathcal{X} \rightarrow \mathbb{R}^+$ is a measurable function

Composition: $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $\kappa' : \mathcal{Y} \rightsquigarrow \mathcal{Z}$.

$$\forall x \in \Omega_{\mathcal{X}} \quad \forall W \in \mathcal{F}_{\mathcal{Z}}, \quad \kappa' \circ \kappa(W|x) = \int_{\mathcal{Y}} \kappa'(W|y) \kappa(dy|x)$$

Markov Kernel (Monad of Finite Measures)

Finite Measures: Let \mathcal{X} be a measurable space.

$\mathcal{G}(\mathcal{X})$ is the set of finite measures μ

Giry Monad

$\mathcal{G}(\mathcal{X})$ is a measurable space when endowed with the σ -algebra generated by the evaluations on measure sets $\delta_U : \mu \mapsto \mu(U)$: $\delta_U^{-1}(a, b)$ for $a, b \in \mathbb{R}^+$.

Remark: $\mathcal{G}(\{0, 1\}) = \{p\delta_0 + (1 - p)\delta_1 \mid 0 \leq p \leq 1\}$

with $\{\delta_1\}$ and $\{p\delta_0 + (1 - p)\delta_1 \mid p \in [0, 1] \cap [a, b]\}$ measurable sets.

Kernel and Giry: $\mathbf{Kern}(\mathcal{X}, \mathcal{Y}) = \mathbf{Meas}(\mathcal{X}, \mathcal{G}(\mathcal{Y}))$

Kernels are measures transformers thanks to the giry monad bind

$$\forall \mu \in \mathcal{G}(\mathcal{X}) \quad \kappa \cdot \mu : W \mapsto \int \kappa(W|x) \mu(dx) \in \mathcal{G}(\mathcal{Y})$$

Markov Kernel (Symmetric Monoidal, Not Closed)

Monoidal unit:

The reals $\Omega_1 = \{*\}$.

Monoidal product

$$\Omega_{\mathcal{X} \otimes \mathcal{Y}} = \Omega_{\mathcal{X}} \times \Omega_{\mathcal{Y}}$$

$\mathcal{F}_{\mathcal{X} \otimes \mathcal{Y}}$ generated by squares of measure sets $U \times V$ for $U \in \mathcal{F}_{\mathcal{X}}$ and $V \in \mathcal{F}_{\mathcal{Y}}$

Product of kernels

$$\kappa \otimes \kappa'(U \times V|x, y) = \kappa(U|x) \kappa'(V|y)$$

Aumann's Lemma implies that Markov kernel is not closed as there is no measurability structures that can be put on real measurable functions such that the evaluation is measurable.

Continuous distributions

Integrable cones

Cones to account for Measures

Following Selinger, we define a cone endowed with the structure inspired by the space of measures $\mathcal{G}(\mathcal{X})$ on a measurable space \mathcal{X} , with a norm $\|\mu\|_{\mathcal{G}(\mathcal{X})} = \mu(\mathcal{X})$.

Algebraic structure: A cone P is an \mathbb{R}^+ semi-module such that:

$$x + y = 0 \Rightarrow x, y = 0 \quad x + y = x + y' \Rightarrow y = y'$$

Total variation: A cone is equipped with a norm $\|-\|_P : P \rightarrow \mathbb{R}^+$ such that:

$$\|x + x'\|_P \leq \|x\|_P + \|x'\|_P, \quad \|\alpha x\|_P = \alpha \|x\|_P \quad \|x\|_P = 0 \Rightarrow x = 0, \quad \|x\|_P \leq \|x + x'\|_P$$

Probability: The unit ball $\mathcal{B}P = \{x \in P \mid \|x\|_P \leq 1\}$ account for (sub)probability distribution.

Complete: A cone is complete for the order: $x \leq_P x'$ if there is $y \in P$ such that $x' = x + y$.
For any monotone $(x_n)_{n \in \mathbb{N}}$ with $\forall n \ \|x_n\|_P \leq 1$ has a lub $x = \sup_{n \in \mathbb{N}} x_n$ such that $\|x\|_P \leq 1$.

Examples: $1 = \mathbb{R}^+$

$\mathcal{G}(\mathcal{X})$ is the cone of finite measures over a measurable space \mathcal{X} .

If \mathcal{X} is a PCS, then $\widehat{\mathcal{X}} = \{x \in (\mathbb{R}^+)^{|\mathcal{X}|} \mid \exists \alpha > 0 \ \alpha x \in \mathcal{P}(\mathcal{X})\}$ is a cone.

Linear maps to account for Kernels: ICone(P, Q)

Following PCSs, we define linear maps inspired by stochastic matrices and kernels.

Linear: a function $f : P \rightarrow Q$ is linear if $f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$.

Stochastic: A substochastic function is a linear function $f : P \rightarrow Q$ such that $\|f\| \leq 1$, where

$$\|f\| = \sup_{x \in \mathcal{B}P} \|f(x)\|_Q \in \mathbb{R}^+$$

Continuous: If $f : P \rightarrow Q$ is linear, then it is bounded. A **continuous** function is such that for all monotone and bounded $(x_n \in P)_{n \in \mathbb{N}}$, $f(\sup_{n \in \mathbb{N}} x_n) = \sup_{n \in \mathbb{N}} f(x_n)$.

Example: Every stochastic kernel $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ between measurable spaces induces a linear stochastic continuous function between the cones of measures:

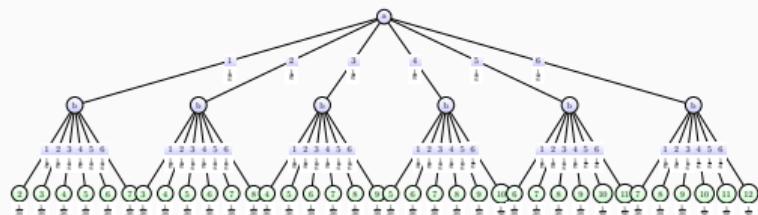
$$\begin{aligned}\hat{\kappa} : \mathcal{G}(\mathcal{X}) &\rightarrow \mathcal{G}(\mathcal{Y}) \\ \mu &\mapsto \lambda U. \int_{x \in \Omega_{\mathcal{X}}} \kappa(U | x) \mu(dx)\end{aligned}$$

Continuous distributions: Random paths

```
def sum(d) → int:  
    a = sample(d)  
    b = sample(d)  
    return a + b
```

```
plt.show()  
  
def SqDist() → float:  
    x = sample(Uniform(-1, 1))
```

Measure: Weighted Random traces:



Sum over all random traces:

$$[\text{Dice}]_k = \sum_{1 \leq a, b \leq 6} \frac{1}{36} \delta_{a+b}(k)$$

Measures^a: Random paths and real measures:

$$\alpha : r_1, r_2 \mapsto \delta_{r_1^2 + r_2^2}$$

$$\mu(dr_1 \otimes dr_2) = \mathbb{1}_{[-1,1]}(r_1)dr_1 \mathcal{N}(0, 1)(r_2)dr_2$$

Integration along the path:

$$[\text{SqDist}] (U) = \int_{\substack{r_1 \in [0,1] \\ r_2 \in \mathbb{R}}} \mathcal{N}(0, 1)(r_2) \delta_{r_1^2 + r_2^2}(U) dr_1 dr_2$$

^aReminiscent of QBS random elements and measures.

Random Paths and Measurability tests

Examples: If \mathcal{X} is a PCS, $P(\mathcal{X}) = P(\mathcal{X})^{\perp\perp}$ implies that measurability is determined by tests against $P(\mathcal{X})^{\perp} = P(\mathcal{X}) \multimap 1$.

Random path: $\text{Path}(\mathbb{R}^\ell, P)$: $\gamma : \mathbb{R}^\ell \rightarrow P$ functions such that for any k -measurability test m :

$$(s, r) \mapsto m(\gamma(r) \mid s) \in \mathbf{Meas}(\mathbb{R}^k \times \mathbb{R}^\ell, 1)$$

Measurability tests: $m : \mathbb{R}^k \times P \multimap 1$ with k random sources such that:

$$\forall r \in \mathbb{R}^k \quad m(_ \mid r) \in \mathbf{Cone}(P, \mathbb{R}^+) \quad \forall x \in P \quad m(x \mid _) \in \mathbf{Meas}(\mathbb{R}^n, \mathbb{R}^+)$$

Measurability structure $\mathcal{M} = (\mathcal{M}_k)_{k \in \mathbb{N}}$ a collection of measurability tests such that:

Precomposition: if $m \in \mathcal{M}_n$ and $\varphi \in \mathbf{Meas}(\mathbb{R}^k, \mathbb{R}^n)$, then $m(\varphi(_) \mid _) \in \mathcal{M}_k$

Separation: if $x \neq y \in P$, there is $m \in \mathcal{M}_0$ such that $m(x) \neq m(y)$

Norm: $\|x\|_P = \sup_{m \in \mathcal{M}_0} \left(\frac{m(x)}{\|m\|} \right)$

Measurable functions and Integration⁹: ICone

Measurable function: linear continuous functions $f : P \rightarrow Q$ preserving random paths:

$$\forall \gamma \in \text{Path}(\mathbb{R}^\ell, P) \quad f \circ \gamma \in \text{Path}(\mathbb{R}^\ell, Q)$$

Measure: $[\mu, \alpha]_d$ where $\mu \in \mathcal{G}(\mathbb{R}^\ell)$ and $\gamma \in \text{Path}(\mathbb{R}^\ell, P)$ is **integrable** iff there is

$$x = \int \alpha(r)\mu(dr) \quad \text{such that} \quad \forall x' \in P \rightharpoonup \mathbb{R}^+ \quad \langle x', x \rangle = \int_{\mathbb{R}^+} x' \circ \alpha(r)\mu(dr)$$

Integrable Cone C : complete cone with measurability structure, where all $[\mu, \alpha]_d$ are integrable.

Integrable linear map of cones: measurable functions $f \in \text{ICone}(C, D)$ preserving integrals:

$$f \left(\int \alpha(r)\mu(dr) \right) = \int f(\alpha(r))\mu(dr)$$

Example: $\mathcal{G}(\mathbb{R}^\ell)$ is integrable. If $\kappa \in \text{Path}(\mathbb{R}^\ell, \mathcal{G}(\mathbb{R}^k)) = \text{Kern}(\mathbb{R}^\ell, \mathbb{R}^k)$ and $\mu \in \mathcal{G}(\mathbb{R}^\ell)$, then $\nu = \int \kappa(- | s)\mu(ds) \in \mathcal{G}(\mathbb{R}^k)$

⁹Pettis 1938, "On Integration in Vector Spaces"

Wish List for Higher-Order Probabilistic Programs with continuous distributions

IConc, a solution to the measurability problem

- Measures:** $\mathcal{G}(\mathbb{R})$ to interpret close programs of type float as measures
- Stochastic kernels** (parameterized measures) to interpret programs sampling in measures given as parameters
- Integration** to interpret sampling in real distributions

$$[\![\text{let } x = \text{sample } d \text{ in } t]\!] = \int_{\mathbb{R}} (f \circ \delta)(r) \mu(dr)$$

- Tensor product** to interpret joint distribution
- SMCC** to account for Higher-Order
- Copy and Discard** of distributions
- Bags of i.i.d. copies** of distributions

Linear, stable and non-linear maps of integrable cones

Let P and Q be integrable cones.

Internal Linear Hom: $P \multimap Q$ the complete cone of linear, continuous, measurable and integrable functions with $\|f\|_{P \multimap Q} = \sup_{x \in \mathcal{B}P} \|f(x)\|_Q$. Equipped with the measurable structure of Q and random paths of P by pre and postcomposition.

Integrals are defined pointwise: for any path $\gamma : \mathbb{R}^k \rightarrow P \multimap Q$, $f = \int_{P \multimap Q} \gamma d\mu$ is such that

$$f(x) = \int_{\mathbb{R}^k} \gamma(x | r) \mu(dr)$$

Stable maps continuous totally monotone (ultra-convex).

Analytic maps $\lambda x. \sum_{n=0}^{\infty} f_n(x, \dots, x)$ such that f_n is n -linear continuous, measurable and separately integrable of norm ≤ 1 .

Internal Non-Linear Hom: $P \Rightarrow_a Q$ the complete cone of analytic maps is an integrable cone.

Linear Logic model:

ICone is an SMCC: Integrable cones and linear maps is complete.

$P \multimap -$ preserves all limits. By the Special Adjoint Functor Theorem, it has a left adjoint, the tensor product $- \otimes P$.

ACone is a CCC as the evaluation and curryfication are analytic.

Exponential: the embedding $\mathbf{ICone} \hookrightarrow \mathbf{ACone}$ preserves all limits. By the SAFT, it has a left adjoint.

The *adjunction* induces a comonad $!$ on **ICone** together with

a unit: $\eta \in \mathbf{ACone}(P, !P)$ a counit: $\varepsilon \in \mathbf{ICone}(!P, P)$ a comultiplication: $\vartheta : !P \rightarrow !!P$

$$\begin{array}{ccc} ! & \curvearrowright & \mathbf{ICone} & \curvearrowright & \mathbf{ACone} \\ & \swarrow & & \downarrow \tau & \searrow \\ & & & & \end{array}$$

The comonad $(!, \varepsilon, \vartheta)$ is strong monoidal, thus:

$$\omega \in \mathbf{ICone}(!P, 1) \quad \Delta \in \mathbf{ICone}(!P, !P \otimes !P)$$

Copy, delete and bags of i.i.d. copies

Eilenberg-Moore !-coalgebras (P, h) are symmetric comonoids: there are coalgebraic morphisms

$$\omega \in \mathbf{ICone}(C, 1) \quad \Delta \in \mathbf{ICone}(C, C \otimes C)$$

Example: Bags of *i.i.d.* copies of distributions.

For any measurable space \mathcal{X} , $\mathcal{G}(\mathcal{X})$ is a !-coalgebra, with

$$\begin{aligned}\mathcal{G}(\mathcal{X}) &\rightarrow !\mathcal{G}(\mathcal{X}) \\ \mu &\mapsto \int_{R^k} \eta \circ \delta(r) \mu(dr)\end{aligned}$$

It is well defined as the dirac $\delta \in \mathbf{Path}(\mathbb{R}^k, \mathcal{G}(\mathcal{X}))$ and $\eta \in \mathbf{ACone}(\mathcal{G}(\mathcal{X}), !\mathcal{G}(\mathcal{X}))$ is measurable, thus $\eta \circ \delta \in \mathbf{Path}(\mathbb{R}^k, !\mathcal{G}(\mathcal{X}))$.

Wish List for Higher-Order Probabilistic Programs

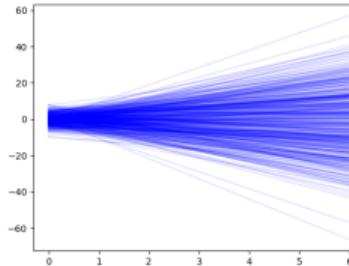
- Measures:** to interpret close programs of ground types unit, booleans, integers, reals
- Stochastic matrices or kernels** (parameterized measures) to interpret programs sampling in measures given as parameters
- Sum or Integration** to interpret sampling

$$[\![\text{let } x = \text{sample } N \text{ in } M]\!] = \sum_{a \in \mathbb{N}} [\![M]\!]_a [\![N]\!]_a \text{ or } \int_{r \in \mathbb{R}} ([\![M]\!] \circ \delta)(r) [\![N]\!](dr)$$

- Tensor product** for joint distributions
- Copy and Discard** to duplicate Samples
- Bags of i.i.d. copies** to duplicate distributions
- Higher-Order** for recursive programs and compositionality
- CPO-enriched** for loops

Example - Higher-Order

```
def model():
    m = sample(Gaussian(0.0, 3.0))
    b = sample(Gaussian(0.0, 3.0))
    f = lambda x: m*x +b
    return f
```



Measure space: \mathbb{R}^2 with borelians

Probability $\mu \in \text{Meas}(\mathbb{R}^2)$: $m, b \sim \mathcal{N}(0, 2) \otimes \mathcal{N}(0, 2)$

Random variable: $\alpha : (m, b) \mapsto \lambda x. m * x + b \in \text{Path}(\mathbb{R}^2, \mathbb{R} \Rightarrow_a \mathbb{R})$

Distribution: $[\![model]\!] = \int_{r \in \mathbb{R}^2} \alpha(r) \mu(dr)$ is in $\mathbb{R} \Rightarrow_a \mathbb{R}$

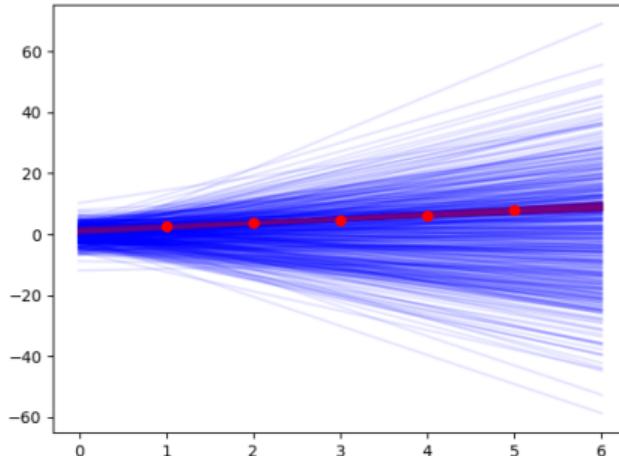
$$[\![\text{let } fs = \text{sample}(\text{dist}) \text{in } 2*fs]\!] = \int_{\mathbb{R}^2} 2 * \alpha(m, b) \mu(dm, db)$$

$$[\![\text{let } fs = \text{sample}(\text{dist}) \text{in } 2*fs]\!](4) = \int_{\mathbb{R}^2} (4 * m + b) \mathcal{N}(0, 2)(dm) \mathcal{N}(0, 2)(db)$$

Linear Regression¹⁰

```
def model(data):
    m = sample(Gaussian(0.0, 3.0))
    b = sample(Gaussian(0.0, 3.0))
    f = lambda x: m*x + b
    for (x, y) in data:
        observe(Gaussian(f(x), 0.5), y)
    return f

data = [(1.0, 2.5), (2.0, 3.8), (3.0, 4.5),
        (4.0, 6.2), (5.0, 8.0)]
```



¹⁰Heunen et al. 2017, “A convenient category for higher-order probability theory”

Aumann's Lemma: Higher-order and effects is subtle mixture.

Probabilistic Coherent Spaces is a great model to study higher-order probabilistic programming with countable distributions.

Cones is a great model to study higher-order probabilistic programming with continuous distributions.

Linear Logic is a great tool box to understand resources.

Semantics helps understanding mathematical structure of computations leading to the design of elegant programming languages.

Bibliography

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