Probabilistic Programming Languages

Guillaume Baudart

MPRI 2025-2026

Warm-up: Hidden Markov Model

Probabilistic Programming Languages

HMM: Hidden Markov Model

Track the position of an agent from noisy observations

- The current position should not be too far from the previous position
- The observations should not be too far from the current position

Probabilistic model: $\forall t \in \mathbb{N}$

- $x_t \sim \mathcal{N}(x_{t-1}, \textit{speed})$
- $y_t \sim \mathcal{N}(x_t, \textit{noise})$

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`Try it in BYO-PPL!`

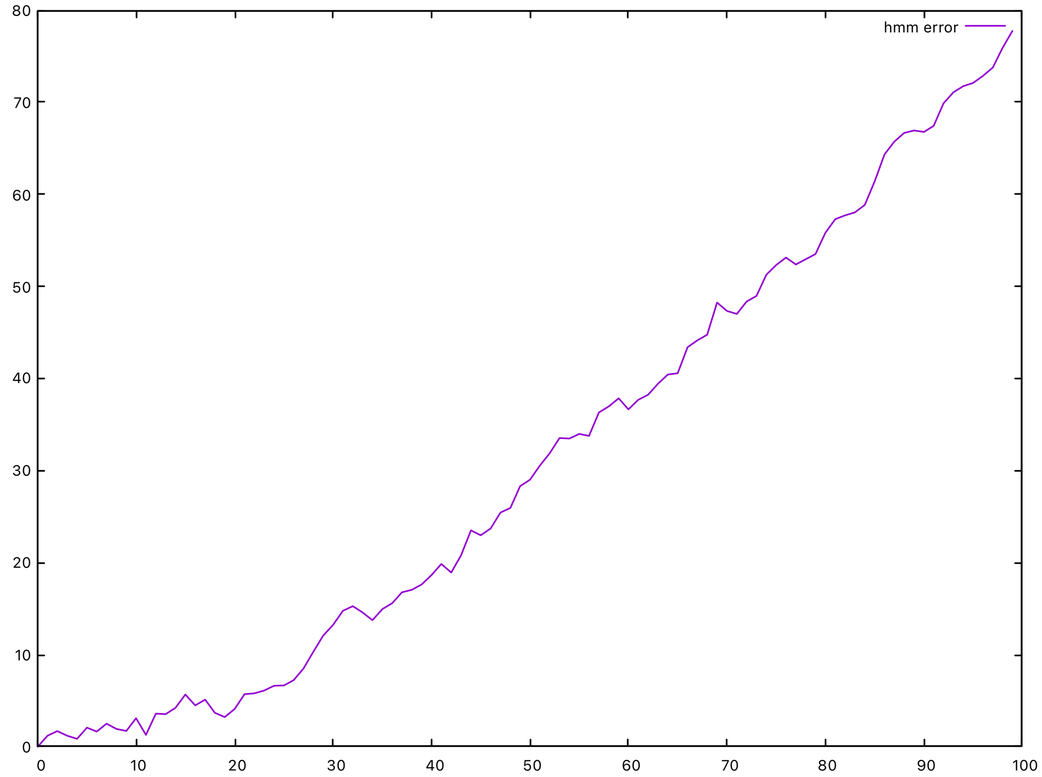
HMM: Hidden Markov Model

hmm.ml

```
open Basic.Importance_sampling

let hmm prob data =
  let rec gen states data =
    match (states, data) with
    | [], y :: data → gen [ y ] data
    | states, [] → states
    | pre_x :: _, y :: data →
        let x = sample prob (gaussian ~mu:pre_x ~sigma:1.0) in
        let () = observe prob (gaussian ~mu:x ~sigma:1.0) y in
        gen (x :: states) data
  in
  gen [] data

let _ =
  let data = List.init 20 (fun i → Float.of_int i) in
  let dist = Distribution.split_list (infer hmm data) in
  let m_x = List.rev (List.map Distribution.mean dist) in
  List.iter2 (Format.printf "%f >> %f@.") data m_x
```

HMM: Hidden Markov Model

```
❯ dune exec ./examples/hmm.exe

0.000000 >> 0.000000
1.052632 >> 0.278989
2.105263 >> 2.923428
3.157895 >> 2.812035
4.210526 >> 2.328341
5.263158 >> 1.742109
6.315789 >> 2.518105
7.368421 >> 3.958375
8.421053 >> 5.946233
9.473684 >> 7.329554
10.526316 >> 9.293653
11.578947 >> 10.181831
12.631579 >> 8.549409
13.684211 >> 9.323073
14.736842 >> 9.280692
15.789474 >> 9.352218
...
```



$\texttt{infer} : (\alpha \to \beta) \to \alpha \to \beta\ \texttt{dist}$

infer : $(\alpha \to \beta) \to \alpha \to \beta$ dist

program

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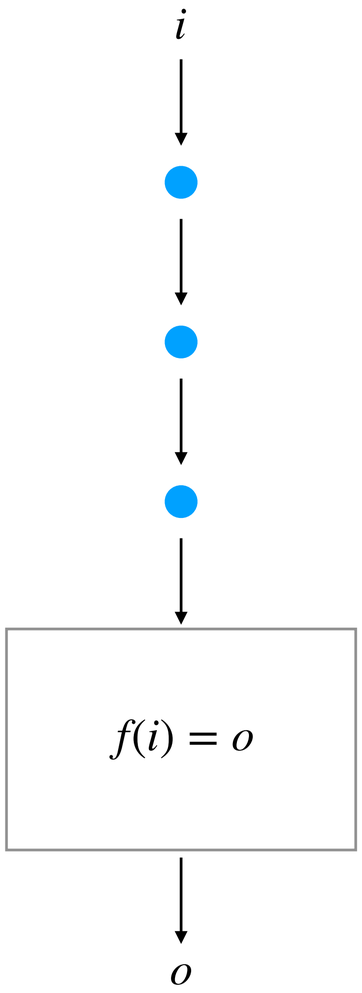

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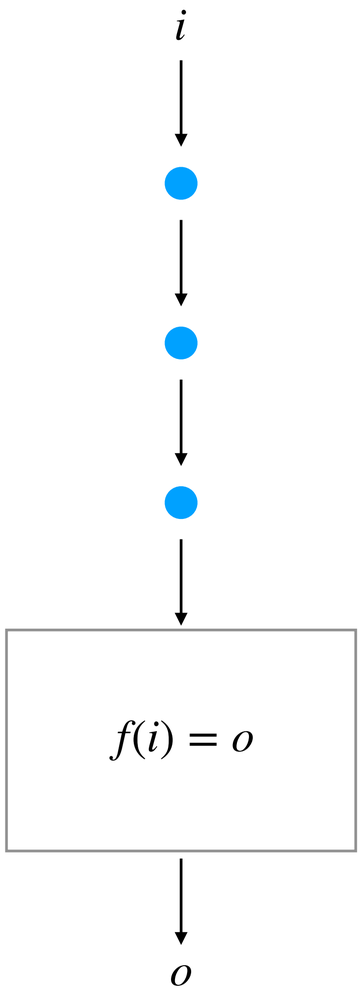

sample

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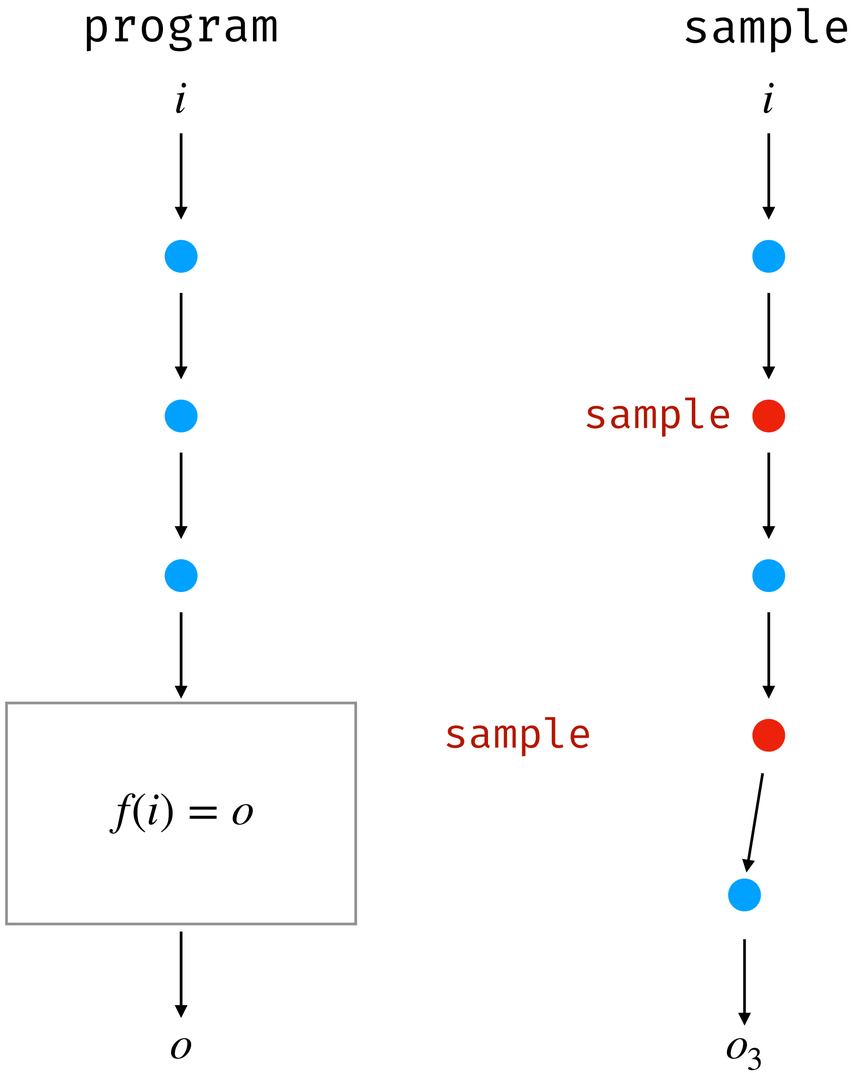

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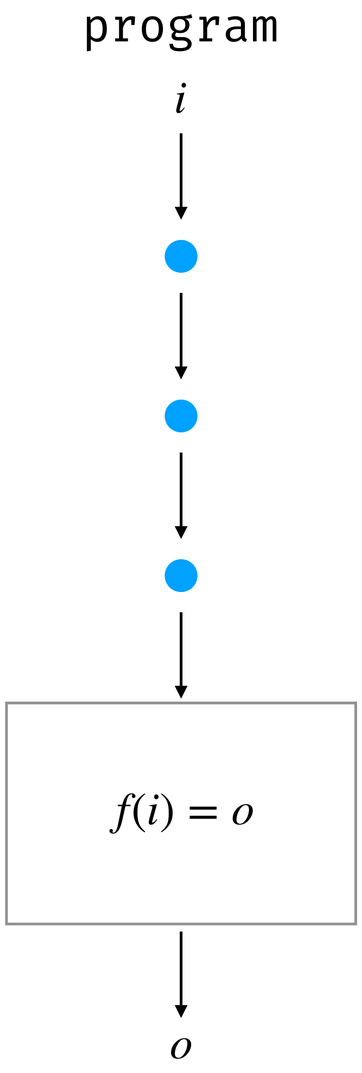

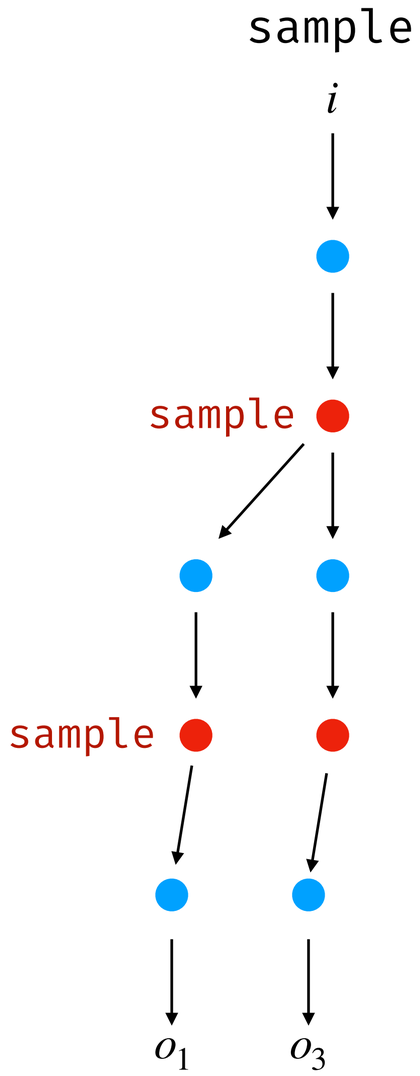

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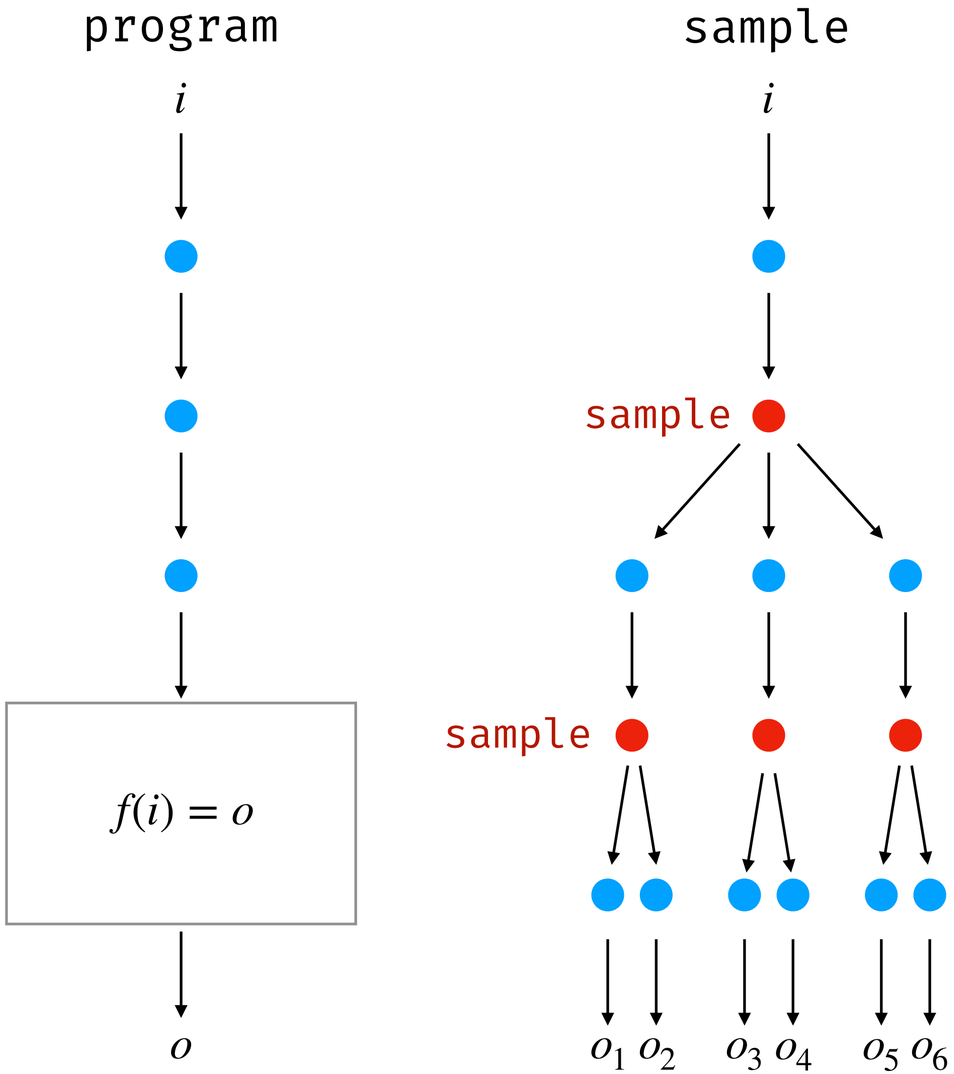

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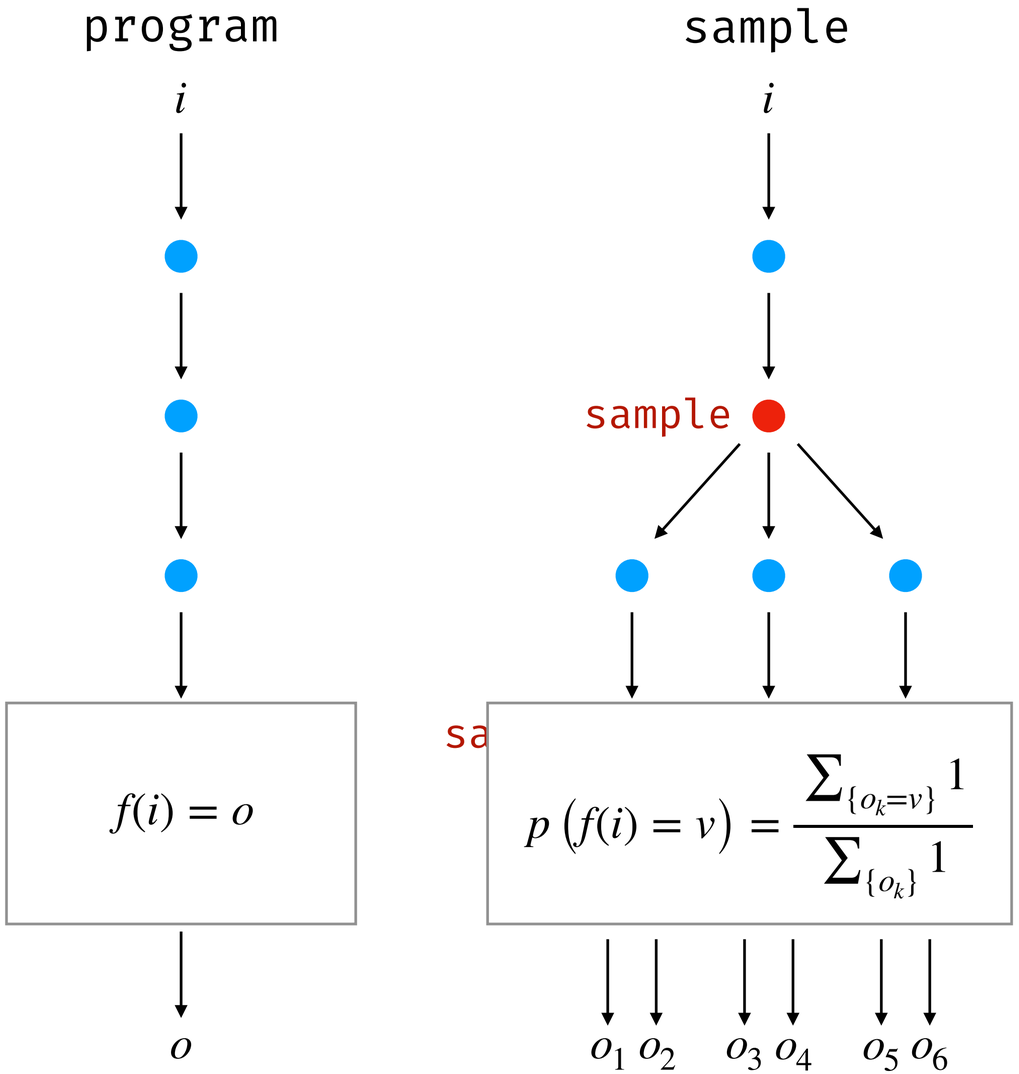

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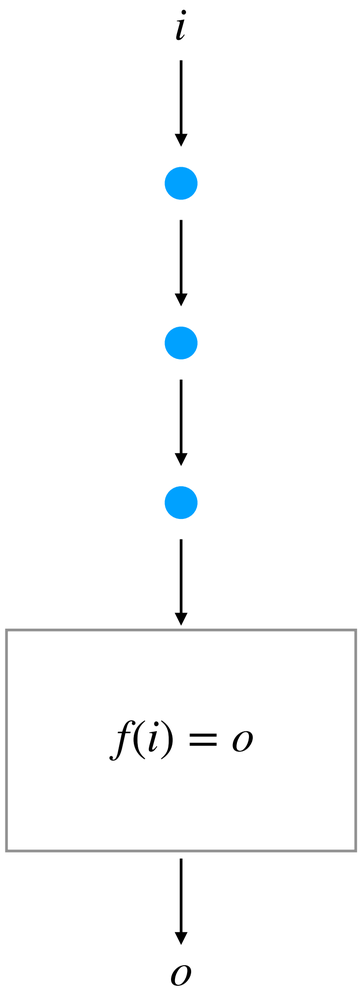
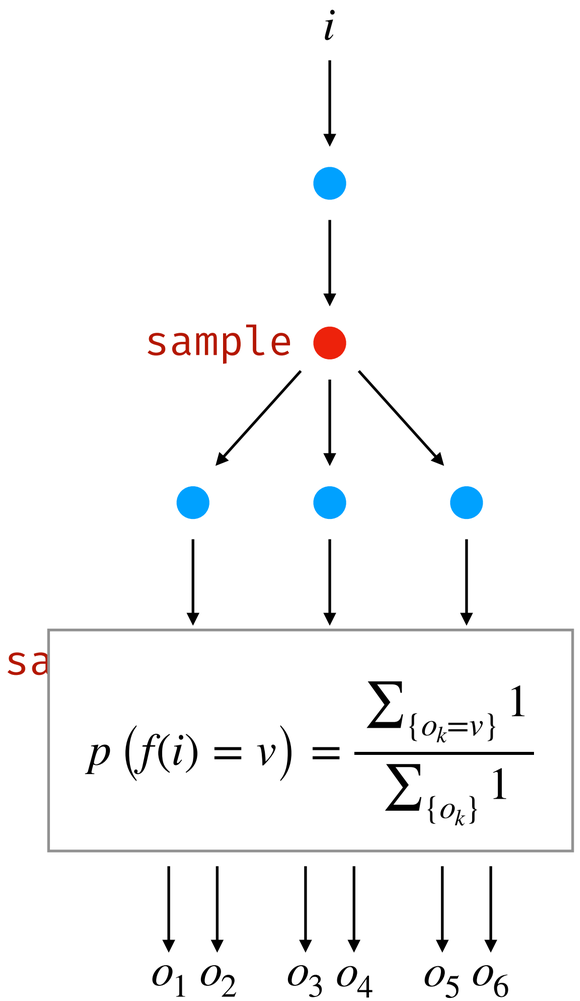


sample



assume

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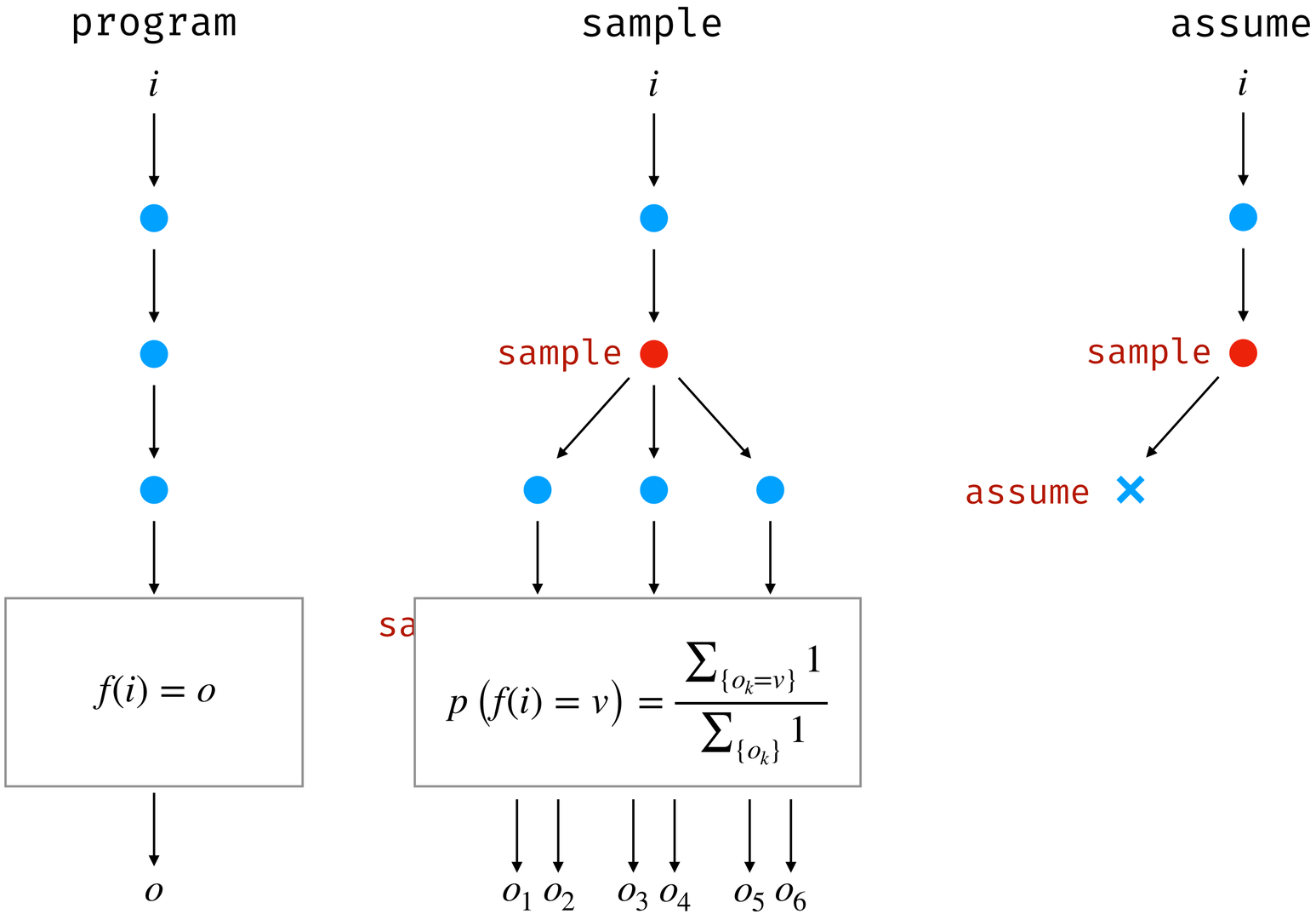

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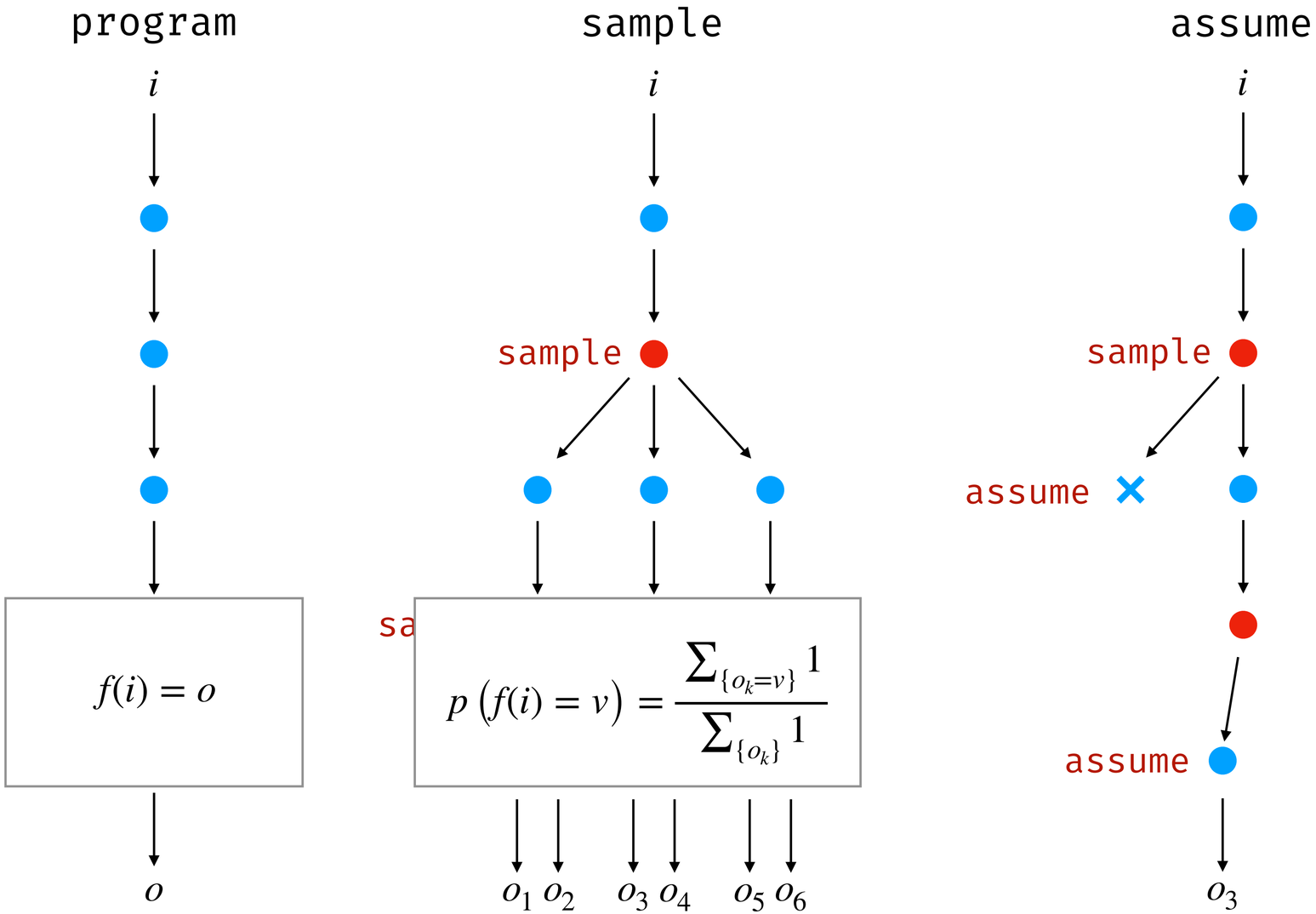

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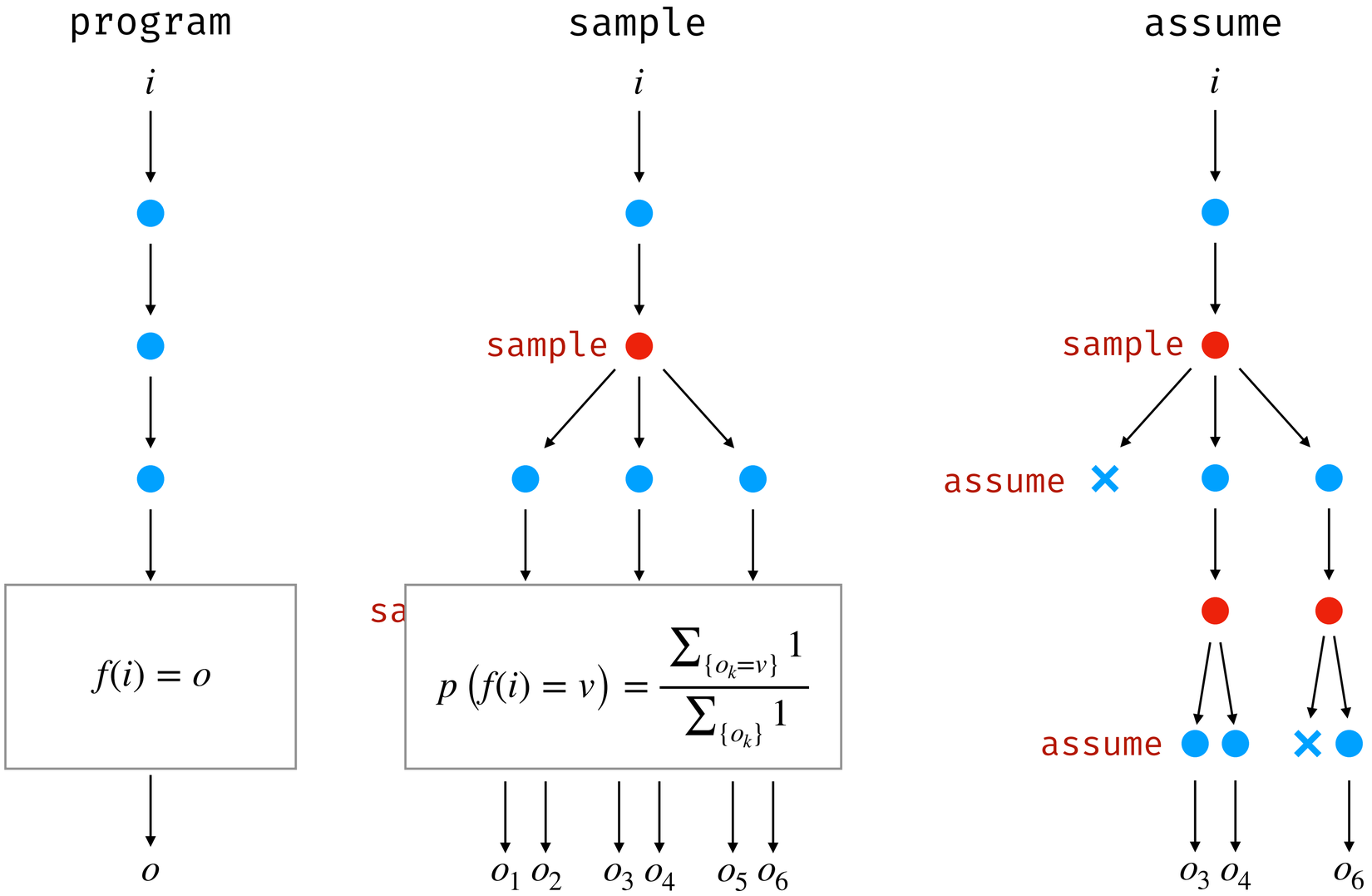

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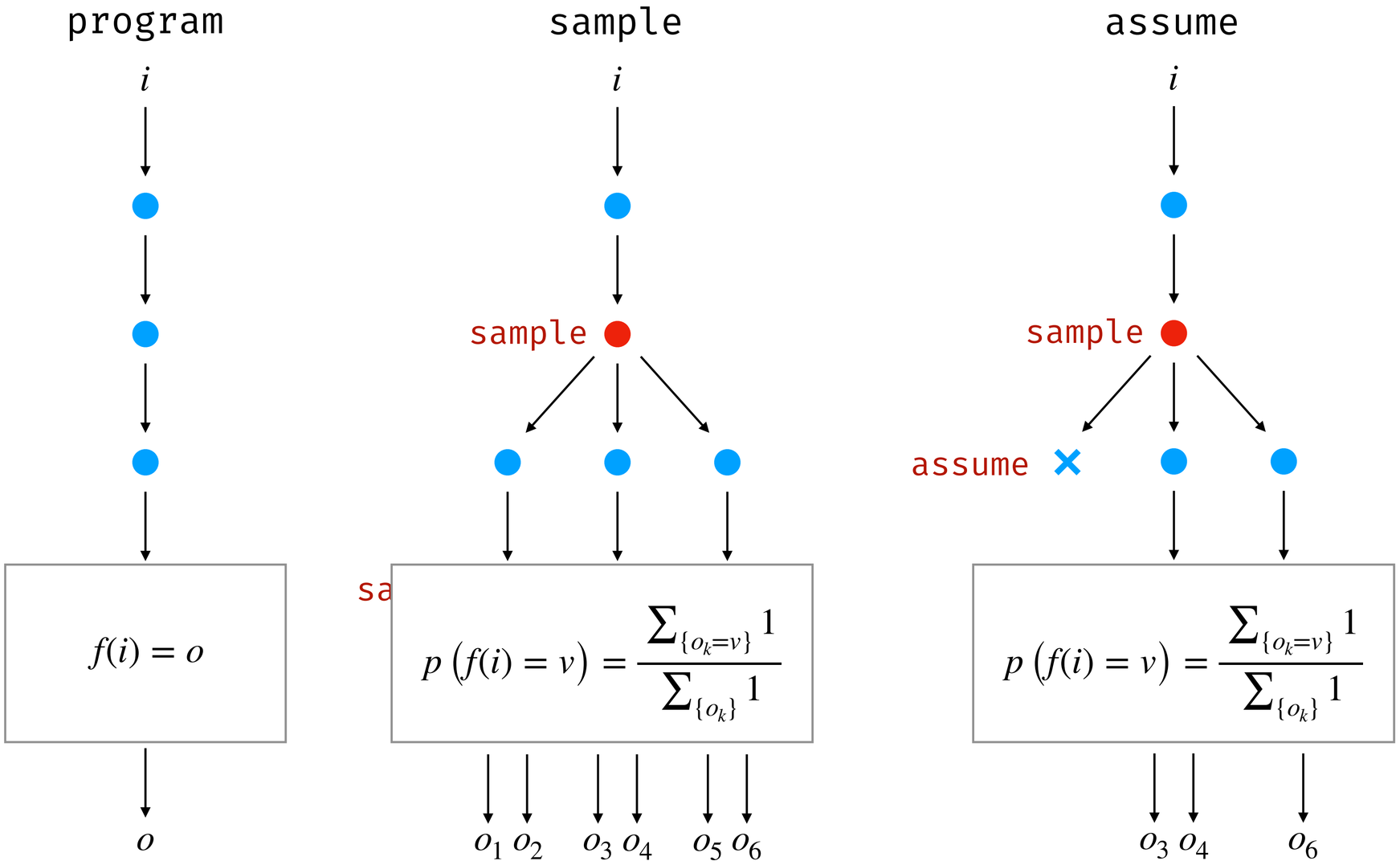

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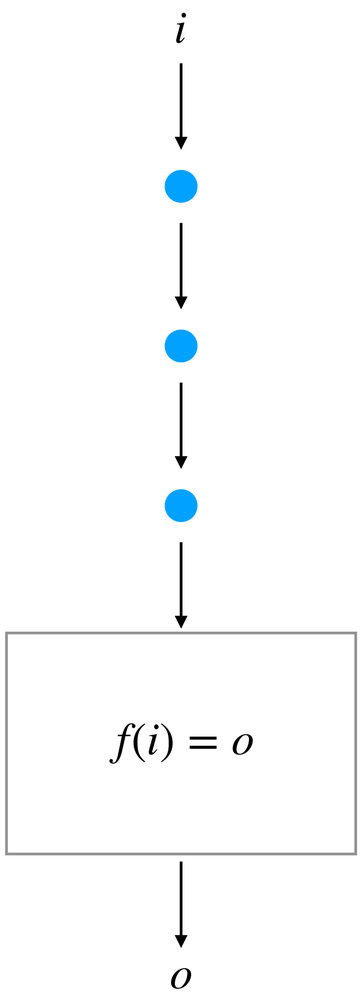

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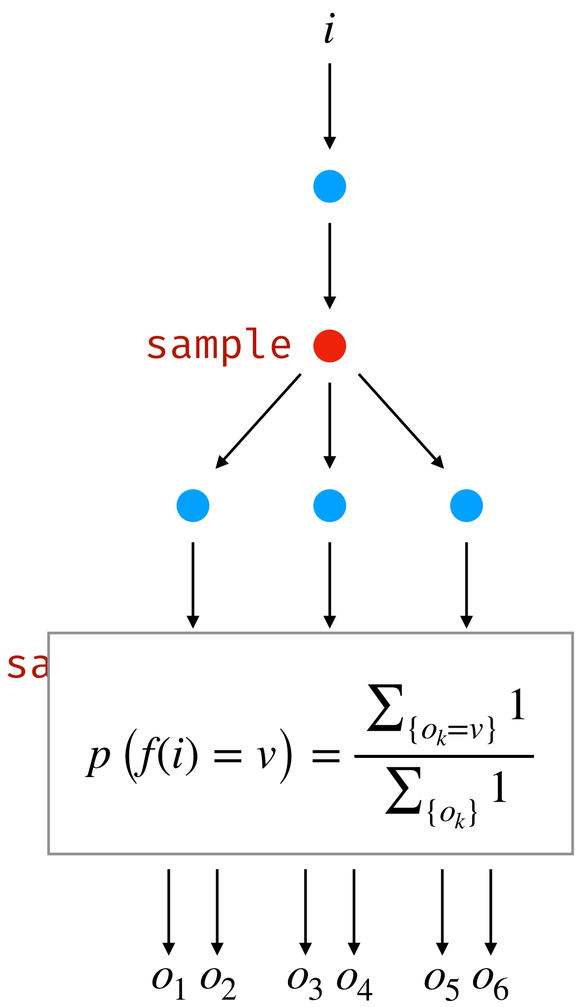

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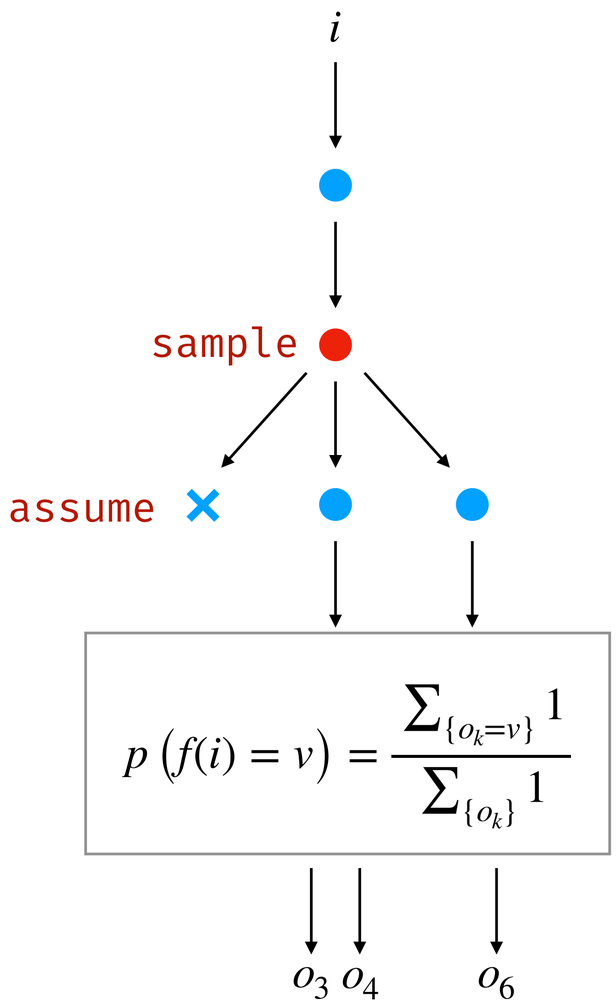
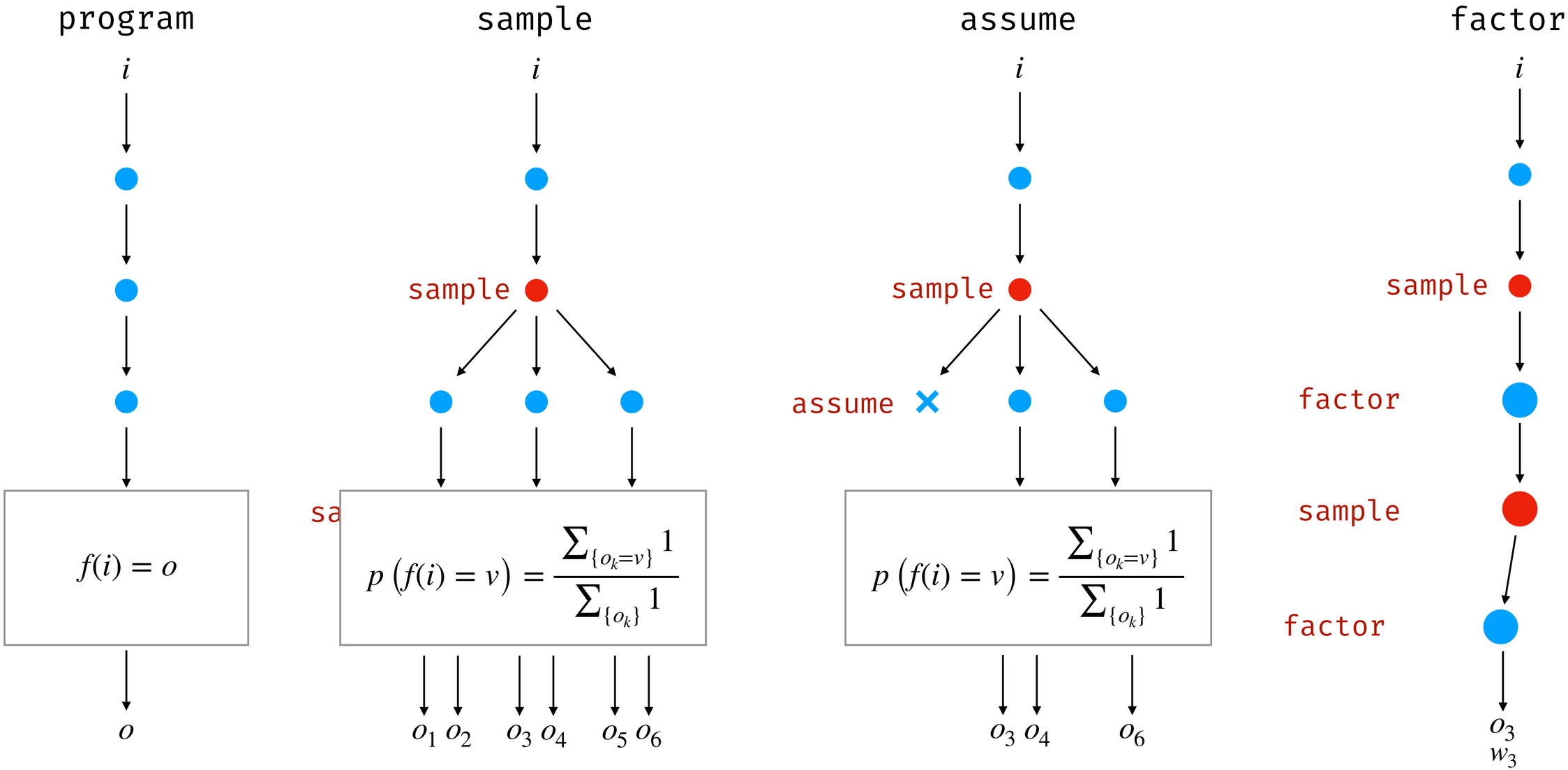


factor

infer : $(\alpha \to \beta) \to \alpha \to \beta$ dist



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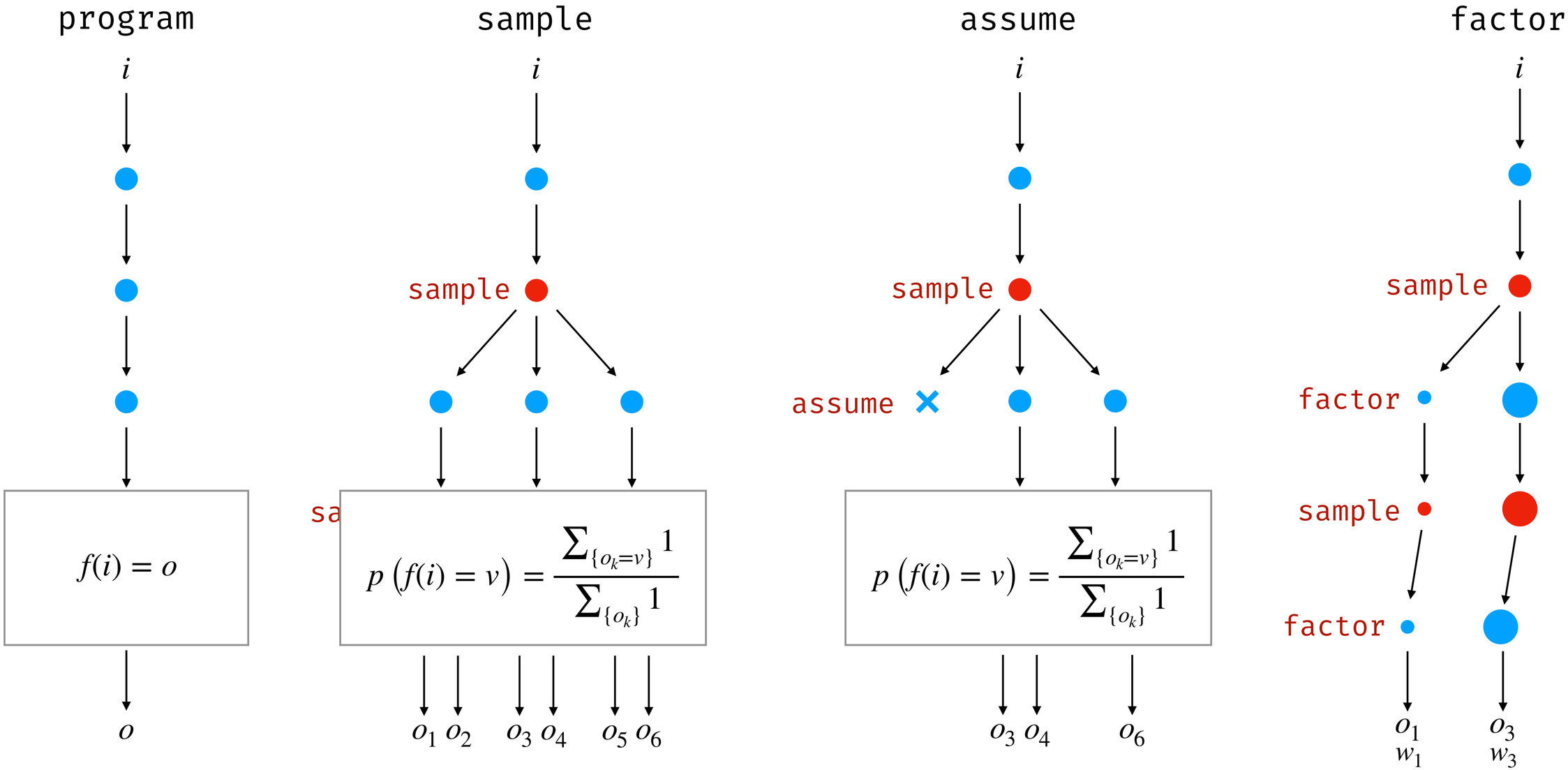

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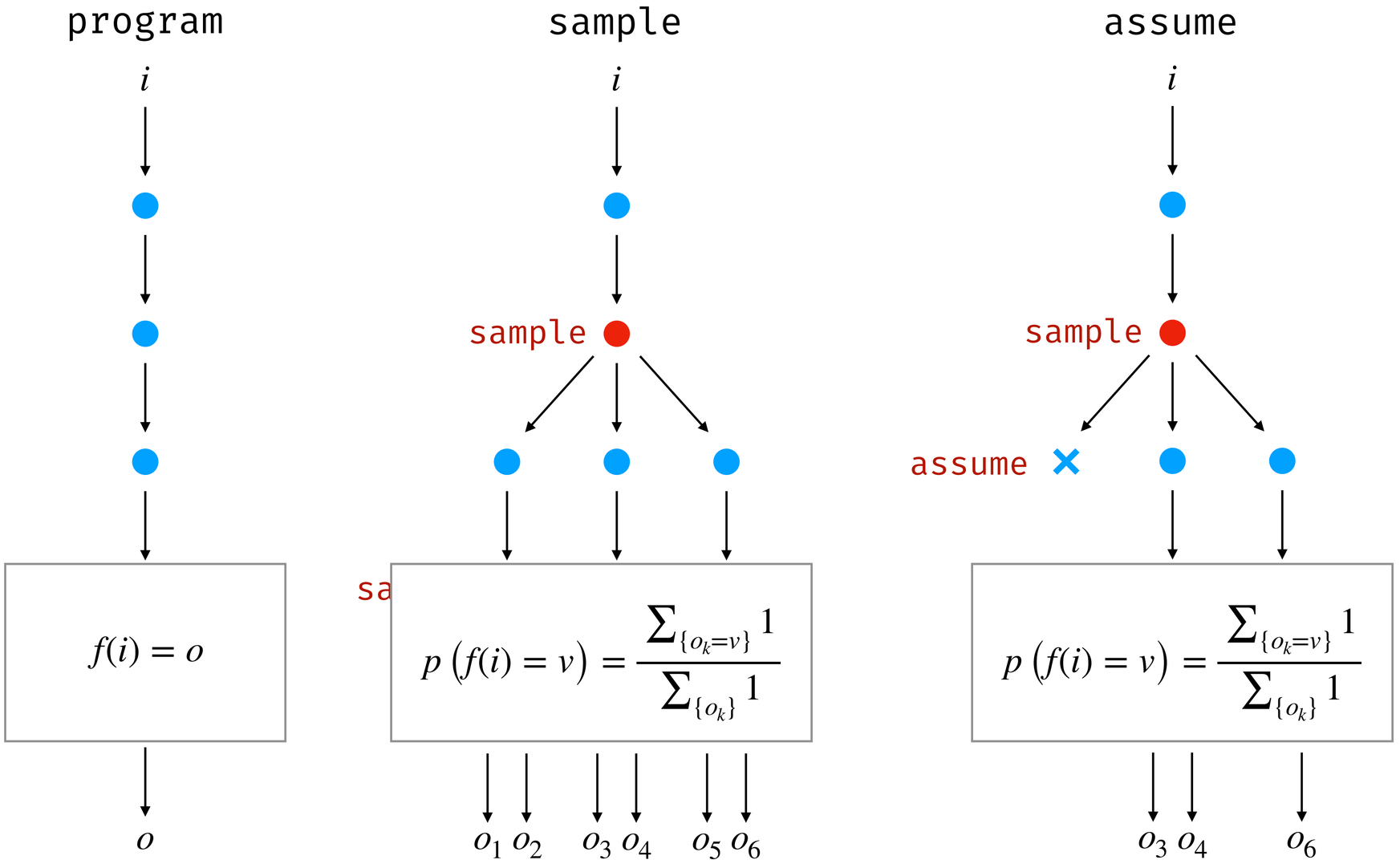

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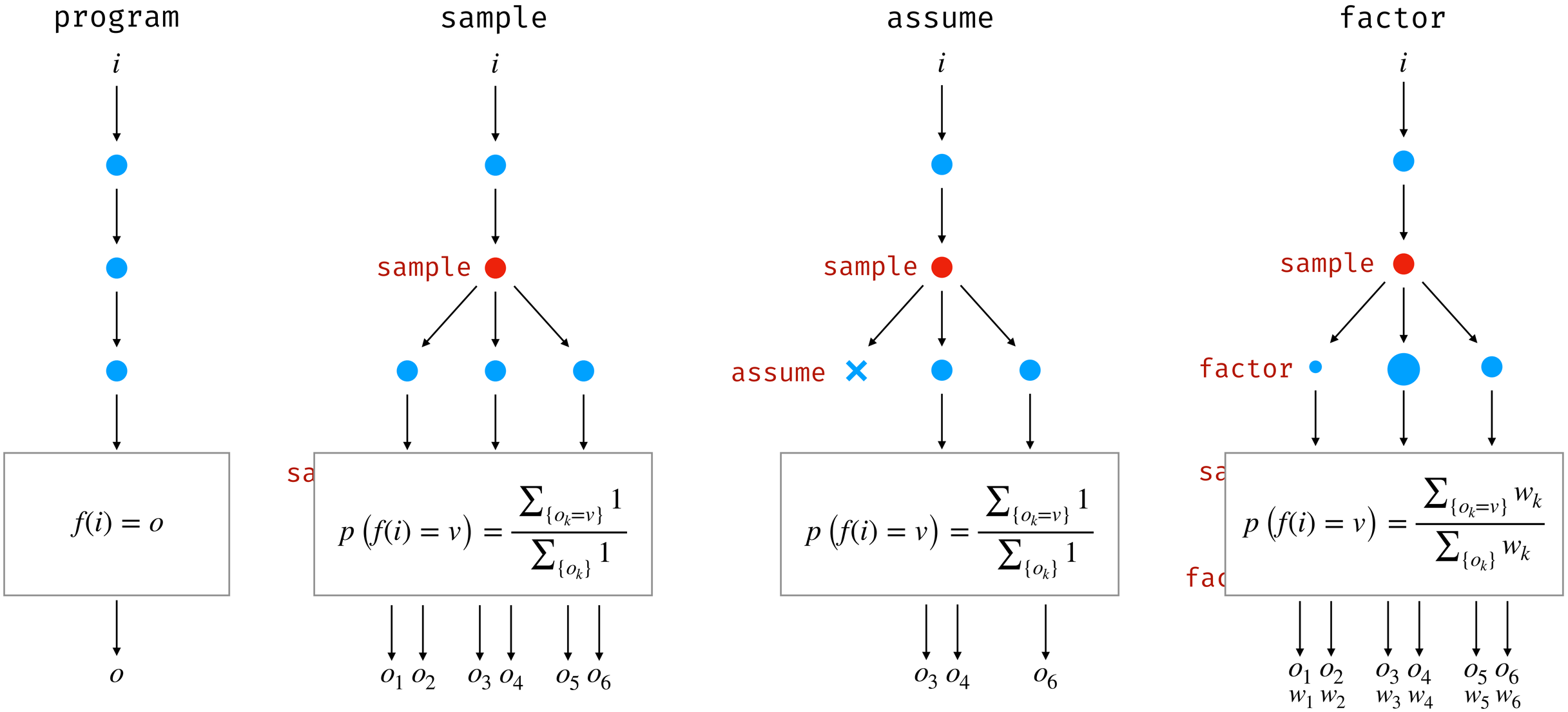

HMM: Importance sampling

Problem:

- Each particle does a random walk and compute a score
- No re-calibration based on observations
- Score decreases at each factor statement

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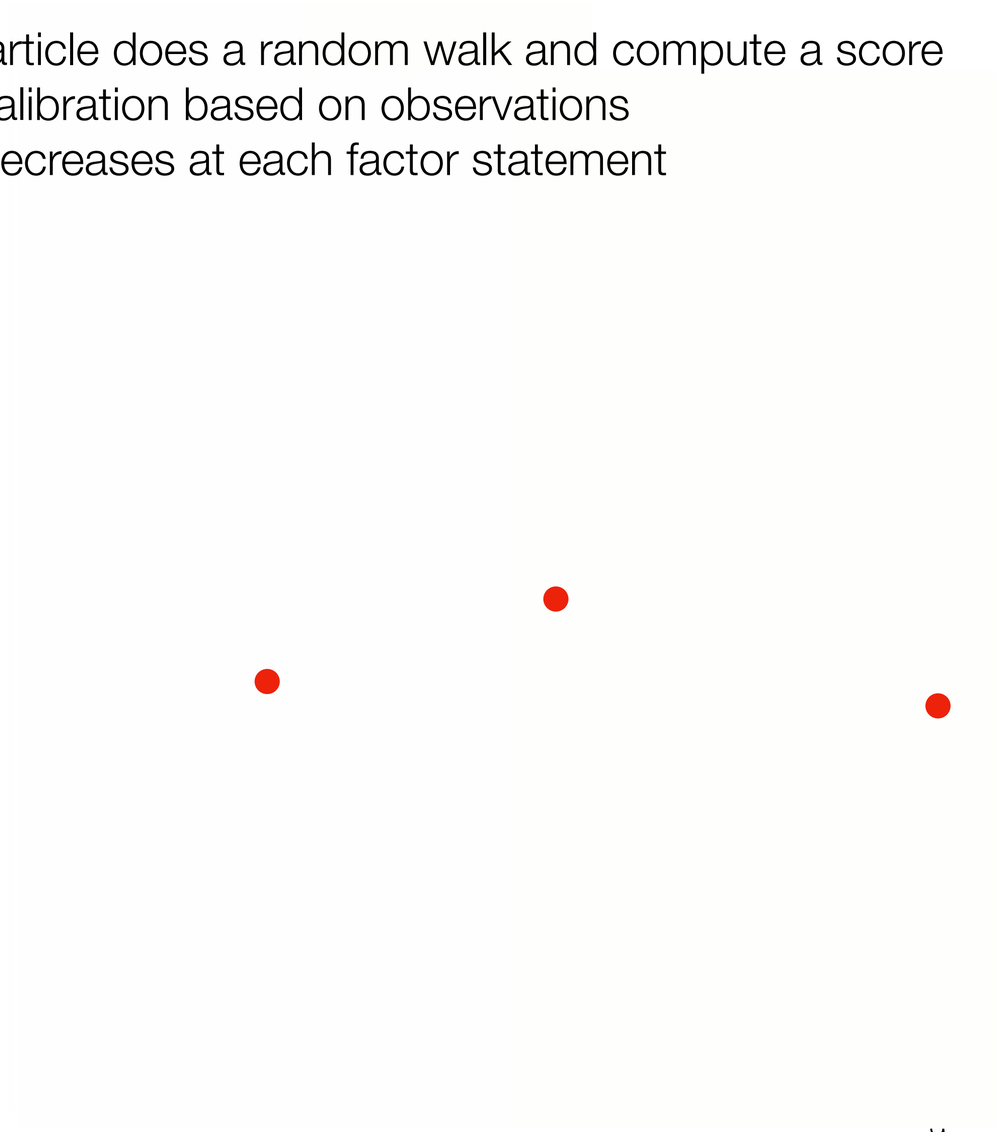

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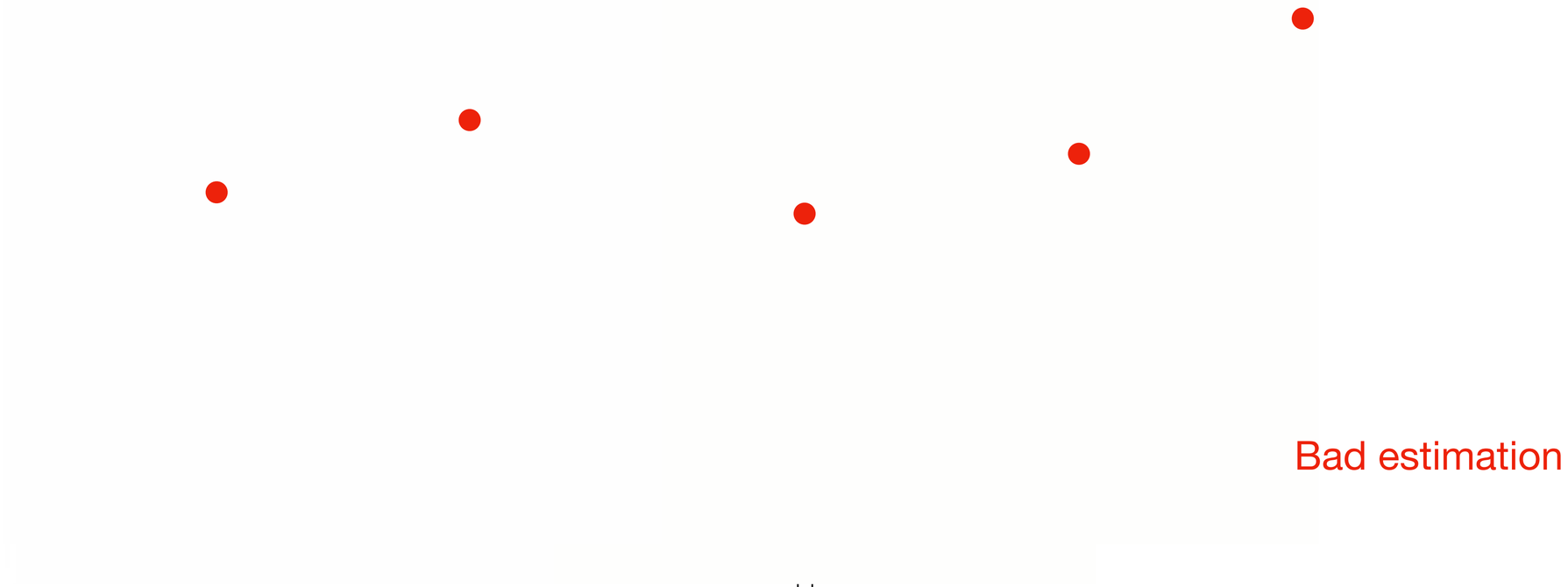

Bad estimation

Remember: The curse of dimensionality

Problem becomes harder as the dimension increases

Basic inference: rejection sampling, importance sampling

- Performances decrease exponentially when the dimension increases
- Only use for low-dimension models

How to mitigate this problem?

- Make assumptions about the posterior distributions
- Break the problem into simpler, smaller problems

17h45mn

Sven Sachsalber, *Looking for a needle in the Haystack*, 2014

HMM: Particle filter

Add a resampling step

- Compute the score of the particles to compute a distribution
- Re-sample a new set of particles from this distribution

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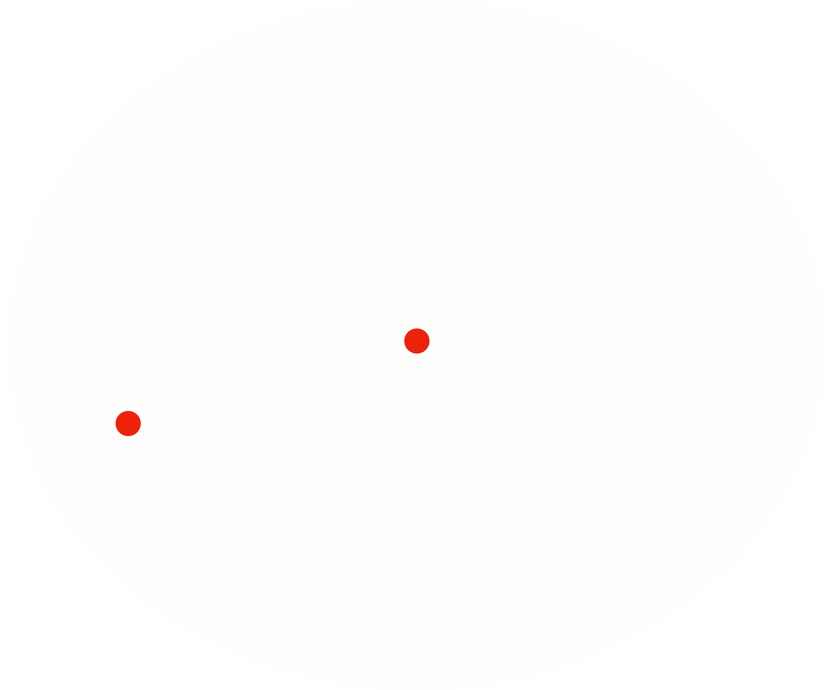

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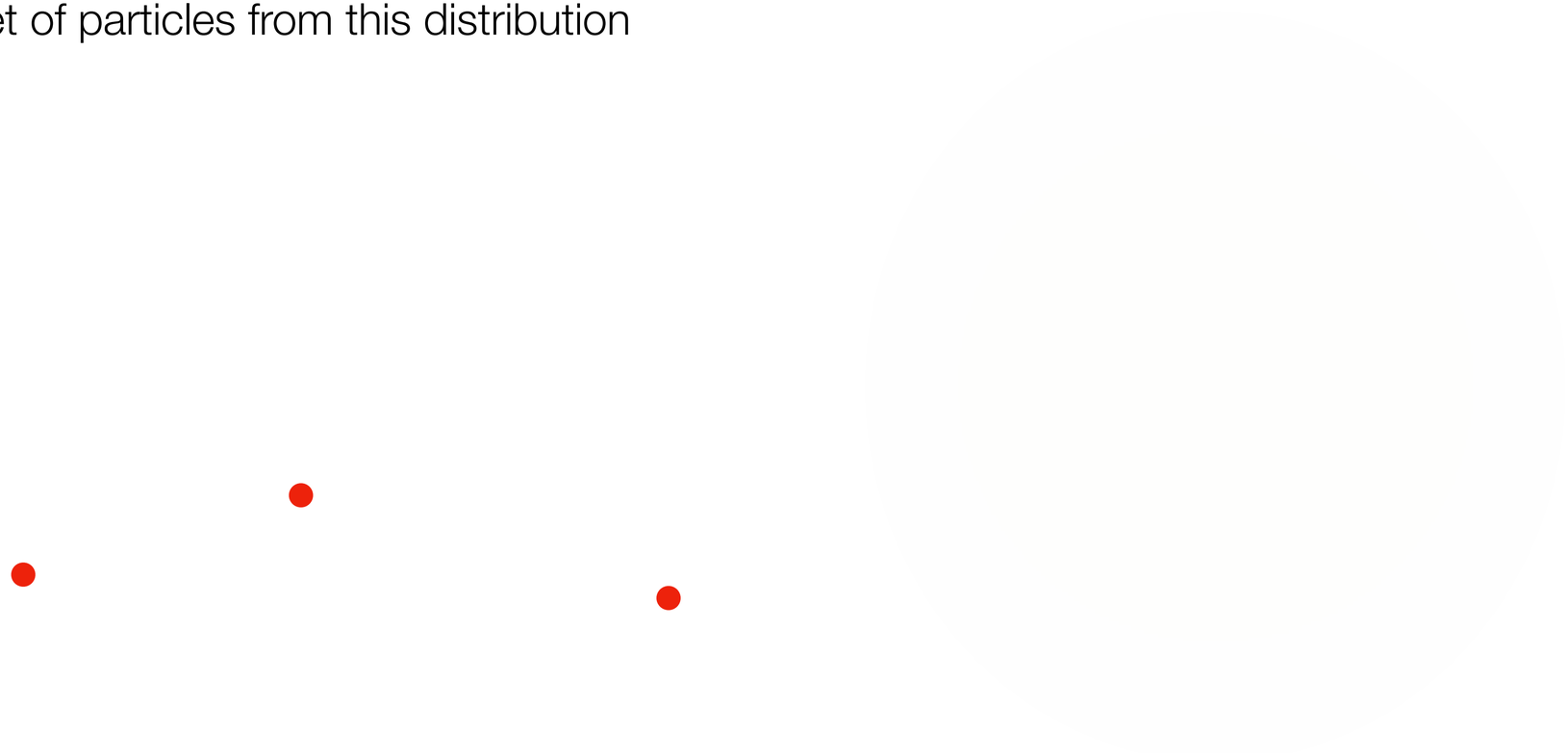

HMM: Particle filter

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Better estimation

Problem: Duplications

Problem: Duplications

How can we duplicate a particle during execution?

- Rerun the particle from the start?
- Force reuse sampled values?
- Clone the memory state?

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Continuation Passing Style

- Functions take an extra argument `k`: the continuation
- `k` implements what should be done with the result of the function
- In our context, we can use continuation to interrupt/restart the execution of a model

Continuation Passing Style (CPS)

Probabilistic Programming Languages

Reminders: CPS

tree.ml

```
let rec tree_height t =
  match t with
  | Empty → 0
  | Node (_, l, r) → 1 + max (tree_height l) (tree_height r)
```

Reminders: CPS

tree.ml

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```
let rec tree_height t =
  match t with
  | Empty → 0
  | Node (_, l, r) →
    let hl = tree_height l in
    let hr = tree_heigh r in
    (1 + max hl hr)
```

1. Add intermediate values

Reminders: CPS

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let rec tree_height t =
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```
let rec tree_height t k =
  match t with
  | Empty → k 0
  | Node (_, l, r) →
    let hl = tree_height l in
    let hr = tree_heigh r in
    k (1 + max hl hr)
```

1. Add intermediate values
2. Add call to continuation

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```
let rec tree_height t k =
  match t with
  | Empty → k 0
  | Node (_, l, r) →
    tree_height l (fun hl →
      tree_heigh r (fun hr →
        k (1 + max hl hr)))
```

1. Add intermediate values
2. Add call to continuation
3. Turn let/in into nested function call

Funny bernoulli CPS

funny_bernoulli.ml

```
let funny_bernoulli () =
  let a = sample (bernoulli ~p:0.5) in
  let b = sample (bernoulli ~p:0.5) in
  let c = sample (bernoulli ~p:0.5) in
  let () = assume (a = 1 || b = 1) in
  a + b + c
```

---

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        assume (a = 1 || b = 1) (fun () →
          k (a + b + c)))
```

1. Add intermediate values
2. Add call to continuation
3. Turn let/in into nested function call

CPS monadic operators

cps_operators.ml

```
let return e k = k e

let ( let* ) e f k = e (fun x → f x k)    (* let* x = e in f(x) *)
```

CPS monadic operators

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  let* c = sample (bernoulli ~p:0.5) in
  let* () = assume (a = 1 || b = 1) in
  return (a + b + c)
```

Sample generation (CPS)

Probabilistic Programming Languages

CPS models

infer.ml

```
module Gen : sig
  type 'a prob
  and 'a next = 'a prob → 'a prob
  and ('a, 'b) model = 'a → ('b → 'b next) → 'b next

  val sample : 'a Distribution.t → ('a → 'b next) → 'b next
  val factor : float → (unit → 'b next) → 'b next
  val draw: ('a, 'b) model → 'a → 'b
end = struct ... end
```

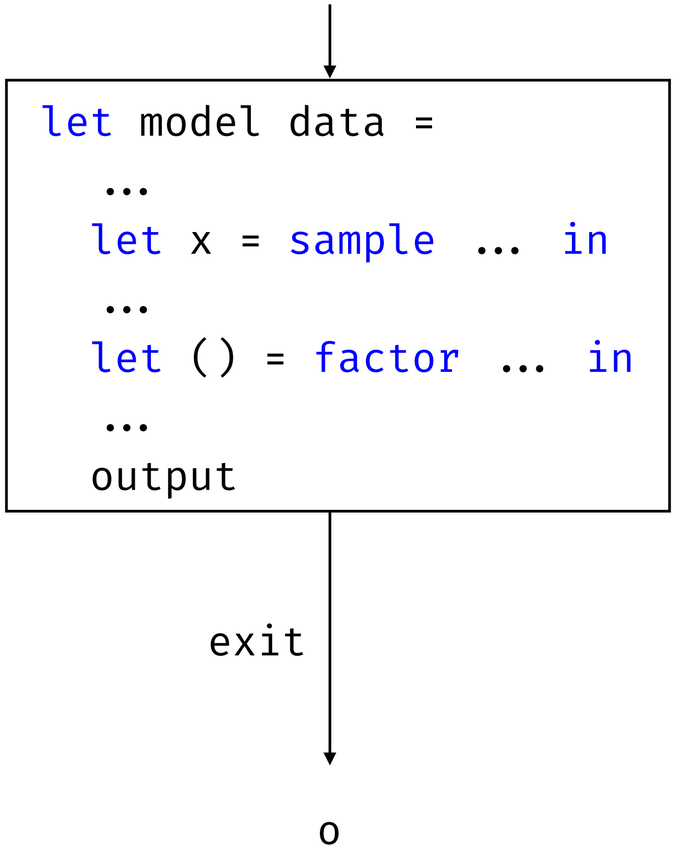
Type 'a prob

- Store all information required for inference
- Type `('a, 'b) model` capture input/output types

Models where probabilistic constructs are CPS functions

- Two arguments: input `'a` and a continuation on the return value `('b → 'b next)`.
- The return value is a continuation `'a next` that updates a probabilistic state of type `'a prob`.

Sample generation



Sample generation

infer.ml

```
module Gen = struct
  type 'a prob = 'a option
  and 'a next = 'a prob → 'a prob
  and ('a, 'b) model = 'a → ('b → 'b next) → 'b next

  let exit v _prob = Some v                                    (* return value *)

  let sample d k prob =
    let v = Distribution.draw d in                                  (* draw *)
    k v prob                                                     (* continue *)

  let factor _s k prob = k () prob                        (* continue, ignore score *)


  let draw model data =
    let v = (model data) exit None in                            (* run model *)
    Option.get v                                              (* extract value *)

end
```

Funny bernoulli

funny_bernoulli.ml

```
open Infer.Gen

let funny_bernoulli () =
  let* a = sample (bernoulli ~p:0.5) in
  let* b = sample (bernoulli ~p:0.5) in
  let* c = sample (bernoulli ~p:0.5) in
  let* () = assume (a || b) in
  return (Bool.to_int a + Bool.to_int b + Bool.to_int c)

let _ =
  for _ = 1 to 10 do
    let v = draw funny_bernoulli () in
    Format.printf "%d " v
  done
```

```
❯ dune exec ./examples/funny_bernoulli.exe

1 1 2 2 2 2 2 1 3 2
```

Importance sampling (CPS)

Probabilistic Programming Languages

Importance sampling

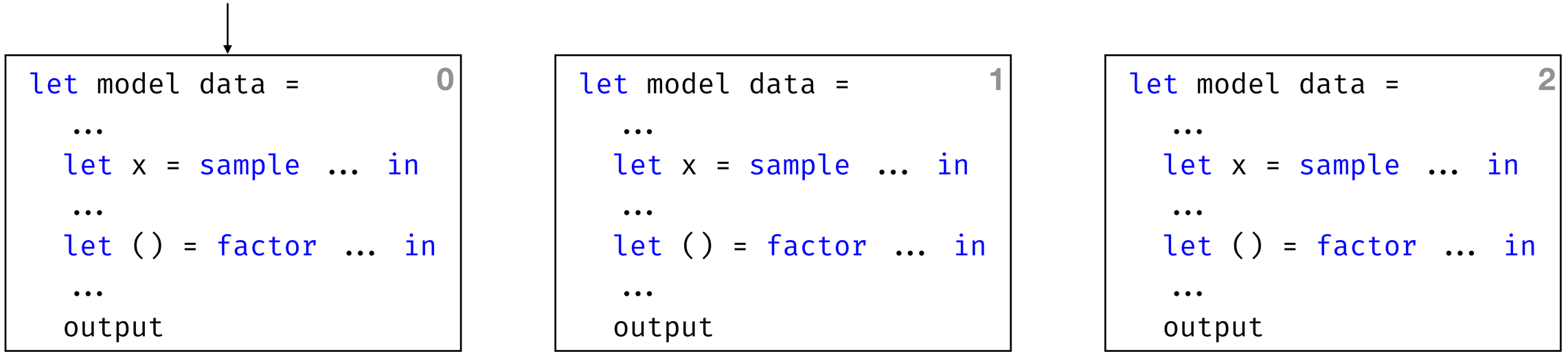
infer.ml

```
module Importance_sampling : sig
  type 'a prob = { score : float; k : 'a next; value : 'a option }
  and 'a next = 'a prob → 'a prob
  and ('a, 'b) model = 'a → ('b → 'b next) → 'b next

  val sample : 'a Distribution.t → ('a → 'b next) → 'b next
  val factor : float → (unit → 'b next) → 'b next
  val infer : ('a, 'b) model → 'a → 'b Distribution.t
end = struct ... end
```

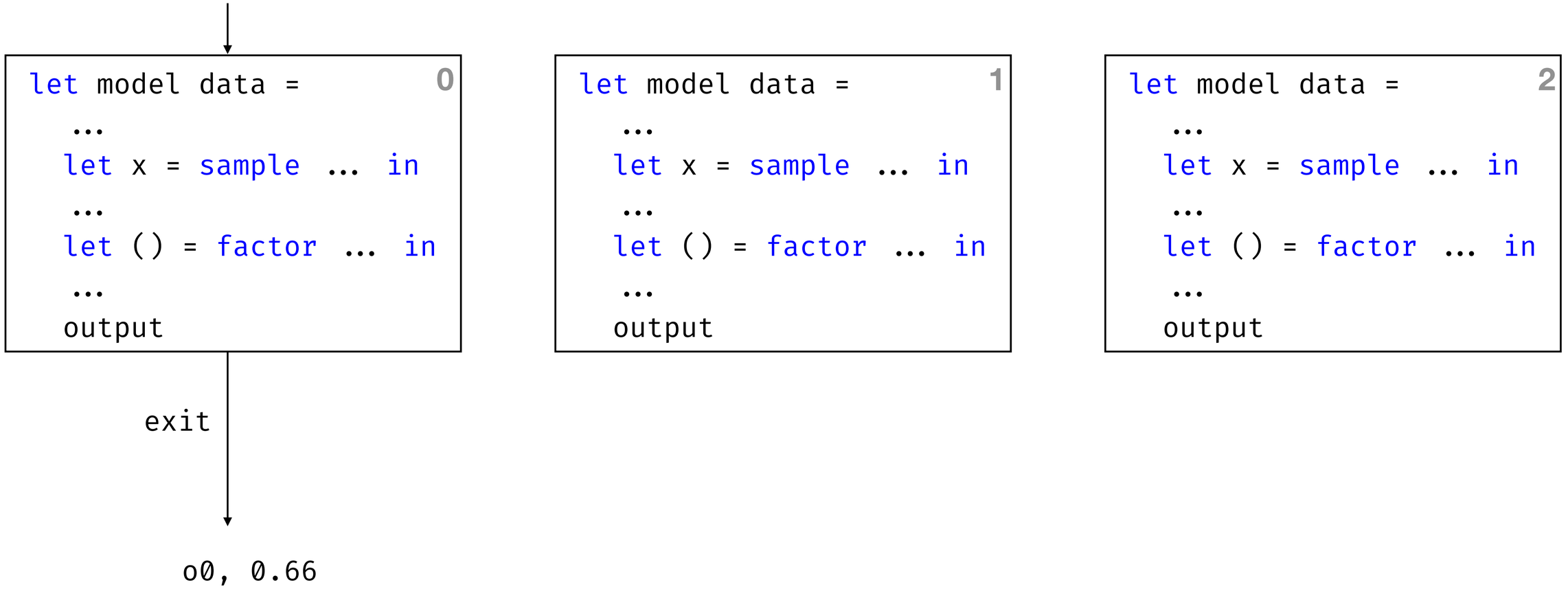
Inference algorithm

- Run a set of n independent executions
- `sample`: draw a sample from a distribution
- `factor`: associate a score to the current execution
- Gather output values and score to approximate the posterior distribution

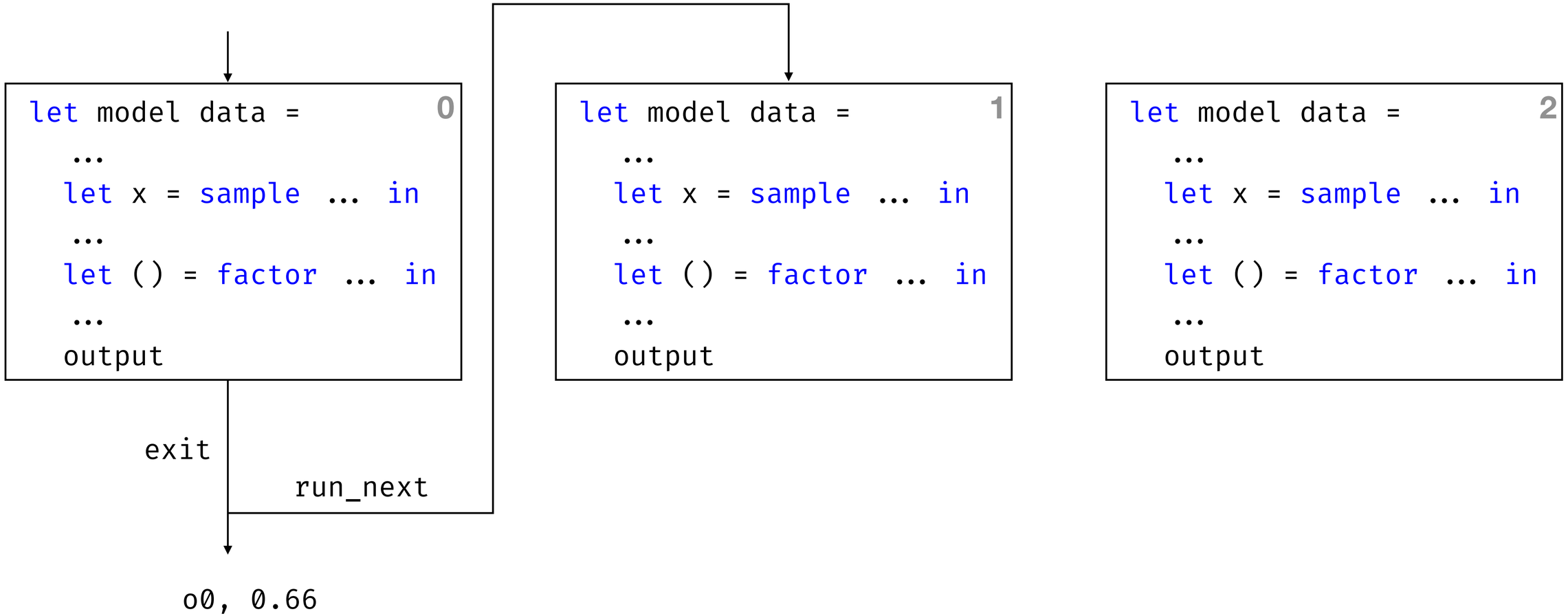
Importance sampling



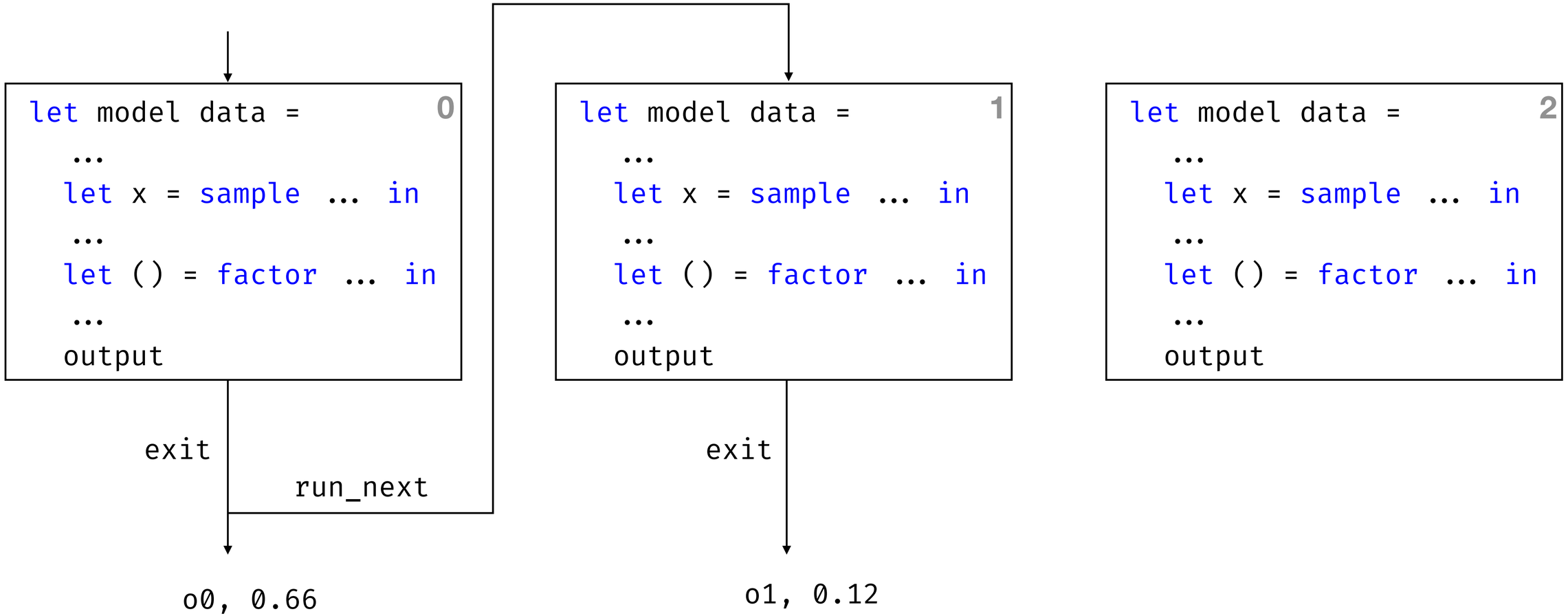
Importance sampling



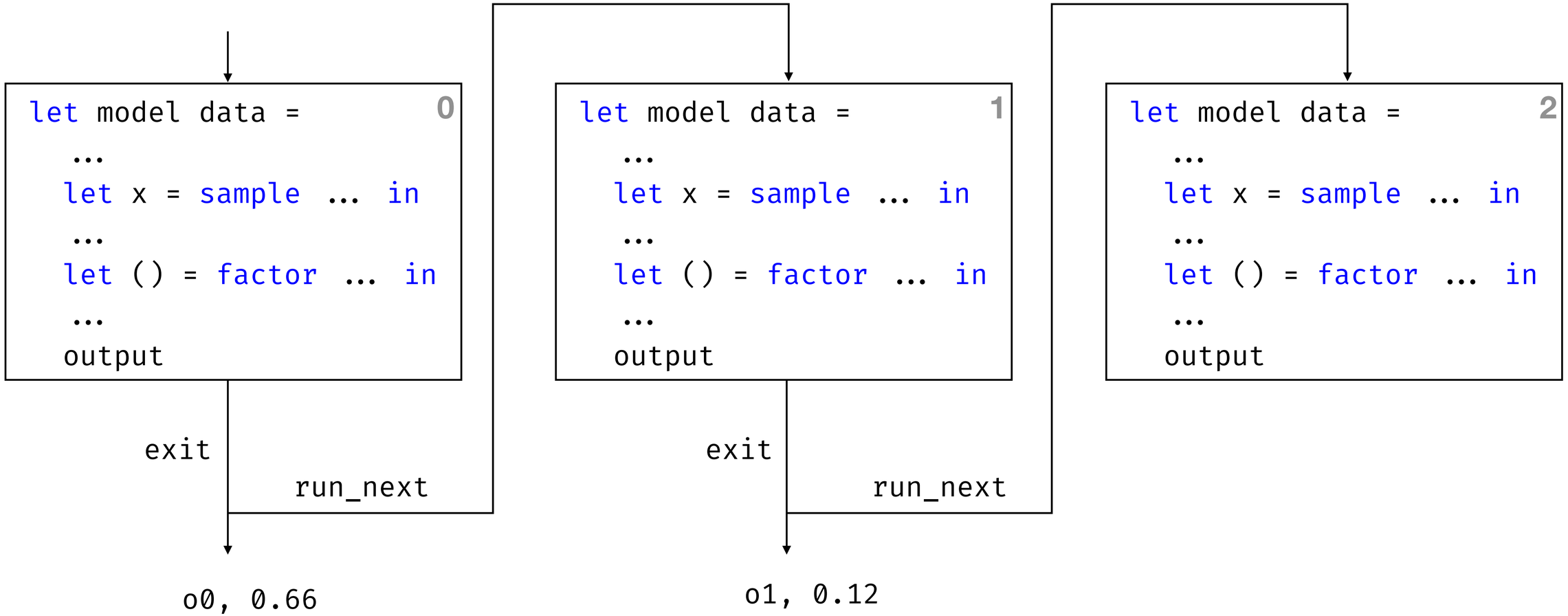
Importance sampling



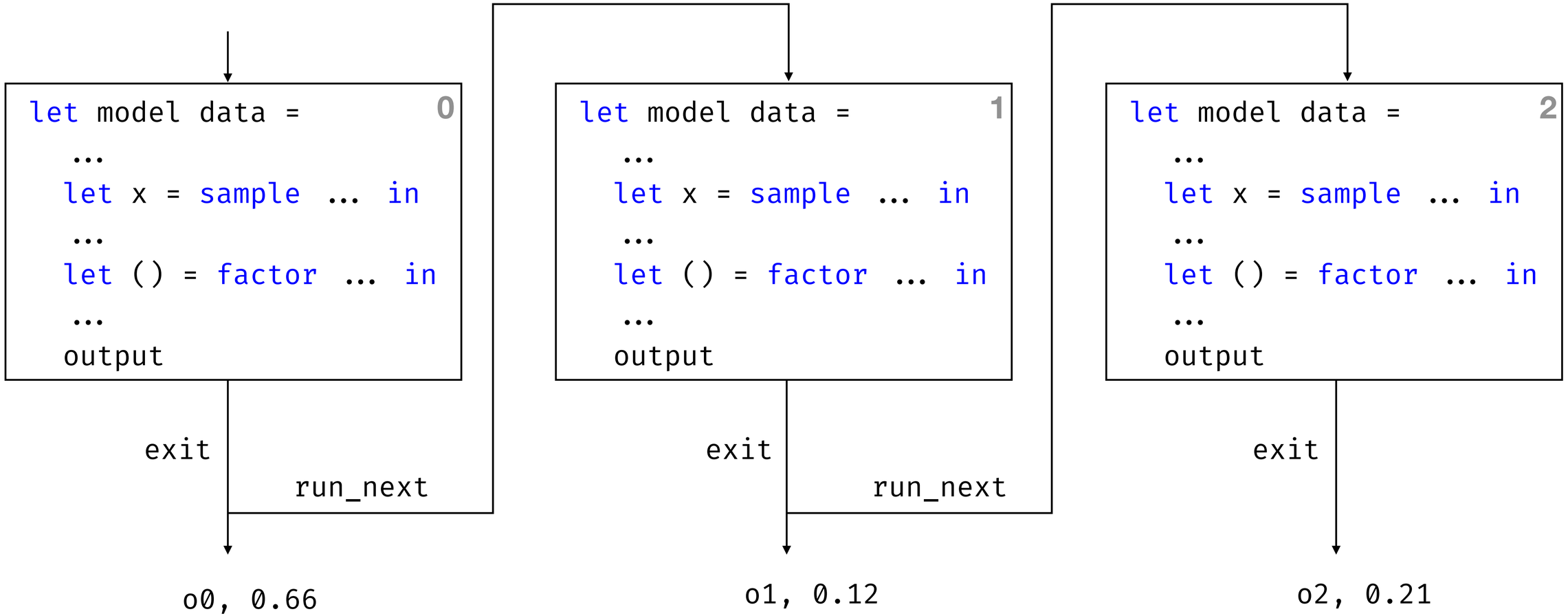
Importance sampling



Importance sampling



Importance sampling



Importance sampling

infer.ml

```
module Importance_sampling = struct
  type 'a prob = { score : float; k : 'a next; value : 'a option }
  and 'a next = 'a prob → 'a prob
  and ('a, 'b) model = 'a → ('b → 'b next) → 'b next

  let sample d k prob = assert false
  let factor s k prob = assert false

  let infer ?(n = 1000) m data = assert false
end
```

Importance sampling

infer.ml

```
module Importance_sampling = struct
  type 'a prob = { score : float; k : 'a next; value : 'a option }
  and 'a next = 'a prob → 'a prob
  and ('a, 'b) model = 'a → ('b → 'b next) → 'b next

  let sample d k prob = assert false
  let factor s k prob = assert false

  let infer ?(n = 1000) m data = assert false
end
```

Try it in BYO-PPL!

Importance sampling

infer.ml

```
module Importance_sampling = struct
  type 'a prob = { score : float; k : 'a next; value : 'a option }
   ...

  let sample d k prob =
    let v = Distribution.draw d in                          (* draw a value *)
    k v prob                                                    (* continue *)

  let factor s k prob =
    k () { prob with score = prob.score +. s }          (* continue with updated score *)
```

Importance sampling

infer.ml

```
module Importance_sampling = struct
  type 'a prob = { score : float; k : 'a next; value : 'a option }
   ...

  let exit v prob = { prob with value = Some v }                    (* return the value *)

  let infer ?(n = 1000) model data =
    let rec gen n support =
      if n = 0 then support
      else
        let k = (model data) exit in                              (* run from the start *)
        let prob = k { score = 0.; value = None; k } in                (* Run the model *)
        let value = Option.get prob.value in                           (* get the value *)
        gen (n - 1) ((value, prob.score) :: support)                  (* add to support *)
    in
    let support = gen n [] in
    Distribution.categorical ~support

end
```

Coin

coin.ml

```
open Infer.Importance_sampling

let coin x =
  let* z = sample (uniform ~a:0. ~b:1.) in
  let* () = Cps_list.iter (observe (bernoulli ~p:z)) x in
  return z

let _ =
  let dist = infer coin [1; 1; 0; 0; 0; 0; 0; 0; 0; 0] in
  let m, s = Distribution.stats dist in
  Format.printf "Coin bias, mean:%f, std:%f@." m s
```

```
❯ dune exec ./examples/coin.exe

Coin bias, mean:0.247876, std:0.118921
Beta(2+1, 8+1), mean:0.250000, std:0.120096
```

Particle filter (CPS)

Probabilistic Programming Languages

Particle filter

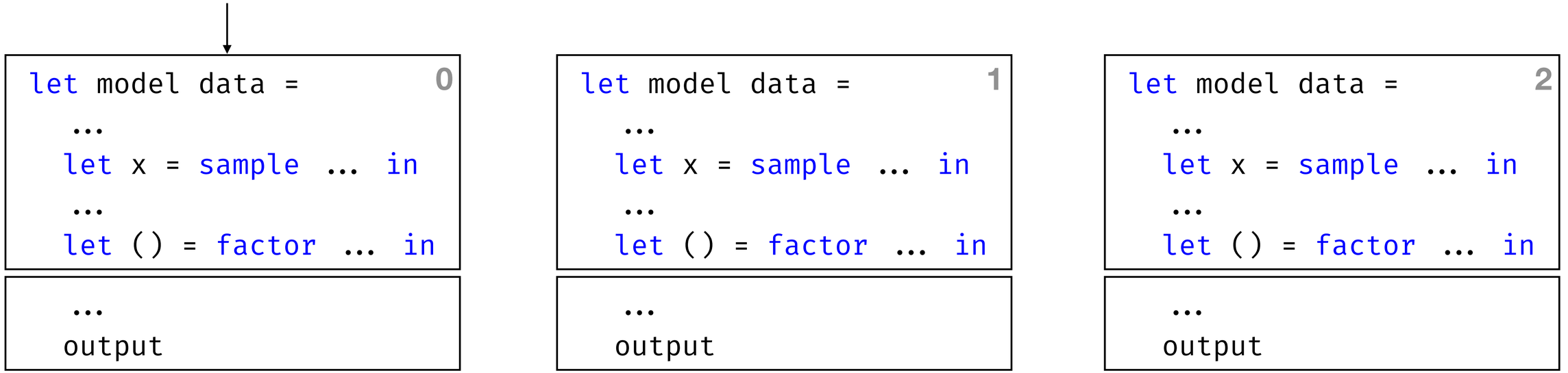
infer.ml

```
module Particle_filter = struct
  include Importance_sampling

  let resample particles = assert false
  let factor s k prob = assert false

  let infer ?(n = 1000) model data = assert false
end
```

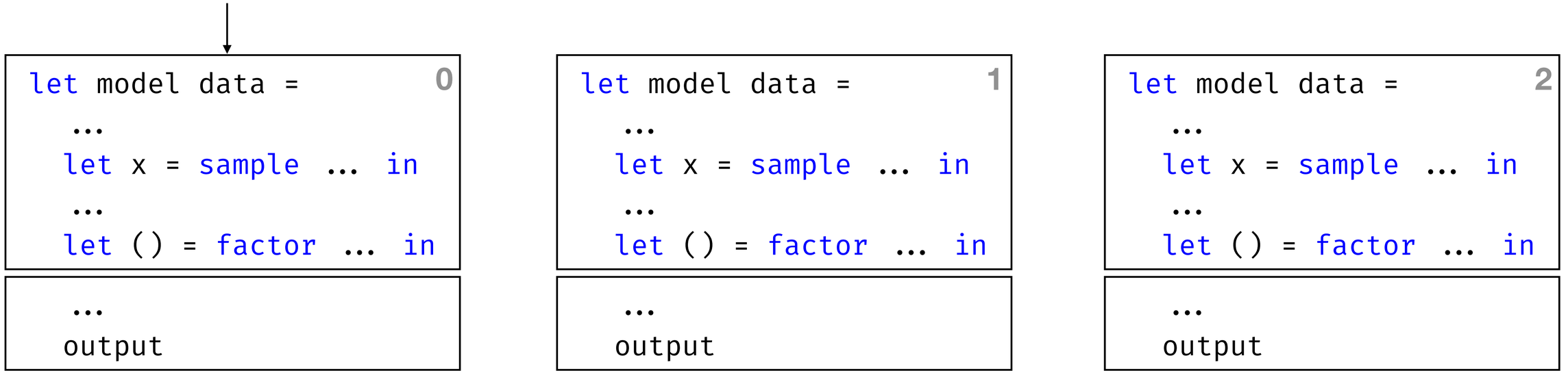
Inference algorithm : importance sampling, but...

- Add a resampling step at each `factor`
- Compute the score of the particles to compute a distribution
- Re-sample a new set of particles from this distribution

Particle filter

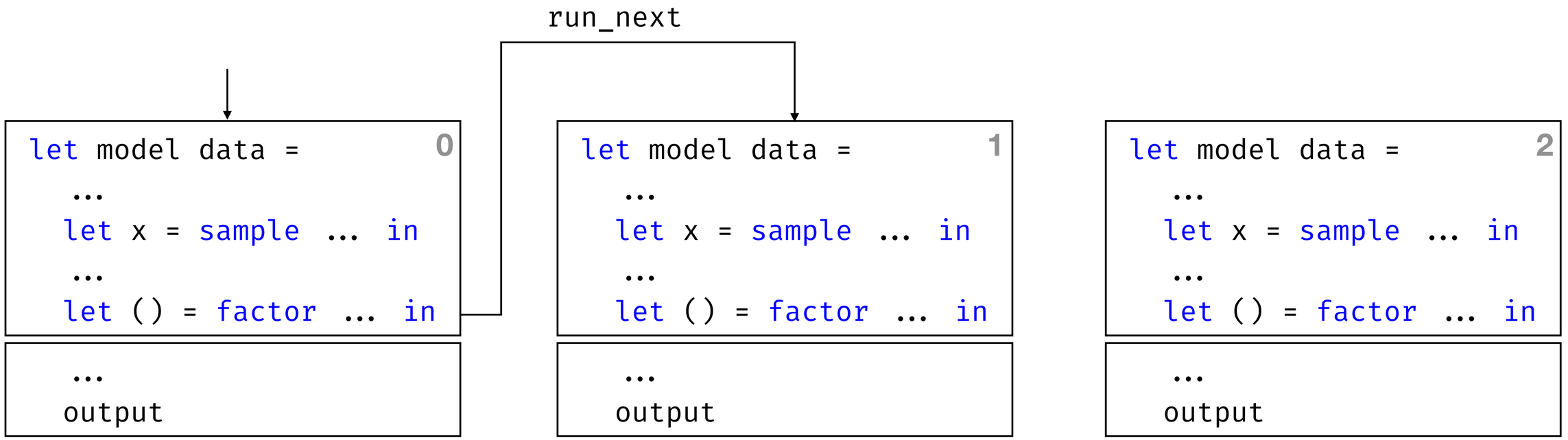


Particle filter



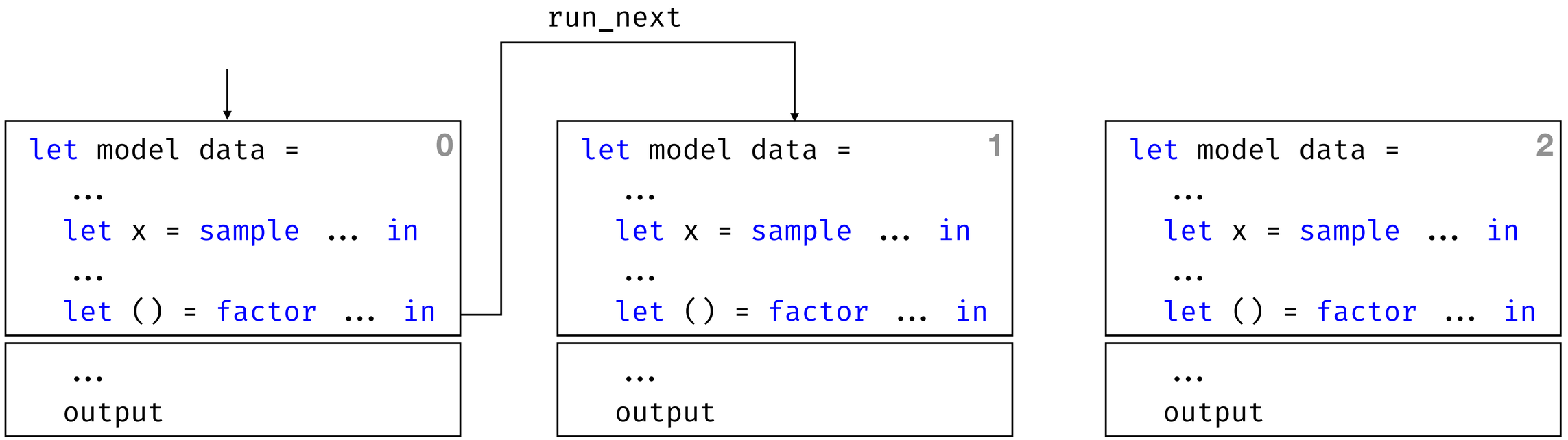
0.66

Particle filter

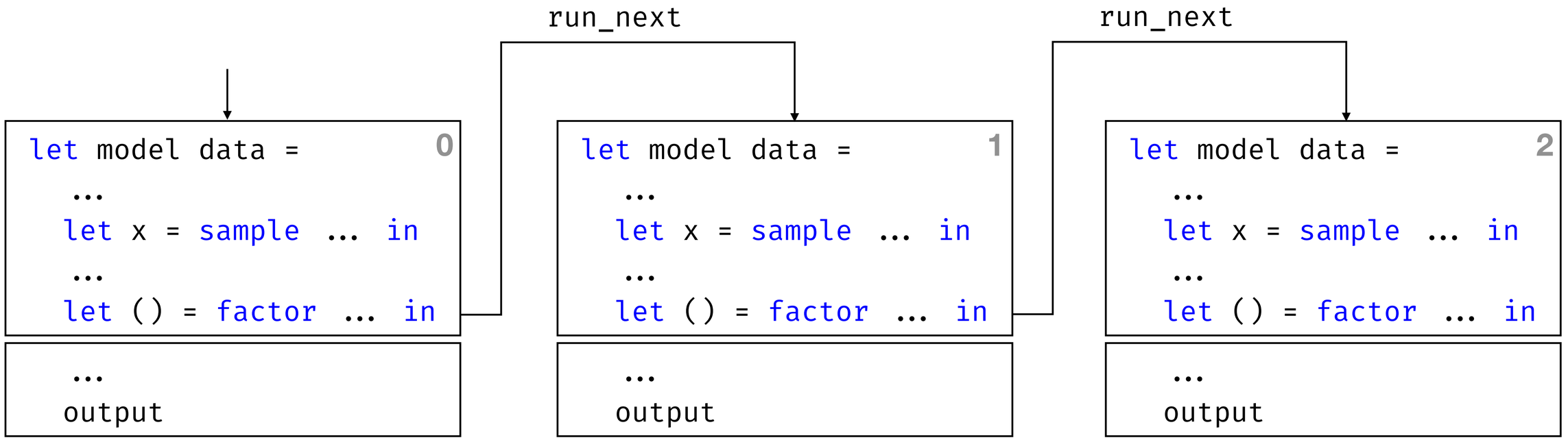


0.66

Particle filter



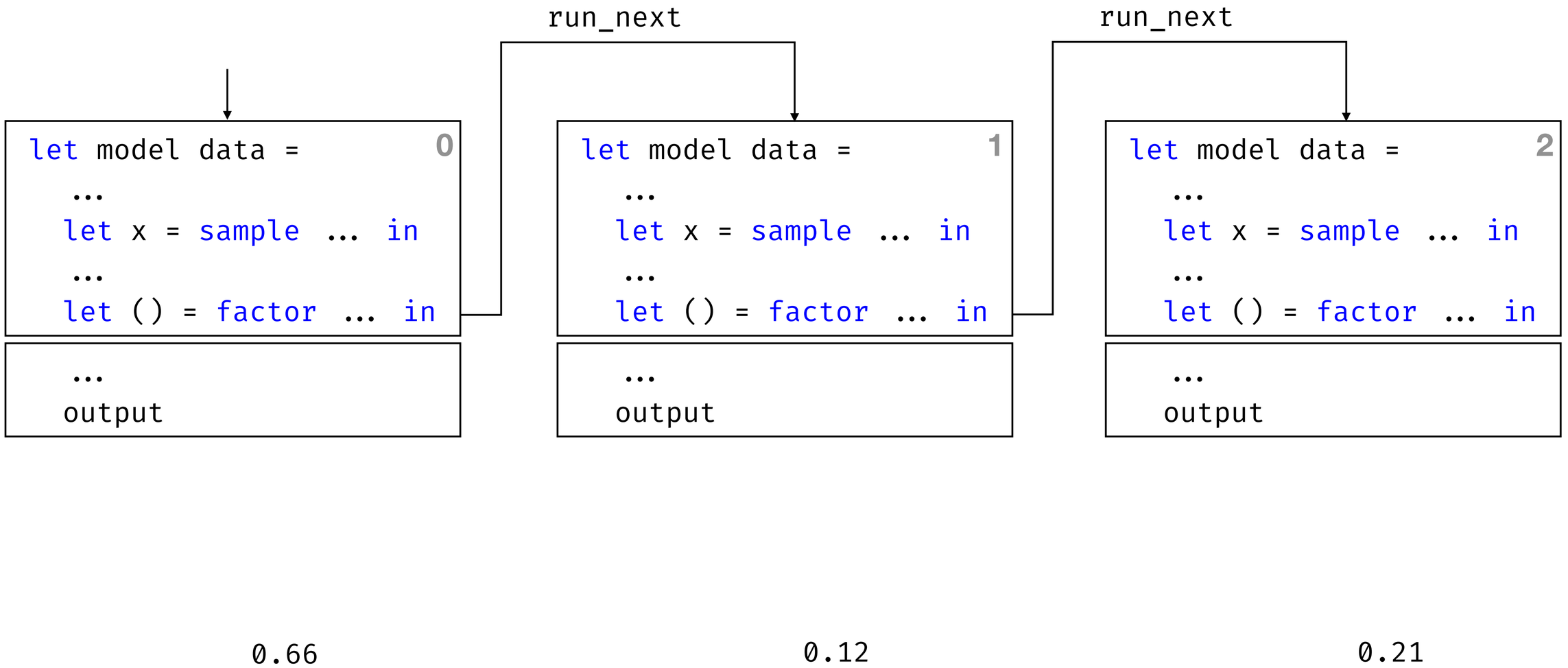
Particle filter



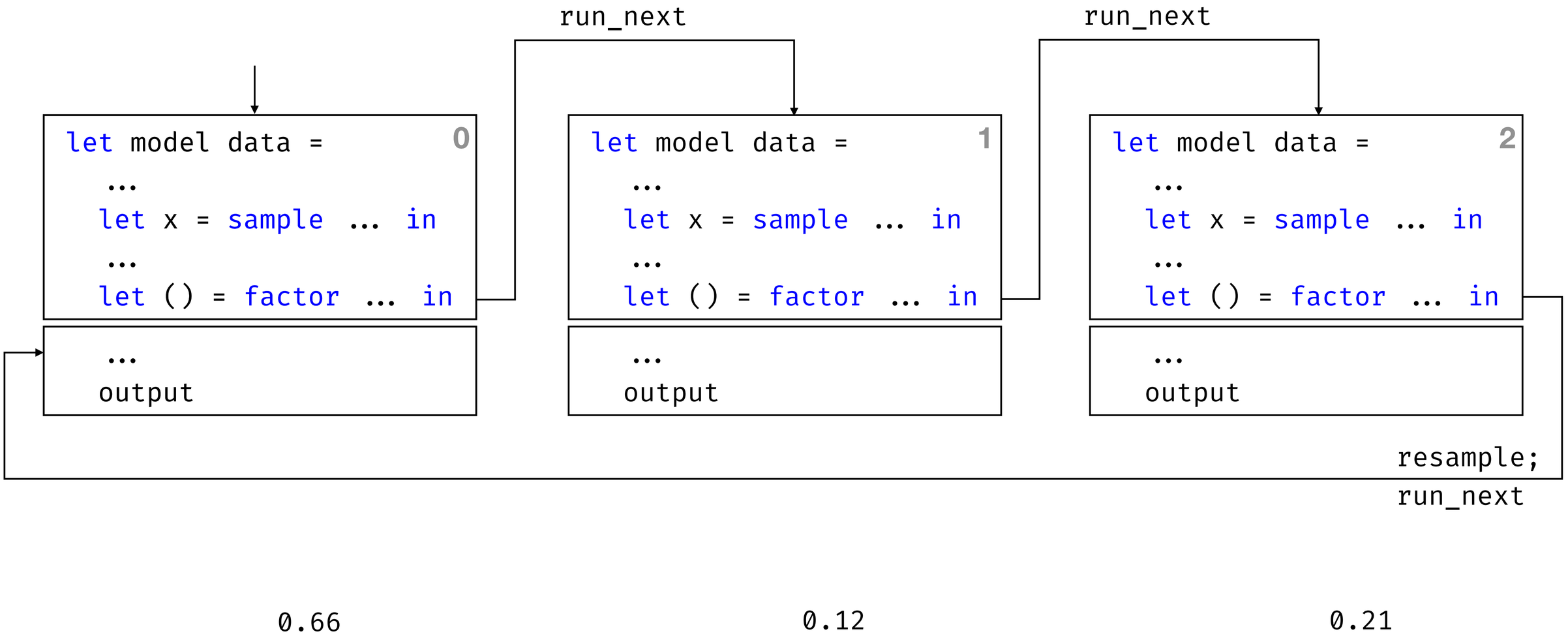
0.66

0.12

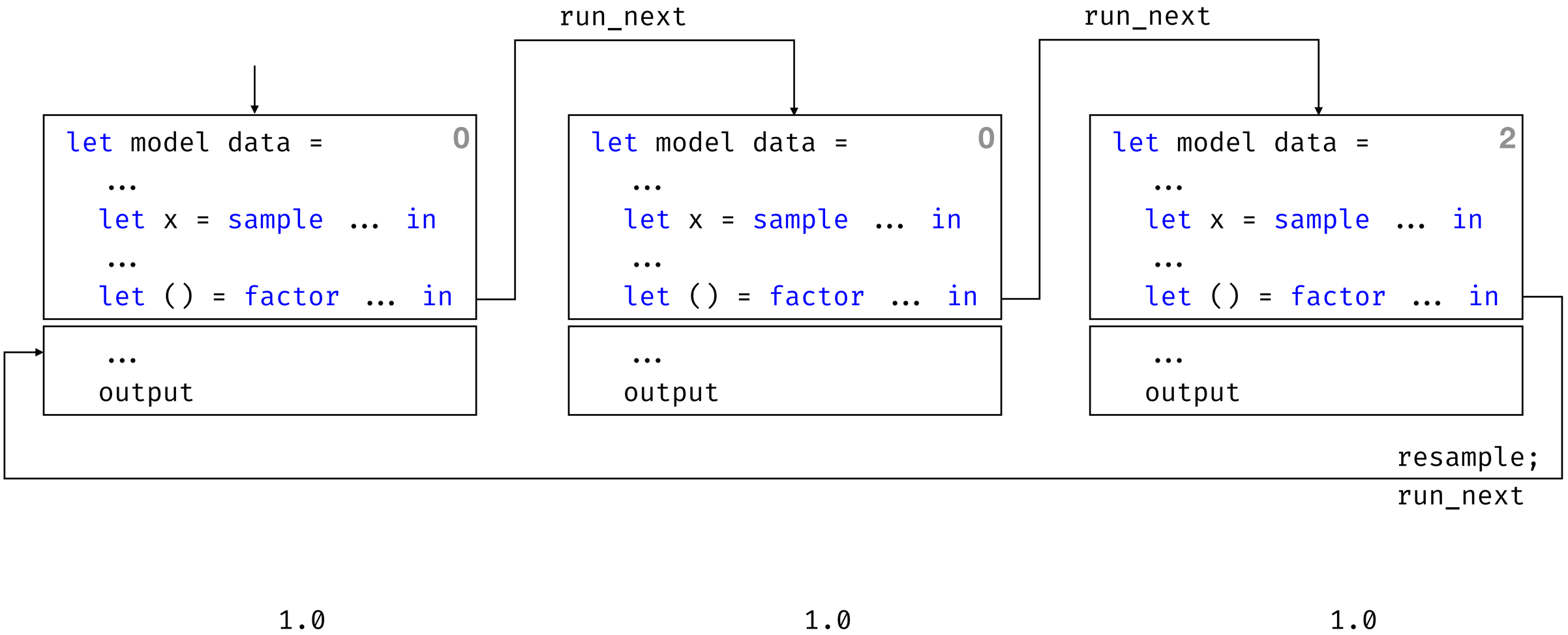
Particle filter



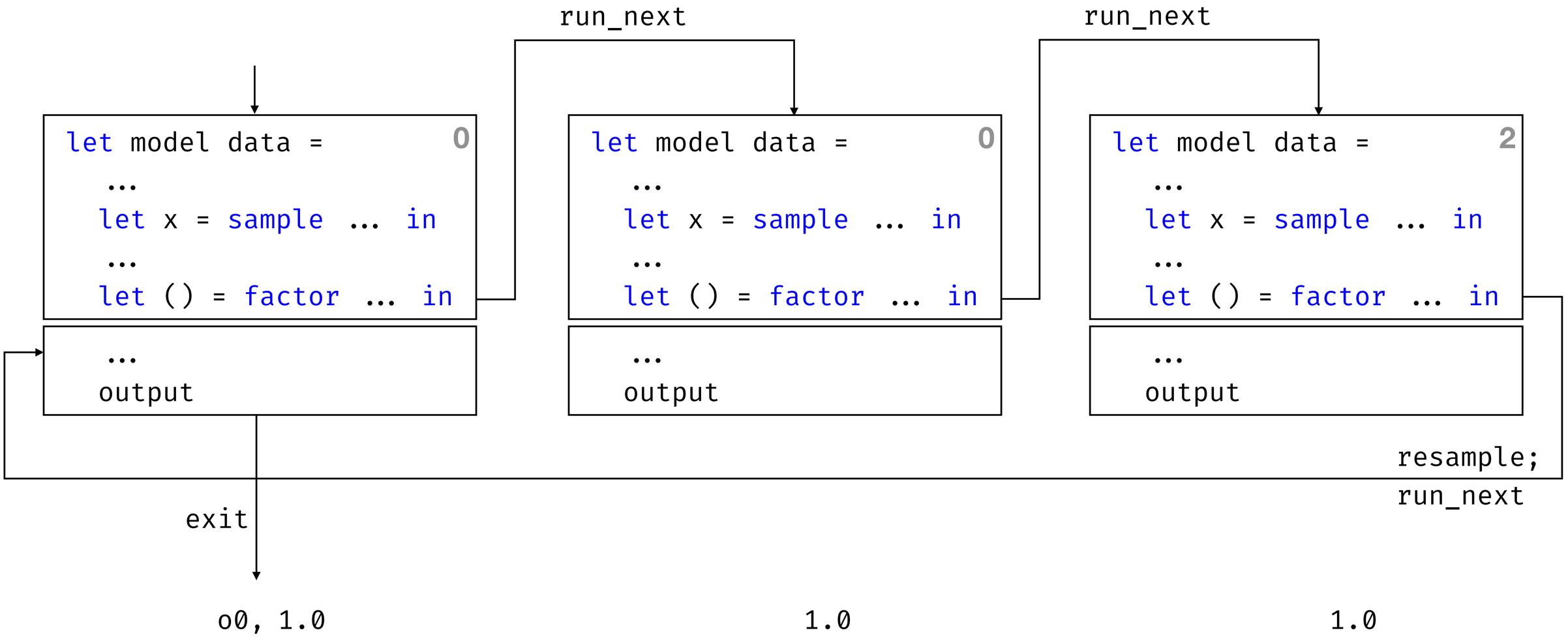
Particle filter



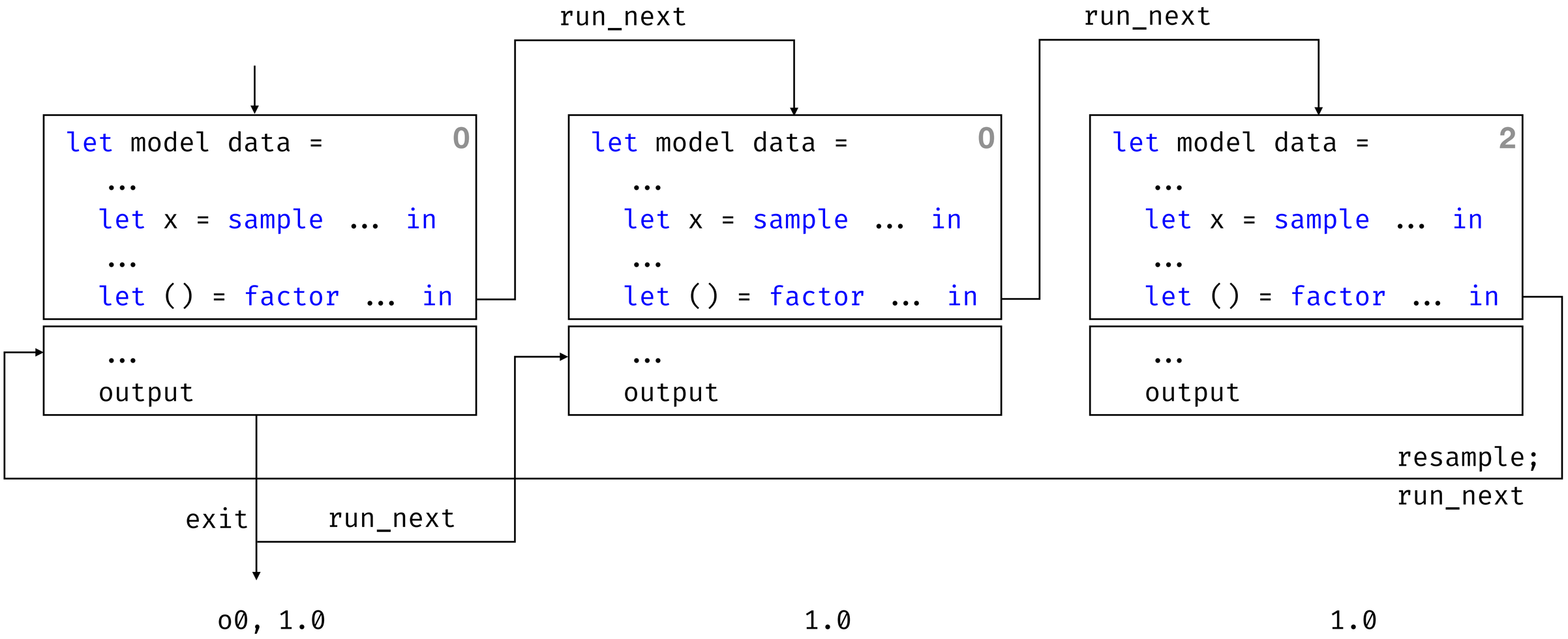
Particle filter



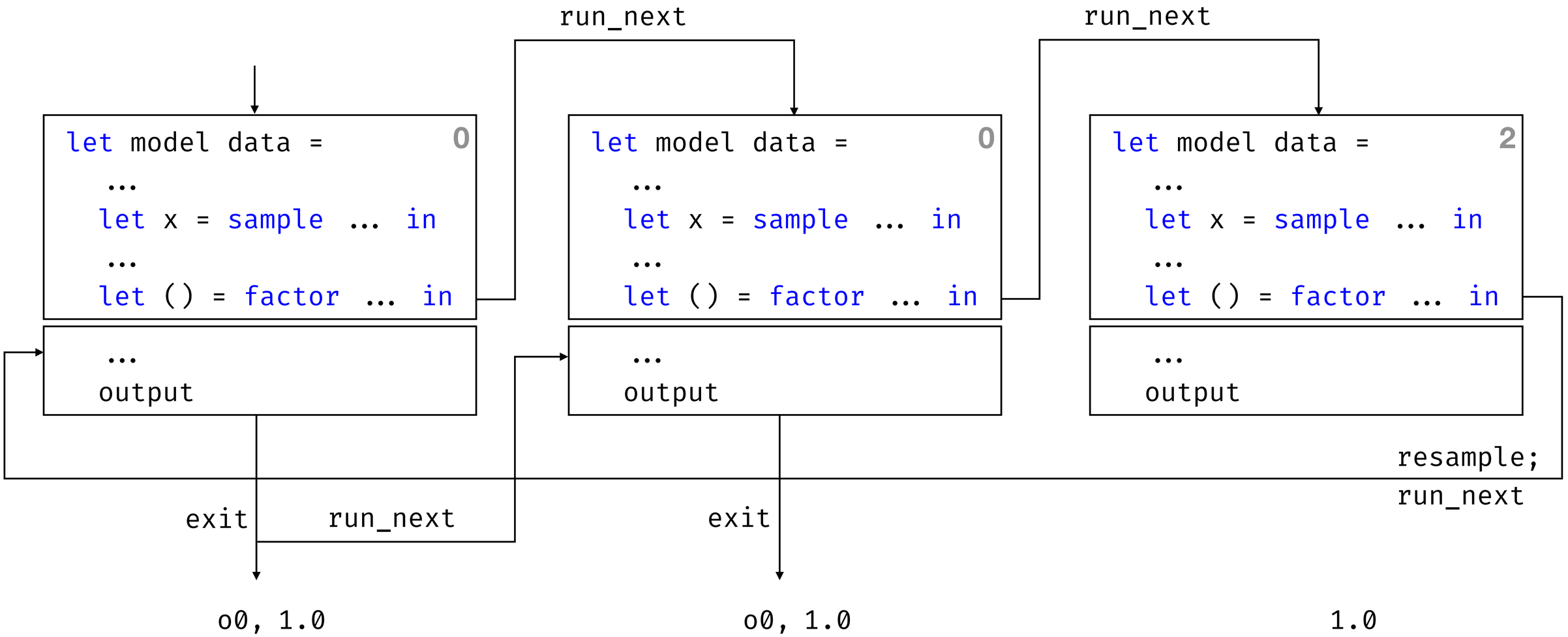
Particle filter



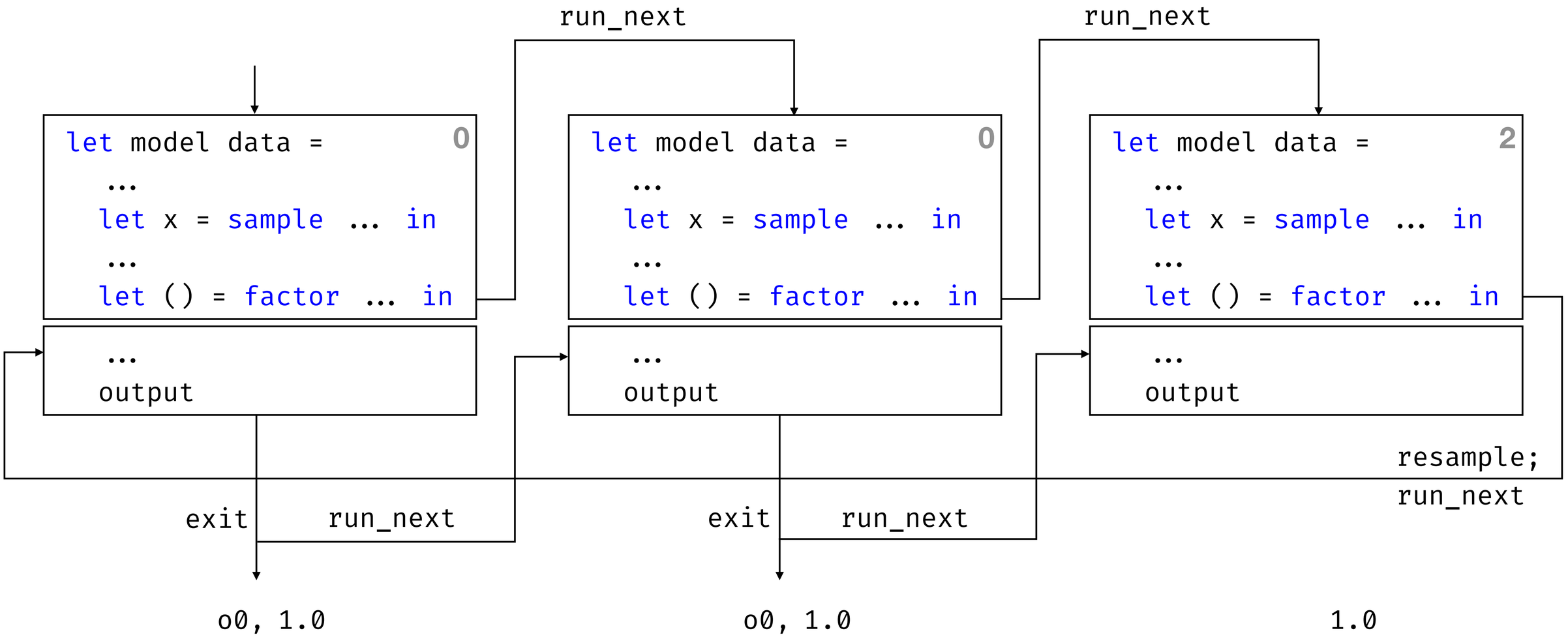
Particle filter



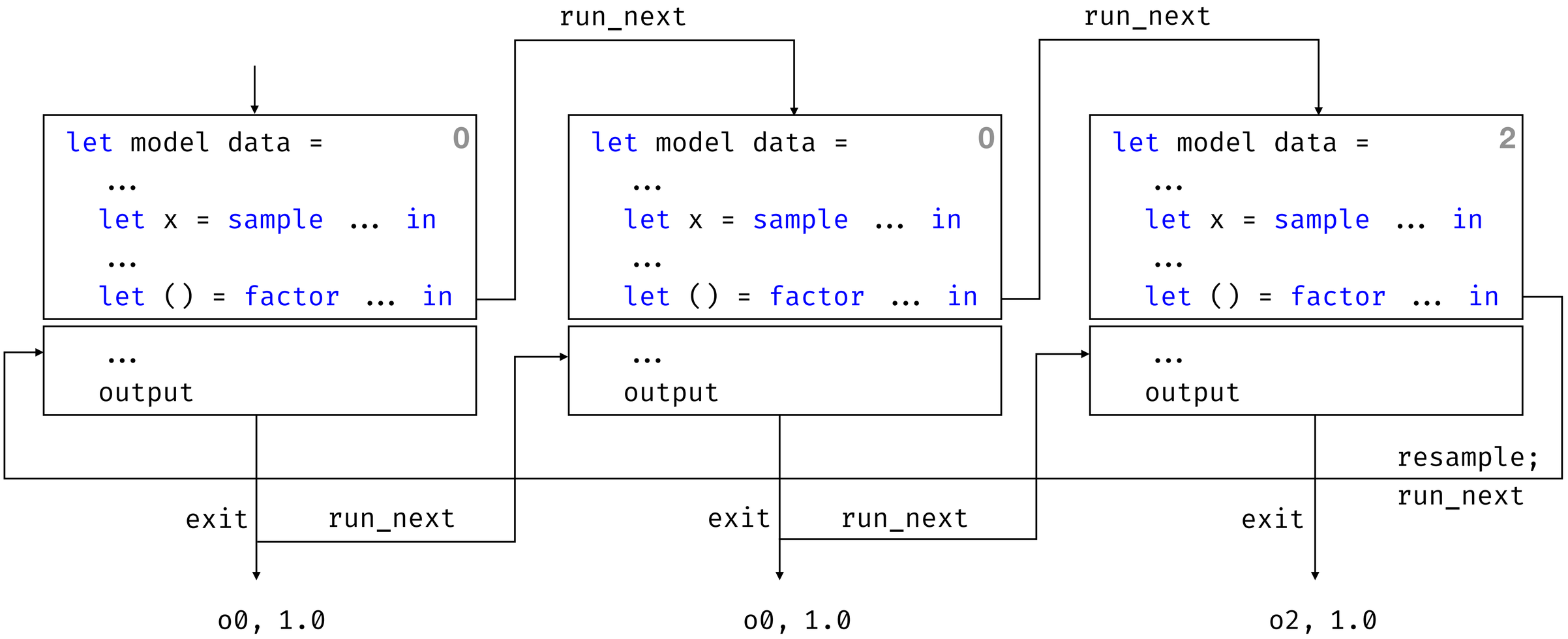
Particle filter



Particle filter



Particle filter



Particle filter

infer.ml

```
module Particle_filter = struct
  include Importance_sampling

  let resample particles =
    let support = List.map (fun p → (p, p.score)) particles in        (* extract support *)
    let dist = Distribution.categorical ~support in
    let particles = List.map (fun _ → Distribution.draw dist) particles in       (* draw *)
    List.map (fun p → {p with score = 0.}) (* reset the scores *)


  let factor s k prob =
    { prob with score = prob.score +. s; k = k () }                    (* break *)
```

Particle filter

infer.ml

```
module Particle_filter = struct
  ...
  let infer ?(n = 1000) model data =
    let rec gen particles =
        if List.exists (fun p → Option.is_none p.value) particles then     (* not finished *)
          let particles = List.map (fun p → p.k p) particles in           (* run next step *)
          let particles = resample particles in                               (* resample *)
          gen particles
        else particles                                                        (* finished *)
      in

    let p_init = { value = None; score = 0.; k = (model data) exit }
    let particles = List.init n (fun _ → p_init) in                  (* initial particles *)
    let support =
      let particles = gen particles in                                      (* run filter *)
      List.map (fun p → (Option.get p.value, p.score)) particles        (* extract results *)
    in
    Distribution.categorical ~support
end
```

HMM: Hidden Markov Model (CPS)

hmm.ml

```
open Infer.Particle_filter

let hmm prob data = assert false
```

Try it in BYO-PPL!

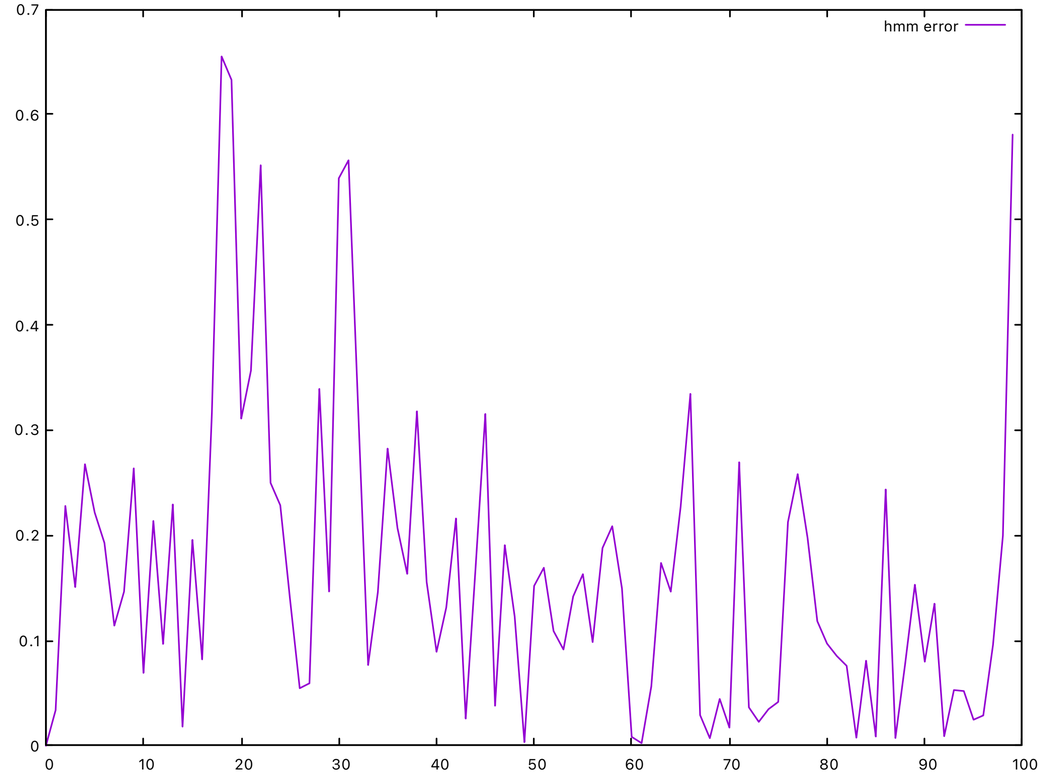
HMM: Hidden Markov Model (CPS)

hmm.ml

```
open Infer.Particle_filter

let hmm data =
  let rec gen states data =
    match (states, data) with
    | [], y :: data → gen [ y ] data
    | states, [] → return states
    | pre_x :: _, y :: data →
        let* x = sample prob (gaussian ~mu:pre_x ~sigma:1.0) in
        let* () = observe prob (gaussian ~mu:x ~sigma:1.0) y in
        gen (x :: states) data
  in
  gen [] data

let _ =
  let data = List.init 20 (fun i → Float.of_int i) in
  let dist = Distribution.split_list (infer ~n:100 hmm data) in
  let m_x = List.map Distribution.mean dist in
  List.iter2 (Format.printf "%f >> %f") data m_x
```

HMM: Hidden Markov Model

```
❯ dune exec ./hmm.exe

0.000000 >> 0.000000
1.052632 >> 0.997546
2.105263 >> 2.300316
3.157895 >> 3.289649
4.210526 >> 4.857555
5.263158 >> 4.907179
6.315789 >> 6.254198
7.368421 >> 7.208341
8.421053 >> 8.432642
9.473684 >> 8.938143
10.526316 >> 9.555007
11.578947 >> 11.098199
12.631579 >> 12.823460
13.684211 >> 13.701444
14.736842 >> 14.934314
15.789474 >> 16.115058
...
```



Exact inference: Enumeration (CPS)

Probabilistic Programming Languages

Enumeration

```
module Enumeration: sig
  type 'a prob
  and 'a next = 'a prob → 'a prob
  and ('a, 'b) model = 'a → ('b → 'b next) → 'b next

  val sample : 'a Distribution.t → ('a → 'b next) → 'b next
  val factor : float → (unit → 'b next) → 'b next
  val infer : ('a, 'b) model → 'a → 'b Distribution.t
end = struct ... end
```

Inference algorithm: depth first exploration

- Maintain a continuation queue
- `sample`: pick one value (and score), push the other in the queue
- `factor`: update the score
- At the end of the model, store the pair (value, score), restart from the top of the queue

Enumeration

infer.ml

```
module Enumeration = struct
  type 'a prob = {
    current_score : float;
    traces : 'a particle list;                    (* executions *)
    support : ('a, float) Hashtbl.t;                 (* results *)
  }

  and 'a particle = { k : 'a next; score : float }
  and 'a next = 'a prob → 'a prob
  and ('a, 'b) model = 'a → ('b → 'b next) → 'b next
```

Enumeration

infer.ml

```
module Enumeration = struct
  type 'a prob = {
    current_score : float;
    traces : 'a particle list;                          (* continuation stack *)
    support : ('a, float) Hashtbl.t;                    (* accumulate results *)
  }

  and 'a particle = { k : 'a next; score : float }
  and 'a next = 'a prob → 'a prob
  and ('a, 'b) model = 'a → ('b → 'b next) → 'b next

  let run_next prob =                                    (* next in the stack *)
    match prob.traces with
    | [] → prob                                         (* nothing left to do *)
    | { k; score } :: traces →
       k { prob with current_score = score; traces }    (* run the first continuation *)
```

Enumeration

infer.ml

```
module Enumeration = struct
  ...
  let exit v prob =                                          (* end, store, run_next *)
    let p = exp prob.current_score in
    (match Hashtbl.find_opt prob.support v with
    | None → Hashtbl.add prob.support v p                        (* add new result *)
    | Some p' → Hashtbl.replace prob.support v (p +. p'));      (* gather same results *)
    run_next prob
```

Enumeration

infer.ml

```
module Enumeration = struct
  ...
  let sample d k prob =
    let sup = Distribution.get_support d in                    (* extract support *)
    let traces =
      List.map
        (fun (v, l) →                                  (* store all continuations *)
          { k = (fun prob → k v prob);
            score = prob.current_score +. l })                 (* update the score! *)
        sup
    in
    run_next { prob with traces = prob.traces @ traces }       (* start first trace *)

  let factor s k prob =
    k () { prob with current_score = prob.current_score +. s }      (* update score *)
```

Enumeration

infer.ml

```
module Enumeration = struct
  ...
  let infer m data =
    let prob =                                                    (* explore all! *)
      (m data) exit
        { current_score = 0.; traces = []; support = Hashtbl.create 11 }
    in
    let support =                                                 (* extract support *)
      prob.support ▷ Hashtbl.to_seq ▷ List.of_seq
      ▷ List.map (fun (v, l) → (v, log l))
    in
    Distribution.categorical ~support                          (* return distribution *)
```

Try it in BYO-PPL!

Inference comparison

Probabilistic Programming Languages

Coin

coin.ml

```
let _ =
  let data = List.init 1000 (fun i → i mod 2) in
  let dist = infer coin data in
  plot dist
```

Coin

coin.ml

```
let _ =
  let data = List.init 1000 (fun i → i mod 2) in
  let dist = infer coin data in
  plot dist
```

Rejection Sampling

Very (very) slow!

Coin

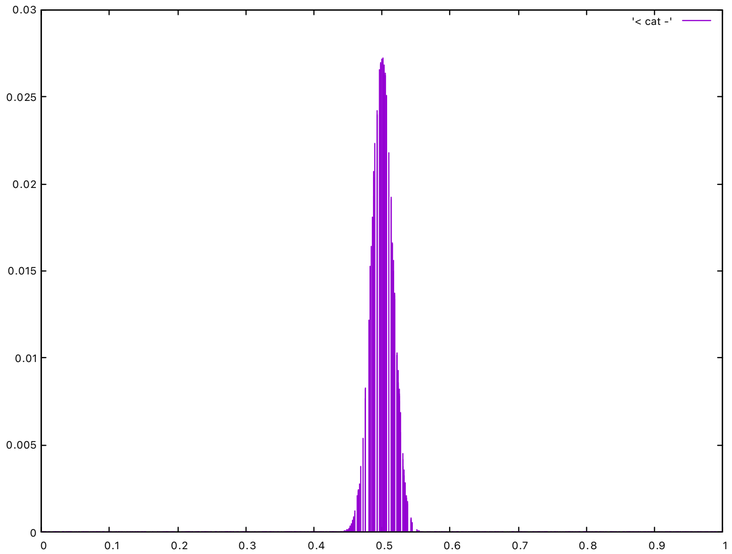
coin.ml

```
let _ =
  let data = List.init 1000 (fun i → i mod 2) in
  let dist = infer coin data in
  plot dist
```

Rejection Sampling

Importance Sampling

Very (very) slow!



Coin

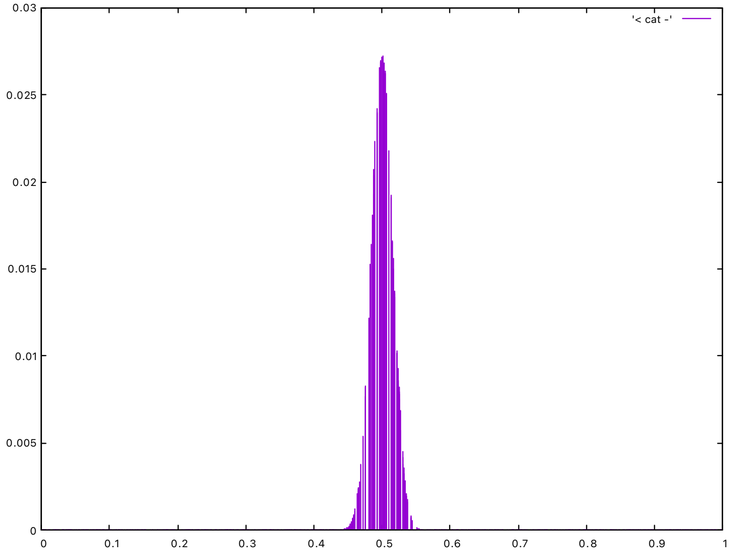
coin.ml

```
let _ =
  let data = List.init 1000 (fun i → i mod 2) in
  let dist = infer coin data in
  plot dist
```

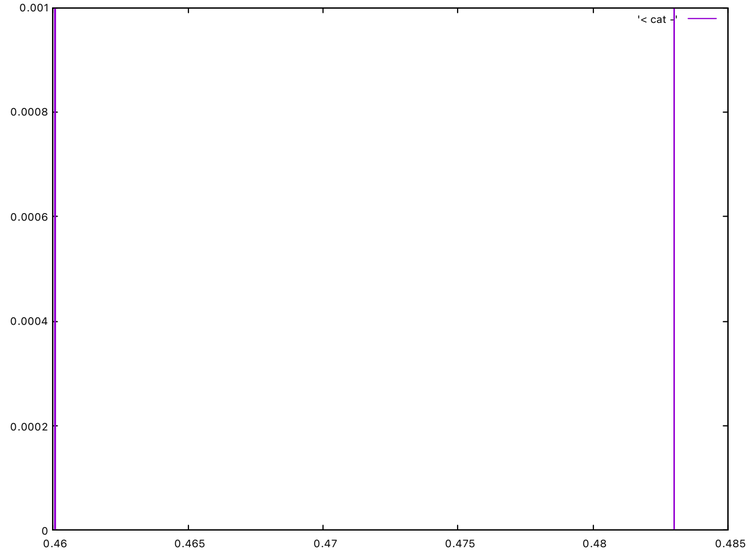
Rejection Sampling

Very (very) slow!

Importance Sampling



Particle Filter



Coin

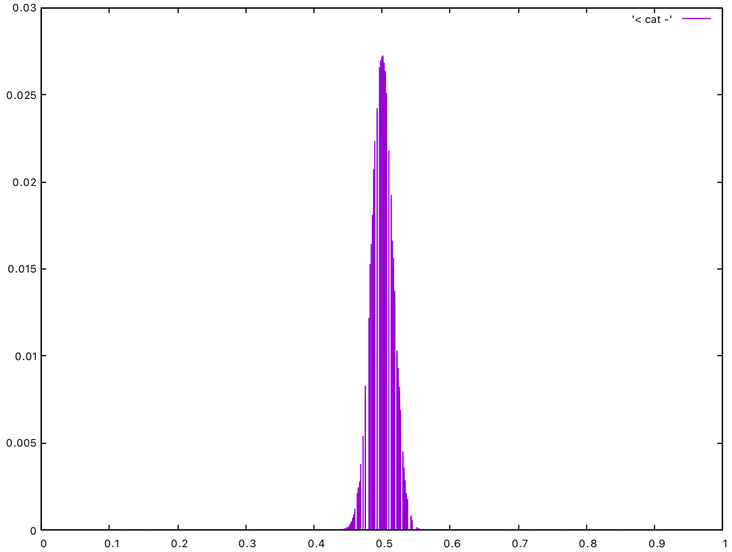
coin.ml

```
let _ =
  let data = List.init 1000 (fun i → i mod 2) in
  let dist = infer coin data in
  plot dist
```

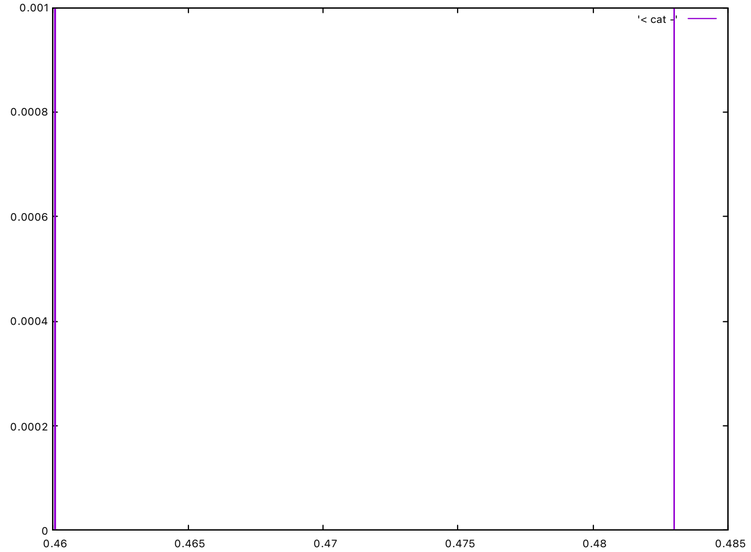
Rejection Sampling

Very (very) slow!

Importance Sampling



Particle Filter



Particle impoverishment

HMM

hmm.ml

```
let _ =
  let data = List.init 100 (fun i → Float.of_int i) in
  let dist = Distribution.split_list (infer hmm data) in
  plot (error (List.rev dist) data)
```

Rejection Sampling

HMM

hmm.ml

```
let _ =
  let data = List.init 100 (fun i → Float.of_int i) in
  let dist = Distribution.split_list (infer hmm data) in
  plot (error (List.rev dist) data)
```

Rejection Sampling

Never terminate!

HMM

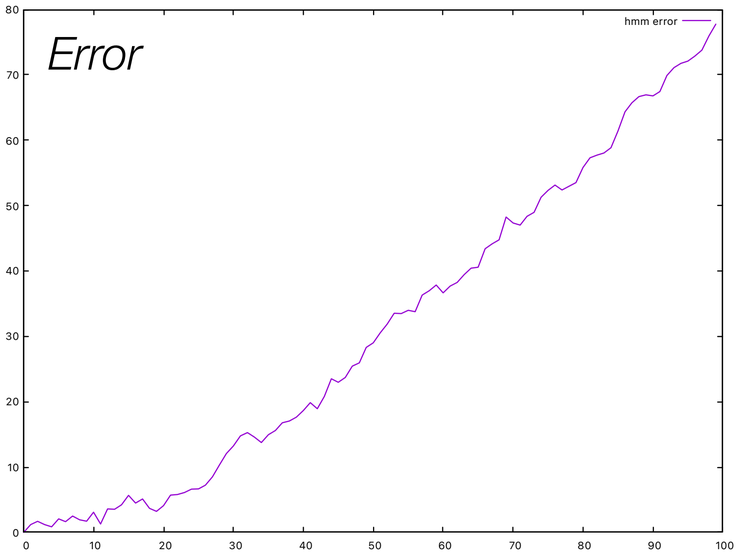
hmm.ml

```
let _ =
  let data = List.init 100 (fun i → Float.of_int i) in
  let dist = Distribution.split_list (infer hmm data) in
  plot (error (List.rev dist) data)
```

Rejection Sampling

Never terminate!

Importance Sampling



HMM

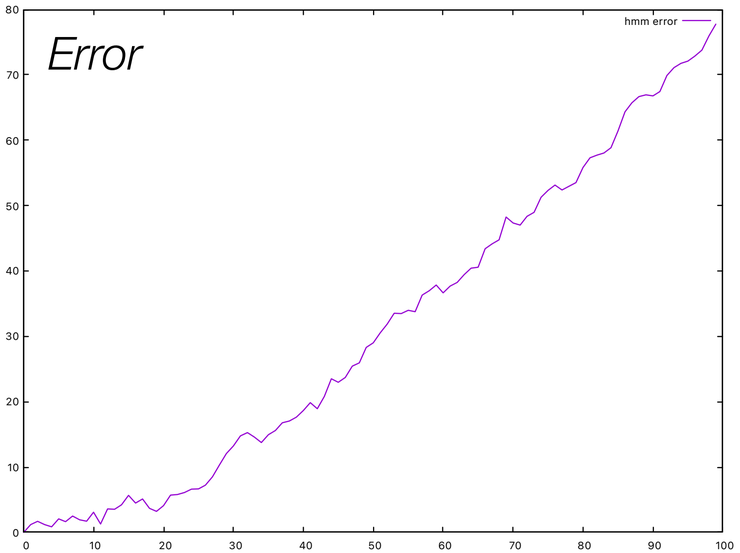
hmm.ml

```
let _ =
  let data = List.init 100 (fun i → Float.of_int i) in
  let dist = Distribution.split_list (infer hmm data) in
  plot (error (List.rev dist) data)
```

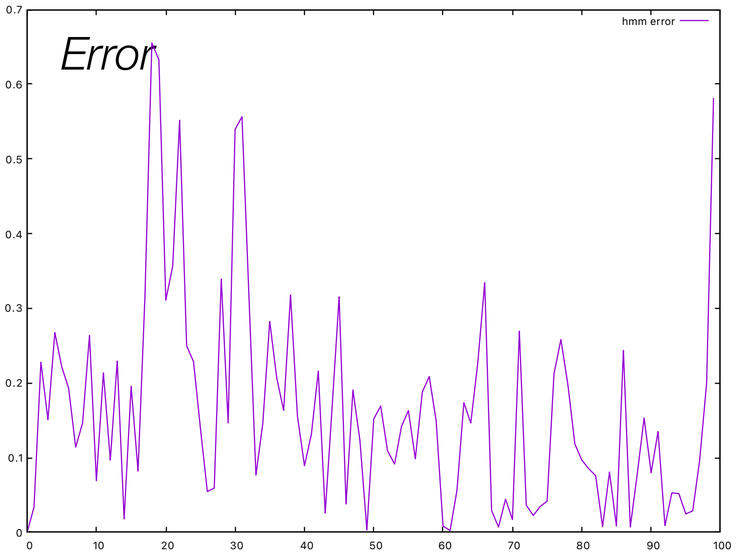
Rejection Sampling

Never terminate!

Importance Sampling



Particle Filter



Inference limitations

Rejection Sampling : Termination

- Only works if valid samples can be generated from the priors
- In practice: simple models with few observations
- E.g., simple discrete models.

Importance Sampling : Weight collapse

- Score strictly decrease at each observation: eventually collapse
- In practice: continuous model to estimate fixed parameters from observations
- E.g., coin, linear regression

Particle Filter: Particle impoverishment

- Duplicate particle without resampling fixed parameter
- In practice: high-dimensional continuous models without fixed parameters
- E.g., hmm, tracker

Inference formalization

BYO-PPL

Sampler

Sampler

Probabilistic semantics $G \vdash^P e : t$

- Expressions are interpreted as weighted samplers
- Draw random samples and track the execution weight
- Given an environment γ, $\{\!|e|\!\}_\gamma = v, w$
- $\{\!|e|\!\} : \Gamma \to V \times [0, \infty)$

$$
\begin{array}{lcl}
\{\!|c|\!\}_\gamma & = & c, 1 \\
\{\!|x|\!\}_\gamma & = & \gamma(x), 1 \\
\{\!|\texttt{sample}(e)|\!\}_\gamma & = & \mathsf{draw}([\![e]\!]_\gamma), 1 \\
\{\!|\texttt{factor}(e)|\!\}_\gamma & = & (), [\![e]\!]_\gamma \\
\{\!|\texttt{let}\ x = e_1\ \texttt{in}\ e_2|\!\} & = & \textit{let}\ v_1, w_1 = \{\!|e_1|\!\}_\gamma\ \textit{in} \\
 & & \textit{let}\ v_2, w_2 = \{\!|e_2|\!\}_{\gamma + [x \leftarrow v_1]}\ \textit{in} \\
 & & v_2, w_1 \times w_2
\end{array}
$$

Sampler

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 & & v_2, w_1 \times w_2
\end{array}
$$

Combine weights

Importance sampling

Inference algorithm

- Run a set of N independent executions
- `sample`: draw a sample from a distribution
- `factor`: associate a score to the current execution
- Gather output values and score to approximate the posterior distribution

$$\llbracket \texttt{infer}(e) \rrbracket_\gamma = \textit{let } \left[v_i, w_i = \{\!\!\{ e \}\!\!\}_\gamma\right]_{1 \leq i \leq N} \textit{ in}$$
$$\textit{let } W = \sum_{i=1}^{N} w_i \textit{ in}$$
$$\lambda U.\ \sum_{i=1}^{N} \frac{w_i}{W} \times \delta_{v_i}(U)$$

Particle filter

Inference algorithm : importance sampling, but...

- Add a resampling step at each `factor`
- Compute the score of the particles to compute a distribution
- Re-sample a new set of particles from this distribution

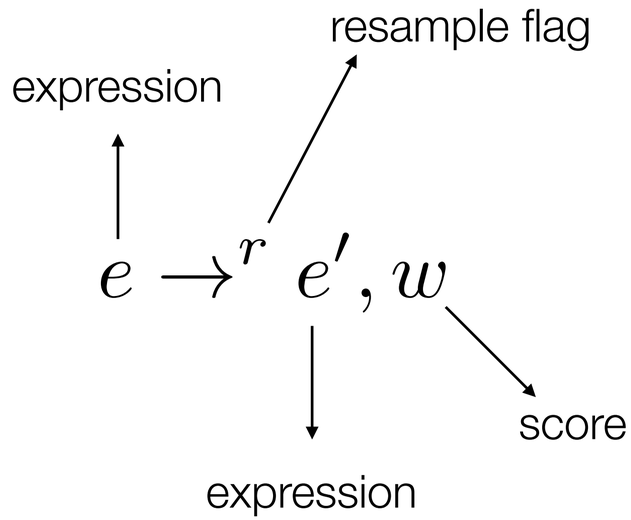
Resampling

$$
\begin{aligned}
\{\!|e|\!\} &\propto \{\!|\texttt{let } d = \texttt{infer}(e) \texttt{ in sample}(d)|\!\}\\
\{\!|\texttt{let } x = e_1 \texttt{ in } e_2|\!\} &\propto \{\!|\texttt{let } d = \texttt{infer}(e_1) \texttt{ in let } x = \texttt{sample}(d) \texttt{ in } e_2|\!\}\\
[\![\texttt{infer}(\texttt{let } x = e_1 \texttt{ in } e_2)]\!] &= [\![\texttt{infer}(\texttt{let } d = \texttt{infer}(e_1) \texttt{ in let } x = \texttt{sample}(d) \texttt{ in } e_2)]\!]
\end{aligned}
$$

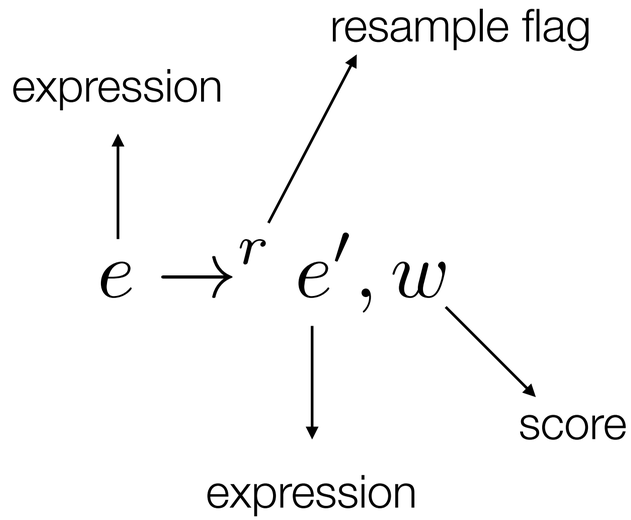
Checkpoints

Probabilistic Programming Languages

Big-step semantics with checkpoints



Big-step semantics with checkpoints



Reduction rules: stop evaluation when r is true

Big-step semantics with checkpoints

$$\frac{e \to^{\mathit{false}} \mathit{true}, 1 \qquad e_1 \to^{r} e_1', w_1}{\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 \to^{r} e_1', w_1}$$

$$\frac{e \to^{\mathit{false}} \mathit{false}, 1 \qquad e_2 \to^{r} e_2', w_2}{\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 \to^{r} e_2', w_2}$$

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$$\frac{e_1 \to^{\mathit{false}} v_1, w_1 \qquad e_2 \to^{r} e_2', w_2}{\texttt{let}\ x = e_1\ \texttt{in}\ e_2 \to^{r} e_2', w_1 \times w_2}$$

$$\frac{e \to^{\mathit{false}} \mu, 1}{\texttt{sample}(e) \to^{\mathit{false}} \mathsf{draw}(\mu), 1}$$

$$\frac{e \to^{\mathit{false}} s, 1}{\texttt{factor}(e) \to^{\mathit{true}} (), s}$$

Big-step semantics with checkpoints

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Combine weights

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Stop!

Particle filter

$$\frac{[e_i \to^{r_i} e'_i, w_i]_{1 \le i \le N} \qquad \bigvee_{1 \le i \le N} r_i \qquad \rho = \mathsf{Cat}\left(\{e'_i, w_i\}_{1 \le i \le N}\right) \qquad [\mathsf{draw}(\rho)]_{1 \le i \le N} \Rightarrow \mu}{[e_i]_{1 \le i \le N} \Rightarrow \mu}$$

$$\frac{\left[e_i \to^{\mathit{false}} v_i, w_i\right]_{1 \le i \le N} \qquad W = \sum_{i=1}^{N} w_i}{[e_i]_{1 \le i \le N} \Rightarrow \lambda U.\ \sum_{i=1}^{N} \frac{w_i}{W}\ \delta_{v_i}}$$

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Resampling

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Gather the results

$$\frac{[e]_{1\le i\le N} \Rightarrow \mu}{e \Rightarrow^{N} \mu}$$

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$$\frac{[e_i \to^{r_i} e'_i, w_i]_{1\le i\le N} \qquad \bigvee_{1\le i\le N} r_i \qquad \rho = \mathsf{Cat}\left(\{e'_i, w_i\}_{1\le i\le N}\right) \qquad [\mathsf{draw}(\rho)]_{1\le i\le N} \Rightarrow \mu}{[e_i]_{1\le i\le N} \Rightarrow \mu}$$

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Gather the results

$$\frac{[e]_{1\le i\le N} \Rightarrow \mu}{e \Rightarrow^{N} \mu}$$

Launch N particles

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