

Ultra light dark matter ringing earth normal modes

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1 Ultra Light Dark Matter

If dark matter is in the form of an ultralight scalar field which can be treated as a classical field, one can represent it in the following form:

$$\phi(t, \mathbf{r}) = \phi_0 e^{i(m_\phi t + \mathbf{p} \cdot \mathbf{r})}$$

This could then have some coupling to normal matter, such as coupling to baryon number or B-L, which would lead to a force in the Fourier domain of the form:

$$\mathbf{f}(\omega, \mathbf{r}) \sim gq\sqrt{2\rho_{DM}} \delta(\omega - \omega_\phi) e^{i\mathbf{p} \cdot \mathbf{r}} \mathbf{v}$$

where g is coupling between the dark matter field and normal matter, q is the test particle “charge” (B, B-L, etc.), ρ_{DM} is the local dark matter density, and ω_ϕ is the frequency corresponding to the dark matter mass.

This can be generalized to be more model agnostic to include any model which gives a weakly coupled plane wave as:

$$\mathbf{f}(\omega, \mathbf{r}) \sim f(\omega)\delta(\omega - \omega_\phi) e^{i\mathbf{p} \cdot \mathbf{r}} \hat{\mathbf{f}}$$

2 Earth Normal Modes

Following the derivation by Aki and Richards [1], to derive the displacement due to a generic force we start by analyzing the equation of motion of the α th particle in a discrete collection of the point particles:

$$m_a \ddot{\mathbf{u}}_\alpha + \gamma \dot{\mathbf{u}}_\alpha + \sum_{\beta}^N c_{\alpha\beta} \mathbf{u}_\beta = \mathbf{f}_\alpha$$

where $c_{\alpha\beta}$ is the spring constant between the α th and β th particle, \mathbf{u}_α is the displacement of the α th particle, γ is the damping constant, and m_α is the

mass of the α th particle. Now decomposing into normal modes and taking the Fourier transform:

$$-m_a \omega^2 \sum_i u_{\alpha,i} \hat{\mathbf{n}}_{\alpha,i} + i\omega \sum_i \gamma_i u_{\alpha,i} \hat{\mathbf{n}}_{\alpha,i} + \sum_{\beta}^N c_{\alpha\beta} \sum_i u_{\beta,i} \hat{\mathbf{n}}_{\beta,i} = \mathbf{f}_{\alpha}$$

where $u_{\beta,i}$ and $\hat{\mathbf{n}}_{\beta,i}$ are the amplitude in the i th mode (i represents a selection of n , l , and m) and its respective normal mode vector. Using the orthogonality and normalization given by Aki and Richards:

$$\mathbf{u}_{\alpha}(\mathbf{r}, \omega) = \sum_i \frac{\sum_{\beta} \hat{\mathbf{n}}(\mathbf{r}_{\beta})_{\beta,i}^* \cdot \mathbf{f}(\mathbf{r}_{\beta}, \omega)_{\beta}}{-\omega^2 + i \frac{\omega \omega_i}{Q} + \omega_i^2} \hat{\mathbf{n}}(\mathbf{r})_{\alpha,i}$$

Generalizing to continuum:

$$\mathbf{u}(\mathbf{r}, \omega) = \sum_i \frac{\int \hat{\mathbf{n}}(\mathbf{r}')_i^* \cdot \mathbf{f}(\mathbf{r}', \omega) dV'}{-\omega^2 + i \frac{\omega \omega_i}{Q} + \omega_i^2} \hat{\mathbf{n}}(\mathbf{r})_i$$

As reference the normal modes for spheroidal motion for $l \neq 0$ are as follows [6] :

$${}_n \hat{\mathbf{n}}(\mathbf{r})_l^m = a_{n,l}(r) Y_l^m(\theta, \phi) \hat{\mathbf{r}} + b_{n,l}(r) R \nabla Y_l^m(\theta, \phi)$$

$$a_{n,l}(r) = c_{n,l} \left[\alpha_{n,l} \frac{dj_l(qr)}{d(qr)} - \beta_{n,l} l(l+1) \frac{j_l(kr)}{kr} \right]$$

$$b_{n,l}(r) = c_{n,l} \frac{r}{R} \left[\alpha_{n,l} \frac{j_l(qr)}{qr} - \beta_{n,l} \left(\frac{j_l(kr)}{kr} + \frac{dj_l(kr)}{d(kr)} \right) \right]$$

$$\alpha_{n,l} = \frac{1}{2} \left[\frac{d^2 j_l(kR)}{d(kR)^2} + (l-1)(l+2) \frac{j_l(kR)}{(kR)^2} \right]$$

$$\beta_{n,l} = \frac{q}{k} \frac{d}{d(qR)} \left(\frac{j_l(qR)}{qR} \right)$$

$$q^2 = \frac{\rho \omega^2}{\lambda + 2\mu}, \quad k^2 = \frac{\rho \omega^2}{\mu}$$

where λ and μ are the Lamé parameters of the material and R is the radius of the earth.

And for $l = 0$:

$${}_n \hat{\mathbf{n}}(\mathbf{r})_0^0 = a_{n,0}(r) \hat{\mathbf{r}}$$

$$\alpha_{n,l} = c_{n,0} \frac{dj_0(qr)}{d(qr)}$$

3 Constraining dark matter with normal modes

Since the earth normal modes have been measured during times with little to no terrestrial excitation [2, 3], we can combine these two results to set limits on the dark matter coupling to normal matter. The picture is that if the earth is being constantly rung with the dark matter (DM) field, then the normal mode amplitudes would never fall below some level. Additionally, since the earth and DM field would have been interacting for an extended period of time, the normal mode and DM field would be phase coherent at the time of measurement. This allows us to ignore any time evolution of the proposed response.

Picking one normal mode and measuring the displacement at the given normal mode frequency would yield the following:

$$\mathbf{u}(\mathbf{r}, \omega) = \frac{-iQ}{\omega^2} \hat{\mathbf{n}}(\mathbf{r}) \int \hat{\mathbf{n}}(\mathbf{r}')^* \cdot \mathbf{f}(\mathbf{r}', \omega) dV'$$

Substituting the plane wave force in:

$$\mathbf{u}(\mathbf{r}, \omega) = \frac{-iQf(\omega)}{\omega^2} \hat{\mathbf{n}}(\mathbf{r}) \int \hat{\mathbf{n}}(\mathbf{r}')^* \cdot \hat{\mathbf{f}} e^{i\mathbf{p} \cdot \mathbf{r}'} dV'$$

Expanding plane wave in spherical harmonics:

$$\mathbf{u}(\mathbf{r}, \omega) = \frac{-iQf(\omega)}{\omega^2} \hat{\mathbf{n}}(\mathbf{r}) \int \hat{\mathbf{n}}(\mathbf{r}')^* \cdot \hat{\mathbf{f}} \left(4\pi \sum_l \sum_m i^l j_l(pr) Y_l^m(\hat{\mathbf{p}}) Y_l^{m*}(\hat{\mathbf{r}}) \right) dV'$$

From here we can substitute the normal mode vector, set up a coordinate system, do the integrals, and (with some assumptions about the placement of the sensor with respect to the nodes of the normal mode) get a relationship between the measured displacement and the amplitude of the plane wave force.

4 Comparison to similar literature

In Ref. 4 and 5, the authors do a similar analysis by relating the proposed dark matter fields to a metric strain amplitude and thus set limits on apparent changes of fundamental constants caused by the DM field. This may be able to be generalized to set limits on the coupling strength of the field but the path to this was not obvious when starting this analysis. The above equations however give a framework to calculate the response given any plane wave weakly interacting with the earth at a normal mode frequency.

5 References

1. Quantitative Seismology 2nd Edition, Keiiti Aki and Paul G. Richards
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