Generalized algebras of Clebsch-Gordan, $6j$, $9j$, F-moves

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1 Introduction

The purpose of this paper is to provide exposition and reference material for constructing 6j and 9j symbols of various algebras, and eventually to also construct projection transformations (eg the conventional Clebsch-Gordan coefficients are associated with the group chain $U(1) \subset SU(2)$, but we want to generalize this to other group subsets). This is somewhat complicated by different notation conventions; historically in $SU(2)$ theory the Clebsch-Gordan/3j, 6j and 9j coefficients are used, which have been generalized in similar notation to point groups (used eg in quantum chemistry). On the other hand anyons are often described in terms of F-symbols and F-moves, of which there are two common variants (a 3-1 move and a 2-2 move). In practical calculations, the F-symbols are probably the most useful; these give directly the coefficients of basis transformations eg $\left[F_j^{j_1,j_2,j_3}\right]_{j_{12},j_{23}} = \langle (j_1 j_2 j_{12}) j_3 j | j_1 (j_2 j_3 j_{23}) j \rangle$, which differs by a normalization and phase factors from a 6j symbol. However the 6j symbols have more symmetry so are useful as an underlying implementation tool in computational algorithms. Hence we want to find generic ways to transform beyween F-symbols and 6j symbols.

The general structure that admits a consistent set of F-symbols is known as a fusion tensor category. It is necessary and sufficient that the F-symbols obey the Biedenharn-Elliott identity, also known as the Pentagon relation.

The F-symbols alone are not sufficient to describe an operator algebra. In particular, the F-symbols do not in general give enough information to be able to determine particle statistics. For this, we need also the braiding statistics, which is encapsulated in the symbol R^c_{ab}, which is a coefficient (or, for non-multiplicity-free algebras, a matrix) that gives the phase change of two charges a and b, which fuse to particle type c, and we braid b over the top of a. For a given fusion algebra, there may be more than one possible set of braiding. The consistency between the R and F symbols is given by the Hexagon relation. This gives a braided tensor fusion category, which is ultimately the object that we are interested in. Mathematically, we need also some other conditions, which I find to be not particularly well specified in the literature, such as K-linear, semi-simplicity, existence of duals, and non-degenerate. Generally the literature also looks at the *finite* case (finite rank, number of charges). We are interested also in the infinite case, which arises from semi-simple Lie agebras. I haven't seen any discussion of infinite braided tensor categories that are *not* derived from a semi-simple Lie algebra.

The classification of *finite* braided tensor fusion categories is more-or-less known, up to categories of rank 5 (where the rank denotes the number of distinct charges in the category). We can subdivide braided tensor fusion categories (BTC for short) into some syb-types,

- *symmetric*: A symmetric braided tensor category (STC) satisfies $R^c_{ab} = R^c_{ba}$ for all possible a, b, c. This implies $(R^c_{ab})^2 = 1$, and hence the charges behave as essentially fermionic or bosonic character. There is a strong connection between STC and finite groups. I am not sure if this is 1–1. For the purposes of this document, we also include infinite groups (ie, semi-simple Lie groups) in this class.
- *modular tensor category* (MTC): this is the 'opposite' of an STC, in the sense that the 'symmetric centre' of the category is trivial.
- *integral*: if the quantum dimension of all objects is an integer, then the BTC is known as integral. All STC's are integral.
- *pointed*: if the quantum dimension of all objects is unity, then the BTC is known as pointed. STC's arising from an abelian group are pointed. But there are many examples of MTC's that are also pointed, for example abelian anyons.

The specific symmetry groups that we want to handle here are

1. $U(1)$
2. $SU(2)$
3. finite cyclic groups Z_n
4. Dihedral groups D_n. THere are two notations for dihedral groups, sometimes D_{2n} is used.
5. Ising anyons $SU(2)_2$
6. Fibonacci anyons $SU(2)_3$
7. $SU(3)_2$ – discussed in a nice paper by Ardonne and Slingerland[1]

Where possible we want to allow projective representations too. Such representations occur as boundary conditions or edge particles in topological states. For the Dihedral groups, this means allowing for half-integer quantum numbers.

2 Literature summary

In this section we give an overview of the primary literature that was used in the development of this document.

- [2] The canonical reference for anyons in physics.
- [3] Introduction to modular categories.

3 Quantum numbers

a quantum number is a representation of some symmetry group. Fusion of quantum numbers is equivalent to taking products of representations, and is given by the Clebsch-Gordan expansion,

$$a \otimes b \simeq \bigoplus N_{ab}^c \, c \tag{1}$$

where N_{ab}^c is the *multiplicity* of the representation c in the expansion of $a \otimes b$. Technically we should distinguish the representation label j from the representation matrices $D(j)$, but we won't bother with this distsinction here, unless it gives a clearer notation. If $N_{ab}^c > 1$ then a representation occurs more than once in the expansion, and this greatly complicates the description of the coupling. This is commonly referred to as the *multiplicity problem*. Symmetry groups that always have $N_{ab}^c \leq 1$ are referred to as *multiplicity free*.

Examples:

- $U(1)$: label the representations by a (half-)integer j, eg representing the particle number or z-component of spin. Then the Clebsch-Gordan expansion corresponds to simple addition of the quantum number, $D(j_1) \otimes D(j_2) = D(j_1 + j_2)$.
- $SU(2)$: The product of $SU(2)$ spins j_1 and j_2 (always non-negative) is a sum of $\min(2j_1{+}1, 2j_2{+}1)$ representations. $N_{j_1,j_2}^j = 1$, if $j \in \{|j_1 - j_2|, |J_1 - j_2| + 1, \cdots, j_1 + j_2\}$, and 0 otherwise.
- Ising anyons, corresponding to $SU(2)_3$, contain spins 1 (the identiy, which has spin 0) and τ (analagous to a spin 1), with the fusion rules $1 \times 1 = 1$, $1 \times \tau = \tau \times 1 = \tau$, $\tau \times \tau = 1 + \tau$.

- $SU(3)_3/Z_3$ is probably the simplest algebra that is *not* multiplicity-free[1], via

$$8 \times 8 = 1 + 8 + 8' + 10 + \bar{10} \tag{2}$$

  where the representation 8 occurs twice on the right hand side.

The 'zero' quantum number always exists. This is often denoted with a 1, since it corresponds to the trivial (identity) representation of the algebra. The identity quantum number is also referred to as *scalar*, since it is the unique representation that is invariant under symmetry transformations. A representation j has a conjugate or dual representation, denoted $\bar{j}$, or in some literature as j^*.

- In $U(1)$ the conjugate correpsonds to negation, so $\bar{j} = -j$.
- $SU(2)$ is self-dual, so $\bar{j} = j$.

The 'conjugate' operation here has nothing to do with complex conjugation. In the Matrix Product Toolkit, the operation $\bar{j}$ is referred to as `adjoint(j)`. Commonly, discussions of $SU(2)$ coupling coefficients omit the quantum number conjugation, but it is still important eg for spinful particles we often have $U(1) \times SU(2)$, and we cannot neglect the quantum number conjugation in the $U(1)$ case.

The *dimension* of a representation j is the dimension of the matrix $D(j)$. This is often denoted d_j. For classical symmetries the dimension is an integer. For non-abelian anyons, the dimension can be fractional and is then known as the *quantum dimension*. The dimensions satisfy the sum rule

$$d_{j_1} d_{j_2} = \sum_j N^j_{j_1,j_2} \, d_j \tag{3}$$

If, for some symmetry group, all of the representations have $d_j = 1$, then the group is abelian. This is equivalent to the statement that for each a, b there is exactly one c such that $N^c_{ab} = 1$, and all other multiplicities are zero.

The quantum dimension q_a can be obtained as the largest eigenvalue of N^c_{ab}, treated as a matrix in the indices b, c. It is also possible to define the quantum dimensions through the F-symbols (see below), as

$$d_a = d_{\bar{a}} = \left| [F^{a\bar{a}a}_a]_{1,1} \right|^{-1} \tag{4}$$

The fusion multiplicities obey the relations

$$N^c_{ab} = N^c_{ba} = N^{\bar{c}}_{\bar{a}\bar{b}} = N^{\bar{a}}_{b\bar{c}} \tag{5}$$

This is 12 relations in total (all permutations of a, b, c plus the dual versions). We also have

$$\sum_e N_{ab}^e N_{ec}^d = \sum_f N_{af}^d N_{bc}^f \tag{6}$$

This is essentially associativity of tensor products; the number of ways of fusing charges a, b, c to a charge d is the same, whether we to the fusion as $(a \otimes b) \otimes c$ or $a \otimes (b \otimes c)$.

4 F-symbols

Hilbert spaces are complex spaces with an in inner product, so we need to distinguish bra's and ket's as separate entities. In diagrammantic notation, we do this with arrows that end at a point, either as an incoming arrow or an outgoing arrow. It doesn't matter whether we regard an incoming arrow as a bra state or as a ket, as long as we are globally consistent. Reversing the orientation of a line requires conjugating the charge label, ie

$$\Big\downarrow a \quad = \quad \Big\uparrow \bar{a} \tag{7}$$

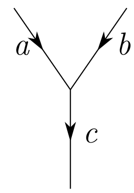
We can think of this as the equivalence $|a\rangle \otimes \langle a| = \langle \bar{a}| \otimes |\bar{a}\rangle$.

Normalization of basis vectors is important. A commonly used normalization convention is known as 'isotopy invariance', which means that the value of a diagram is invariant under continuous deformations, as long as end points are fixed and lines are not passed through each other or around open end points. This is equivalent to normalization of basis vectors, $\langle a|a\rangle = 1$. Note that this convention isn't as obvious as it looks: in $SU(2)$ it is common to consider a spin j as a 'reduced ket', $\|j\rangle$ which stands for a vector of $2j+1$ basis vectors (ie, a mixed state)

$$\|j\rangle \equiv \begin{pmatrix} |j; -j\rangle \\ |j; -j+1\rangle \\ \vdots \\ |j; j\rangle \end{pmatrix} \tag{8}$$

in which case we have $\langle j\|j\rangle = 2j + 1 = d_j$.

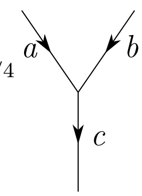
Fusion of two quantum numbers looks like



(9)

where we fuse two incoming states a, b into an outgoing state c. Isotopy invariance means that we need a normalization factor to relate this to basis vectors,

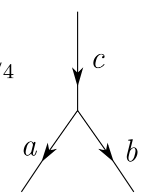
$$\left(\frac{d_c}{d_a d_b}\right)^{1/4}$$



$$= |c\rangle\langle a|\langle b| \tag{10}$$

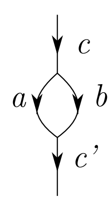
Or we can split the charge c into a and b,

$$\left(\frac{d_c}{d_a d_b}\right)^{1/4}$$

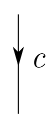


$$= |a\rangle|b\rangle\langle c| \tag{11}$$

Putting these together, we can form the inner product,

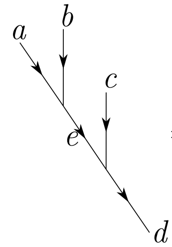


$$= \delta_{c,c'}\sqrt{\frac{d_a d_b}{d_c}}$$

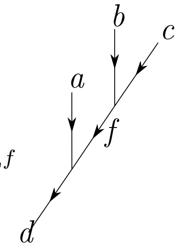


(12)

When fusing three particles a, b, c together, there is an ambiguity as to whether we fuse a, b first and then c, or fuse a with the fusion of b, c. The difference is the intermediate quantum number, here denoted e or f. The difference between these forms of fusion is given by the F symbol.



$$= \sum_f \left[F_d^{a,b,c}\right]_{e,f}$$



(13)

Alternatively, we can write

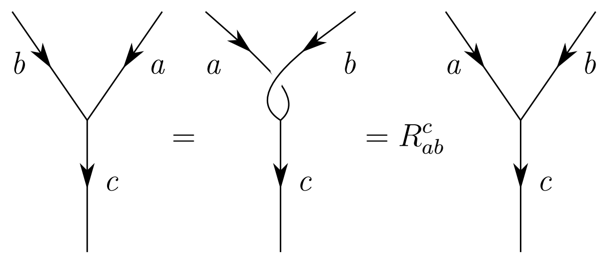
$$\|((abe)c)d\rangle = \sum_f \left[F_d^{a,b,c}\right]_{e,f} \|(a(bcf))d\rangle \tag{14}$$

or as

$$\left[F_j^{j_1,j_2,j_3}\right]_{j_{12},j_{23}} = \langle (j_1 j_2 j_{12}) j_3 j \| j_1 (j_2 j_3 j_{23}) j\rangle \tag{15}$$

where we use the notation $\|a\rangle$ to denote a state vector, as a reminder that we may have $d_a > 1$, so it isn't an ordinary vector. But we need to be careful with normalization of the inner product, maybe we should have a $1/d_j$ here? For given exterior legs a, b, c, d we can view the F-symbol as the basis transformation that maps between two different choices of internal labelling.

Note that the left-right ordering of spins a, b, c, d is important here. Although the Clebsch-Gordan series for $a \otimes b$ is the same as that for $b \otimes a$, the F-symbol is *not* symmetric under this interchange. We can represent this with an R matrix,



(16)

5 Symmetries of 6j symbols

See Butler page 59

6 Irreducible Tensors

Using the normalization convention of Beidenharn, we define

$$\langle\, j'm' \,|\, T^{[k]}_M \,|\, jm \,\rangle = \langle\, j' \,\|\, \boldsymbol{T}^{[k]} \,\|\, j \,\rangle\; C^{j\;\,k\;\,j'}_{mMm'} \tag{17}$$

Using the orthogonality of the CG coefficients, this defines the reduced matrix elements[1],

$$\langle\, j' \,\|\, \boldsymbol{T}^{[k]} \,\|\, j \,\rangle = \sum_{mM} C^{j\;\,k\;\,j'}_{mMm'} \langle\, j'm' \,|\, T^{[k]}_M \,|\, jm \,\rangle \tag{18}$$

[1] That this factorization exists is precisely the celebrated Wigner-Eckart theorem.

where m' is arbitrary. Alternatively, we can sum over m' and divide by $(2m'+1)$,

$$\langle\, j' \,\|\, \boldsymbol{T}^{[k]} \,\|\, j \,\rangle = \sum_{mMm'} \frac{1}{2j'+1} C^{j\ k\ j'}_{mMm'} \langle\, j'm' \,|\, T^{[k]}_M \,|\, jm \,\rangle \tag{19}$$

This is a hint that this normalization of the reduced matrix elements is not as symmetric as one might like; however using this normalization the reduced and full matrix elements of scalar operators coincide. In particular, the reduced matrix elements of the identity operator are

$$\langle\, j' \,\|\, I \,\|\, j \,\rangle = \delta_{j'j} \tag{20}$$

and those of the angular momentum operator are

$$\langle\, j' \,\|\, J \,\|\, j \,\rangle = \sqrt{j(j+1)}\; \delta_{j'j} \tag{21}$$

But the trace of a scalar operator needs to be normalized properly;

$$\mathrm{Tr}\, X = \sum_j (2j+1) \langle\, j \,\|\, X \,\|\, j \,\rangle \tag{22}$$

The normalization used here is *not* equivalent to that of Edmonds, who instead defines

$$\langle\, j'm' \,|\, T^{[k]}_M \,|\, jm \,\rangle = \langle\, j' \,\|\, \boldsymbol{T}^{[k]} \,\|\, j \,\rangle_{\text{Edmonds}} \frac{(-1)^{j-m}}{\sqrt{2k+1}} C^{j'\ j\ \ \ k}_{m'-mM} \tag{23}$$

This normalization is also used by Varshalovich *et al.* [?], thus it is important to distinguish properties that explicitly depend on the normalization choice for the reduced matrix elements versus intrinsic properties.

6.1 Tensor multiplication

The coupling of two operators is just as for ordinary spins;

$$\left[\boldsymbol{S}^{[k_1]} \times \boldsymbol{T}^{[k_2]} \right]^{[k]} \tag{24}$$

which denotes the set of operators with components

$$\left[\boldsymbol{S}^{[k_1]} \times \boldsymbol{T}^{[k_2]} \right]^{[k]}_{\mu} = \sum_{\mu_1\mu_2} C^{k_1\,k_2\,k}_{\mu_1\mu_2\mu} S^{[k_1]}_{\mu_1} T^{[k_2]}_{\mu_2} \tag{25}$$

Applying the Wigner-Eckart gives, after a few lines of algebra,

$$\begin{aligned}
&\langle\, j' \,\|\, \left[\boldsymbol{S}^{[k_1]} \times \boldsymbol{T}^{[k_2]}\right]^{[k]} \,\|\, j\,\rangle \\
&= (-1)^{j+j'+k} \sum_{j''} \sqrt{(2j''+1)(2k+1)} \begin{Bmatrix} j' & k_1 & j'' \\ k_2 & j & k \end{Bmatrix} \\
&\times \langle\, j' \,\|\, \boldsymbol{S}^{[k_1]} \,\|\, j''\,\rangle \langle\, j'' \,\|\, \boldsymbol{T}^{[k_2]} \,\|\, j\,\rangle
\end{aligned} \tag{26}$$

Note that multiplication of a tensor operator by a *scalar* is a special case where $k_2 = 0$, and Eq. (43) implies the coupling coefficient is identically 1.

A Coupling coefficients for $SU(2)$

A.1 Clebsch-Gordan coefficients

We use the notation of Biedenharn:

$$C^{j_1\ j_2\ j_3}_{m_1 m_2 m_3} \tag{27}$$

For fixed j_1 and j_2, these coefficients form a unitary matrix of dimension $(2j_1+1)(2j_2+1)$, with rows labelled by m_1, m_2 and columns labelled by j, m. An explicit form is:

$$\begin{aligned}
C^{j_1\ j_2\ j}_{m_1 m_2 m} &= \delta_{m_1+m_2,m} \\
&\times \left[\frac{(2j+1)(j+j_1-j_2)!(j-j_1+j_2)!(j_1+j_2-j)!}{(j+j_1+j_2+1)!}\right]^{\frac{1}{2}} \\
&\times \left[\frac{(j+m)!(j-m!)}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\right]^{\frac{1}{2}} \\
&\times \sum_s \frac{(-1)^{j_2+m_2+s}(j_2+j+m_1+s)!(j_1-m_1+s)!}{s!(j-j_1+j_2-s)!(j+m-s)!(j_1-j_2-m+s)!}\ .
\end{aligned} \tag{28}$$

Orthogonality of rows

$$\sum_{m_1 m_2} C^{j_1\ j_2\ j}_{m_1 m_2 m} C^{j_1\ j_2\ j'}_{m_1 m_2 m'} = \delta_{jj'}\delta_{mm'} \tag{29}$$

and orthogonality of columns

$$\sum_{jm} C^{j_1\ j_2\ j}_{m_1 m_2 m} C^{j_1\ j_2\ j}_{m_1' m_2' m} = \delta_{m_1 m_1'}\delta_{m_2 m_2'} \tag{30}$$

The 'classical' symmetries form a group of order 12 and until the work of Regge[**?**] it was believed that these exhausted the symmetries. The true symmetry group is of order 72.

The symmetry relations are (the group is generated by the first 4, the remainder are for reference),

$$\begin{aligned}
C^{j_1\;j_2\;j}_{m_1 m_2 m} &= (-1)^{j_1+j_2-j} C^{j_1\quad j_2\quad j}_{-m_1,-m_2,-m}\,, \\
C^{j_1\;j_2\;j}_{m_1 m_2 m} &= (-1)^{j_1+j_2-j} C^{j_2\;j_1\;j}_{m_2 m_1 m}\,, \\
C^{j_1\;\;j_2\;\;j}_{m_1,m_2,m_1+m_2} &= C^{\frac{1}{2}(j_1+j_2+m_1+m_2)\;\frac{1}{2}(j_1+j_2-m_1-m_2)\;j}_{\frac{1}{2}(j_1-j_2+m_1-m_2),\frac{1}{2}(j_1-j_2-m_1+m_2),j_1-j_2}\,, \\
C^{j_1\;j_2\;j}_{m_1 m_2 m} &= (-1)^{j_2+m_2}\sqrt{\tfrac{2j+1}{2j_1+1}}\, C^{j\quad j_2\quad j_1}_{-m,m_2,-m_1}\,, \\
C^{j_1\;j_2\;j}_{m_1 m_2 m} &= (-1)^{j_1-m_1}\sqrt{\tfrac{2j+1}{2j_2+1}}\, C^{j_1\quad j\quad j_2}_{m_1,-m,-m_2}\,, \\
C^{j_1\;j_2\;j}_{m_1 m_2 m} &= (-1)^{j_2+m_2}\sqrt{\tfrac{2j+1}{2j_1+1}}\, C^{j_2\quad j\;\;j_1}_{-m_2,m,m_1}\,, \\
C^{j_1\;j_2\;j}_{m_1 m_2 m} &= (-1)^{j_1-m_1}\sqrt{\tfrac{2j+1}{2j_2+1}}\, C^{j\;\;j_1\quad j_2}_{m\,-m_1\,m_2}
\end{aligned} \tag{31}$$

A useful special case is

$$C^{j_1\;0j}_{m_1 0m} = \delta_{j_1 j}\delta_{m_1 m} \tag{32}$$

Alternate notation (Edmonds 1957):

$$C^{j_1\;j_2\;j}_{m_1 m_2 m} \equiv \langle j_1 m_1 j_2 m_2 | j_1 j_2 j m\rangle \tag{33}$$

Alternate notation (Varshalovich *et al.*) [**?**]:

$$C^{j_1\;j_2\;j}_{m_1 m_2 m} \equiv C^{jm}_{j_1 m_1 j_2 m_2} \tag{34}$$

As is apparant from the phase factors and normalizations arising in the symmetry relations, the Clebsch-Gordan coefficients have a mixed symmetry. This is also clear from the notation Eq. (**??**) where j_1 and j_2 transform as bras, but j transforms as a ket. To see how this works, we can interpret the special coefficient $C^{j'\;j\;0}_{m' m 0}$ as a metric tensor $\eta_{j'j}$, that couples spins j' and j to a scalar. Here, j' is the conjugate spin to j. For $SU(2)$, we have numerically $j' = j$, but for $U(1)$ we have $m' = -m$. In this form, the Clebsh-Gordan coefficient has mixed symmetry $C_{j_1 j_1}{}^{j}$. This is convenient because Eq. (33) is usually exactly what we want, but it is important to keep this detail in mind. The 'symmetric' version of the Clebsch-Gordan coefficient is the $3j$ coefficient (an unfortunate name because it is a rather different object to the $N-j$ coefficients for $N > 3$), and has the form

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{j+m+2j_1}\frac{1}{\sqrt{2j+1}} C^{j_1\quad j_2\quad j}_{-m_1 -m_2 m} \tag{35}$$

The inverse form is

$$C^{j_1\ j_2\ j}_{m_1 m_2 m} = (-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} \tag{36}$$

The phase factors are chosen so that any cyclic permutation of columns leaves the $3j$ symbol unchanged. Under an odd permutation of columns, the symbol picks up a phase factor $(-1)^{j_1+j_2+j}$, which is the same phase factor from the transformation $(m_1, m_2, m) \to (-m_1, -m_2, -m)$.

A.2 $6j$ Symbols

The simplest known explitit form is due to Racah[**?**, **?**],

$$\begin{aligned} \begin{Bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \end{Bmatrix} &= \Delta(j_1 j_2 j)\Delta(k_1 k_2 j)\Delta(j_1 k_2 k)\Delta(k_1 j_2 k) \\ \times \sum_z & \frac{(-1)^z (z+1)!}{(z-j_1-j_2-j)!(z-k_1-k_2-j)!(z-j_1-k_2-k)!(z-k_1-j_2-k)!} \\ & \times \frac{1}{(j_1+j_2+k_1+k_2-z)!(j_1+k_1+j+k-z)!(j_2+k_2+j+k-z)!}, \end{aligned} \tag{37}$$

where $\Delta(abc)$ is the *triangle coefficient*,

$$\Delta(abc) = \epsilon_{abc}\left[\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}\right]^{\frac{1}{2}} . \tag{38}$$

Here ϵ_{abc} enforces the *triangle condition*,

$$\epsilon_{abc} = \begin{cases} 1, & \text{if } c \in \{|a-b|, |a-b|+1, \ldots, a+b\} \\ 0, & \text{otherwise} \end{cases} . \tag{39}$$

This is, despite the apparant asymmetry, in fact symmetric in all permutations of a, b, c. The definition of the $6j$ symbols seems to be universal, with just one common alternate, the Racah coefficient W, that differs by a simple phase factor (note also the change in ordering of the indices),

$$W(j_1, j_2, j_5, j_4; j_3, j_6) = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} (-1)^{j_1+j_2+j_3+j_4} \tag{40}$$

The $6j$ symbol is invariant under permutations of its columns, and swapping the elements of *two* columns. *i.e.*

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5 & j_4 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_3 & j_2 \\ j_4 & j_6 & j_5 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_5 & j_3 \\ j_1 & j_2 & j_6 \end{Bmatrix} \tag{41}$$

and so on. The full symmetry group is bigger, 144 elements, but the remaining symmetries are not pure permutations. They also satisfy an orthogonality constraint,

$$\sum_{j_3}(2j_3+1)\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6' \end{Bmatrix} = \frac{\delta_{j_6 j_6'}}{2j_6+1} \tag{42}$$

and a special case where one of the j's is zero,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & 0 \end{Bmatrix} = \delta_{j_2 j_4}\delta_{j_1 j_5}\frac{(-1)^{j_1+j-2+j_3}}{\sqrt{(2j_1+1)(2j_2+1)}} \ . \tag{43}$$

The $6j$ symbol gives the recoupling of three angular momenta,

$$\begin{gathered}\langle j_1(j_2j_3)j_{23}; j'm'|(j_1j_2)j_{12}j_3; jm\rangle \\ = \delta_{j'j}\delta_{m'm}(-1)^{j_1+j_2+j_3+j}\sqrt{(2j_{12}+1)(2j_{23}+1)}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}\end{gathered} \tag{44}$$

This is different to Eq. (2.121) in my thesis, I think the ordering of the $j's$ in that equation is incorrect.

Alternative couplings:

$$\begin{gathered}\langle (j_1j_2)j_{12}j_3; j'm'|(j_1j_3)j_{13}j_2; jm\rangle \\ = \delta_{j'j}\delta_{m'm}(-1)^{j_2+j_3+j_{12}+j_{13}}\sqrt{(2j_{12}+1)(2j_{23}+1)}\begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{Bmatrix}\end{gathered} \tag{45}$$

$$\begin{gathered}\langle j_1(j_2j_3)j_{23}; j'm'|(j_1j_3)j_{13}j_2; jm\rangle \\ = \delta_{j'j}\delta_{m'm}(-1)^{j_1+j_3+j_{23}}\sqrt{(2j_{13}+1)(2j_{23}+1)}\begin{Bmatrix} j_1 & j_3 & j_{13} \\ j_3 & j & j_{23} \end{Bmatrix}\end{gathered} \tag{46}$$

This is simply a notation change from the recoupling of tensor operators,

$$\begin{gathered}((\boldsymbol{P}^{[j_1]}\times\boldsymbol{Q}^{[j_2]})^{[j_{12}]}\times\boldsymbol{R}^{[j_3]})^{[j]} = (-1)^{j_1+j_2+j_3+j}\textstyle\sum_{j_{23}}\sqrt{(2j_{12}+1)(2j_{23}+1)} \\ \times\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}(\boldsymbol{P}^{[j_1]}\times(\boldsymbol{Q}^{[j_2]}\times\boldsymbol{R}^{[j_3]})^{[j_{23}]})^{[j]}\end{gathered} \tag{47}$$

and so on.

For tensors that commute, there is a phase factor from the coupling;

$$(\boldsymbol{P}^{[j_1]}\times\boldsymbol{Q}^{[j_2]})^{[j]} = (-1)^{j_1+j_2-j}(\boldsymbol{Q}^{[j_2]}\times\boldsymbol{P}^{[j_1]})^{[j]} \tag{48}$$

The definition of the $6j$ symbols in terms of the Clebsch-Gordan coefficients is

$$\sum_{mm_i m_{ij}} C^{j_{12}\, j_3\, j}_{m_{12} m_3 m} C^{j_1\, j_2\, j_{12}}_{m_1 m_2 m_{12}} C^{j_1\, j_{23}\, j'}_{m_1 m_{23} m'} C^{j_2\, j_3\, j_{23}}_{m_2 m_3 m_{23}} = \delta_{j'j}\delta_{m'm}(-1)^{j_1+j_2+j_3+j}\sqrt{(2j_{12}+1)(2j_{23}+1)}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \tag{49}$$

When one argument is zero, the $6j$ coefficients reduce to a simple form,

$$\begin{Bmatrix} a & b & c \\ d & e & 0 \end{Bmatrix} = (-1)^{a+b+c}\frac{\delta_{ae}\delta_{bd}}{\sqrt{(2a+1)(2b+1)}} \tag{50}$$

The symmetry relations can be used to shift the zero to any position. There are many formulas for other special cases listed in Varshalovich *et al.* [**?**].

A.3 $9j$ Symbols

A practical formula for evaluation of $9j$ coefficients is in terms of a summation over $6j$ coeffecients:

$$\begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix} = \sum_k (-1)^{2k}(2k+1)\begin{Bmatrix} j_{11} & j_{21} & j_{31} \\ j_{32} & j_{33} & k \end{Bmatrix}\begin{Bmatrix} j_{12} & j_{22} & j_{32} \\ j_{21} & k & j_{23} \end{Bmatrix}\begin{Bmatrix} j_{13} & j_{23} & j_{33} \\ k & j_{11} & j_{12} \end{Bmatrix}. \tag{51}$$

From this, it can be shown that the $9j$ coefficient is zero unless the triangle conditions are fulfilled by the entries in each row and each column. There are 72 known symmetries of the $9j$ coefficent. The $9j$ coefficient is invarant under even permutations of its rows, even permutation of its columns and under interchange of rows and columns (transposition). It is multiplied by a factor $(-1)^{\sum_{ik} j_{ik}}$ under an odd permutation of its rows or columns.

The $9j$ symbols are related to the recoupling of 4 angular momenta,

$$\langle (j_1 j_2) j_{12} (j_3 j_4) j_{34}\; j'm' | (j_1 j_3) j_{13} (j_2 j_4) j_{24}\; jm\rangle = \delta_{jj'}\delta_{mm'}\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} \tag{52}$$

where

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = \sqrt{(2j_{12}+1)(2j_{13}+1)(2j_{24}+1)(2j_{34}+1)}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} \tag{53}$$

As follows from the definition, a $9j$ symbol can also be expressed as a sum of products of Clebsch-Gordan coefficients,

$$\sum_{m_i, m_{ik}} C^{j_1\ j_2\ j_{12}}_{m_1 m_2 m_{12}} C^{j_3\ j_4\ j_{34}}_{m_3 m_4 m_{34}} C^{j_{12}\ j_{34}\ j}_{m_{12} m_{34} m} C^{j_1\ j_3\ j_{13}}_{m_1 m_3 m_{13}} C^{j_2\ j_4\ j_{24}}_{m_2 m_4 m_{24}} C^{j_{13}\ j_{24}\ j'}_{m_{13} m_{24} m'}$$
$$= \delta_{jj'}\delta_{mm'} \begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} \tag{54}$$

If we sum over all of the m's, then we get an additional factor $2j+1$ on the right hand side.

We will encounter this $[\cdots]$ symbol quite a lot, so it is worthwhile to examine a bit its symmetries. From the symmetries of the $9j$ coefficient, the $[\cdots]$ is symmetric under transpose of all indices.

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = \begin{bmatrix} j_1 & j_3 & j_{13} \\ j_2 & j_4 & j_{24} \\ j_{12} & j_{34} & j \end{bmatrix} \tag{55}$$

Under a swap of the first two rows or first two columns, it picks up a phase factor of $(-1)^{\sum j}$, being the sum of all 9 quantum numbers. Hence it is invariant under the combined swap of the first two rows and first two columns,

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = \begin{bmatrix} j_4 & j_3 & j_{34} \\ j_2 & j_1 & j_{12} \\ j_{24} & j_{13} & j \end{bmatrix} \tag{56}$$

Other symmetries are more complicated as they involve non-unitary prefactors.

References

[1] `http://dx.doi.org/10.1088/1751-8113/43/39/395205`

[2] Parsa Bonderson, PhD thesis, CalTech 2007. `http://thesis.library.caltech.edu/2447/`

[3] M. Müger, Modular Categories. http://arxiv.org/abs/1201.6593v2