

Generalized algebras of Clebsch-Gordan, $6j$, $9j$, F-moves

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April 30, 2017

1 Introduction

The purpose of this paper is to provide exposition and reference material for constructing $6j$ and $9j$ symbols of various algebras, and eventually to also construct projection transformations (eg the conventional Clebsch-Gordan coefficients are associated with the group chain $U(1) \subset SU(2)$, but we want to generalize this to other group subsets). This is somewhat complicated by different notation conventions; historically in $SU(2)$ theory the Clebsch-Gordan/ $3j$, $6j$ and $9j$ coefficients are used, which have been generalized in similar notation to point groups (used eg in quantum chemistry). On the other hand anyons are often described in terms of F-symbols and F-moves, of which there are two common variants (a 3-1 move and a 2-2 move). In practical calculations, the F-symbols are probably the most useful; these give directly the coefficients of basis transformations eg $(F_j^{j_1, j_2, j_3})_{j_{12}, j_{23}} = \langle (j_1 j_2 j_{12}) j_3 j | j_1 (j_2 j_3 j_{23}) j \rangle$, which differs by a normalization and phase factors from a $6j$ symbol. However the $6j$ symbols have more symmetry so are useful as an underlying implementation tool in computational algorithms. Hence we want to find generic ways to transform between F-symbols and $6j$ symbols.

We also hope to unify this picture with braiding statistics and particle exchange statistics; it is not a priori clear that these are the same thing! In particular, I haven't seen yet any good discussion on the different types of exchange statistics that are possible for the same particle algebra (eg spinless fermions versus bosons, or even abelian anyons, all appear to have the same $U(1)$ fusion algebra). For non-abelian anyons, the R symbols are generally presented as fixed quantities, but clearly for simple spinless (eg $U(1)$) particles we have a choice of particle exchange statistics, which appears to be possible to encode into the R matrix, but this then implies that the R matrix is *not* fixed by the fusion algebra alone. Is this different in the anyonic case? Eg can we give non-abelian anyons an additional fermionic phase?

The specific symmetry groups that we want to handle here are

1. $U(1)$
2. $SU(2)$
3. finite cyclic groups Z_n
4. Dihedral groups D_n . There are two notations for dihedral groups, sometimes D_{2n} is used.
5. Ising anyons $SU(2)_2$
6. Fibonacci anyons $SU(2)_3$
7. $SU(3)_2$ – discussed in a nice paper by Ardonne and Slingerland[1]

Where possible we want to allow projective representations too. Such representations occur as boundary conditions or edge particles in topological states. For the Dihedral groups, this means allowing for half-integer quantum numbers.

2 Quantum numbers

a quantum number is a representation of some symmetry group. Fusion of quantum numbers is equivalent to taking products of representations, and is given by the Clebsch-Gordan expansion,

$$a \otimes b = \bigoplus n_{ab}^c c \quad (1)$$

where n_{ab}^c is the *multiplicity* of the representation c in the expansion of $a \otimes b$. Technically we should distinguish the representation label j from the representation matrices $D(j)$, but we won't bother with this distinction here, unless it gives a clearer notation. If $n_{j_1, j_2}^j > 1$ then a representation occurs more than once in the expansion, and this greatly complicates the description of the coupling. This is commonly referred to as the *multiplicity problem*. Symmetry groups that always have $n_{j_1, j_2}^j \leq 1$ are referred to as *multiplicity free*.

Examples:

- $U(1)$: label the representations by a (half-)integer j , eg representing the particle number or z -component of spin. Then the Clebsch-Gordan expansion corresponds to simple addition of the quantum number, $D(j_1) \otimes D(j_2) = D(j_1 + j_2)$.
- $SU(2)$: The product of $SU(2)$ spins j_1 and j_2 (always non-negative) is a sum of $\min(2j_1+1, 2j_2+1)$ representations. $n_{j_1, j_2}^j = 1$, if $j \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2\}$, and 0 otherwise.
- Ising anyons, corresponding to $SU(2)_2$, contain spins 1 (the identity, which has spin 0) and τ (analogous to a spin 1), with the fusion rules $1 \times 1 = 1$, $1 \times \tau = \tau \times 1 = \tau$,

$$\tau \times \tau = 1 + \tau.$$

- $SU(3)_3/Z_3$ is probably the simplest algebra that is *not* multiplicity-free[1], via

$$8 \times 8 = 1 + 8 + 8' + 10 + \bar{10} \quad (2)$$

where the representation 8 occurs twice on the right hand side.

The ‘zero’ quantum number always exists. This is often denoted with a 1, since it corresponds to the trivial (identity) representation of the algebra. But in practice the spin is 0. The identity quantum number is also referred to as *scalar*, since it is the unique representation that is invariant under symmetry transformations. A representation j has a conjugate or dual representation, denoted \bar{j} .

- In $U(1)$ the conjugate corresponds to negation, so $\bar{j} = -j$.
- $SU(2)$ is self-dual, so $\bar{j} = j$.

The ‘conjugate’ operation here has nothing to do with complex conjugation. In the Matrix Product Toolkit, the operation \bar{j} is referred to as `adjoint(j)`. Commonly, discussions of $SU(2)$ coupling coefficients omit the quantum number conjugation, but it is still important eg for spinful particles we often have $U(1) \times SU(2)$, and we cannot neglect the quantum number conjugation in the $U(1)$ case.

The *dimension* of a representation j is the dimension of the matrix $D(j)$. This is often denoted d_j . For classical symmetries the dimension is an integer. For non-abelian anyons, the dimension can be fractional and is then known as the *quantum dimension*. The dimensions satisfy the sum rule

$$d_{j_1} d_{j_2} = \sum_j n_{j_1, j_2}^j d_j \quad (3)$$

If, for some symmetry group, all of the representations have $d_j = 1$, then the group is abelian. This is equivalent to the statement that for each a, b there is exactly one c such that $n_{ab}^c = 1$, and all other multiplicities are zero.

3 Fusion

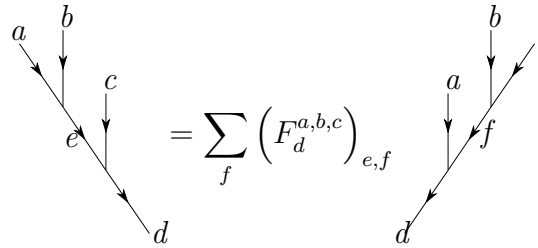
Hilbert spaces are complex spaces with an inner product, so we need to distinguish bra’s and ket’s as separate entities. In diagrammatic notation, we do this with incoming or outgoing arrows. It doesn’t matter whether we regard an incoming arrow as a bra state or

as a ket, as long as we are globally consistent. Fusion of two quantum numbers looks like



$$(4)$$

where we fuse two incoming states a, b into an outgoing state c . When fusing three particles a, b, c together, there is an ambiguity as to whether we fuse a, b first and then c , or fuse a with the fusion of b, c . The difference is the intermediate quantum number, here denoted e or f . The difference between these forms of fusion is given by the F symbol.



$$(5)$$

Alternatively, we can write

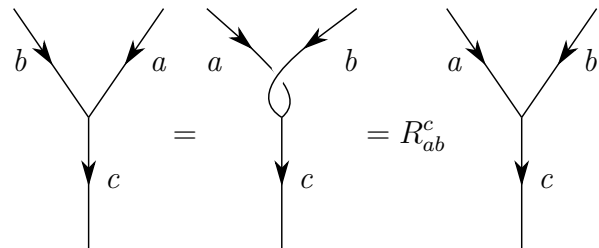
$$\|((abe)c)d\rangle = \sum_f \left(F_d^{a,b,c} \right)_{e,f} \| (a(bcf))d \rangle \quad (6)$$

or as

$$(F_j^{j_1, j_2, j_3})_{j_{12}, j_{23}} = \langle (j_1 j_2 j_{12}) j_3 j \| j_1 (j_2 j_3 j_{23}) j \rangle \quad (7)$$

where we use the notation $\|a\rangle$ to denote a state vector, as a reminder that we may have $d_a > 1$, so it isn't an ordinary vector. But we need to be careful with normalization of the inner product, maybe we should have a $1/d_j$ here? For given exterior legs a, b, c, d we can view the F-symbol as the basis transformation that maps between two different choices of internal labelling.

Note that the left-right ordering of spins a, b, c, d is important here. Although the Clebsch-Gordan series for $a \otimes b$ is the same as that for $b \otimes a$, the F-symbol is *not* symmetric under this interchange. We can represent this with an R matrix,



$$(8)$$

4 Irreducible Tensors

Using the normalization convention of Beidenharn, we define

$$\langle j'm' | T_M^{[k]} | jm \rangle = \langle j' \| \mathbf{T}^{[k]} \| j \rangle C_{mMm'}^{j \ k \ j'} \quad (9)$$

Using the orthogonality of the CG coefficients, this defines the reduced matrix elements¹,

$$\langle j' \| \mathbf{T}^{[k]} \| j \rangle = \sum_{mM} C_{mMm'}^{j \ k \ j'} \langle j'm' | T_M^{[k]} | jm \rangle \quad (10)$$

where m' is arbitrary. Alternatively, we can sum over m' and divide by $(2m' + 1)$,

$$\langle j' \| \mathbf{T}^{[k]} \| j \rangle = \sum_{mMm'} \frac{1}{2j' + 1} C_{mMm'}^{j \ k \ j'} \langle j'm' | T_M^{[k]} | jm \rangle \quad (11)$$

This is a hint that this normalization of the reduced matrix elements is not as symmetric as one might like; however using this normalization the reduced and full matrix elements of scalar operators coincide. In particular, the reduced matrix elements of the identity operator are

$$\langle j' \| I \| j \rangle = \delta_{j'j} \quad (12)$$

and those of the angular momentum operator are

$$\langle j' \| J \| j \rangle = \sqrt{j(j+1)} \delta_{j'j} \quad (13)$$

But the trace of a scalar operator needs to be normalized properly;

$$\text{Tr } X = \sum_j (2j+1) \langle j \| X \| j \rangle \quad (14)$$

The normalization used here is *not* equivalent to that of Edmonds, who instead defines

$$\langle j'm' | T_M^{[k]} | jm \rangle = \langle j' \| \mathbf{T}^{[k]} \| j \rangle_{\text{Edmonds}} \frac{(-1)^{j-m}}{\sqrt{2k+1}} C_{m'-mM}^{j' \ j \ k} \quad (15)$$

This normalization is also used by Varshalovich *et al.* [?], thus it is important to distinguish properties that explicitly depend on the normalization choice for the reduced matrix elements versus intrinsic properties.

¹That this factorization exists is precisely the celebrated Wigner-Eckart theorem.

4.1 Tensor multiplication

The coupling of two operators is just as for ordinary spins;

$$\left[\mathbf{S}^{[k_1]} \times \mathbf{T}^{[k_2]} \right]^{[k]} \quad (16)$$

which denotes the set of operators with components

$$\left[\mathbf{S}^{[k_1]} \times \mathbf{T}^{[k_2]} \right]_{\mu}^{[k]} = \sum_{\mu_1 \mu_2} C_{\mu_1 \mu_2 \mu}^{k_1 k_2 k} S_{\mu_1}^{[k_1]} T_{\mu_2}^{[k_2]} \quad (17)$$

Applying the Wigner-Eckart gives, after a few lines of algebra,

$$\begin{aligned} & \langle j' \parallel \left[\mathbf{S}^{[k_1]} \times \mathbf{T}^{[k_2]} \right]^{[k]} \parallel j \rangle \\ &= (-1)^{j+j'+k} \sum_{j''} \sqrt{(2j''+1)(2k+1)} \left\{ \begin{matrix} j' & k_1 & j'' \\ k_2 & j & k \end{matrix} \right\} \\ & \quad \times \langle j' \parallel \mathbf{S}^{[k_1]} \parallel j'' \rangle \langle j'' \parallel \mathbf{T}^{[k_2]} \parallel j \rangle \end{aligned} \quad (18)$$

Note that multiplication of a tensor operator by a *scalar* is a special case where $k_2 = 0$, and Eq. (35) implies the coupling coefficient is identically 1.

A Coupling coefficients for $SU(2)$

A.1 Clebsch-Gordan coefficients

We use the notation of Biedenharn:

$$C_{m_1 m_2 m}^{j_1 j_2 j_3} \quad (19)$$

For fixed j_1 and j_2 , these coefficients form a unitary matrix of dimension $(2j_1+1)(2j_2+1)$, with rows labelled by m_1, m_2 and columns labelled by j, m . An explicit form is:

$$\begin{aligned} C_{m_1 m_2 m}^{j_1 j_2 j} &= \delta_{m_1+m_2, m} \\ & \times \left[\frac{(2j+1)(j+j_1-j_2)!(j-j_1+j_2)!(j_1+j_2-j)!}{(j+j_1+j_2+1)!} \right]^{\frac{1}{2}} \\ & \times \left[\frac{(j+m)!(j-m)!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!} \right]^{\frac{1}{2}} \\ & \times \sum_s \frac{(-1)^{j_2+m_2+s} (j_2+j+m_1+s)!(j_1-m_1+s)!}{s!(j-j_1+j_2-s)!(j+m-s)!(j_1-j_2-m+s)!} . \end{aligned} \quad (20)$$

Orthogonality of rows

$$\sum_{m_1 m_2} C_{m_1 m_2 m}^{j_1 j_2 j} C_{m_1 m_2 m'}^{j_1 j_2 j'} = \delta_{jj'} \delta_{mm'} \quad (21)$$

and orthogonality of columns

$$\sum_{jm} C_{m_1 m_2 m}^{j_1 j_2 j} C_{m'_1 m'_2 m}^{j_1 j_2 j} = \delta_{m_1 m'_1} \delta_{m_2 m'_2} \quad (22)$$

The ‘classical’ symmetries form a group of order 12 and until the work of Regge[?] it was believed that these exhausted the symmetries. The true symmetry group is of order 72. The symmetry relations are (the group is generated by the first 4, the remainder are for reference),

$$\begin{aligned} C_{m_1 m_2 m}^{j_1 j_2 j} &= (-1)^{j_1+j_2-j} C_{-m_1, -m_2, -m}^{j_1 j_2 j} , \\ C_{m_1 m_2 m}^{j_1 j_2 j} &= (-1)^{j_1+j_2-j} C_{m_2 m_1 m}^{j_2 j_1 j} , \\ C_{m_1, m_2, m_1+m_2}^{j_1 j_2 j} &= C_{\frac{1}{2}(j_1+j_2+m_1+m_2), \frac{1}{2}(j_1+j_2-m_1-m_2), j}^{\frac{1}{2}(j_1-j_2+m_1-m_2), \frac{1}{2}(j_1-j_2-m_1+m_2), j_1-j_2} , \\ C_{m_1 m_2 m}^{j_1 j_2 j} &= (-1)^{j_2+m_2} \sqrt{\frac{2j+1}{2j_1+1}} C_{-m, m_2, -m_1}^j , \\ C_{m_1 m_2 m}^{j_1 j_2 j} &= (-1)^{j_1-m_1} \sqrt{\frac{2j+1}{2j_2+1}} C_{m_1, -m, -m_2}^{j_1 j j_2} , \\ C_{m_1 m_2 m}^{j_1 j_2 j} &= (-1)^{j_2+m_2} \sqrt{\frac{2j+1}{2j_1+1}} C_{-m_2, m, m_1}^{j_2 j j_1} , \\ C_{m_1 m_2 m}^{j_1 j_2 j} &= (-1)^{j_1-m_1} \sqrt{\frac{2j+1}{2j_2+1}} C_{m, -m_1, m_2}^j \end{aligned} \quad (23)$$

A useful special case is

$$C_{m_1 0 m}^{j_1 0 j} = \delta_{j_1 j} \delta_{m_1 m} \quad (24)$$

Alternate notation (Edmonds 1957):

$$C_{m_1 m_2 m}^{j_1 j_2 j} \equiv \langle j_1 m_1 j_2 m_2 | j_1 j_2 j m \rangle \quad (25)$$

Alternate notation (Varshalovich *et al.*) [?]:

$$C_{m_1 m_2 m}^{j_1 j_2 j} \equiv C_{j_1 m_1 j_2 m_2}^{j m} \quad (26)$$

As is apparant from the phase factors and normalizations arising in the symmetry relations, the Clebsch-Gordan coefficients have a mixed symmetry. This is also clear from the notation Eq. (??) where j_1 and j_2 transform as bras, but j transforms as a ket. To see how this works, we can interpret the special coefficient $C_{m'_1 m'_2 0}^{j' j 0}$ as a metric tensor $\eta_{j' j}$, that couples spins j'

and j to a scalar. Here, j' is the conjugate spin to j . For $SU(2)$, we have numerically $j' = j$, but for $U(1)$ we have $m' = -m$. In this form, the Clebsh-Gordan coefficient has mixed symmetry $C_{j_1 j_1}^j$. This is convenient because Eq. (25) is usually exactly what we want, but it is important to keep this detail in mind. The ‘symmetric’ version of the Clebsch-Gordan coefficient is the $3j$ coefficient (an unfortunate name because it is a rather different object to the $N - j$ coefficients for $N > 3$), and has the form

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{j+m+2j_1} \frac{1}{\sqrt{2j+1}} C_{-m_1 -m_2 m}^{j_1 j_2 j} \quad (27)$$

The inverse form is

$$C_{m_1 m_2 m}^{j_1 j_2 j} = (-1)^{j_1-j_2+m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} \quad (28)$$

The phase factors are chosen so that any cyclic permutation of columns leaves the $3j$ symbol unchanged. Under an odd permutation of columns, the symbol picks up a phase factor $(-1)^{j_1+j_2+j}$, which is the same phase factor from the transformation $(m_1, m_2, m) \rightarrow (-m_1, -m_2, -m)$.

A.2 $6j$ Symbols

The simplest known explicit form is due to Racah[?, ?],

$$\begin{aligned} \left\{ \begin{matrix} j_1 & j_2 & j \\ k_1 & k_2 & k \end{matrix} \right\} &= \Delta(j_1 j_2 j) \Delta(k_1 k_2 j) \Delta(j_1 k_2 k) \Delta(k_1 j_2 k) \\ &\times \sum_z \frac{(-1)^z (z+1)!}{(z-j_1-j_2-j)!(z-k_1-k_2-j)!(z-j_1-k_2-k)!(z-k_1-j_2-k)!} \\ &\times \frac{1}{(j_1+j_2+k_1+k_2-z)!(j_1+k_1+j+k-z)!(j_2+k_2+j+k-z)!}, \end{aligned} \quad (29)$$

where $\Delta(abc)$ is the *triangle coefficient*,

$$\Delta(abc) = \epsilon_{abc} \left[\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!} \right]^{\frac{1}{2}}. \quad (30)$$

Here ϵ_{abc} enforces the *triangle condition*,

$$\epsilon_{abc} = \begin{cases} 1, & \text{if } c \in \{|a-b|, |a-b|+1, \dots, a+b\} \\ 0, & \text{otherwise} \end{cases}. \quad (31)$$

This is, despite the apparant asymmetry, in fact symmetric in all permutations of a, b, c . The definition of the $6j$ symbols seems to be universal, with just one common alternate, the Racah coefficient W , that differs by a simple phase factor (note also the change in ordering of the indices),

$$W(j_1, j_2, j_5, j_4; j_3, j_6) = \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} (-1)^{j_1+j_2+j_3+j_4} \quad (32)$$

The $6j$ symbol is invariant under permutations of its columns, and swapping the elements of *two* columns. *i.e.*

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} = \left\{ \begin{matrix} j_2 & j_1 & j_3 \\ j_5 & j_4 & j_6 \end{matrix} \right\} = \left\{ \begin{matrix} j_1 & j_3 & j_2 \\ j_4 & j_6 & j_5 \end{matrix} \right\} = \left\{ \begin{matrix} j_4 & j_5 & j_3 \\ j_1 & j_2 & j_6 \end{matrix} \right\} \quad (33)$$

and so on. The full symmetry group is bigger, 144 elements, but the remaining symmetries are not pure permutations. They also satisfy an orthogonality constraint,

$$\sum_{j_3} (2j_3 + 1) \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j'_6 \end{matrix} \right\} = \frac{\delta_{j_6 j'_6}}{2j_6 + 1} \quad (34)$$

and a special case where one of the j 's is zero,

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & 0 \end{matrix} \right\} = \delta_{j_2 j_4} \delta_{j_1 j_5} \frac{(-1)^{j_1+j-2+j_3}}{\sqrt{(2j_1+1)(2j_2+1)}}. \quad (35)$$

The $6j$ symbol gives the recoupling of three angular momenta,

$$\begin{aligned} & \langle j_1(j_2 j_3) j_{23}; j' m' | (j_1 j_2) j_{12} j_3; j m \rangle \\ &= \delta_{j' j} \delta_{m' m} (-1)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\} \end{aligned} \quad (36)$$

This is different to Eq. (2.121) in my thesis, I think the ordering of the j 's in that equation is incorrect.

Alternative couplings:

$$\begin{aligned} & \langle (j_1 j_2) j_{12} j_3; j' m' | (j_1 j_3) j_{13} j_2; j m \rangle \\ &= \delta_{j' j} \delta_{m' m} (-1)^{j_2+j_3+j_{12}+j_{13}} \sqrt{(2j_{12}+1)(2j_{23}+1)} \left\{ \begin{matrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{matrix} \right\} \end{aligned} \quad (37)$$

$$\begin{aligned} & \langle j_1(j_2 j_3) j_{23}; j' m' | (j_1 j_3) j_{13} j_2; j m \rangle \\ &= \delta_{j' j} \delta_{m' m} (-1)^{j_1+j_3+j_{23}} \sqrt{(2j_{13}+1)(2j_{23}+1)} \left\{ \begin{matrix} j_1 & j_3 & j_{13} \\ j_3 & j & j_{23} \end{matrix} \right\} \end{aligned} \quad (38)$$

This is simply a notation change from the recoupling of tensor operators,

$$\begin{aligned} ((\mathbf{P}^{[j_1]} \times \mathbf{Q}^{[j_2]})^{[j_{12}]} \times \mathbf{R}^{[j_3]})^{[j]} &= (-1)^{j_1+j_2+j_3+j} \sum_{j_{23}} \sqrt{(2j_{12}+1)(2j_{23}+1)} \\ &\times \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\} (\mathbf{P}^{[j_1]} \times (\mathbf{Q}^{[j_2]} \times \mathbf{R}^{[j_3]})^{[j_{23}]})^{[j]} \end{aligned} \quad (39)$$

and so on.

For tensors that commute, there is a phase factor from the coupling;

$$(\mathbf{P}^{[j_1]} \times \mathbf{Q}^{[j_2]})^{[j]} = (-1)^{j_1+j_2-j} (\mathbf{Q}^{[j_2]} \times \mathbf{P}^{[j_1]})^{[j]} \quad (40)$$

The definition of the $6j$ symbols in terms of the Clebsch-Gordan coefficients is

$$\begin{aligned} &\sum_{mm_i m_{ij}} C_{m_{12} m_3 m}^{j_{12} j_3 j} C_{m_1 m_2 m_{12}}^{j_1 j_2 j_{12}} C_{m_1 m_{23} m'}^{j_1 j_{23} j'} C_{m_2 m_3 m_{23}}^{j_2 j_3 j_{23}} \\ &= \delta_{j'j} \delta_{m'm} (-1)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\} \end{aligned} \quad (41)$$

When one argument is zero, the $6j$ coefficients reduce to a simple form,

$$\left\{ \begin{matrix} a & b & c \\ d & e & 0 \end{matrix} \right\} = (-1)^{a+b+c} \frac{\delta_{ac} \delta_{bd}}{\sqrt{(2a+1)(2b+1)}} \quad (42)$$

The symmetry relations can be used to shift the zero to any position. There are many formulas for other special cases listed in Varshalovich *et al.* [?].

A.3 $9j$ Symbols

A practical formula for evaluation of $9j$ coefficients is in terms of a summation over $6j$ coefficients:

$$\left\{ \begin{matrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{matrix} \right\} = \sum_k (-1)^{2k} (2k+1) \left\{ \begin{matrix} j_{11} & j_{21} & j_{31} \\ j_{32} & j_{33} & k \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{22} & j_{32} \\ j_{21} & k & j_{23} \end{matrix} \right\} \left\{ \begin{matrix} j_{13} & j_{23} & j_{33} \\ k & j_{11} & j_{12} \end{matrix} \right\}. \quad (43)$$

From this, it can be shown that the $9j$ coefficient is zero unless the triangle conditions are fulfilled by the entries in each row and each column. There are 72 known symmetries of the $9j$ coefficient. The $9j$ coefficient is invariant under even permutations of its rows, even permutation of its columns and under interchange of rows and columns (transposition). It is multiplied by a factor $(-1)^{\sum_{ik} j_{ik}}$ under an odd permutation of its rows or columns.

The $9j$ symbols are related to the recoupling of 4 angular momenta,

$$\begin{aligned} & \langle (j_1 j_2) j_{12} (j_3 j_4) j_{34} j' m' | (j_1 j_3) j_{13} (j_2 j_4) j_{24} j m \rangle \\ &= \delta_{jj'} \delta_{mm'} \begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} \end{aligned} \quad (44)$$

where

$$\begin{aligned} & \begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = \\ & \sqrt{(2j_{12} + 1)(2j_{13} + 1)(2j_{24} + 1)(2j_{34} + 1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} \end{aligned} \quad (45)$$

As follows from the definition, a $9j$ symbol can also be expressed as a sum of products of Clebsch-Gordan coefficients,

$$\begin{aligned} & \sum_{m_i, m_{ik}} C_{m_1 m_2 m_{12}}^{j_1 j_2 j_{12}} C_{m_3 m_4 m_{34}}^{j_3 j_4 j_{34}} C_{m_{12} m_{34} m}^{j_{12} j_{34} j} C_{m_1 m_3 m_{13}}^{j_1 j_3 j_{13}} C_{m_2 m_4 m_{24}}^{j_2 j_4 j_{24}} C_{m_{13} m_{24} m'}^{j_{13} j_{24} j'} \\ &= \delta_{jj'} \delta_{mm'} \begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} \end{aligned} \quad (46)$$

If we sum over all of the m 's, then we get an additional factor $2j + 1$ on the right hand side.

We will encounter this $[\dots]$ symbol quite a lot, so it is worthwhile to examine a bit its symmetries. From the symmetries of the $9j$ coefficient, the $[\dots]$ is symmetric under transpose of all indices.

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = \begin{bmatrix} j_1 & j_3 & j_{13} \\ j_2 & j_4 & j_{24} \\ j_{12} & j_{34} & j \end{bmatrix} \quad (47)$$

Under a swap of the first two rows or first two columns, it picks up a phase factor of $(-1)^{\Sigma j}$, being the sum of all 9 quantum numbers. Hence it is invariant under the combined swap of the first two rows and first two columns,

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = \begin{bmatrix} j_4 & j_3 & j_{34} \\ j_2 & j_1 & j_{12} \\ j_{24} & j_{13} & j \end{bmatrix} \quad (48)$$

Other symmetries are more complicated as they involve non-unitary prefactors.

References

- [1] <http://dx.doi.org/10.1088/1751-8113/43/39/395205>