Generalized algebras of Clebsch-Gordan, $6j$, $9j$, F-moves

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1 Introduction

The purpose of this paper is to provide exposition and reference material for constructing 6j and 9j symbols of various algebras, and eventually to also construct projection transformations (eg the conventional Clebsch-Gordan coefficients are associated with the group chain $U(1) \subset SU(2)$, but we want to generalize this to other group subsets). This is somewhat complicated by different notation conventions; historically in $SU(2)$ theory the Clebsch-Gordan/3j, 6j and 9j coefficients are used, which have been generalized in similar notation to point groups (used eg in quantum chemistry). On the other hand anyons are often described in terms of F-symbols and F-moves, of which there are two common variants (a 3-1 move and a 2-2 move). In practical calculations, the F-symbols are probably the most useful; these give directly the coefficients of basis transformations eg $\left[F_j^{j_1,j_2,j_3}\right]_{j_{12},j_{23}} = \langle (j_1 j_2 j_{12}) j_3 j | j_1 (j_2 j_3 j_{23}) j \rangle$, which differs by a normalization and phase factors from a 6j symbol. However the 6j symbols have more symmetry so are useful as an underlying implementation tool in computational algorithms. Hence we want to find generic ways to transform beyween F-symbols and 6j symbols.

The general structure that admits a consistent set of F-symbols is known as a fusion tensor category. It is necessary and sufficient that the F-symbols obey the Biedenharn-Elliott identity, also known as the Pentagon relation.

The F-symbols alone are not sufficient to describe an operator algebra. In particular, the F-symbols do not in general give enough information to be able to determine particle statistics. For this, we need also the braiding statistics, which is encapsulated in the symbol R^c_{ab}, which is a coefficient (or, for non-multiplicity-free algebras, a matrix) that gives the phase change of two charges a and b, which fuse to particle type c, and we braid b over the top of a. For a given fusion algebra, there may be more than one possible set of braiding. The consistency between the R and F symbols is given by the Hexagon relation. This gives a braided tensor fusion category, which is ultimately the object that we are interested in. Mathematically, we need also some other conditions, which I find to be not particularly well specified in the literature, such as K-linear, semi-simplicity, existence of duals, and non-degenerate. Generally the literature also looks at the *finite* case (finite rank, number of charges). We are interested also in the infinite case, which arises from semi-simple Lie agebras. I haven't seen any discussion of infinite braided tensor categories that are *not* derived from a semi-simple Lie algebra.

The classification of *finite* braided tensor fusion categories is more-or-less known, up to categories of rank 5 (where the rank denotes the number of distinct charges in the category). We can subdivide braided tensor fusion categories (BTC for short) into some syb-types,

- *symmetric*: A symmetric braided tensor category (STC) satisfies $R^c_{ab} = R^c_{ba}$ for all possible a, b, c. This implies $(R^c_{ab})^2 = 1$, and hence the charges behave as essentially fermionic or bosonic character. There is a strong connection between STC and finite groups. I am not sure if this is 1–1. For the purposes of this document, we also include infinite groups (ie, semi-simple Lie groups) in this class.
- *modular tensor category* (MTC): this is the 'opposite' of an STC, in the sense that the 'symmetric center' of the category is trivial.
- *integral*: if the quantum dimension of all objects is an integer, then the BTC is known as integral. All STC's are integral.
- *pointed*: if the quantum dimension of all objects is unity, then the BTC is known as pointed. STC's arising from an abelian group are pointed. But there are many examples of MTC's that are also pointed, for example abelian anyons. There are anyon algebras that are integral but not pointed (ie, all of the quantum dimensions are integral but there is at least one member that has dimension ¿ 1), Bonderson gives an example based on the fusion rules for D_5, but in principle any finite group could also have some non-trivial braid solutions.

The specific symmetry groups that we want to handle here are

1. $U(1)$
2. $SU(2)$
3. finite cyclic groups Z_n
4. Dihedral groups D_n. THere are two notations for dihedral groups, sometimes D_{2n} is used.
5. Ising anyons $SU(2)_2$
6. Fibonacci anyons $SU(2)_3$

7. $SU(3)_2$ – discussed in a nice paper by Ardonne and Slingerland[1]

Where possible we want to allow projective representations too. Such representations occur as boundary conditions or edge particles in topological states. For the Dihedral groups, this means allowing for half-integer quantum numbers.

2 Literature summary

In this section we give an overview of the primary literature that was used in the development of this document.

- [2] A very good introdutcion to anyons in physics. Most of the diagrams and equations for F-symbols are from this thesis.
- [3] Introduction to modular categories.
- [4] classifies, and provides coupling coefficients, for all modular tensor categories of rank ≤ 4.
- [5] Discusses in detail the effect of surface topology on the construction of fusion diagrams, in particular how to handle boundary conditions for 1D anyon chains.
- [6] DMRG for anyon chains, good discussion of normalization convention.

3 Quantum numbers

A quantum number is a representation of some symmetry group. Fusion of quantum numbers is equivalent to taking products of representations, and is given by the Clebsch-Gordan expansion,

$$a \otimes b \simeq \bigoplus N_{ab}^c \, c \tag{1}$$

where N_{ab}^c is the *multiplicity* of the representation c in the expansion of $a \otimes b$. Technically we should distinguish the representation label j from the representation matrices $D(j)$, but we won't bother with this distsinction here, unless it gives a clearer notation. If $N_{ab}^c > 1$ then a representation occurs more than once in the expansion, and this greatly complicates the description of the coupling. This is commonly referred to as the *multiplicity problem.* Symmetry groups that always have $N_{ab}^c \leq 1$ are referred to as *multiplicity free.*

Examples:

- $U(1)$: label the representations by a (half-)integer j, eg representing the particle number or z-component of spin. Then the Clebsch-Gordan expansion corresponds to simple addition of the quantum number, $D(j_1) \otimes D(j_2) = D(j_1 + j_2)$.

- $SU(2)$: The product of $SU(2)$ spins j_1 and j_2 (always non-negative) is a sum of $\min(2j_1{+}1, 2j_2{+}1)$ representations. $N^j_{j_1,j_2} = 1$, if $j \in \{|j_1{-}j_2|, |J_1{-}j_2|{+}1, \cdots, j_1{+}j_2\}$, and 0 otherwise.
- Ising anyons, corresponding to $SU(2)_3$, contain spins 1 (the identiy, which has spin 0) and τ (analagous to a spin 1), with the fusion rules $1 \times 1 = 1$, $1 \times \tau = \tau \times 1 = \tau$, $\tau \times \tau = 1 + \tau$.
- $SU(3)_3/Z_3$ is probably the simplest algebra that is *not* multiplicity-free[1], via

$$8 \times 8 = 1 + 8 + 8' + 10 + \bar{1}0 \tag{2}$$

where the representation 8 occurs twice on the right hand side.

The 'zero' quantum number always exists. This is often denoted with a 1, since it corresponds to the trivial (identity) representation of the algebra. The identity quantum number is also referred to as *scalar*, since it is the unique representation that is invariant under symmetry transformations. A representation j has a conjugate or dual representation, denoted $\bar{j}$, or in some literature as j^*.

- In $U(1)$ the conjugate correpsonds to negation, so $\bar{j} = -j$.
- $SU(2)$ is self-dual, so $\bar{j} = j$.

The 'conjugate' operation here has nothing to do with complex conjugation. In the Matrix Product Toolkit, the operation $\bar{j}$ is referred to as `adjoint(j)`. Commonly, discussions of $SU(2)$ coupling coefficients omit the quantum number conjugation, but it is still important eg for spinful particles we often have $U(1) \times SU(2)$, and we cannot neglect the quantum number conjugation in the $U(1)$ case.

The *dimension* of a representation j is the dimension of the matrix $D(j)$. This is often denoted d_j. For classical symmetries the dimension is an integer. For non-abelian anyons, the dimension can be fractional and is then known as the *quantum dimension*. The dimensions satisfy the sum rule

$$d_{j_1} d_{j_2} = \sum_j N^j_{j_1,j_2} \; d_j \tag{3}$$

If, for some symmetry group, all of the representations have $d_j = 1$, then the group is abelian. This is equivalent to the statement that for each a, b there is exactly one c such that $N^c_{ab} = 1$, and all other multiplicities are zero.

The quantum dimension q_a can be obtained as the largest eigenvalue of N^c_{ab}, treated as a matrix in the indices b, c. It is also possible to define the quantum dimensions through the

F-symbols (see below), as

$$d_a = d_{\bar{a}} = |[F_a^{a\bar{a}a}]_{1,1}|^{-1} \tag{4}$$

The fusion multiplicities obey the relations

$$N_{ab}^c = N_{ba}^c = N_{\bar{a}\bar{b}}^{\bar{c}} = N_{b\bar{c}}^{\bar{a}} \tag{5}$$

This is 12 relations in total (all permutations of a, b, c plus the dual versions). We also have

$$\sum_e N_{ab}^e N_{ec}^d = \sum_f N_{af}^d N_{bc}^f \tag{6}$$

This is essentially associativity of tensor products; the number of ways of fusing charges a, b, c to a charge d is the same, whether we to the fusion as $(a \otimes b) \otimes c$ or $a \otimes (b \otimes c)$.

4 F-symbols

Hilbert spaces are complex spaces with an in inner product, so we need to distinguish bra's and ket's as separate entities. In diagrammantic notation, we do this with arrows that end at a point, either as an incoming arrow or an outgoing arrow. It doesn't matter whether we regard an incoming arrow as a bra state or as a ket, as long as we are globally consistent. Reversing the orientation of a line requires conjugating the charge label, ie

$$\big\downarrow a \quad = \quad \big\uparrow \bar{a} \tag{7}$$

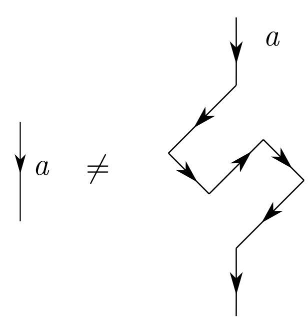
In these diagrams, there is an implicit 'arrow of time', which flows from the top to the bottom, irrespective of the direction of the arrow. Often it is convenient to use the above relation such that the direction of 'time' and the direction of the arrows coincide. There are some subtleties involved in bending the leg of a tensor in such a way that it points backwards, see below.

Normalization of basis vectors is important. We can choose a normalization such that some particular operation becomes easier to express. A commonly used normalization convention is known as 'isotopy invariance', which means that the value of a diagram is invariant under continuous deformations, as long as end points are fixed and lines are not passed through each other or around open end points. Note however there is a subtlety that we cannot bend Now we want to connect our diagrams to physical states in Hilbert space. This is a useful normalization when considering an operator point of view, because then the matrix elements of the identity operator are 1. This means that composing operators is very simple, we just join the corresponding lines of the tensor diagrams and there are no additional normalization factors to consider. However, a consequence of this is that the trace operation has a normalization factor, $Tr a = d_a$, where d_a is the dimension of the representation. This corresponds to a common normalization convention in $SU(2)$; it is common to consider a spin j as a 'reduced ket', $\|j\rangle$ which stands for a vector of $2j+1$ basis vectors (ie, a mixed state)

$$\|j\rangle \equiv \begin{pmatrix} |j;-j\rangle \\ |j;-j+1\rangle \\ \vdots \\ |j;j\rangle \end{pmatrix} \tag{8}$$

in which case we have $\langle j\|j\rangle = 2j+1 = d_j$. This is the normalization convention that us used by Bonderson [2], and we will use it for most of these notes. However it might be computationally more convenient to use an alternative normalization convention whereby $\langle j\|j\rangle = 1$. Ref [6] calls this 'implicit' normalization, and uses a mix of both conventions in numerical calculations. The Matrix Product Toolkit uses the isotopy convention, which is certainly more natural for constructing matrix elements. The Matrix Product Toolkit reuses the tensor arithmetic for states and operators, and it would be a major complication to mix normalization conventions. However the implicit normalization has some computational advantages, especially the inner product operation has *no* additional normalization factors, so as far as external algorithms are concerned we can treat anyonic states as ordinary vectors, which simplifies interfacing to numerical algorithms. This will be discussed in more detail later.

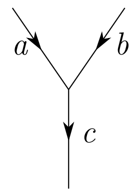
Tensor diagrams have some subleties which are overlooked by some authors. In particular, although we can freely bend tensor legs to the left and right, we *cannot* freely bend them up and down. This is discussed in Bonderson [2], also by Pfeifer et al[6]. In particular, the following two diagrams are *not* equivaent,



(9)

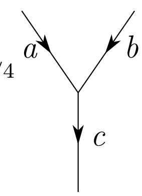
and we will see below that they are related by a phase factor.

Fusion of two quantum numbers looks like



$$(10)$$

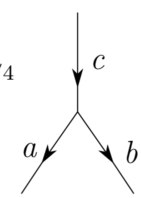
where we fuse two incoming states a, b into an outgoing state c. Isotopy invariance means that we need a normalization factor to relate this to basis vectors,

$$\left(\frac{d_c}{d_a d_b}\right)^{1/4}$$


$$= |c\rangle\langle a|\langle b| \qquad (11)$$

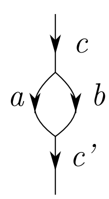
In alternate notation, we can express the right hand side as $|a, b; c\rangle$.

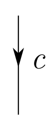
We can turn this diagram around, to represent splitting the charge c into a and b,

$$\left(\frac{d_c}{d_a d_b}\right)^{1/4}$$


$$= |a\rangle|b\rangle\langle c| \qquad (12)$$

Putting these together, we can form the inner product,



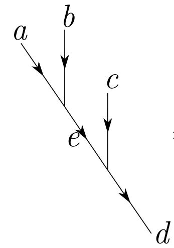
$$= \delta_{c,c'}\sqrt{\frac{d_a d_b}{d_c}}$$


$$(13)$$

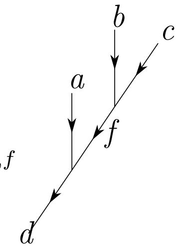
This is the basic operation that we use when contracting tensor networks.

When fusing three particles a, b, c together, there is an ambiguity as to whether we fuse a, b first and then c, or fuse a with the fusion of b, c. The difference is the intermediate quantum number, here denoted e or f. The difference between these forms of fusion is given by the

F symbol.



$$= \sum_f \left[F_d^{a,b,c}\right]_{e,f} \tag{14}$$



Alternatively, we can write, in Bonderson's notation.

$$|\,a,b;e\rangle|\,e,c;d\rangle = \sum_f \left[F_d^{a,b,c}\right]_{e,f} |\,b,c;f\rangle|\,a,f;d\rangle \tag{15}$$

Or as

$$|\,((abe)c)d\rangle = \sum_f \left[F_d^{a,b,c}\right]_{e,f} |\,(a(bcf))d\rangle \tag{16}$$

or as

$$\left[F_j^{j_1,j_2,j_3}\right]_{j_{12},j_{23}} = \langle (j_1 j_2 j_{12}) j_3 j \| j_1 (j_2 j_3 j_{23}) j \rangle \tag{17}$$

For given exterior legs a, b, c, d we can view the F-symbol as the basis transformation that maps between two different choices of internal labelling.

Note that the left-right ordering of spins a, b, c, d is important here. This is essentially a normal ordering of particles (or operators). We haven't said anything so far about how we re-order particles. The F-symbols have some symmetries under interchange of charges, but that isn't enough, by itself, to determine particle statistics. For that, we need the R matrix, which is discussed below.

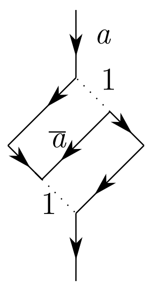
Treating $\left[F_d^{a,b,c}\right]_{e,f}$ as a matrix with respect to (e, f) charges, it is unitary:

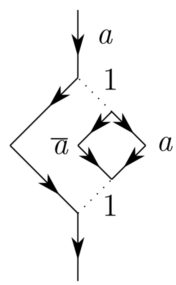
$$\left[\left(F_d^{abc}\right)^\dagger\right]_{fe} = \left[\left(F_d^{abc}\right)^*\right]_{ef} = \left[\left(F_d^{abc}\right)^{-1}\right]_{fe} \tag{18}$$

Consistency of F-moves is given by the Pentagon relation, which we can express as

$$\left[F_e^{f,c,d}\right]_{g,l} \left[F_e^{a,b,l}\right]_{f,k} = \sum_h \left[F_g^{a,b,c}\right]_{f,g} \left[F_e^{a,h,d}\right]_{g,k} \left[F_k^{b,c,d}\right]_{h,l} \tag{19}$$

We are now in a position to examine further the diagrams from Eq. (9). Firstly, we flip the arrows so that we are consistent with the direction of time. Now the bends in the curve look like fusion and splitting of anyons a and $\bar{a}$ to and from the vacuum,



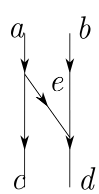
$= [F_a^{a,\bar{a},a}]_{1,1}$


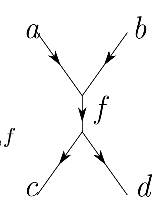
$= d_a\, [F_a^{a,\bar{a},a}]_{1,1}$


(20)

The quantity $d_a\, [F_a^{a,\bar{a},a}]_{1,1} = \chi_a$ is a non-trivial phase which is known as the Frobenius-Schur indicator. For charges that are not self-dual, it is possible to fix a gauge such that $\chi_a = 1$. However if a is self-dual then $\chi_a = \pm 1$ is a gauge-invariant quantity and we need to keep track of diagrams that bend in the fashion of Eq. (20). Up to rank 4, only two modular tensor categories have particles with Frobenius-Schur indicator -1, namely the semion and the σ particle in $(A_1, 2)$ (this has the same S-matrix as Ising anyons[4]; the same fusion rules have two different possible braid representations, with ± 1 Frobenius-Schur indicator). In $SU(2)$, half-integer spins also have -1 Frobenius-Schur indicator. For STC, the Frobenius-Schur indicator determines whether the representation is orthogonal or symplectic. It is essentially the same quantity as the $2j$ phase (see below). In *most* cases where we want to deform lines, it turns out that the phase factors cancel so we don't need to incorporate χ_a, however we should keep it in mind.

The F-symbol that we have used so far is known as a 3-1 symbol, since it represents the fusion (or splitting) of 3 particles into 1. There is also a 2-2 F-move, which corresponds to the change in labelling



$= \sum_f \left[F_{c,d}^{a,b}\right]_{e,f}$


(21)

Note it is important here that both e and f legs are pointing in the same direction.

5 Symmetries of 6j symbols

See Butler page 59

6 Braiding

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